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An economical finite element approximation of generalized Newtonian flows

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Abstract

We consider an economical bilinear rectangular mixed finite element scheme on regular mesh for generalized Newtonian flows, where the viscosity obeys a Carreau type law for a pseudo-plastic. The key issue in the scheme is that the two components of the velocity and the pressure are defined on different meshes. Optimal error bounds for both the velocity and pressure are obtained by proving a discrete Babuška–Brezzi inf–sup condition on the regular quadrangulation. Finally, we perform some numerical experiments, including an example in a unit square with exact solutions, a backward-facing step and a four-to-one abrupt contraction generalized Newtonian flows. Numerical experiments confirm our error bounds. © 2002 Published by Elsevier Science B.V.

Keywords: Economical finite element; Error bound; Generalized Newtonian flow; Carreau law

1. Introduction

Let Ω be a rectangular domain or L-shaped domain in the plane with a boundary $\Gamma = \partial\Omega$. Consider the following generalized Newtonian flow problem:

($\mathscr{P}$) Find $(\mathbf{u}, p)$ such that

$$-\sum_{j=1}^{2} \frac{\partial}{\partial x_j}[\mu(|D(\mathbf{u})|)D_{ij}(\mathbf{u})] + \frac{\partial p}{\partial x_i} = f_i, \quad \text{in } \Omega, \ i = 1,2, \tag{1.1}$$

$$\operatorname{div} \mathbf{u} = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \Gamma, \quad \text{and} \quad \int_\Omega p \, d\mathbf{x} = 0; \tag{1.3}$$

where $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ is the velocity, p is the pressure, $\mathbf{f} = (f_1, f_2)^{\mathrm{T}}$ is the applied body force and $D(\mathbf{u})$ is the rate of deformation tensor with entries

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$$D_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2 \tag{1.4}$$

and

$$|D(\mathbf{u})|^2 = \sum_{i,j=1}^{2} [D_{ij}(\mathbf{u})]^2. \tag{1.5}$$

We will assume throughout for ease of exposition that the viscosity μ satisfies the following assumption:

(A) $\mu \in C^1[0,\infty)$ and that there exist positive constants C_μ and M_μ such that

$$M_\mu(t-s) \leqslant \mu(t)t - \mu(s)s \leqslant C_\mu(t-s), \quad \forall t \geqslant s \geqslant 0. \tag{1.6}$$

For example the Carreau law

$$\mu(t) = \mu_\infty + (\mu_0 - \mu_\infty)(1 + (\lambda t)^2)^{(n-1)/2}, \tag{1.7}$$

where $\lambda > 0$ is a constant, μ_∞ and μ_0 are the shear viscosities at infinity and zero shear rate, respectively ($0 < \mu_\infty < \mu_0$), n is the power-law index ($0 < n \leqslant 1$). Not that $n = 1$ refers to Newtonian fluids. The inequality (1.6) is satisfied with $C_\mu = \mu_0$ and $M_\mu = \mu_\infty$.

During recent years, many authors have proposed and analyzed the finite element approximation of generalized Newtonian flow problems. For error bounds of using conforming elements, see Refs. [3–5,12, 20]. For error bounds of using nonconforming linear elements, see Refs. [1,2].

Due to the restricted regularity of solutions of generalized Newtonian flow problems, low order finite element method is often used in practice. When the low-order finite element method is applied to generalized Newtonian flow problems, Stokes or Navier–Stokes equations, some special treatments are usually needed in order to keep the scheme stable. The key issue for low order finite element approximation is how to design the finite element subspaces for the velocity and pressure such that the Babuška–Brezzi (BB) inf–sup condition is satisfied. This topic has attracted the attention of many authors [6,7,11,15]. There are two typical ways to design low order finite element approximation. One is designing nonconforming linear-constant velocity–pressure element. For example, Crouzeix and Raviart [10] proposed a nonconforming linear element which has been proven very effective for the Stokes and Navier–Stokes equations. Unfortunately, Falk and Morley [13] have shown that this simple nonconforming element does not satisfy a discrete Korn inequality and therefore cannot be used with confidence to approximate the generalized Newtonian flow problem ($\mathscr{P}$). Recently Kouhia and Stenberg [18] constructed another linear nonconforming element for nearly incompressible linear elasticity and Stokes flow in two dimensions. This element consists of a conforming linear approximation for one velocity component, a nonconforming linear approximation for the other component with a piecewise constant approximation for the pressure. Bao and Barrett [1,2] used the element to solve the generalized Newtonian flow problem ($\mathscr{P}$) with the viscosity obeys a general law including the Carreau law or power law by proving a generalized BB condition and a generalized discrete Korn inequality. Optimal error estimates were proven. Another way is to construct conforming bilinear-constant velocity–pressure element by using different quadrangulations. Han [16,17] presented a bilinear-constant velocity–pressure finite element for the Stokes equations by using three different quadrangulations for constructing three finite dimensional subspaces of the pressure p, the velocity components u_1 and u_2 on a rectangular physical domain with uniform mesh. This finite element approximation is called economical element in the sense that the degree of freedom is highly reduced. In [16,17] optimal error bounds, $\mathrm{O}(h)$, are proved for the velocity approximation in $H^1(\Omega)$ norm and the pressure approximation in $L^2(\Omega)$ norm, on assuming that $u \in [H^2(\Omega)]^2$ and $p \in H^1(\Omega)$ for Stokes equations. In this paper we extend these results to the generalized Newtonian flow problem $(\mathscr{P})$ on a rectangular domain or L-shaped domain with regular quadrangulation. Optimal error bounds for both the velocity and pressure are obtained by proving a discrete BB inf–sup condition on the regular quadrangulation. In addition we apply this economical finite element to simulate a generalized Newtonian flow problem on a unit square with exact solution, which is used to test our error bounds. Furthermore we apply it to simulate a backward-facing step generalized Newtonian flow and a four-to-one abrupt contraction generalized Newtonian flow. We note that more general assumptions than (1.6) on the viscosity, which allow for singular/degenerate power laws as in [5], lead to a number of technical difficulties with this economical element. These will be addressed elsewhere.

