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AN EXPONENTIAL WAVE INTEGRATOR SINE PSEUDOSPECTRAL METHOD FOR THE KLEIN–GORDON–ZAKHAROV SYSTEM*

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Abstract. An exponential wave integrator sine pseudospectral method is presented and analyzed for discretizing the Klein–Gordon–Zakharov (KGZ) system with two dimensionless parameters $0 < \varepsilon \leq 1$ and $0 < \gamma \leq 1$ which are inversely proportional to the plasma frequency and the speed of sound, respectively. The main idea in the numerical method is to apply the sine pseudospectral discretization for spatial derivatives followed by using an exponential wave integrator for temporal derivatives in phase space. The method is explicit, symmetric in time, and it is of spectral accuracy in space and second-order accuracy in time for any fixed $\varepsilon = \varepsilon_0$ and $\gamma = \gamma_0$. In the $\mathcal{O}(1)$-plasma frequency and speed of sound regime, i.e., $\varepsilon = \mathcal{O}(1)$ and $\gamma = \mathcal{O}(1)$, we establish rigorously error estimates for the numerical method in the energy space $H^1 \times L^2$. We also study numerically the resolution of the method in the simultaneous high-plasma-frequency and subsonic limit regime, i.e., $(\varepsilon, \gamma) \to 0$ under $\varepsilon \lesssim \gamma$. In fact, in this singular limit regime, the solution of the KGZ system is highly oscillating in time, i.e., there are propagating waves with wavelength of $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(1)$ in time and space, respectively. Our extensive numerical results suggest that, in order to compute "correct" solutions in the simultaneous high-plasma-frequency and subsonic limit regime, the meshing strategy (or ε-scalability) is time step $\tau = \mathcal{O}(\varepsilon^2)$ and mesh size $h = \mathcal{O}(1)$ independent of ε. Finally, we also observe numerically that the method has the property of near conservation of the energy over long time in practical computations.

Key words. Klein–Gordon–Zakharov system, high-plasma-frequency limit, subsonic limit, exponential wave integrator, sine pseudospectral method, error bounds, ε-scalability

AMS subject classifications. 35L70, 65N12, 65N15, 65N35

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1. Introduction.

The Klein–Gordon–Zakharov (KGZ) system is a classical model describing the interaction between the Langmuir waves and ion acoustic waves in a plasma; see, e.g., [8, 13] for its physical relevance. Under a proper nondimensionalization [25], the dimensionless KGZ system in d dimensions ($d = 1, 2, 3$) reads

$$\varepsilon^2 \partial_{tt}\psi(\mathbf{x}, t) - \Delta\psi(\mathbf{x}, t) + \frac{1}{\varepsilon^2}\psi(\mathbf{x}, t) + \psi(\mathbf{x}, t)\phi(\mathbf{x}, t) = 0, \tag{1.1}$$

$$\gamma^2 \partial_{tt}\phi(\mathbf{x}, t) - \Delta\phi(\mathbf{x}, t) - \Delta\left(\psi^2(\mathbf{x}, t)\right) = 0, \qquad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \tag{1.2}$$

with initial conditions

$$\psi(\mathbf{x}, 0) = \psi^{(0)}(\mathbf{x}),\ \partial_t\psi(\mathbf{x}, 0) = \psi^{(1)}(\mathbf{x}),\ \phi(\mathbf{x}, 0) = \phi^{(0)}(\mathbf{x}),\ \partial_t\phi(\mathbf{x}, 0) = \phi^{(1)}(\mathbf{x}). \tag{1.3}$$

Here, the real-valued functions $\psi = \psi(\mathbf{x}, t)$ and $\phi = \phi(\mathbf{x}, t)$ are the fast time scale component of electric field raised by electrons and the derivation of ion density from its equilibrium, respectively; $0 < \varepsilon \leq 1$ and $0 < \gamma \leq 1$ are two dimensionless param-

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eters which are inversely proportional to the plasma frequency and speed of sound, respectively. When $\varepsilon = \mathcal{O}(1)$, it corresponds to an $\mathcal{O}(1)$-plasma-frequency regime, and respectively, $0 < \varepsilon \ll 1$ to the high-plasma-frequency limit regime; and when $\gamma = \mathcal{O}(1)$, it corresponds to $\mathcal{O}(1)$-sound-wave regime, and respectively, $0 < \gamma \ll 1$ to the subsonic limit regime. It is well known that the KGZ system (1.1)–(1.3) is time symmetric or time reversible, and conserves the total *energy* [25, 26], i.e., for $t \geq 0$

$$E(t) := \int_{\mathbb{R}^d} \left[\varepsilon^2 \left(\partial_t \psi\right)^2 + |\nabla \psi|^2 + \frac{1}{\varepsilon^2}\psi^2 + \frac{\gamma^2}{2}|\nabla \varphi|^2 + \frac{1}{2}\phi^2 + \phi\psi^2\right] d\mathbf{x} \equiv E(0), \tag{1.4}$$

where φ is defined via $\Delta\varphi = \partial_t \phi$ with $\lim_{|\mathbf{x}|\to\infty} \varphi = 0$.

For fixed $\varepsilon = \varepsilon_0 > 0$ and $\gamma = \gamma_0 > 0$, i.e., $\mathcal{O}(1)$-plasma-frequency and speed of sound regime, the above KGZ system has been studied in both analytical and numerical aspects. Along the analytical front, the local well-posedness of the Cauchy problem (1.1)–(1.3) in the energy space $H^1 \times L^2$ was performed by Ozawa, Tsutaya, and Tsutsumi [27] when $\varepsilon = \mathcal{O}(1)$ and $\gamma = \mathcal{O}(1)$ with $\varepsilon \neq \gamma$. In addition, when $\varepsilon = \gamma$, as pointed out in [25, 26] that there is no null form structure for the KGZ system [23], this suggests that the KGZ system (1.1)–(1.3) may be locally ill-posed in the energy space. Along the numerical front, Wang, Chen, and Zhang [32] presented an energy conservative finite difference method and established its error estimate.

However, in the simultaneous high-plasma-frequency and subsonic limit regime, i.e., if $(\varepsilon, \gamma) \to 0$ under $\varepsilon \lesssim \gamma$ or the plasma frequency and the speed of sound go to infinity, the analysis and efficient computation of the KGZ system are rather mathematically and numerically complicated issues. The analysis difficulty is mainly due to that the energy $E(t)$ in (1.4) is unbounded when $(\varepsilon, \gamma) \to 0$ under $\varepsilon \lesssim \gamma$. In this regime, Masmoudi and Nakanishi [25] showed that the energy $E(t)$ is at least $\mathcal{O}(\varepsilon^{-2})$ when $\varepsilon \to 0$. They investigated the convergence in $H^s \times H^{s-1}$ $(s > 3/2)$ as $(\varepsilon, \gamma) \to 0$ under $\varepsilon \lesssim \gamma$ and subsequently showed the convergence results in the energy space $H^1 \times L^2$ [26]. Based on their results [25, 26], in the subsonic limit, i.e., $\gamma \to 0$, the KGZ system converges to the Klein–Gordon equation; and in the high-plasma-frequency limit, the KGZ system converges to the Zakharov system [7, 25, 26]. In addition, the solution (ψ, ϕ) of the KGZ system propagates waves with wavelength $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(\varepsilon)$, respectively, in time when $0 < \varepsilon \ll 1$ under $\varepsilon \lesssim \gamma$. This highly oscillatory nature in time brings severe numerical burdens, which is similar to the case of the numerical solution for the Klein–Gordon (KG) equation in the nonrelativistic limit regime [5], making the computation in the simultaneous high-plasma-frequency and subsonic limit regime extremely challenging. To our knowledge, so far there are few results on the numerics of the KGZ system in this regime.

Recently, we rigorously carried out error estimates and compared numerically temporal/spatial resolution of various numerical methods for solving the KG equation, e.g., (1.1), in the nonrelativistic limit regime where the solution has the similar oscillatory behavior as that of the above KGZ system. Based on our results [5], for the KG equation in the nonrelativistic limit regime, i.e., $0 < \varepsilon \ll 1$, in order to compute "correct" solutions, the frequently used finite difference time domain (FDTD) methods [1, 14, 24, 30] share the same ε-scalability: time step $\tau = \mathcal{O}(\varepsilon^3)$ and mesh size $h = \mathcal{O}(1)$ [5]. In addition, our results demonstrated that the exponential wave integrator sine pseudospectral (EWI-SP) method for the KG equation can improve the ε-scalability to $\tau = \mathcal{O}(1)$ and $\tau = \mathcal{O}(\varepsilon^2)$ for linear and nonlinear KG equation, respectively [5]. This suggests that, when the solution has a highly oscillatory nature in time, the EWI-SP method has much better temporal resolution than the FDTD methods.

The main aim of this paper is to extend the EWI-SP method for discretizing the KG equation [5] to the KGZ system; establish rigorously error estimates for the method in the energy space $H^1 \times L^2$ when $\varepsilon = \mathcal{O}(1)$ and $\gamma = \mathcal{O}(1)$; and study numerically the resolution capacity and ε-scalability of the method in the simultaneous high-plasma-frequency and subsonic limit regime. Based on our results, we will show that the ε-scalability of the EWI-SP method for the KGZ system in the simultaneous high-plasma-frequency and subsonic limit regime is $\tau = \mathcal{O}(\varepsilon^2)$ and $h = \mathcal{O}(1)$. In addition, we also observe that the EWI-SP method nearly conserves the total energy over a long time in practical computation for the dynamics of the KGZ system, which is a favorable property for numerical time integrators and has been extensively studied for second-order ordinary different equations (ODEs) and dispersive partial differential equations (PDEs) in the literatures; see, e.g., [4, 5, 10, 11, 12] and references therein.

The paper is organized as follows. In section 2, we present an EWI-SP method for discretizing the KGZ system and mention two finite difference sine pseudospectral methods for comparison. In section 3, we establish error bounds of the EWI-SP method for the KGZ system when $\varepsilon = \mathcal{O}(1)$ and $\gamma = \mathcal{O}(1)$. In section 4 extensive numerical results are reported towards the understanding of temporal/spatial resolution capacity of the EWI-SP method in the simultaneous high-plasma-frequency and subsonic limit regime. Finally, some concluding remarks are drawn in section 5. Throughout the paper we adopt the standard Sobolev spaces and the corresponding norms, and denote $p \lesssim q$ to represent that there exists a generic constant C which is independent of τ, h, ε, and γ such that $|p| \le Cq$.

2. Numerical methods. For the simplicity of notation, we shall only present the methods in one space dimension. Generalizations to higher dimensions are straightforward and results remain valid with tensor products. In practice we truncate the whole space problem (1.1)–(1.3) in one dimension, i.e., $d = 1$, into an interval $\Omega = (a, b)$ with homogeneous Dirichlet boundary conditions. In one dimension, the problem collapses to

$$\varepsilon^2\partial_{tt}\psi(x,t) - \partial_{xx}\psi(x,t) + \frac{1}{\varepsilon^2}\psi(x,t) + \psi(x,t)\phi(x,t) = 0, \quad x \in \Omega, \quad t > 0, \tag{2.1}$$

$$\gamma^2\partial_{tt}\phi(x,t) - \partial_{xx}\phi(x,t) - \partial_{xx}\left(\psi^2(x,t)\right) = 0, \quad x \in \Omega, \quad t > 0, \tag{2.2}$$

$$\psi(a,t) = \psi(b,t) = 0, \quad \phi(a,t) = \phi(b,t) = 0, \quad t \ge 0, \tag{2.3}$$

$$\psi(x,0) = \psi^{(0)}(x), \quad \partial_t\psi(x,0) = \psi^{(1)}(x), \quad x \in \overline{\Omega} = [a,b], \tag{2.4}$$

$$\phi(x,0) = \phi^{(0)}(x), \quad \partial_t\phi(x,0) = \phi^{(1)}(x), \quad x \in \overline{\Omega}. \tag{2.5}$$

We remark here that the boundary conditions considered here are inspired by the inherent physical nature of the system and they have been widely used in the literature for dealing with analysis and computation of the KGZ system (see, e.g., [32] and references therein).

2.1. Exponential wave integrator sine pseudospectral method. Here we present an exponential wave integrator sine spectral/pseudospectral discretization for (2.1)–(2.5), which applies the sine spectral/pseudospectral discretization to spatial derivatives followed by using an exponential wave integrator (EWI) [4, 5, 18, 19, 21, 22] for temporal discretization in phase space.

Choose a mesh size $h := (b-a)/M$ with M an even positive integer, time step $\tau = \Delta t > 0$, and denote grid points with coordinates $(x_j, t_n) := (a + jh, n\tau)$ for

$j = 0, \ldots, M$ and $n = 0, 1, \ldots$. Let

$$Y_M = \text{span}\left\{\Phi_l(x),\ l = 1, \ldots, M-1\right\} \tag{2.6}$$

with

$$\mu_l = \frac{\pi l}{b-a}, \qquad \Phi_l(x) = \sin\left(\mu_l(x-a)\right), \quad x \in \Omega, \qquad l = 1, 2, \ldots, M-1.$$

Denote by $\mathcal{P}_M : Y = \{u(x) \in C(\Omega) : u(a) = u(b) = 0\} \to Y_M$ the standard projection operator [17, 20, 29], i.e.,

$$\mathcal{P}_M u(x) = \sum_{l=1}^{M-1} \widehat{u}_l \Phi_l(x), \quad a \le x \le b, \quad \forall\, u \in Y,$$

with $\widehat{u}_l$ the sine transform coefficient

$$\widehat{u}_l = \frac{2}{b-a}\int_a^b u(x)\Phi_l(x)dx, \quad l = 1, 2, \ldots. \tag{2.7}$$

Now, the sine spectral discretization [17, 20, 29] for (2.1)–(2.2) is as follows: Find $\psi_M(x,t)$ and $\phi_M(x,t) \in Y_M$, i.e.,

$$\psi_M(x,t) = \sum_{l=1}^{M-1} \widehat{\psi}_l(t)\Phi_l(x), \quad \phi_M(x,t) = \sum_{l=1}^{M-1} \widehat{\phi}_l(t)\Phi_l(x), \qquad x \in \overline{\Omega}, \quad t \ge 0, \tag{2.8}$$

such that

$$\varepsilon^2 \partial_{tt}\psi_M - \partial_{xx}\psi_M + \frac{1}{\varepsilon^2}\psi_M + \mathcal{P}_M\left(\psi_M\phi_M\right) = 0, \tag{2.9}$$

$$\gamma^2 \partial_{tt}\phi_M - \partial_{xx}\phi_M - \partial_{xx}\left(\mathcal{P}_M\left((\psi_M)^2\right)\right) = 0, \qquad x \in \overline{\Omega}, \quad t \ge 0. \tag{2.10}$$

Plugging (2.8) into (2.9)–(2.10), noticing the orthogonality of the sine basis functions, for $l = 1, \ldots, M-1$ and $w \in \mathbb{R}$ and when t is near t_n $(n = 0, 1, \ldots)$, we have

$$\frac{d^2}{dw^2}\widehat{\psi}_l(t_n + w) + \beta_l^2 \widehat{\psi}_l(t_n + w) + \frac{1}{\varepsilon^2}\widehat{f}_l^n(w) = 0, \tag{2.11}$$

$$\frac{d^2}{dw^2}\widehat{\phi}_l(t_n + w) + \theta_l^2 \widehat{\phi}_l(t_n + w) + \theta_l^2 \widehat{g}_l^n(w) = 0, \tag{2.12}$$

where

$$\theta_l = \frac{\mu_l}{\gamma},\ \beta_l = \frac{\sqrt{1+\varepsilon^2\mu_l^2}}{\varepsilon^2},\ \widehat{f}_l^n(w) = (\widehat{\psi_M\phi_M})_l(t_n + w),\ \widehat{g}_l^n(w) = (\widehat{(\psi_M)^2})_l(t_n + w).$$

Using the variation-of-constants formula [5, 16, 18, 19], for $n \ge 0$ and $w \in \mathbb{R}$, the general solutions of the above second-order ODEs are

$$\widehat{\psi}_l(t_n + w) = \cos(\beta_l w)\,\widehat{\psi}_l(t_n) + \frac{\sin(\beta_l w)}{\beta_l}\widehat{\psi}_l'(t_n) - \int_0^w \widehat{f}_l^n(s)\frac{\sin(\beta_l(w-s))}{\varepsilon^2\beta_l}ds, \tag{2.13}$$

$$\widehat{\phi}_l(t_n + w) = \cos(\theta_l w)\,\widehat{\phi}_l(t_n) + \frac{\sin(\theta_l w)}{\theta_l}\widehat{\phi}_l'(t_n) - \theta_l\int_0^w \widehat{g}_l^n(s)\sin(\theta_l(w-s))\,ds. \tag{2.14}$$

Differentiating (2.13) and (2.14) with respect to w, we obtain, with $t = t_n + w$,

$$\widehat{\psi}_l'(t) = -\beta_l \sin(\beta_l w)\,\widehat{\psi}_l(t_n) + \cos(\beta_l w)\,\widehat{\psi}_l'(t_n) - \int_0^w \widehat{f}_l^n(s)\frac{\cos(\beta_l(w-s))}{\varepsilon^2}ds, \tag{2.15}$$

$$\widehat{\phi}_l'(t) = -\theta_l \sin(\theta_l w)\,\widehat{\phi}_l(t_n) + \cos(\theta_l w)\,\widehat{\phi}_l'(t_n) - \theta_l^2 \int_0^w \widehat{g}_l^n(s)\cos(\theta_l(w-s))\,ds. \tag{2.16}$$

When $n = 0$, from the initial conditions (2.4)–(2.5), we have

$$\widehat{\psi}_l(0) = \widehat{(\psi^{(0)})}_l, \quad \widehat{\psi}_l'(0) = \widehat{(\psi^{(1)})}_l, \quad \widehat{\phi}_l(0) = \widehat{(\phi^{(0)})}_l, \quad \widehat{\phi}_l'(0) = \widehat{(\phi^{(1)})}_l. \tag{2.17}$$

Evaluating (2.13)–(2.14) and (2.15)–(2.16) with $w = \tau$ and $n = 0$, we get

$$\widehat{\psi}_l(t_1) = \cos(\beta_l\tau)\,\widehat{(\psi^{(0)})}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widehat{(\psi^{(1)})}_l - \int_0^\tau \widehat{f}_l^0(s)\frac{\sin(\beta_l(\tau-s))}{\varepsilon^2\beta_l}ds, \tag{2.18}$$

$$\widehat{\phi}_l(t_1) = \cos(\theta_l\tau)\,\widehat{(\phi^{(0)})}_l + \frac{\sin(\theta_l\tau)}{\theta_l}\widehat{(\phi^{(1)})}_l - \theta_l \int_0^\tau \widehat{g}_l^0 \sin(\theta_l(\tau-s))\,ds, \tag{2.19}$$

$$\widehat{\psi}_l'(t_1) = -\beta_l \sin(\beta_l\tau)\,\widehat{(\psi^{(0)})}_l + \cos(\beta_l\tau)\,\widehat{(\psi^{(1)})}_l - \int_0^\tau \widehat{f}_l^0(s)\frac{\cos(\beta_l(\tau-s))}{\varepsilon^2}ds, \tag{2.20}$$

$$\widehat{\phi}_l'(t_1) = -\theta_l \sin(\theta_l\tau)\,\widehat{(\phi^{(0)})}_l + \cos(\theta_l\tau)\,\widehat{(\phi^{(1)})}_l - \theta_l^2 \int_0^\tau \widehat{g}_l^0(s)\cos(\theta_l(\tau-s))\,ds. \tag{2.21}$$

For $n \geq 1$, choosing $w = \tau$ and $w = -\tau$, respectively, in (2.13)–(2.14), and then summing the corresponding equations together [5, 16, 18, 19], we obtain

$$\widehat{\psi}_l(t_{n+1}) = 2\cos(\beta_l\tau)\,\widehat{\psi}_l(t_n) - \widehat{\psi}_l(t_{n-1}) - \int_0^\tau \widehat{(f^+)}_l^n(s)\frac{\sin(\beta_l(\tau-s))}{\varepsilon^2\beta_l}ds, \tag{2.22}$$

$$\widehat{\phi}_l(t_{n+1}) = 2\cos(\theta_l\tau)\,\widehat{\phi}_l(t_n) - \widehat{\phi}_l(t_{n-1}) - \theta_l \int_0^\tau \widehat{(g^+)}_l^n(s)\sin(\theta_l(\tau-s))\,ds, \tag{2.23}$$

where

$$\widehat{(f^+)}_l^n(s) := \widehat{f}_l^n(s) + \widehat{f}_l^n(-s), \qquad \widehat{(g^+)}_l^n(s) := \widehat{g}_l^n(s) + \widehat{g}_l^n(-s).$$

Carrying out the procedure similar to (2.15)–(2.16) with subtraction instead of summation [21, 22], we get

$$\widehat{\psi}_l'(t_{n+1}) = \widehat{\psi}_l'(t_{n-1}) - 2\beta_l \sin(\beta_l\tau)\,\widehat{\psi}_l(t_n) - \int_0^\tau \widehat{(f^+)}_l^n(s)\frac{\cos(\beta_l(\tau-s))}{\varepsilon^2}ds, \tag{2.24}$$

$$\widehat{\phi}_l'(t_{n+1}) = \widehat{\phi}_l'(t_{n-1}) - 2\theta_l \sin(\theta_l\tau)\,\widehat{\phi}_l(t_n) - \theta_l^2 \int_0^\tau \widehat{(g^+)}_l^n(s)\cos(\theta_l(\tau-s))\,ds. \tag{2.25}$$

In order to design an explicit scheme, we will adopt the following quadrature with $A \in C([0,\tau])$ and $0 \neq \delta \in \mathbb{R}$ [16]:

$$\int_0^\tau A(s)\sin(\delta(\tau-s))\,ds \approx A(0)\int_0^\tau \sin(\delta(\tau-s))\,ds = \frac{1-\cos(\delta\tau)}{\delta}A(0), \tag{2.26}$$

$$\int_0^\tau A(s)\cos(\delta(\tau-s))\,ds \approx A(0)\int_0^\tau \cos(\delta(\tau-s))\,ds = \frac{\sin(\delta\tau)}{\delta}A(0), \tag{2.27}$$

to approximate all the integrals in (2.18)–(2.25).