The layout of this paper is as follows. In Section 2 we introduce our finite element approximation for the generalized Newtonian flow problem $(\mathscr{P})$. In Section 3 we establish error bounds for the finite element approximation. Finally in Section 4 we report on some numerical experiments. Throughout we adopt the standard notation for Sobolev spaces. For any open bounded set G of $\mathbb{R}^2$, with Lipschitz boundary ∂G, we denote the norm and standard semi-norm of $W^{m,v}(G)$ for any nonnegative integer m and $v \in [1,\infty]$ by $\|\cdot\|_{m,v,G}$ and $|\cdot|_{m,v,G}$, respectively. For $v = 2$ we adopt the standard notation $H^m(G) \equiv W^{m,2}(G)$, $\|\cdot\|_{m,2,G} \equiv \|\cdot\|_{m,G}$ and $|\cdot|_{m,2,G} \equiv |\cdot|_{m,G}$. We adopt similar notation for the product spaces $[W^{m,v}(G)]^2 \equiv W^{m,v}(G) \times W^{m,v}(G)$ and the trace space $W^{m,v}(\gamma)$, where $\gamma \subseteq \partial G$. Finally C and C_i denote positive generic constants independent of the mesh size h.

2. An economical finite element approximation

We introduce the following spaces:

$$X = H_0^1(\Omega) \times H_0^1(\Omega) \quad \text{with norm } \|\mathbf{v}\|_X = \sqrt{|v_1|_{1,\Omega}^2 + |v_2|_{1,\Omega}^2},$$

where $\mathbf{v} = (v_1, v_2)^{\mathrm{T}}$, and

$$M = \left\{ q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}\mathbf{x} = 0 \right\} \quad \text{with norm } \|q\|_M = \|q\|_{0,\Omega}.$$

Then the weak formulation of problem $(\mathscr{P})$ is:

$(\mathscr{L})$ Find $(\mathbf{u}, p) \in X \times M$ such that

$$A(\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) = f(\mathbf{v}), \quad \forall \mathbf{v} \in X, \tag{2.1}$$

$$B(\mathbf{u},q) = 0, \quad \forall q \in M; \tag{2.2}$$

where

$$A(\mathbf{w},\mathbf{v}) = \int_\Omega \mu(|D(\mathbf{w})|) D(\mathbf{w}) : D(\mathbf{v}) \, \mathrm{d}\mathbf{x}, \tag{2.3}$$

$$B(\mathbf{v},q) = -\int_\Omega q \, \mathrm{div}\, \mathbf{v} \, \mathrm{d}\mathbf{x}, \qquad f(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \tag{2.4}$$

and

$$D(\mathbf{w}) : D(\mathbf{v}) = \sum_{i,j=1}^{2} D_{ij}(\mathbf{w}) D_{ij}(\mathbf{v}).$$

When the viscosity μ satisfies (1.6) it is easy to show, see [5, Lemma 2.1], that there exist two positive constants C_1 and C_2 such that for all 2×2 real symmetric matrices Y and Z

$$|\mu(|Y|)Y - \mu(|Z|)Z| \leqslant C_1|Y-Z|, \tag{2.5}$$

$$C_2|Y-Z|^2 \leqslant (\mu(|Y|)Y - \mu(|Z|)Z) : (Y-Z). \tag{2.6}$$

Here and throughout $|\cdot|$ is the Euclidean norm as in (1.5) above. Noting the Korn inequality

$$C\|\mathbf{v}\|_X^2 \leqslant \int_\Omega |D(\mathbf{v})|^2\,\mathrm{d}\mathbf{x} \leqslant \|\mathbf{v}\|_X^2, \quad \forall v\in X, \tag{2.7}$$

see for example [19]; it follows from (2.3), (2.5) and (2.6) that there exist two positive constants C_3 and C_4 such that

$$|A(\mathbf{w},\mathbf{v}) - A(\mathbf{z},\mathbf{v})| \leqslant C_3\|\mathbf{w}-\mathbf{z}\|_X\|\mathbf{v}\|_X, \quad \forall \mathbf{w},\mathbf{z},\mathbf{v}\in X, \tag{2.8}$$

$$C_4\|\mathbf{w}-\mathbf{v}\|_X^2 \leqslant A(\mathbf{w},\mathbf{w}-\mathbf{v}) - A(\mathbf{v},\mathbf{w}-\mathbf{v}), \quad \forall \mathbf{w},\mathbf{v}\in X. \tag{2.9}$$

Furthermore it is easy to show that the bilinear form $B(\cdot,\cdot)$ is bounded on $X\times M$ and satisfies the BB inf–sup condition, see [15, p. 81], i.e. there exist positive constants C_5 and C_6 such that

$$|B(\mathbf{v},q)| \leqslant C_5\|\mathbf{v}\|_X\|q\|_M, \quad \forall \mathbf{v}\in X, \ \forall q\in M, \tag{2.10}$$

$$C_6\|q\|_M \leqslant \sup_{\mathbf{v}\in X\setminus\{0\}} \frac{B(\mathbf{v},q)}{\|\mathbf{v}\|_X}, \quad \forall q\in M. \tag{2.11}$$

It follows immediately from (2.8)–(2.11) that the problem $(\mathscr{L})$ is well-posed; that is, for all $\mathbf{f}\in X'$, the dual of X, there exists a unique solution $(\mathbf{u},p)\in X\times M$ solving $(\mathscr{L})$ and

$$\|\mathbf{u}\|_X + \|p\|_M \leqslant C\|\mathbf{f}\|_{X'}. \tag{2.12}$$

We now use the economical conforming bilinear-constant velocity–pressure finite element of Han and Wu [17] to discretize the problem $(\mathscr{L})$. For ease of exposition, we assume that Ω is a rectangular domain, i.e. $\Omega = (a,b)\times(c,d)$. The finite element method discussed below can be easily generated to the case in which the domain Ω is L-shaped domain. Let M and N be two positive integers. First we divide Ω into MN rectangles (see Fig. 1(a))

$$T_{ij} = \{(x_1,x_2)\colon x_1^{(i-1)} \leqslant x_1 \leqslant x_1^{(i)},\ x_2^{(j-1)} \leqslant x_2 \leqslant x_2^{(j)}\}, \quad 1\leqslant i\leqslant M,\ 1\leqslant j\leqslant N,$$

where $a = x_1^{(0)} < x_1^{(1)} < \cdots < x_1^{(M)} = b$ is a partition of $[a,b]$, and $c = x_2^{(0)} < x_2^{(1)} < \cdots < x_2^{(N)} = d$ is a partition of $[c,d]$. The corresponding quadrangulation is denoted by $\mathscr{T}^h$. Let $h_1^{(i)} = x_1^{(i)} - x_1^{(i-1)}$, $i=1,2,\ldots,M$,

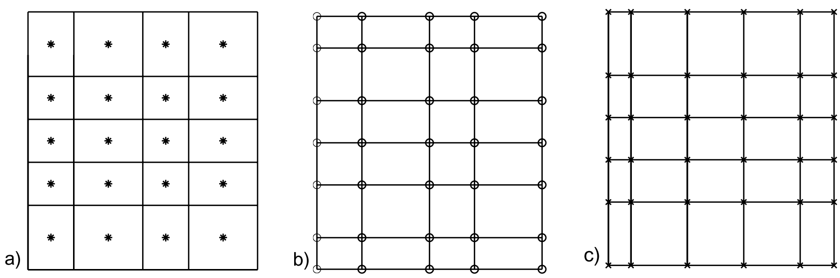

Fig. 1. Quadrangulations of Ω: (a) $\mathscr{T}^h$, '$*$'—nodes for pressure p; (b) $\mathscr{T}_1^h$, '○'—nodes for first component of velocity u_1 and (c) $\mathscr{T}_2^h$, '$\times$'—nodes for second component of velocity u_2.

$h_2^{(j)} = x_2^{(j)} - x_2^{(j-1)}$, $j = 1, 2, \ldots, N$ and $h = h_{\max}$ and $h_{\min}$ be the maximum value and minimum value of the set $\{h_1^{(1)}, h_1^{(2)}, \ldots, h_1^{(M)}, h_2^{(1)}, h_2^{(2)}, \ldots, h_2^{(N)}\}$, respectively. We say that $\mathscr{T}^h$ is a regular quadrangulation of Ω if the following assumption is true [8]:

(B) There exists a constant $\sigma_0 > 0$ independent of h such that $h_{\max}/h_{\min} \leqslant \sigma_0$.

Then for each $T_{ij} \in \mathscr{T}^h$ we connect the two midpoints of the two vertical sides of T_{ij} by a straight horizontal line segment. Then Ω is divided into $M(N+1)$ rectangles (see Fig. 1(b)), the corresponding quadrangulation is denoted by $\mathscr{T}_1^h$. Similarly, for each $T_{ij} \in \mathscr{T}^h$ we connect the two midpoints of the two horizontal sides of T_{ij} by a straight vertical line segment. Then Ω is divided into $(M+1)N$ rectangles (see Fig. 1(c)), the corresponding quadrangulation is denoted by $\mathscr{T}_2^h$.