Denote $\psi_M^n(x)$, $\phi_M^n(x)$, $(\psi_t)_M^n(x)$, and $(\phi_t)_M^n(x)$ as the approximations of $\psi_M(x,t_n)$, $\phi_M(x,t_n)$, $\partial_t\psi_M(x,t_n)$, and $\partial_t\phi_M(x,t_n)$, respectively, for $x\in\overline{\Omega}$ and $n\ge 0$; and $\widehat{\psi}_l^n$, $\widehat{\phi}_l^n$, $\widehat{(\psi_t)}_l^n$, and $\widehat{(\phi_t)}_l^n$ as the approximations of $\widehat{\psi}_l(t_n)$, $\widehat{\phi}_l(t_n)$, $\widehat{\psi}_l'(t_n)$, and $\widehat{\phi}_l'(t_n)$, respectively. Choosing $\psi_M^0=\mathcal{P}_M\psi^{(0)}$, $(\psi_t)_M^0=\mathcal{P}_M\psi^{(1)}$, $\phi_M^0=\mathcal{P}_M\phi^{(0)}$, and $(\phi_t)_M^0=\mathcal{P}_M\phi^{(1)}$, then an exponential wave integrator sine spectral discretization for computing ψ_M^{n+1}, ϕ_M^{n+1}, $(\psi_t)_M^{n+1}$, and $(\phi_t)_M^{n+1}$ for $n\ge 0$ reads

$$\psi_M^{n+1}(x)=\sum_{l=1}^{M-1}\widehat{\psi}_l^{n+1}\,\Phi_l(x),\quad \phi_M^{n+1}(x)=\sum_{l=1}^{M-1}\widehat{\phi}_l^{n+1}\,\Phi_l(x),\tag{2.28}$$

$$(\psi_t)_M^{n+1}(x)=\sum_{l=1}^{M-1}\widehat{(\psi_t)}_l^{n+1}\,\Phi_l(x),\quad (\phi_t)_M^{n+1}(x)=\sum_{l=1}^{M-1}\widehat{(\phi_t)}_l^{n+1}\,\Phi_l(x),\quad n\ge 0,\tag{2.29}$$

where, for $n=0$,

$$\begin{aligned}
\widehat{\psi}_l^1&=\cos(\beta_l\tau)\,\widehat{(\psi^{(0)})}_l+\frac{\sin(\beta_l\tau)}{\beta_l}\widehat{(\psi^{(1)})}_l+\frac{\cos(\beta_l\tau)-1}{(\varepsilon\beta_l)^2}(\widehat{\psi_M^0\phi_M^0})_l\,,\\
\widehat{\phi}_l^1&=\cos(\theta_l\tau)\,\widehat{(\phi^{(0)})}_l+\frac{\sin(\theta_l\tau)}{\theta_l}\widehat{(\phi^{(1)})}_l+[\cos(\theta_l\tau)-1]\left(\widehat{(\psi_M^0)^2}\right)_l\,,\\
\widehat{(\psi_t)}_l^1&=-\beta_l\sin(\beta_l\tau)\,\widehat{(\psi^{(0)})}_l+\cos(\beta_l\tau)\,\widehat{(\psi^{(1)})}_l-\frac{\sin(\beta_l\tau)}{\varepsilon^2\beta_l}(\widehat{\psi_M^0\phi_M^0})_l\,,\\
\widehat{(\phi_t)}_l^1&=-\theta_l\sin(\theta_l\tau)\,\widehat{(\phi^{(0)})}_l+\cos(\theta_l\tau)\,\widehat{(\phi^{(1)})}_l-\theta_l\sin(\theta_l\tau)\left(\widehat{(\psi_M^0)^2}\right)_l\,;
\end{aligned}$$

and for $n\ge 1$,

$$\begin{aligned}
\widehat{\psi}_l^{n+1}&=-\widehat{\psi}_l^{n-1}+2\cos(\beta_l\tau)\,\widehat{\psi}_l^n+\frac{2\,[\cos(\beta_l\tau)-1]}{(\varepsilon\beta_l)^2}(\widehat{\psi_M^n\phi_M^n})_l\,,\\
\widehat{\phi}_l^{n+1}&=-\widehat{\phi}_l^{n-1}+2\cos(\theta_l\tau)\,\widehat{\phi}_l^n+2\,[\cos(\theta_l\tau)-1]\left(\widehat{(\psi_M^n)^2}\right)_l\,,\\
\widehat{(\psi_t)}_l^{n+1}&=\widehat{(\psi_t)}_l^{n-1}-2\beta_l\sin(\beta_l\tau)\,\widehat{\psi}_l^n-\frac{2\sin(\beta_l\tau)}{\varepsilon^2\beta_l}(\widehat{\psi_M^n\phi_M^n})_l\,,\\
\widehat{(\phi_t)}_l^{n+1}&=\widehat{(\phi_t)}_l^{n-1}-2\theta_l\sin(\theta_l\tau)\,\widehat{\phi}_l^n-2\theta_l\sin(\theta_l\tau)\left(\widehat{(\psi_M^n)^2}\right)_l\,.
\end{aligned}$$

We remark here that the above EWI for solving second-order wave-type ODEs has been widely used in the literature and it can be traced back to Gautschi [16]. As demonstrated in the literature [18, 19, 21, 22], it gives exact solutions to linear second-order ODEs and shows favorable properties on solving the oscillatory second-order ODEs compared to standard finite difference time integrators. Thus we can anticipate that the above method will be favorable over FDTD methods for solving the KGZ system in the simultaneous high-plasma-frequency and subsonic limit regime.

The above procedure is not suitable in practice due to the difficulty in evaluating the integrals defining the sine transform coefficients. Here, we adopt an efficient implementation by choosing $\psi_M^0(x)$, $(\psi_t)_M^0(x)$, $\phi_M^0(x)$, and $(\phi_t)_M^0(x)$ as the interpolations of $\psi^{(0)}(x)$, $\psi^{(1)}(x)$, $\phi^{(0)}(x)$, and $\phi^{(1)}(x)$ on the grids, respectively, and approximating the integrals by a quadrature rule on the grids [15, 29]. Let ψ_j^n, ϕ_j^n, $(\psi_t)_j^n$, and $(\phi_t)_j^n$ be the approximations of $\psi(x_j,t_n)$, $\phi(x_j,t_n)$, $\partial_t\psi(x_j,t_n)$, and $\partial_t\phi(x_j,t_n)$, respectively, for $j=0,1,\dots,M$ and $n\ge 0$, and denote ψ^n, ϕ^n, $(\psi_t)^n$, and $(\phi_t)^n$ as the vectors with components ψ_j^n, ϕ_j^n $(\psi_t)_j^n$, and $(\phi_t)_j^n$, respectively. Choosing $\psi_j^0=\psi^{(0)}(x_j)$, $\phi_j^0=\phi^{(0)}(x_j)$, $(\psi_t)_j^0=\psi^{(1)}(x_j)$, and $(\phi_t)_j^0=\phi^{(1)}(x_j)$ for $j=0,1,\dots,M$, then an EWI-SP discretization for computing ψ^{n+1}, ϕ^{n+1}, ψ_t^{n+1}, and ϕ_t^{n+1} for $n\ge 0$ reads

$$\psi_j^{n+1}=\sum_{l=1}^{M-1}\widetilde{(\psi^{n+1})}_l\,\Phi_l(x_j),\quad \phi_j^{n+1}=\sum_{l=1}^{M-1}\widetilde{(\phi^{n+1})}_l\,\Phi_l(x_j),\quad 0\le j\le M, \tag{2.30}$$

$$(\psi_t)_j^{n+1}=\sum_{l=1}^{M-1}\widetilde{(\psi_t)}_l^{n+1}\,\Phi_l(x_j),\quad (\phi_t)_j^{n+1}=\sum_{l=1}^{M-1}\widetilde{(\phi_t)}_l^{n+1}\,\Phi_l(x_j), \tag{2.31}$$

where, for $n=0$,

$$\begin{aligned}
\widetilde{(\psi^1)}_l&=\cos(\beta_l\tau)\,\widetilde{(\psi^0)}_l+\frac{\sin(\beta_l\tau)}{\beta_l}\widetilde{(\psi_t)}_l^0+\frac{\cos(\beta_l\tau)-1}{(\varepsilon\beta_l)^2}\widetilde{(\xi^0)}_l,\\
\widetilde{(\phi^1)}_l&=\cos(\theta_l\tau)\,\widetilde{(\phi^0)}_l+\frac{\sin(\theta_l\tau)}{\theta_l}\widetilde{(\phi_t)}_l^0+[\cos(\theta_l\tau)-1]\,\widetilde{(\rho^0)}_l,\\
\widetilde{(\psi_t)}_l^1&=-\beta_l\sin(\beta_l\tau)\,\widetilde{(\psi^{(0)})}_l+\cos(\beta_l\tau)\,\widetilde{(\psi^{(1)})}_l-\frac{\sin(\beta_l\tau)}{\varepsilon^2\beta_l}\widetilde{(\xi^0)}_l\,,\\
\widetilde{(\phi_t)}_l^1&=-\theta_l\sin(\theta_l\tau)\,\widetilde{(\phi^{(0)})}_l+\cos(\theta_l\tau)\,\widetilde{(\phi^{(1)})}_l-\theta_l\sin(\theta_l\tau)\,\widetilde{(\rho^0)}_l\,;
\end{aligned}$$

and for $n\ge 1$,

$$\begin{aligned}
\widetilde{(\psi^{n+1})}_l&=-\widetilde{(\psi^{n-1})}_l+2\cos(\beta_l\tau)\,\widetilde{(\psi^n)}_l+\frac{2[\cos(\beta_l\tau)-1]}{(\varepsilon\beta_l)^2}\widetilde{(\xi^n)}_l\,,\\
\widetilde{(\phi^{n+1})}_l&=-\widetilde{(\phi^{n-1})}_l+2\cos(\theta_l\tau)\,\widetilde{(\phi^n)}_l+2[\cos(\theta_l\tau)-1]\,\widetilde{(\rho^n)}_l\,,\\
\widetilde{(\psi_t)}_l^{n+1}&=\widetilde{(\psi_t)}_l^{n-1}-2\beta_l\sin(\beta_l\tau)\,\widetilde{(\psi^n)}_l-\frac{2\sin(\beta_l\tau)}{\varepsilon^2\beta_l}\widetilde{(\xi^n)}_l\,,\\
\widetilde{(\phi_t)}_l^{n+1}&=\widetilde{(\phi_t)}_l^{n-1}-2\theta_l\sin(\theta_l\tau)\,\widetilde{(\phi^n)}_l-2\theta_l\sin(\theta_l\tau)\,\widetilde{(\rho^n)}_l
\end{aligned}$$

with $\xi^n=(\psi_0^n\phi_0^n,\dots,\psi_M^n\phi_M^n)^T$, $\rho^n=((\psi_0^n)^2,\dots,(\psi_M^n)^2)^T$, and $\widetilde{U}_l$ are the discrete sine transform coefficients for a vector $U\in X_M:=\{U=(U_0,\dots,U_M)^T|U_0=U_M=0\}\subset\mathbb{R}^{M+1}$ defined as

$$\widetilde{U}_l=\frac{2}{M}\sum_{j=1}^{M-1}U_j\Phi_l(x_j),\quad l=1,\dots,M-1.$$

In addition, the energy $E(t_n)$ in (1.4) can be discretized as

$$E^n=h\sum_{j=1}^{M-1}\left[\varepsilon^2|(\psi_t)_j^n|^2+\frac{1+\varepsilon^2\phi_j^n}{\varepsilon^2}\left|\psi_j^n\right|^2+\frac{1}{2}\left|\phi_j^n\right|^2+\frac{M\mu_j^2}{2}\left|\widetilde{(\psi^n)}_j\right|^2+\frac{M\gamma^2}{4\mu_j^2}\left|\widetilde{(\phi_t^n)}_j\right|^2\right].$$

The above EWI-SP discretization for the KGZ system is explicit, time symmetric by noticing that it is unchanged by interchanging $\tau \leftrightarrow -\tau$ and $n+1 \leftrightarrow n-1$ and easy to extend to two and three dimensions. The memory cost is $\mathcal{O}(M)$ and computational cost per time step is $\mathcal{O}(M \ln M)$ thanks to the discrete sine transform (DST).

Remark 2.1. Another way to approximate the integrals in (2.18)–(2.25) is to use the trapezoidal rule

$$\int_0^\tau A(s) \sin(\delta(\tau - s))\, ds \approx \frac{\tau}{2} A(0) \sin(\delta\tau),$$
$$\int_0^\tau A(s) \cos(\delta(\tau - s))\, ds \approx \frac{\tau}{2} \left[A(0) \cos(\delta\tau) + A(\tau)\right].$$

Then the EWI-SP method is still valid after some minor changes and we omit the details here for brevity.

2.2. Finite difference integrator sine pseudospectral methods. For comparison purpose, here we recall two frequently used finite difference time integrators for wave-type equations, which will still be coupled with the sine pseudospectral approximation to the spatial derivatives for solving the KGZ system (2.1)–(2.5). Let u_j^n be the numerical approximation of $u(x_j, t_n)$ $(j = 0, 1, \ldots, M,\ n = 0, 1, \ldots)$ for a general function $u(x,t)$ on $[a,b] \times [0,\infty)$, introduce the finite difference discretization operators as

$$\delta_t^+ u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}, \quad \delta_t^- u_j^n = \frac{u_j^n - u_j^{n-1}}{\tau}, \quad \delta_t^2 u_j^n = \delta_t^+ \delta_t^- u_j^n = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2},$$

and denote the sine pseudospectral approximation to ∂_{xx} as

$$D_{xx} u_j^n = -\sum_{l=1}^{M-1} \mu_l^2 \widetilde{(u^n)}_l \Phi_l(x_j), \qquad j = 0, 1, \ldots, M, \quad n \geq 0.$$

Then an implicit energy conservative finite difference sine pseudospectral (ECFD-SP) scheme for the KGZ system (2.1)–(2.2) reads, for $j = 1, \ldots, M-1$ and $n = 1, 2, \ldots,$

$$\varepsilon^2 \delta_t^2 \psi_j^n - D_{xx} \frac{\psi_j^{n+1} + \psi_j^{n-1}}{2} + \left(\psi_j^{n+1} + \psi_j^{n-1}\right) \left[\frac{1}{2\varepsilon^2} + \frac{\phi_j^{n+1} + \phi_j^{n-1}}{4}\right] = 0, \tag{2.32}$$

$$\gamma^2 \delta_t^2 \phi_j^n - \frac{1}{2} D_{xx} \left[\phi_j^{n+1} + \phi_j^{n-1} + \left(\psi_j^{n+1}\right)^2 + \left(\psi_j^{n-1}\right)^2\right] = 0. \tag{2.33}$$

The initial and boundary conditions in (2.3)–(2.5) are discretized as

$$\psi_j^0 = \psi^{(0)}(x_j), \quad \phi_j^0 = \phi^{(0)}(x_j), \quad j = 0, 1, \ldots, M, \tag{2.34}$$

$$\psi_0^n = \psi_M^n = 0, \quad \phi_0^n = \phi_M^n = 0, \quad n \geq 0, \tag{2.35}$$

$$\psi_j^1 = \psi_j^0 + \tau \psi^{(1)}(x_j) + \frac{\tau^2}{2\varepsilon^2} \left[D_{xx} \psi_j^0 - \frac{1}{\varepsilon^2} \psi_j^0 - \psi_j^0 \phi_j^0\right], \quad j = 1, \ldots, M-1, \tag{2.36}$$

$$\phi_j^1 = \phi_j^0 + \tau \phi^{(1)}(x_j) + \frac{\tau^2}{2\gamma^2} D_{xx} \left[\phi_j^0 + \left(\psi_j^0\right)^2\right], \quad j = 1, \ldots, M-1. \tag{2.37}$$

The above ECFD-SP scheme for the KGZ system is time symmetric and implicit. In addition, following the standard techniques used in [9, 17, 20, 29, 31, 32] for proving energy conservation of FDTD methods, it is easy to show that it conserves the energy in the discretized level.

LEMMA 2.1. *The ECFD-SP scheme* (2.32)–(2.33) *for the KGZ system conserves the energy in the discretized level, i.e.,*

$$E_h^n = h\sum_{j=1}^{M-1}\left[\frac{|\psi_j^n|^2(1+\varepsilon^2\phi_j^n)+|\psi_j^{n+1}|^2(1+\varepsilon^2\phi_j^{n+1})}{2\varepsilon^2}+\frac{|\phi_j^n|^2+|\phi_j^{n+1}|^2}{4}+\varepsilon^2|\delta_t^+\psi_j^n|^2\right]$$
$$+\frac{b-a}{4}\sum_{l=1}^{M-1}\mu_l^2\left[\left(\widetilde{(\psi^n)}_l\right)^2+\left(\widetilde{(\psi^{n+1})}_l\right)^2+\frac{\gamma^2|\widetilde{(\phi^{n+1})}_l-\widetilde{(\phi^n)}_l|^2}{\tau^2\mu_l^4}\right]\equiv E_h^0,\ n\geq 0.$$

However, at every time step, a fully nonlinear coupled system must be solved which is time consuming in practical computation, especially in two and three dimensions. This motivates us to consider the following semi-implicit finite difference sine pseudospectral (SIFD-SP) method for the KGZ system:

$$\varepsilon^2\delta_t^2\psi_j^n-\frac{1}{2}D_{xx}\left(\psi_j^{n+1}+\psi_j^{n-1}\right)+\frac{1}{2\varepsilon^2}\left(\psi_j^{n+1}+\psi_j^{n-1}\right)+\psi_j^n\phi_j^n=0,\quad n\geq 1, \tag{2.38}$$

$$\gamma^2\delta_t^2\phi_j^n-\frac{1}{2}D_{xx}\left[\phi_j^{n+1}+\phi_j^{n-1}+\left(\psi_j^{n+1}\right)^2+\left(\psi_j^{n-1}\right)^2\right]=0,\quad 1\leq j\leq M-1. \tag{2.39}$$

Again, this SIFD-SP for the KGZ system is time symmetric and implicit. However, at every time step, only a linear system needs to be solved and it can be solved very efficiently via DST at the cost of $\mathcal{O}(M\ln M)$. Thus this scheme is more efficient than ECFD-SP in practical computation, especially in two and three dimensons. Finally, we also consider here an explicit finite difference sine pseudospectral (EXFD-SP) method, which is widely used in the literature when $\varepsilon=\mathcal{O}(1)$ and $\gamma=\mathcal{O}(1)$ [14, 24, 30, 32],

$$\varepsilon^2\delta_t^2\psi_j^n-D_{xx}\psi_j^n+\frac{1}{\varepsilon^2}\psi_j^n+\psi_j^n\phi_j^n=0, \tag{2.40}$$

$$\gamma^2\delta_t^2\phi_j^n-D_{xx}\phi_j^n-D_{xx}\left(\left(\psi_j^n\right)^2\right)=0,\quad 1\leq j\leq M-1,\quad n\geq 1. \tag{2.41}$$

The above EXFD-SP is time symmetric, explicit, and very easy to implement. However, this scheme has a stability condition as $\tau\lesssim\min\{\varepsilon^2,\varepsilon h,\gamma h\}$, which is standard and acceptable when $\varepsilon=\mathcal{O}(1)$ and $\gamma=\mathcal{O}(1)$, but it is very severe in the high-plasma-frequency and/or subsonic limit regime, i.e., $0<\varepsilon\ll 1$ and/or $0<\gamma\ll 1$.

Remark 2.2. Based on our numerical results, the way to compute the first step approximation ψ^1 and ϕ^1 in (2.36)–(2.37) for ECFD-SP, SIFD-SP, and EXFD-SP gives reasonably good results when τ is small enough for fixed ε and/or γ, and it is numerically unstable for fixed $\tau=\mathcal{O}(1)$ when ε and/or γ are small enough. Thus in our practical computation, we always use (2.30) with $n=0$ to compute the first step approximation ψ^1 and ϕ^1 for ECFD-SP, SIFD-SP, and EXFD-SP, which doesn't cause any numerical instability with any $\tau>0$, $\varepsilon>0$, and $\gamma>0$ for ECFD-SP and SIFD-SP.

Remark 2.3. The above numerical methods can be easily extended to the KGZ system under other boundary conditions instead of the homogeneous Dirichlet boundary condition, such as periodic boundary condition or homogeneous Neumann boundary condition which are the most popular boundary conditions used in the literature for solving the KGZ system numerically.

3. Error bounds in energy space for EWI-SP. In this section, we will rigorously establish error bounds in the energy space $H^1 \times L^2$ for the EWI-SP method (2.28)–(2.29) for fixed $\varepsilon = \varepsilon_0 = \mathcal{O}(1)$ and $\gamma = \gamma_0 = \mathcal{O}(1)$. Without loss of generality and for the simplicity of notation, we set $\varepsilon = \gamma = 1$ in this section.

3.1. Main results. Let T^* be the maximum existence time for the solution of the KGZ system [25, 26, 27] and denote $0 < T < T^*$. In order to establish error estimates for the EWI-SP method, we make the following assumptions on the exact solution (ψ, ϕ) of (2.1)–(2.5):

$$\text{(A1)} \qquad \psi \in C\left([0,T]; W^{1,\infty} \cap H^{m_0} \cap H_0^2\right) \cap C^1\left([0,T]; W^{1,4}\right) \cap C^2\left([0,T]; H^1\right),$$

$$\text{(A2)} \qquad \phi \in C\left([0,T]; L^{\infty}(\Omega) \cap H^{m_0} \cap H_0^1\right) \cap C^1\left([0,T]; L^4\right) \cap C^2\left([0,T]; L^2\right)$$

for some integer $m_0 \geq 3$. Under the assumptions (A1) and (A2), for $\Omega_T := \Omega \times [0,T]$, we denote

$$K_1 := \|\psi\|_{L^\infty(\Omega_T)}, \qquad K_2 := \|\partial_x \psi\|_{L^\infty(\Omega_T)}, \qquad K_3 := \|\phi\|_{L^\infty(\Omega_T)}.$$

Define the "error" functions as

$$e_\psi^n(x) := \psi(x,t_n) - \psi_M^n(x), \qquad e_\phi^n(x) := \phi(x,t_n) - \phi_M^n(x), \qquad x \in \Omega, \quad n \geq 0,$$

and assume the stability condition

$$\tau \leq \frac{\pi h}{3\sqrt{h^2+\pi^2}}, \tag{3.1}$$

then we have the following theorem.