Corresponding to the quadrangulation $\mathscr{T}^h$, let

$$M^h = \left\{ q^h \colon q^h|_T = \text{constant}, \ \forall T \in \mathscr{T}^h \text{ and } \int_\Omega q^h \, \mathrm{d}\mathbf{x} = 0 \right\}.$$

Then M^h is a subspace of M. Furthermore using the quadrangulation $\mathscr{T}_1^h$ and $\mathscr{T}_2^h$, we construct two subspaces of $H_0^1(\Omega)$. Set

$$S_1^h = \left\{ v^h \in C^{(0)}(\bar{\Omega}) \colon v^h|_{T_1} \in Q_1(T_1), \ \forall T_1 \in \mathscr{T}_1^h, \text{ and } v^h|_\Gamma = 0 \right\},$$

$$S_2^h = \left\{ v^h \in C^{(0)}(\bar{\Omega}) \colon v^h|_{T_2} \in Q_1(T_2), \ \forall T_2 \in \mathscr{T}_2^h, \text{ and } v^h|_\Gamma = 0 \right\},$$

where Q_1 denotes the space of all polynomials of degree $\leqslant 1$ with respect to each of the two variables x_1 and x_2. Let $X^h = S_1^h \times S_2^h$, obviously X^h is a conforming subspace of X. Then the standard approximation theory, see e.g. [8, p. 123], yields for $v \in (1, 2]$ that

$$\inf_{\mathbf{v}^h \in X^h} \|\mathbf{u} - \mathbf{v}^h\|_X \leqslant Ch^{2-(2/v)} |\mathbf{u}|_{2,v,\Omega}, \qquad \inf_{q^h \in M^h} \|p - q^h\|_M \leqslant Ch^{2-(2/v)} |p|_{1,v,\Omega}, \tag{2.13}$$

where the unique solution $(\mathbf{u}, p)$ of $(\mathscr{L})$ is assumed to be such that $\mathbf{u} \in [W^{2,v}(\Omega)]^2$ and $p \in W^{1,v}(\Omega)$. We note that such regularity has been proved for $v = 2$ if $\mathbf{f} \in [L^2(\Omega)]^2$ and $\Gamma \in C^2$, see [14]. See also [14] for some regularity results in the case when Ω is generalized convex.

Then the corresponding finite element approximation of problem $(\mathscr{L})$ is:

$(\mathscr{L}^h)$ Find $(\mathbf{u}^h, p^h) \in X^h \times M^h$ such that

$$A(\mathbf{u}^h, \mathbf{v}^h) + B(\mathbf{v}^h, p^h) = f(\mathbf{v}^h), \quad \forall \mathbf{v}^h \in X^h, \tag{2.14}$$

$$B(\mathbf{u}^h, q^h) = 0, \quad \forall q^h \in M^h. \tag{2.15}$$

3. Error estimates

In order to get error estimates of the finite element approximation $(\mathscr{L}^h)$, we first prove the bilinear form $B(\cdot,\cdot)$ satisfies the discrete BB condition on $X^h \times M^h$, i.e. there exists a positive constant β_0 independent of h such that

$$C\|q^h\|_M \leqslant \sup_{\mathbf{v}^h \in X^h \setminus \{0\}} \frac{B(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_X}, \quad \forall q^h \in M^h. \tag{3.1}$$

The proof is an extension of that in [17] from a uniform mesh $\mathscr{T}^h$ to a regular mesh $\mathscr{T}^h$. For the quadrangulation $\mathscr{T}^h$, we divided the edges of all rectangles into two sets, the set containing all vertical edges is denoted by L_V, the set containing all horizontal edges is denoted by L_H. We define the operator $I^h : X \to X^h$ by $I^h\mathbf{v} = (I_1^h v_1, I_2^h v_2) \in S_1^h \times S_2^h$ satisfying

$$\int_l I_1^h v_1 \, dx_2 = \int_l v_1 \, dx_2, \quad \forall l \in L_V, \qquad \int_l I_2^h v_2 \, dx_1 = \int_l v_2 \, dx_1, \quad \forall l \in L_H. \tag{3.2}$$

It is straightforward to check that for any $\mathbf{v} \in X$, $I^h\mathbf{v} \in X^h$ is uniquely determined by (3.2).

Lemma 3.1. *Suppose that the quadrangulation $\mathscr{T}^h$ is regular, i.e. assumption (B) holds, then the following results hold*:

(*i*) *For any $\mathbf{v} \in X$, we have*

$$B(\mathbf{v}, q^h) = B(I^h\mathbf{v}, q^h), \quad \forall q^h \in M^h. \tag{3.3}$$

(*ii*) *There is a constant C independent of* $\mathbf{v}$ *and mesh size h, such that*

$$\|I^h\mathbf{v}\|_{1,\Omega} \leqslant C\|\mathbf{v}\|_{1,\Omega}, \quad \forall \mathbf{v} \in X. \tag{3.4}$$

Proof. (i) The equality (3.3) is a combination of (2.4) and (3.2) by using integration by parts and noting that q^h is piecewise constant.