THEOREM 3.1 ($H^1 \times L^2$-norm error estimates). *Let $(\psi_M^n(x), \phi_M^n(x))$ be the approximation obtained from the EWI-SP* (2.28). *Assuming $\tau \lesssim h$, under the assumptions* (A1) *and* (A2), *there exist $h_0 > 0$ and $\tau_0 > 0$ sufficiently small and independent of h and τ, such that for any $0 < h \leq h_0$ and $0 < \tau \leq \tau_0$ satisfying* (3.1), *we have*

$$\left\|e_\psi^n\right\|_{H^1(\Omega)} + \left\|e_\phi^n\right\|_{L^2(\Omega)} \lesssim \tau^2 + h^{m_0-1}, \quad 0 \leq n \leq \frac{T}{\tau}, \tag{3.2}$$

$$\|\psi_M^n\|_{L^\infty(\Omega)} \leq K_1 + 1, \quad \left\|\frac{d}{dx}\psi_M^n\right\|_{L^\infty(\Omega)} \leq K_2 + 1, \quad \|\phi_M^n\|_{L^\infty(\Omega)} \leq K_3 + 1. \tag{3.3}$$

Define the operator

$$L := \left(-\partial_{xx}\right)^{-1/2};$$

if, in addition, we make additional assumptions on the exact solution (ψ, ϕ) as

$$\text{(A3)} \qquad \partial_t \psi \in C\left([0,T]; L^\infty \cap H^{m_0}\right), \ \partial_t \phi \in C\left([0,T]; H^{m_0}\right), \ L\partial_t \phi \in C\left([0,T]; L^\infty\right),$$

and denote

$$K_4 := \|\partial_t \psi\|_{L^\infty(\Omega_T)}, \quad K_5 := \|L\partial_t \phi\|_{L^\infty(\Omega_T)},$$

then we have the energy convergence result.

THEOREM 3.2 (energy convergence). *Let $((\psi_t)_M^n(x), (\phi_t)_M^n(x))$ be the approximation obtained from* (2.29). *Under the same assumptions as in Theorem* 3.1 *and* (A3), *there exist $h_0 > 0$ and $\tau_0 > 0$ sufficiently small and independent of h and τ, such that for any $0 < h \le h_0$ and $0 < \tau \le \tau_0$ satisfying* (3.1), *we have*

$$\|\partial_t \psi(x, t_n) - (\psi_t)_M^n(x)\|_{L^2} + \|L\partial_t \phi(x, t_n) - L(\phi_t)_M^n(x)\|_{L^2} \lesssim \tau^2 + h^{m_0-1}, \tag{3.4}$$

$$\|(\psi_t)_M^n\|_{L^\infty} \le K_4 + 1, \quad \|L(\phi_t)_M^n\|_{L^\infty} \le K_5 + 1, \quad 0 \le n \le \frac{T}{\tau} - 1. \tag{3.5}$$

These imply that the EWI-SP method (2.28)–(2.29) *is energy convergent, i.e.,*

$$|E(t_n) - E_M^n| = |E(0) - E_M^n| \lesssim \tau^2 + h^{m_0-1}, \quad 0 \le n \le \frac{T}{\tau} - 1, \tag{3.6}$$

where E_M^n is the energy associated with the EWI-SP (2.28)–(2.29) *defined as*

$$E_M^n := \int_\Omega \left[\varepsilon^2 \left((\psi_t)_M^n\right)^2 + |\nabla \psi_M^n|^2 + \frac{1 + \varepsilon^2 \phi_M^n}{\varepsilon^2} (\psi_M^n)^2 + \frac{\gamma^2}{2} |\nabla \varphi_M^n|^2 + \frac{1}{2} (\phi_M^n)^2\right] d\mathbf{x}$$

with φ_M^n given by $\Delta \varphi_M^n = (\phi_t)_M^n$ and $\varphi_M^n|_{\partial\Omega} = 0$.

3.2. Proof of Theorem 3.1. The proof of (3.2)–(3.3) proceeds by means of the classical discrete energy method with suitable defined error energy combined with the method of mathematical induction (similar to the technique used in [2, 3, 4, 5, 6]). For simplicity of notation, we denote $\|u(\cdot, t_s)\|_{W^{p,q}(\Omega)}$ with any fixed $0 \le t_s \le T$ and $\|u(\cdot, t)\|_{W^{p,q}(\Omega_T)}$ by $\|u\|_{W^{p,q}}$ for short, provided that no confusion occurs.

The case when $n = 0$ is trivial. From the discretization of the initial data, i.e., $\psi_M^0 = \mathcal{P}_M \psi^{(0)}$ and $\phi_M^0 = \mathcal{P}_M \phi^{(0)}$, we have

$$\begin{aligned}
\|e_\psi^0\|_{H^1} &= \left\|\psi(x, 0) - \psi_M^0\right\|_{H^1} = \left\|\psi^{(0)} - \mathcal{P}_M \psi^{(0)}\right\|_{H^1} \lesssim h^{m_0-1}, \\
\|e_\phi^0\|_{L^2} &= \left\|\phi(x, 0) - \phi_M^0\right\|_{L^2} = \left\|\phi^{(0)} - \mathcal{P}_M \phi^{(0)}\right\|_{L^2} \lesssim h^{m_0}, \\
\left\|\psi_M^0\right\|_{L^\infty} &\le \left\|\psi^{(0)}\right\|_{L^\infty} + \left\|\psi^{(0)} - \mathcal{P}_M \psi^{(0)}\right\|_{L^\infty} \lesssim K_1 + h^{m_0-1}, \\
\left\|\frac{d}{dx} \psi_M^0\right\|_{L^\infty} &\le \left\|\frac{d}{dx} \psi^{(0)}\right\|_{L^\infty} + \left\|\frac{d}{dx} \left(\psi^{(0)} - \mathcal{P}_M \psi^{(0)}\right)\right\|_{L^\infty} \lesssim K_2 + h^{m_0-2}, \\
\left\|\phi_M^0\right\|_{L^\infty} &\le \left\|\phi^{(0)}\right\|_{L^\infty} + \left\|\phi^{(0)} - \mathcal{P}_M \phi^{(0)}\right\|_{L^\infty} \lesssim K_3 + h^{m_0-1}.
\end{aligned}$$

Therefore, there exists a constant $h_1 > 0$ sufficiently small such that, when $0 < h \le h_1$, the estimates (3.2)–(3.3) are valid for $n = 0$.

Now we prove for the case when $n \ge 1$ and the proof is divided into three main steps.

Step 1. *"Error" equations.* First we find the equations for properly defined "error" functions. To this purpose, we write the exact solution of the KGZ system as

$$\psi(x, t) = \sum_{l=1}^{\infty} \widehat{\psi}_l(t) \Phi_l(x), \quad \phi(x, t) = \sum_{l=1}^{\infty} \widehat{\phi}_l(t) \Phi_l(x), \quad x \in \overline{\Omega}, \quad 0 \le t \le T, \tag{3.7}$$

where $\widehat{\psi}_l$ and $\widehat{\phi}_l$ are the sine transform coefficients of ψ and ϕ, respectively, which are defined as in (2.7). Similarly to (2.18)–(2.19) and (2.22)–(2.23), for $l = 1, 2, \ldots,$ and $n \geq 1$, we have

$$\widehat{\psi}_l(t_1) = \cos(\beta_l\tau)\,\widehat{(\psi^{(0)})}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widehat{(\psi^{(1)})}_l - \int_0^\tau \widehat{F}_l^0(s)\frac{\sin(\beta_l(\tau-s))}{\beta_l}ds, \tag{3.8}$$

$$\widehat{\phi}_l(t_1) = \cos(\mu_l\tau)\,\widehat{(\phi^{(0)})}_l + \frac{\sin(\mu_l\tau)}{\mu_l}\widehat{(\phi^{(1)})}_l - \mu_l\int_0^\tau \widehat{G}_l^0\sin(\mu_l(\tau-s))\,ds, \tag{3.9}$$

$$\widehat{\psi}_l(t_{n+1}) = 2\cos(\beta_l\tau)\,\widehat{\psi}_l(t_n) - \widehat{\psi}_l(t_{n-1}) - \int_0^\tau \widehat{(F^+)}_l^n(s)\frac{\sin(\beta_l(\tau-s))}{\beta_l}ds, \tag{3.10}$$

$$\widehat{\phi}_l(t_{n+1}) = 2\cos(\mu_l\tau)\,\widehat{\phi}_l(t_n) - \widehat{\phi}_l(t_{n-1}) - \mu_l\int_0^\tau \widehat{(G^+)}_l^n(s)\sin(\mu_l(\tau-s))\,ds, \tag{3.11}$$

where $\beta_l = \sqrt{1+\mu_l^2}$ and $\theta_l = \mu_l$ for $1 \leq l \leq M-1$ when $\varepsilon = \gamma = 1$ and

$$\widehat{F}_l^n(s) = \widehat{(\psi\phi)}_l(t_n+s), \qquad \widehat{G}_l^n(s) = \widehat{(\psi^2)}_l(t_n+s),$$
$$\widehat{(F^+)}_l^n(s) := \widehat{F}_l^n(s) + \widehat{F}_l^n(-s), \qquad \widehat{(G^+)}_l^n(s) := \widehat{G}_l^n(s) + \widehat{G}_l^n(-s), \qquad n \geq 0.$$

Denote

$$e^n(x) := \mathcal{P}_M\psi(x,t_n) - \psi_M^n(x) = \sum_{l=1}^{M-1}\widehat{e}_l^n\Phi_l(x), \quad x \in \overline{\Omega}, \quad 0 \leq n \leq \frac{T}{\tau},$$
$$\eta^n(x) := \mathcal{P}_M\phi(x,t_n) - \phi_M^n(x) = \sum_{l=1}^{M-1}\widehat{\eta}_l^n\Phi_l(x), \quad x \in \overline{\Omega}, \quad 0 \leq n \leq \frac{T}{\tau},$$

then we have

$$\widehat{e}_l^n = \widehat{\psi}_l(t_n) - \widehat{\psi}_l^n, \quad \widehat{\eta}_l^n = \widehat{\phi}_l(t_n) - \widehat{\phi}_l^n \quad l = 1, \ldots, M-1, \quad 0 \leq n \leq \frac{T}{\tau}.$$

With the help of the triangle inequality and Parseval's identity, we have

$$\begin{aligned}\|\psi(\cdot,t_n) - \psi_M^n\|_{H^1} &\leq \|\psi(\cdot,t_n) - \mathcal{P}_M\psi(\cdot,t_n)\|_{H^1} + \|e^n\|_{H^1}\\ &\lesssim h^{m_0-1} + \sqrt{\sum_{l=1}^{M-1}(1+\mu_l^2)\,(\widehat{e}_l^n)^2}, \quad 0 \leq n \leq \frac{T}{\tau},\end{aligned} \tag{3.12}$$

$$\begin{aligned}\|\phi(\cdot,t_n) - \phi_M^n\|_{L^2} &\leq \|\phi(\cdot,t_n) - \mathcal{P}_M\phi(\cdot,t_n)\|_{L^2} + \|\eta^n\|_{L^2}\\ &\lesssim h^{m_0} + \sqrt{\sum_{l=1}^{M-1}(\widehat{\eta}_l^n)^2}, \quad 0 \leq n \leq \frac{T}{\tau}.\end{aligned} \tag{3.13}$$

Hence, in the following we only need to estimate the errors $\|e^n\|_{H^1}$ and $\|\eta^n\|_{L^2}$.

For each fixed $l = 1, \ldots, M-1$, subtracting (3.8)–(3.11) from (2.18)–(2.23), respectively, we obtain the equations for the "error" functions $\widehat{e}_l^n$ and $\widehat{\eta}_l^n$ as

$$\widehat{e}_l^{n+1} = 2\cos(\beta_l\tau)\,\widehat{e}_l^n - \widehat{e}_l^{n-1} - \widehat{U}_l^n, \quad 1 \le n \le \frac{T}{\tau} - 1, \tag{3.14}$$

$$\widehat{\eta}_l^{n+1} = 2\cos(\mu_l\tau)\,\widehat{\eta}_l^n - \widehat{\eta}_l^{n-1} - \widehat{V}_l^n, \quad 1 \le n \le \frac{T}{\tau} - 1, \tag{3.15}$$

$$\widehat{e}_l^0 = 0, \quad \widehat{e}_l^1 = \widehat{U}_l^0, \quad \widehat{\eta}_l^0 = 0, \quad \widehat{\eta}_l^1 = \widehat{V}_l^0, \tag{3.16}$$

where

$$\widehat{U}_l^n = \frac{1}{\beta_l}\int_0^\tau \widehat{P}_l^n(s)\sin(\beta_l(\tau-s))\,ds, \quad \widehat{V}_l^n = \mu_l\int_0^\tau \widehat{Q}_l^n(s)\sin(\mu_l(\tau-s))\,ds, \quad n \ge 0,$$

with

$$\widehat{P}_l^n(s) = \begin{cases} \widehat{F}_l^0(s) - (\widehat{\psi_M^0\phi_M^0})_l, & n = 0, \\ \widehat{F}_l^n(s) + \widehat{F}_l^n(-s) - 2(\widehat{\psi_M^n\phi_M^n})_l, & 1 \le n \le \frac{T}{\tau} - 1, \end{cases}$$
$$\widehat{Q}_l^n(s) = \begin{cases} \widehat{G}_l^0(s) - \left(\widehat{(\psi_M^0)^2}\right)_l, & n = 0, \\ \widehat{G}_l^n(s) + \widehat{G}_l^n(-s) - 2\left(\widehat{(\psi_M^n)^2}\right)_l, & 1 \le n \le \frac{T}{\tau} - 1. \end{cases}$$

We also need the following estimates throughout the rest of the proof. Under the stability condition (3.1), we have for $0 \le s \le \tau$,

$$0 < \mu_l\tau < \beta_l\tau \le \frac{\pi}{3}, \qquad \frac{1}{2} \le \cos(\beta_l\tau) < \cos(\mu_l\tau) < 1,$$
$$0 \le \sin(\beta_l(\tau-s)) \le \sin(\beta_l\tau) < 1, \qquad 0 \le \sin(\mu_l(\tau-s)) \le \sin(\mu_l\tau) < 1.$$

With the help of the Hölder inequality, we have

$$\begin{aligned} \left(1+\mu_l^2\right)\left(\widehat{U}_l^n\right)^2 &= \frac{1+\mu_l^2}{(\beta_l)^2}\left(\int_0^\tau \widehat{P}_l^n(s)\sin(\beta_l(\tau-s))\,ds\right)^2 \\ &= \left(\int_0^\tau \widehat{P}_l^n(s)\sin(\beta_l(\tau-s))\,ds\right)^2 \\ &\le \int_0^\tau \sin(\beta_l(\tau-s))\,ds \cdot \int_0^\tau \left(\widehat{P}_l^n(s)\right)^2 \sin(\beta_l(\tau-s))\,ds \\ &\le (1-\cos(\beta_l\tau))\,\frac{\sin(\beta_l\tau)}{\beta_l}\int_0^\tau \left(\widehat{P}_l^n(s)\right)^2 ds \\ &\le \tau\,(1-\cos(\beta_l\tau))\int_0^\tau \left(\widehat{P}_l^n(s)\right)^2 ds, \quad 0 \le n \le \frac{T}{\tau} - 1. \end{aligned} \tag{3.17}$$

Similarly, we obtain

$$\left(\widehat{V}_l^n\right)^2 \le \tau\,(1-\cos(\mu_l\tau))\int_0^\tau \mu_l^2\left(\widehat{Q}_l^n(s)\right)^2 ds, \quad 0 \le n \le \frac{T}{\tau} - 1. \tag{3.18}$$

Step 2. *Estimates for the case* $n = 1$. Summing up (3.17) and (3.18) with $n = 1$ for $l = 1, \ldots, M-1$, thanks to Parseval's identity, we have

$$\begin{aligned}\|e^1\|_{H^1}^2 &= \frac{b-a}{2}\sum_{l=1}^{M-1}\left(1+\mu_l^2\right)\left(\widehat{e}_l^1\right)^2 = \frac{b-a}{2}\sum_{l=1}^{M-1}\left(1+\mu_l^2\right)\left(\widehat{U}_l^0\right)^2\\ &\le \frac{\tau(b-a)}{2}\sum_{l=1}^{M-1}\int_0^\tau \left(\widehat{P}_l^0(s)\right)^2 ds,\\ \|\eta^1\|_{L^2}^2 &\le \frac{\tau(b-a)}{2}\sum_{l=1}^{M-1}\int_0^\tau \mu_l^2\left(\widehat{Q}_l^0(s)\right)^2 ds.\end{aligned}$$

Using Parseval's identity and the triangle inequality, we get

$$\begin{aligned}\|e^1\|_{H^1}^2 &\le \frac{\tau(b-a)}{2}\sum_{l=1}^{M-1}\int_0^\tau \left(\widehat{(\psi\phi)}_l(s) - (\widehat{\psi_M^0\phi_M^0})_l\right)^2 ds\\ &= \tau\int_0^\tau \left\|\mathcal{P}_M\left[(\psi\phi)(\cdot,s)-\psi_M^0\phi_M^0\right]\right\|_{L^2}^2 ds \le \tau\int_0^\tau \left\|(\psi\phi)(\cdot,s)-\psi_M^0\phi_M^0\right\|_{L^2}^2 ds\\ &\le 2\tau\int_0^\tau \left\|(\psi\phi)(\cdot,s)-\psi^{(0)}\phi^{(0)}\right\|_{L^2}^2 ds + 2\tau^2\left\|\psi^{(0)}\phi^{(0)}-\psi_M^0\phi_M^0\right\|_{L^2}^2,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\|\eta^1\|_{L^2}^2 &\le \frac{\tau(b-a)}{2}\sum_{l=1}^{M-1}\int_0^\tau \left(\mu_l^2\left[\left(\widehat{(\psi)^2}\right)_l(s) - \left(\widehat{(\psi_M^0)^2}\right)_l\right]\right)^2 ds\\ &\le 2\tau\int_0^\tau \left\|\partial_x\psi^2(\cdot,s)-\partial_x\left(\psi^{(0)}\right)^2\right\|_{L^2}^2 ds + 2\tau^2\left\|\frac{d}{dx}\left(\left(\psi^{(0)}\right)^2-\left(\psi_M^0\right)^2\right)\right\|_{L^2}^2.\end{aligned} \tag{3.20}$$

Under the regularity of the solution (ψ,ϕ), using the Hölder inequality, we get

$$\begin{aligned}&\left\|(\psi\phi)(\cdot,s)-\psi^{(0)}\phi^{(0)}\right\|_{L^2}^2\\ &\quad= \int_a^b \left((\psi\phi)(\cdot,s)-\psi^{(0)}\phi^{(0)}\right)^2 dx\\ &\quad= \int_a^b\left(\int_0^s \partial_w(\psi\phi)(x,w)dw\right)^2 dx \le \int_a^b s\int_0^s \left(\partial_w(\psi\phi)(x,w)\right)^2 dwdx\\ &\quad\le 2s\int_0^s \left(\|(\psi_w\phi)(\cdot,w)\|_{L^2}^2 + \|(\psi\phi_w)(\cdot,w)\|_{L^2}^2\right)dw\\ &\quad\le 2s^2\left(\|\phi\|_{L^\infty}^2\|\partial_t\psi\|_{L^\infty([0,T];L^2)}^2 + \|\psi\|_{L^\infty}^2\|\partial_t\phi\|_{L^\infty([0,T];L^2)}^2\right)\lesssim s^2,\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\left\|\psi^{(0)}\phi^{(0)}-\psi_M^0\phi_M^0\right\|_{L^2}^2\\ &\quad\le 2\left(\left\|\phi^{(0)}\right\|_{L^\infty}^2\left\|\psi^{(0)}-\psi_M^0\right\|_{L^2}^2 + \left\|\psi_M^0\right\|_{L^\infty}^2\left\|\phi^{(0)}-\phi_M^0\right\|_{L^2}^2\right)\\ &\quad= 2\left(\left\|\phi^{(0)}\right\|_{L^\infty}^2\left\|\psi^{(0)}-\mathcal{P}_M\psi^{(0)}\right\|_{L^2}^2 + \left\|\psi_M^0\right\|_{L^\infty}^2\left\|\phi^{(0)}-\mathcal{P}_M\phi^{(0)}\right\|_{L^2}^2\right)\\ &\quad\lesssim h^{2m_0}.\end{aligned} \tag{3.22}$$

Similarly to (3.21) and (3.22), we obtain

$$\left\|\partial_x\left((\psi)^2(\cdot,s)-\left(\psi^{(0)}\right)^2\right)\right\|_{L^2}^2$$
$$\le 8s^2\left(\|\psi\|_{L^\infty}^2\|\partial_{tx}\psi\|_{L^\infty([0,T];L^2)}^2+\|\partial_x\psi\|_{L^\infty}^2\|\partial_t\psi\|_{L^\infty([0,T];L^2)}^2\right)\lesssim s^2, \tag{3.23}$$
$$\left\|\frac{d}{dx}\left(\left(\psi^{(0)}\right)^2-\left(\psi_M^0\right)^2\right)\right\|_{L^2}^2$$
$$\le 4\left(\left\|\frac{d}{dx}\psi^{(0)}\right\|_{L^\infty}^2\left\|\psi^{(0)}-\mathcal{P}_M\psi^{(0)}\right\|_{L^2}^2+\left\|\psi^{(0)}\right\|_{L^\infty}^2\left\|\frac{d}{dx}\left(\psi^{(0)}-\mathcal{P}_M\psi^{(0)}\right)\right\|_{L^2}^2\right.$$
$$\left.+\left\|\frac{d}{dx}\psi_M^0\right\|_{L^\infty}^2\left\|\psi^{(0)}-\mathcal{P}_M\psi^{(0)}\right\|_{L^2}^2+\left\|\psi_M^0\right\|_{L^\infty}^2\left\|\frac{d}{dx}\left(\psi^{(0)}-\mathcal{P}_M\psi^{(0)}\right)\right\|_{L^2}^2\right)$$
$$\lesssim h^{2(m_0-1)}. \tag{3.24}$$