(ii) Let $\mathbf{w}^h \in X^h$ be the standard generalized interpolant of $\mathbf{v} \in X$, see [8,9]. We set $\mathbf{e}^h = I^h\mathbf{v} - \mathbf{w}^h$, $\mathbf{e} = \mathbf{v} - \mathbf{w}^h$, then

$$\|I^h\mathbf{v}\|_X \leqslant \|\mathbf{w}^h\|_X + \|\mathbf{e}^h\|_X \leqslant C_6\|\mathbf{v}\|_X + \|\mathbf{e}^h\|_X. \tag{3.5}$$

Let $\Omega_j = \{(x_1, x_2)\colon a \leqslant x_1 \leqslant b,\ x_2^{(j-1)} \leqslant x_2 \leqslant x_2^{(j)}\}$ be a subdomain of Ω, and $T_{ij}^2 \subset \Omega_j$, $i = 0, 1, 2, \ldots, M$ denote the rectangles in $\mathscr{T}_2^h$, see Fig. 2, then for the second component of $\mathbf{e}^h$ we have on noting that e_2^h is bilinear polynomial on each T_{ij}^2

$$|e_2^h|_{1,\Omega_j}^2 = \sum_{i=0}^{M} \int_{T_{ij}^2} \left[\left(\frac{\partial e_2^h}{\partial x_1}\right)^2 + \left(\frac{\partial e_2^h}{\partial x_2}\right)^2\right] d\mathbf{x} = (\widetilde{E}_j)^{\mathrm{T}} A_j (\widetilde{E}_j), \tag{3.6}$$

where $\widetilde{E}_j$ denotes the vector whose components are the node values of e_2^h in the domain Ω_j and A_j is a symmetric matrix. A direct computation shows that

$$\rho(A_j) \leqslant \frac{8}{3\sigma_0}, \tag{3.7}$$

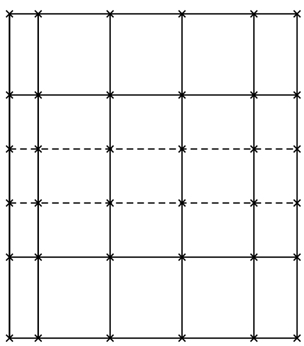

Fig. 2. Subdomain Ω_j—bounded by the dash lines.

where $\rho(A)$ denotes the spectral radius of a matrix A. Decomposing A_j as $A_j = P_j^{\mathrm{T}} P_j$, (3.6) implies

$$|e_2^h|_{1,\Omega_j}^2 = \|P_j \widetilde{E}_j\|_2^2. \tag{3.8}$$

By using the fact that

$$\int_l e_2^h \, \mathrm{d}x_1 = \int_l e_2 \, \mathrm{d}x_1, \quad \forall\, l \in L_{\mathrm{H}}, \tag{3.9}$$

we get

$$B_j \widetilde{E}_j = E_j, \tag{3.10}$$

where B_j is a symmetric positive definite matrix and E_j is a vector whose components are the integrals of the right-hand side of (3.9) divided by h_l with h_l the length of the segment l. A direct computation shows that

$$\rho(B_j^{-1}) \leqslant \frac{2}{\sigma_0}. \tag{3.11}$$

Combining (3.7), (3.8), (3.10) and (3.11), one gets

$$\begin{aligned} |e_2^h|_{1,\Omega_j}^2 &= \|P_j B_j^{-1} E_j\|_2^2 \leqslant \|P_j\|_2^2 \|B_j^{-1}\|_2^2 \|E_j\|_2^2 = \rho(P_j^{\mathrm{T}} P_j)\rho((B_j^{-1})^{\mathrm{T}} B_j^{-1})\|E_j\|_2^2 \\ &= \rho(A_j)\rho(B_j^{-1})^2 \|E_j\|_2^2 \leqslant \frac{8}{3\sigma_0}\frac{4}{\sigma_0^2}\|E_j\|_2^2 = \frac{32}{3\sigma_0^3}\|E_j\|_2^2. \end{aligned} \tag{3.12}$$

On the other hand, using the Hölder's inequality, trace theorem and interpolation error between $\mathbf{w}^h$ and $\mathbf{v}$, one gets

$$\begin{aligned} \|E_j\|_2^2 &= \sum_{l \in L_{\mathrm{H}} \cap \Omega_j} \frac{1}{h_l^2}\left[\int_l e_2 \, \mathrm{d}x_1\right]^2 \leqslant \sum_{l \in L_{\mathrm{H}} \cap \Omega_j} \frac{1}{h_{\min}} \int_l (e_2)^2 \, \mathrm{d}x_1 \leqslant C_7 \sum_{i=0}^{M} \left(h_{\min}^{-2}\|e_2\|_{0,T_{ij}^2}^2 + |e_2|_{1,T_{ij}^2}^2\right) \\ &\leqslant C_7 \sum_{i=0}^{M}\left(\sigma_0^2 |v_2|_{1,T_{ij}^2}^2 + |v_2|_{1,T_{ij}^2}^2\right) \leqslant C|v_2|_{1,\Omega_j}^2. \end{aligned} \tag{3.13}$$

Summing (3.12) for $j = 1, 2, \ldots, N$, noting (3.13), one obtains

$$|e_2^h|_{1,\Omega}^2 \leqslant C|v_2|_{1,\Omega}^2. \tag{3.14}$$

Similarly, for the first component, we have

$$|e_1^h|_{1,\Omega}^2 \leqslant C|v_1|_{1,\Omega}^2. \tag{3.15}$$

The required inequality (3.4) is a combination of (3.5), (3.14) and (3.15). □

The inf–sup condition (3.1) can be obtained from Lemma 3.1 and (2.11), see detail in Ref. [15]. It follows immediately from (2.8)–(2.10) and (3.1) that the problem $(\mathscr{L}^h)$ is well-posed; that is, for all $\mathbf{f} \in [L^2(\Omega)]^2$, there exists a unique solution $(\mathbf{u}^h, p^h) \in X^h \times M^h$ solving $(\mathscr{L}^h)$ and

$$\|\mathbf{u}^h\|_X + \|p^h\|_M \leqslant C[\|f_1\|_{0,\Omega} + \|f_2\|_{0,\Omega}]. \tag{3.16}$$