Plugging (3.21)–(3.24) into (3.19)–(3.20), noticing (3.12)–(3.13) with $n=1$, we have

$$\left\|e^1\right\|_{H^1}^2\lesssim\tau\int_0^\tau s^2ds+\tau^2h^{2m_0}\lesssim\tau^4+h^{2m_0}\ \Rightarrow\ \left\|e_\psi^1\right\|_{H^1}\lesssim\tau^2+h^{m_0-1}, \tag{3.25}$$
$$\left\|\eta^1\right\|_{L^2}^2\lesssim\tau^4+h^{2(m_0-1)}\ \Rightarrow\ \left\|e_\phi^1\right\|_{L^2}\lesssim\tau^2+h^{m_0-1}, \tag{3.26}$$

which establish the error estimate (3.2) for $n=1$. In addition, (3.25), together with the triangle inequality and inverse inequality (cf. [5]), implies

$$\begin{aligned}\left\|\psi_M^1\right\|_{L^\infty}&\le\|\psi(\cdot,t_1)\|_{L^\infty}+\|\mathcal{P}_M\psi(\cdot,t_1)-\psi(\cdot,t_1)\|_{L^\infty}+\left\|\mathcal{P}_M\psi(\cdot,t_1)-\psi_M^1\right\|_{L^\infty}\\&\lesssim\|\psi(\cdot,t_1)\|_{L^\infty}+\|\mathcal{P}_M\psi(\cdot,t_1)-\psi(\cdot,t_1)\|_{L^\infty}+\left\|\mathcal{P}_M\psi(\cdot,t_1)-\psi_M^1\right\|_{H^1}\\&\lesssim\|\psi(\cdot,t_1)\|_{L^\infty}+h^{m_0-1}+\left\|e^1\right\|_{H^1}\lesssim K_1+h^{m_0-1}+\tau^2.\end{aligned} \tag{3.27}$$

Similarly, using the inverse inequality, we obtain

$$\begin{aligned}\left\|\frac{d}{dx}\psi_M^1\right\|_{L^\infty}&\lesssim\|\partial_x\psi(\cdot,t_1)\|_{L^\infty}+\|\partial_x\left(\mathcal{P}_M\psi(\cdot,t_1)-\psi(\cdot,t_1)\right)\|_{L^\infty}+\left\|\frac{d}{dx}e^1\right\|_{L^\infty}\\&\lesssim\|\partial_x\psi(\cdot,t_1)\|_{L^\infty}+h^{m_0-2}+\frac{1}{h^{1/2}}\left\|e^1\right\|_{L^\infty}\lesssim K_2+h^{m_0-2}+\frac{\tau^2}{h^{1/2}},\end{aligned} \tag{3.28}$$

$$\begin{aligned}\left\|\phi_M^1\right\|_{L^\infty}&\lesssim\|\phi(\cdot,t_1)\|_{L^\infty}+\|\mathcal{P}_M\phi(\cdot,t_1)-\phi(\cdot,t_1)\|_{L^\infty}+\left\|\eta^1\right\|_{L^\infty}\\&\lesssim K_3+h^{m_0-1}+\frac{1}{h^{1/2}}\left\|\eta^1\right\|_{L^2}\lesssim K_3+h^{m_0-2}+\frac{\tau^2}{h^{1/2}}.\end{aligned} \tag{3.29}$$

Thus, under the condition $\tau\lesssim h$, there exist $\tau_1>0$ and $h_2>0$ sufficiently small, such that when $0<\tau\le\tau_1$ and $0<h\le h_2$, we have

$$\left\|\psi_M^1\right\|_{L^\infty}\le K_1+1,\quad\left\|\frac{d}{dx}\psi_M^1\right\|_{L^\infty}\le K_2+1,\quad\left\|\phi_M^1\right\|_{L^\infty}\le K_3+1,$$

which immediately implies that (3.3) is valid for $n=1$.

Step 3. *Estimates for the case* $n \geq 2$. Assuming that (3.2)–(3.3) are valid for all $1 \leq n \leq m-1 \leq \frac{T}{\tau}-1$, we now prove that they are still valid when $n=m$. Denote

$$\mathcal{E}_\psi^n = \sum_{l=1}^{M-1} \widehat{(\mathcal{E}_\psi)}_l^n, \quad \widehat{(\mathcal{E}_\psi)}_l^n = \left(\widehat{e}_l^{n+1}\right)^2 + \left(\widehat{e}_l^n\right)^2 + \frac{\cos(\beta_l\tau)}{1-\cos(\beta_l\tau)}\left(\widehat{e}_l^{n+1}-\widehat{e}_l^n\right)^2, \tag{3.30}$$

$$\mathcal{E}_\phi^n = \sum_{l=1}^{M-1} \widehat{(\mathcal{E}_\phi)}_l^n, \quad \widehat{(\mathcal{E}_\phi)}_l^n = \left(\widehat{\eta}_l^{n+1}\right)^2 + \left(\widehat{\eta}_l^n\right)^2 + \frac{\cos(\mu_l\tau)}{1-\cos(\mu_l\tau)}\left(\widehat{\eta}_l^{n+1}-\widehat{\eta}_l^n\right)^2, \tag{3.31}$$

$$\mathcal{E}^n = \sum_{l=1}^{M-1} \widehat{\mathcal{E}}_l^n, \quad \widehat{\mathcal{E}}_l^n = \left(1+\mu_l^2\right)\widehat{(\mathcal{E}_\psi)}_l^n + \widehat{(\mathcal{E}_\phi)}_l^n, \quad n \geq 0. \tag{3.32}$$

Multiplying both sides of (3.14) by $(1+\mu_l^2)(\widehat{e}_l^{n+1}-\widehat{e}_l^{n-1})$ and then dividing by $1-\cos(\beta_l\tau)$, noticing (3.17) and $\cos(\beta_l\tau) \geq 1/2$, we have

$$\begin{aligned}
&\left(1+\mu_l^2\right)\left[\widehat{(\mathcal{E}_\psi)}_l^n - \widehat{(\mathcal{E}_\psi)}_l^{n-1}\right] \leq \frac{1+\mu_l^2}{1-\cos(\beta_l\tau)}\widehat{U}_l^n\left(\widehat{e}_l^{n+1}-\widehat{e}_l^{n-1}\right)\\
&\quad\leq \frac{1+\mu_l^2}{1-\cos(\beta_l\tau)}\left[\frac{1}{\tau}\left(\widehat{U}_l^n\right)^2 + 2\tau\left(\widehat{e}_l^{n+1}-\widehat{e}_l^n\right)^2 + 2\tau\left(\widehat{e}_l^n-\widehat{e}_l^{n-1}\right)^2\right]\\
&\quad\leq \int_0^\tau \left(\widehat{P}_l^n(s)\right)^2 ds + \frac{4\tau\cos(\beta_l\tau)}{1-\cos(\beta_l\tau)}\left(1+\mu_l^2\right)\left[\left(\widehat{e}_l^{n+1}-\widehat{e}_l^n\right)^2 + \left(\widehat{e}_l^n-\widehat{e}_l^{n-1}\right)^2\right]\\
&\quad\leq 4\tau\left(1+\mu_l^2\right)\left[\widehat{(\mathcal{E}_\psi)}_l^n + \widehat{(\mathcal{E}_\psi)}_l^{n-1}\right] + \int_0^\tau \left(\widehat{P}_l^n(s)\right)^2 ds, \quad n \geq 1. \qquad (3.33)
\end{aligned}$$

Similarly, multiplying both sides of (3.15) by $\widehat{\eta}_l^{n+1}-\widehat{\eta}_l^{n-1}$ and then dividing by $1-\cos(\mu_l\tau)$, noticing (3.18) and $\cos(\mu_l\tau) \geq 1/2$, we get

$$\widehat{(\mathcal{E}_\phi)}_l^n - \widehat{(\mathcal{E}_\phi)}_l^{n-1} \leq 4\tau\left[\widehat{(\mathcal{E}_\phi)}_l^n + \widehat{(\mathcal{E}_\phi)}_l^{n-1}\right] + \int_0^\tau \mu_l^2\left(\widehat{Q}_l^n(s)\right)^2 ds, \quad n \geq 1. \tag{3.34}$$

From (3.33) and (3.34), we have

$$\widehat{\mathcal{E}}_l^n - \widehat{\mathcal{E}}_l^{n-1} \leq 4\tau\left[\widehat{\mathcal{E}}_l^n + \widehat{\mathcal{E}}_l^{n-1}\right] + \int_0^\tau \left[\left(\widehat{P}_l^n(s)\right)^2 + \mu_l^2\left(\widehat{Q}_l^n(s)\right)^2\right] ds, \quad n \geq 1.$$

Summing up the above inequalities for $l=1,\ldots,M-1$, we obtain

$$\mathcal{E}^n - \mathcal{E}^{n-1} \leq 4\tau\left(\mathcal{E}^n + \mathcal{E}^{n-1}\right) + \int_0^\tau \sum_{l=1}^{M-1}\left[\left(\widehat{P}_l^n(s)\right)^2 + \mu_l^2\left(\widehat{Q}_l^n(s)\right)^2\right] ds, \quad n \geq 1.$$

Summing up the above inequalities for $n=1,\ldots,m-1$, when $\tau \leq 1/8$, we get

$$\mathcal{E}^{m-1} \leq 2\mathcal{E}^0 + 8\tau\sum_{n=0}^{m-2}\mathcal{E}^n + 2\sum_{n=1}^{m-1}\int_0^\tau \sum_{l=1}^{M-1}\left[\left(\widehat{P}_l^n(s)\right)^2 + \mu_l^2\left(\widehat{Q}_l^n(s)\right)^2\right] ds.$$

Using the discrete Gronwall's inequality [5, 28], we have

$$\mathcal{E}^{m-1} \leq C_1\left\{\mathcal{E}^0 + \sum_{n=1}^{m-1}\int_0^\tau \sum_{l=1}^{M-1}\left[\left(\widehat{P}_l^n(s)\right)^2 + \mu_l^2\left(\widehat{Q}_l^n(s)\right)^2\right] ds\right\},$$

where C_1 is a constant independent of h (or l) and τ (or m). From the above inequality and (3.30)–(3.32), with the help of the triangle inequality and Parseval's identity, we have

$$\|e^m\|_{H^1}^2 + \|\eta^m\|_{L^2}^2 = \frac{b-a}{2} \sum_{l=1}^{M-1} \left[\left(1+\mu_l^2\right) (\widehat{e}_l^m)^2 + (\widehat{\eta}_l^m)^2 \right] \le \frac{b-a}{2}\, \mathcal{E}^{m-1}$$

$$\le \frac{(b-a)C_1}{2} \left\{ \mathcal{E}^0 + \sum_{n=1}^{m-1} \int_0^\tau \sum_{l=1}^{M-1} \left[\left(\widehat{P}_l^n(s)\right)^2 + \mu_l^2 \left(\widehat{Q}_l^n(s)\right)^2 \right] ds \right\}. \tag{3.35}$$