Theorem 3.1. *Let $(\mathbf{u}, p)$ be the unique solution of problem $(\mathscr{L})$ and $(\mathbf{u}^h, p^h)$ be the unique solution of problem $(\mathscr{L}^h)$. Then we have the following abstract error bound*:

$$\|\mathbf{u} - \mathbf{u}^h\|_X + \|p - p^h\|_M \leqslant Ch^{2-(2/v)}[|\mathbf{u}|_{2,v,\Omega} + |p|_{1,v,\Omega}]. \tag{3.17}$$

Proof. Let

$$\widetilde{X}^h := \{\mathbf{v}^h \in X^h\colon B(\mathbf{v}^h, q^h) = 0,\ \forall q^h \in M^h\}. \tag{3.18}$$

For any $\mathbf{v}^h \in \widetilde{X}^h$ and $q^h \in M^h$, let

$$\begin{aligned} &\mathbf{e} := \mathbf{u} - \mathbf{u}^h, \qquad \mathbf{e}^a := \mathbf{u} - \mathbf{v}^h, \qquad \mathbf{e}^h := \mathbf{v}^h - \mathbf{u}^h, \\ &\xi := p - p^h, \qquad \xi^a := p - q^h, \qquad \xi^h := q^h - p^h. \end{aligned} \tag{3.19}$$

Subtracting (2.14) and (2.15) from (2.1) and (2.2) with $\mathbf{v} = \mathbf{v}^h$, respectively, we obtain

$$A(\mathbf{u}, \mathbf{v}^h) - A(\mathbf{u}^h, \mathbf{v}^h) + B(\mathbf{v}^h, p - p^h) = 0, \quad \forall \mathbf{v}^h \in X^h, \tag{3.20}$$

$$B(\mathbf{u} - \mathbf{u}^h, q^h) = 0, \quad \forall q^h \in M^h. \tag{3.21}$$

Then from (2.8)–(2.10), (3.19) and (3.20), noting the triangle inequality, we have that

$$\begin{aligned} C_4\|\mathbf{e}\|_X^2 &\leqslant A(\mathbf{u}, \mathbf{u} - \mathbf{u}^h) - A(\mathbf{u}^h, \mathbf{u} - \mathbf{u}^h) \\ &= A(\mathbf{u}, \mathbf{e}^a) - A(\mathbf{u}^h, \mathbf{e}^a) + A(\mathbf{u}, \mathbf{e}^h) - A(\mathbf{u}^h, \mathbf{e}^h) \\ &= A(\mathbf{u}, \mathbf{e}^a) - A(\mathbf{u}^h, \mathbf{e}^a) - B(\mathbf{e}^h, p - p^h) \\ &= A(\mathbf{u}, \mathbf{e}^a) - A(\mathbf{u}^h, \mathbf{e}^a) - B(\mathbf{e}^h, p - q^h) \\ &= A(\mathbf{u}, \mathbf{e}^a) - A(\mathbf{u}^h, \mathbf{e}^a) - B(\mathbf{e}, p - q^h) + B(\mathbf{e}^a, p - q^h) \\ &\leqslant C_8[\|\mathbf{e}\|_X\|\mathbf{e}^a\|_X + \|\mathbf{e}\|_X\|\xi^a\|_M + \|\mathbf{e}^a\|_X\|\xi^a\|_M] \\ &\leqslant C[\|\mathbf{e}^a\|_X + \|\xi^a\|_M]^2. \end{aligned} \tag{3.22}$$

Furthermore following the standard argument, see [15, p. 155], we have from (2.10) and (3.1) that

$$\inf_{\mathbf{v}^h \in \widetilde{X}^h} \|\mathbf{u} - \mathbf{v}^h\|_X \leqslant C \inf_{\mathbf{v}^h \in X^h} \|\mathbf{u} - \mathbf{v}^h\|_X. \tag{3.23}$$

Therefore combining (2.13), (3.22) and (3.23) we obtain the desired result for $\|\mathbf{u} - \mathbf{u}^h\|_X$ in (3.17).

We now estimate $\|p - p^h\|_M$. From (3.20) we have for any $\mathbf{v}^h \in X^h$ and $q^h \in M^h$ on adopting the notation (3.19) and noting (2.8), (2.10) and (3.20) that

$$\begin{aligned} B(\mathbf{v}^h, \xi^h) &= -B(\mathbf{v}^h, \xi^a) + B(\mathbf{v}^h, \xi) \\ &= -B(\mathbf{v}^h, \xi^a) + A(\mathbf{u}^h, \mathbf{v}^h) - A(\mathbf{u}, \mathbf{v}^h) \leqslant C[\|\mathbf{v}^h\|_X\|\xi^a\|_M + \|\mathbf{u} - \mathbf{u}^h\|_X\|\mathbf{v}^h\|_X]. \end{aligned} \tag{3.24}$$

Therefore combining (3.1) and (3.24), using the triangle inequality, we obtain the desired result for $\|p - p^h\|_M$ in (3.17). The error estimate (3.17) is proved completely. □

4. Numerical results

In this section, we use the economical finite element approximation (2.14), (2.15) to solve a few generalized Newtonian flow problems, including an example with exact solution which is used to test the error estimate (3.17), a backward-facing step and a four-to-one abrupt contraction generalized Newtonian flows.

Example 1. A model problem.

We consider the case of the Carreau law, (1.7) with $\lambda = \mu_0 = 1$, $\mu_\infty = 0.5$ and various choices of $n \in (0, 1]$. We set $\Omega = (0, 1) \times (0, 1)$ and choose $\mathbf{f}$, for each choice of n, such that the unique solution of $(\mathscr{P})$ is

$$u_1(\mathbf{x}) = x_1^2(1-x_1)^2x_2(1-x_2)(1-2x_2), \tag{4.1}$$

$$u_2(\mathbf{x}) = -x_1(1-x_1)(1-2x_1)x_2^2(1-x_2)^2, \tag{4.2}$$

$$p(\mathbf{x}) = 4x_1x_2 - 1. \tag{4.3}$$

We computed the finite element approximation $(\mathbf{u}^h, p^h)$, see (2.14) and (2.15). For computational ease, we employed numerical integration on the right hand side of (2.14). We chose a quadrature rule over each rectangle, which integrated a cubic polynomial exactly. It is a simple matter to show that the error estimate (3.17), with $v = 2$, remains valid; as it would do even for a cruder rule. Four meshes were used in our computations. The quadrangulation $\mathscr{T}^h$ in mesh A consisted of four similar rectangles obtained by subdividing Ω using the lines $x_1 = 0.5$, $x_2 = 0.5$. Mesh B was generated by subdividing every rectangle in mesh A into four similar rectangles. Meshes C and D were similarly generated from meshes B and C, respectively. We note that these meshes are regular, in fact they are uniform, i.e. $\sigma_0 = 1$ in assumption (B).

Table 1 shows, the maximum of the error $|\mathbf{u} - \mathbf{u}^h|$ over the mesh points, $\|\mathbf{u} - \mathbf{u}^h\|_{0,\Omega}$, $\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}$ and $\|p - p^h\|_{0,\Omega}$ for $n = 1$, 0.5 and 0.001, respectively. The integral norms were approximated using the same quadrature rule as described above. We see that numerical results confirm our error bound (3.17) with $v = 2$ for this simple model problem. It should be noted that mesh A is very crude; which have only two degree of freedoms for u_1^h and u_2^h, respectively.

Example 2. Backward-facing step generalized Newtonian flow.

As shown in Fig. 3(a), we consider a generalized Newtonian flow in the case of Carreau law, (1.7) with $\lambda = \mu_0 = 1$, $\mu_\infty = 0.5$, in a backward-facing step channel. We take $L = 3.3$, $H = 1.0$, $s = 0.8$, $\mathbf{u}_{\text{in}}(x_2) = (16(x_2 - H/2)(H - x_2), 0)^{\mathrm{T}}$, $\mathbf{u}_{\text{out}}(x_2) = (2x_2(1-x_2), 0)^{\mathrm{T}}$. Fig. 4 shows contours of streamfunction for $n = 1.0$, 0.5 and 0.001.

Table 1
Errors of the finite element approximation in Example 1 for different values of n

	Mesh			
	A	B	C	D
$n = 1.0$				
$\max|\mathbf{u} - \mathbf{u}^h|$	4.464E−4	2.526E−4	9.242E−5	2.367E−5
$\|\mathbf{u} - \mathbf{u}^h\|_{0,\Omega}$	1.172E−3	8.162E−4	2.594E−4	7.144E−5
$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}$	1.531E−2	1.200E−2	6.850E−3	3.637E−3
$\|p - p^h\|_{0,\Omega}$	4.639E−1	2.348E−1	1.177E−1	5.891E−2
$n = 0.5$				
$\max|\mathbf{u} - \mathbf{u}^h|$	4.457E−4	2.524E−4	9.233E−5	2.364E−5
$\|\mathbf{u} - \mathbf{u}^h\|_{0,\Omega}$	1.172E−3	8.164E−4	2.595E−4	7.145E−5
$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}$	1.531E−2	1.200E−2	6.850E−3	3.637E−3
$\|p - p^h\|_{0,\Omega}$	4.639E−1	2.348E−1	1.177E−1	5.891E−2
$n = 0.001$				
$\max|\mathbf{u} - \mathbf{u}^h|$	4.451E−4	2.521E−4	9.224E−5	2.361E−5
$\|\mathbf{u} - \mathbf{u}^h\|_{0,\Omega}$	1.173E−3	8.165E−4	2.595E−4	7.146E−5
$\|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}$	1.531E−2	1.200E−2	6.850E−3	3.637E−3
$\|p - p^h\|_{0,\Omega}$	4.639E−1	2.348E−1	1.177E−1	5.891E−2

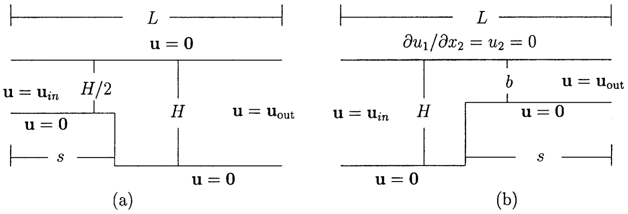

Fig. 3. Setup for (a) Example 2 and (b) Example 3.

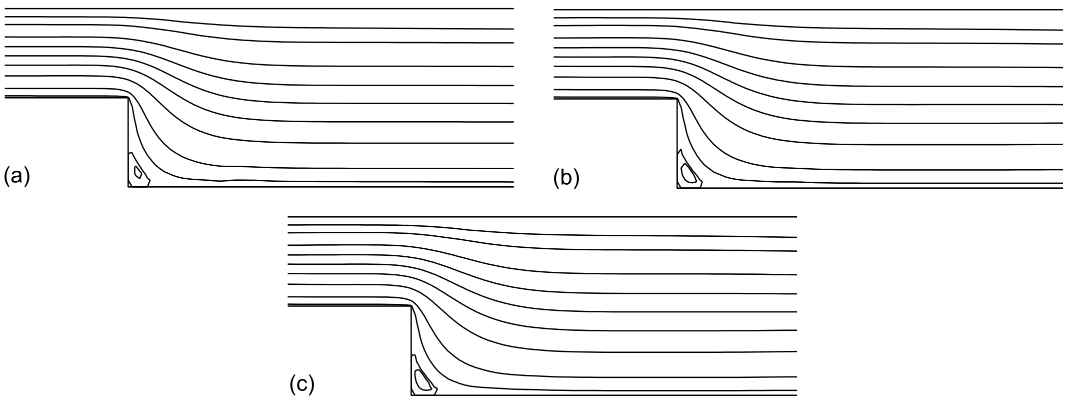

Fig. 4. Contours of streamfunction in Example 2: (a) $n = 1.0$, (b) $n = 0.5$ and (c) $n = 0.001$.

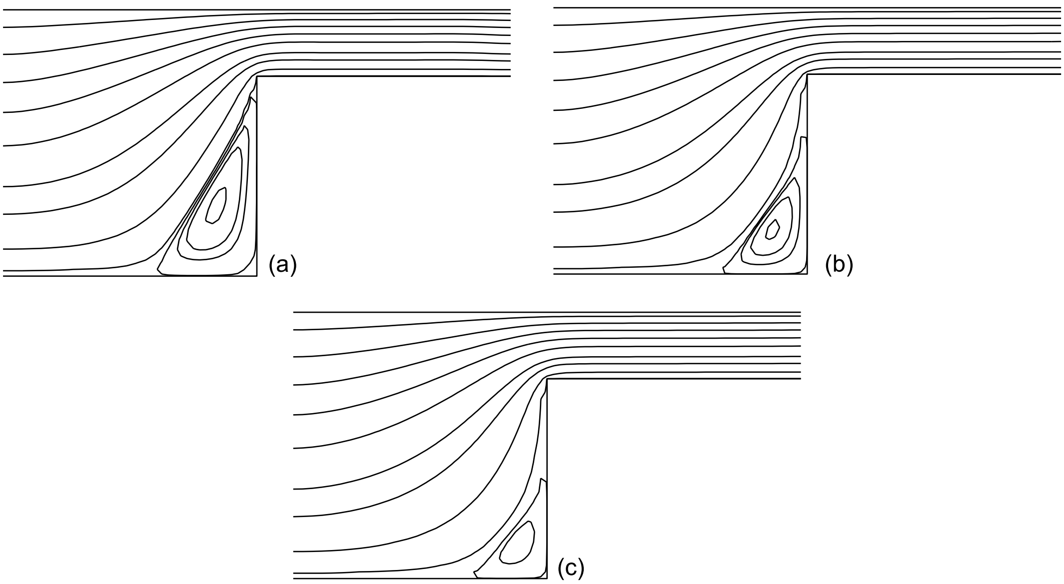

Fig. 5. Contours of streamfunction in Example 3: (a) $n = 2.0$, (b) $n = 1.0$ and (c) $n = 0.001$.

Table 2
Votex length along x-axis versus n

	$n = 2.0$	$n = 1$	$n = 0.5$	$n = 0.001$
Votex length w_x	0.4005	0.3340	0.3087	0.2908

Example 3. A four-to-one abrupt contraction generalized Newtonian flow.

As shown in Fig. 3(b), we consider a generalized Newtonian flow in the case of Carreau law, (1.7) with $\lambda = \mu_0 = 1$, $\mu_\infty = 0.5$, in a four-to-one abrupt contraction channel. We take $L = 2.0$, $H = 1.0$, $s = 1.0$, $b = 0.15$, $\mathbf{u}_{\text{in}}(x_2) = (x_2(2 - x_2), 0)^{\text{T}}$, $\mathbf{u}_{\text{out}}(x_2) = (64(x_2 - 3H/4)(5H/4 - x_2), 0)^{\text{T}}$. Fig. 5 shows contours of streamfunction for $n = 2.0$, 1.0 and 0.001. Furthermore Table 2 shows the votex length along x-axis for different n.

From Figs. 4 and 5, we can see that the economical finite element approximation (2.14), (2.15) can be used to simulate generalized Newtonian flows.

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