From (3.30)–(3.32) with $n = 0$ and (3.16), noticing (3.17), (3.18), (3.21)–(3.24), we obtain

$$\begin{aligned}
\mathcal{E}^0 &= \sum_{l=1}^{M-1} \frac{1+\mu_l^2}{1-\cos(\beta_l\tau)} \left(\widehat{e}_l^1\right)^2 + \frac{1}{1-\cos(\mu_l\tau)} \left(\widehat{\eta}_l^1\right)^2 \\
&= \sum_{l=1}^{M-1} \frac{1+\mu_l^2}{1-\cos(\beta_l\tau)} \left(\widehat{U}_l^0\right)^2 + \frac{1}{1-\cos(\mu_l\tau)} \left(\widehat{V}_l^0\right)^2 \\
&\le \tau \sum_{l=1}^{M-1} \int_0^\tau \left[\left(\widehat{P}_l^0(s)\right)^2 + \mu_l^2 \left(\widehat{Q}_l^0(s)\right)^2 \right] ds \lesssim \tau^4 + h^{2(m_0-1)}.
\end{aligned} \tag{3.36}$$

Also, using Parseval's identity, we have

$$\begin{aligned}
\sum_{l=1}^{M-1} \left(\widehat{P}_l^n(s)\right)^2 &= \sum_{l=1}^{M-1} \left[\widehat{(\psi\phi)}_l(t_n+s) + \widehat{(\psi\phi)}_l(t_n-s) - 2\widehat{(\psi_M^n\phi_M^n)}_l \right]^2 \\
&\le \frac{2}{b-a} \left\| (\psi\phi)(\cdot,t_n+s) + (\psi\phi)(\cdot,t_n-s) - 2\psi_M^n\phi_M^n \right\|_{L^2}^2, \\
\sum_{l=1}^{M-1} \mu_l^2 \left(\widehat{Q}_l^n(s)\right)^2 &= \sum_{l=1}^{M-1} \mu_l^2 \left[\widehat{(\psi^2)}_l(t_n+s) + \widehat{(\psi^2)}_l(t_n-s) - 2\left(\widehat{(\psi_M^n)^2}\right)_l \right]^2 \\
&\le \frac{2}{b-a} \left\| \partial_x \left[\psi^2(\cdot,t_n+s) + \psi^2(\cdot,t_n-s) - 2\left(\psi_M^n\right)^2 \right] \right\|_{L^2}^2.
\end{aligned}$$

Due to the regularity of (ψ,ϕ), with the help of the triangle inequality and the Hölder inequality, we get

$$\begin{aligned}
&\left\| (\psi\phi)(\cdot,t_n+s) + (\psi\phi)(\cdot,t_n-s) - 2\psi_M^n\phi_M^n \right\|_{L^2}^2 \\
&\le 2\left\| (\psi\phi)(\cdot,t_n+s) + (\psi\phi)(\cdot,t_n-s) - 2(\psi\phi)(\cdot,t_n) \right\|_{L^2}^2 + 8\left\| (\psi\phi)(\cdot,t_n) - \psi_M^n\phi_M^n \right\|_{L^2}^2 \\
&\le 2\int_a^b \left(\int_0^s \int_{-r}^r \partial_{ww}(\psi\phi)(x,t_n+w)\,dwdr \right)^2 dx + 8\left\| (\psi\phi)(\cdot,t_n) - \psi_M^n\phi_M^n \right\|_{L^2}^2 \\
&\le 2\int_0^s s \int_a^b \left(\int_{-r}^r \partial_{ww}(\psi\phi)(x,t_n+w)\,dw \right)^2 dxdr + 8\left\| (\psi\phi)(\cdot,t_n) - \psi_M^n\phi_M^n \right\|_{L^2}^2 \\
&\le 4\int_0^s sr \int_{-r}^r \left\| \partial_{ww}(\psi\phi)(x,t_n+w) \right\|_{L^2}^2 dwdr + 8\left\| (\psi\phi)(\cdot,t_n) - \psi_M^n\phi_M^n \right\|_{L^2}^2 \\
&\le \frac{8s^4}{3} \left\| \partial_{tt}(\psi\phi)(\cdot,t) \right\|_{L^\infty([0,T];L^2)}^2 + 8\left\| (\psi\phi)(\cdot,t_n) - \psi_M^n\phi_M^n \right\|_{L^2}^2.
\end{aligned}$$

Similarly to (3.22), we have

$$\begin{aligned}&\left\|(\psi\phi)(\cdot,t_n)-\psi_M^n\phi_M^n\right\|_{L^2}\\&\quad\le\left\|\phi(\cdot,t_n)\right\|_{L^\infty}\left\|\psi(\cdot,t_n)-\psi_M^n\right\|_{L^2}+\left\|\psi_M^n\right\|_{L^\infty}\left\|\phi(\cdot,t_n)-\phi_M^n\right\|_{L^2}.\end{aligned}$$

Combining the above inequality and

$$\begin{aligned}&\left\|\partial_{tt}(\psi\phi)(\cdot,t)\right\|_{L^\infty([0,T];L^2)}\\&\quad=\left\|\partial_{tt}\psi(\cdot,t)\phi+2\partial_t\psi(\cdot,t)\partial_t\phi(\cdot,t)+\partial_{tt}\phi(\cdot,t)\psi\right\|_{L^\infty([0,T];L^2)}\\&\quad\le\left\|\psi\right\|_{L^\infty}\left\|\partial_{tt}\phi(\cdot,t)\right\|_{L^\infty([0,T];L^2)}+2\left\|\partial_t\psi(\cdot,t)\right\|_{L^\infty([0,T];L^4)}\left\|\partial_t\phi(\cdot,t)\right\|_{L^\infty([0,T];L^4)}\\&\qquad+\left\|\phi\right\|_{L^\infty}\left\|\partial_{tt}\psi(\cdot,t)\right\|_{L^\infty([0,T];L^2)},\end{aligned}$$

we get

$$\left\|(\psi\phi)(\cdot,t_n+s)+(\psi\phi)(\cdot,t_n-s)-2\psi_M^n\phi_M^n\right\|_{L^2}^2\lesssim\tau^4+\left(\tau^2+h^{m_0-1}\right)^2. \tag{3.37}$$

Similarly, we can obtain

$$\begin{aligned}&\left\|\partial_x\left[\psi^2(\cdot,t_n+s)+\psi^2(\cdot,t_n-s)-2\left(\psi_M^n\right)^2\right]\right\|_{L^2}^2\\&\quad\le\frac{8s^4}{3}\left\|\partial_{ttx}\left(\psi(\cdot,t)\right)^2\right\|_{L^\infty([0,T];L^2)}^2+8\left\|\partial_x\left(\left(\psi(\cdot,t_n)\right)^2-\left(\psi_M^n\right)^2\right)\right\|_{L^2}^2\\&\quad\lesssim s^4\left[\left\|\psi\right\|_{L^\infty}^2\left\|\partial_{ttx}\psi\right\|_{L^\infty([0,T];L^2)}^2+\left\|\partial_x\psi\right\|_{L^\infty}^2\left\|\partial_{tt}\psi\right\|_{L^\infty([0,T];L^2)}^2\right.\\&\qquad\left.+\left\|\partial_{tx}\psi\right\|_{L^\infty([0,T];L^4)}^2\left\|\partial_t\psi\right\|_{L^\infty([0,T];L^4)}^2\right]\\&\qquad+\left(\left\|\partial_x\psi\right\|_{L^\infty}^2+\left\|\frac{d}{dx}\psi_M^n\right\|_{L^\infty}^2\right)\left\|\psi(\cdot,t_n)-\psi_M^n\right\|_{L^2}^2\\&\qquad+\left(\left\|\psi\right\|_{L^\infty}^2+\left\|\psi_M^n\right\|_{L^\infty}^2\right)\left\|\partial_x\left(\psi(\cdot,t_n)-\psi_M^n\right)\right\|_{L^2}^2\\&\quad\lesssim\tau^4+\left(\tau^2+h^{m_0-1}\right)^2.\end{aligned}\tag{3.38}$$

Plugging the estimates (3.36), (3.37), and (3.38) into (3.35), we have

$$\begin{aligned}&\left\|e^m\right\|_{H^1}^2+\left\|\eta^m\right\|_{L^2}^2\\&\quad\lesssim\tau^4+h^{2(m_0-1)}+\tau(m-1)\left[\tau^4+\left(\tau^2+h^{m_0-1}\right)^2\right]\\&\quad\lesssim\tau^4+h^{2(m_0-1)}+T\left[\tau^4+\left(\tau^2+h^{m_0-1}\right)^2\right]\le C_2\left(\tau^2+h^{m_0-1}\right)^2,\end{aligned}$$

where C_2 is a constant independent of h (or l) and τ (or m). This, together with (3.12)–(3.13), implies that (3.2) is valid for $n=m$. In addition, similarly to (3.27)–(3.29), we have

$$\begin{aligned}&\left\|\psi_M^m\right\|_{L^\infty}\lesssim K_1+h^{m_0-1}+\tau^2,\quad\left\|\frac{d}{dx}\psi_M^m\right\|_{L^\infty}\lesssim K_2+h^{m_0-2}+\frac{\tau^2}{h^{1/2}},\\&\left\|\phi_M^m\right\|_{L^\infty}\lesssim K_3+h^{m_0-2}+\frac{\tau^2}{h^{1/2}}.\end{aligned}$$

Again, under the assumption $\tau \lesssim h$, there exist $\tau_2 > 0$ and $h_3 > 0$ sufficiently small and independent of $2 \le m \le T/\tau$, such that when $0 < \tau \le \tau_2$ and $0 < h \le h_3$, we have

$$\|\psi_M^m\|_{L^\infty} \le K_1 + 1, \qquad \left\|\frac{d}{dx}\psi_M^m\right\|_{L^\infty} \le K_2 + 1, \qquad \|\phi_M^m\|_{L^\infty} \le K_3 + 1.$$

Thus, (3.3) is also valid for $n = m$. Therefore, the proof of (3.2)–(3.3) is completed by the method of mathematical induction (noting that C_1 and C_2 are chosen uniformly for all m) with the choice of $\tau_0 = \min\{1/8, \tau_1, \tau_2\}$ and $h_0 = \min\{h_1, h_2, h_3\}$. ∎

3.3. Proof of Theorem 3.2. The proof follows the same procedures as those for Theorem 3.1 and we will use the same notation. When $n = 0$, the proof of (3.4)–(3.5) is trivial. For $1 \le n \le \frac{T}{\tau} - 1$, similarly to (2.20)–(2.21) and (2.24)–(2.25), differentiating the exact solution (3.7) with respect to time t and then setting $t = t_n$, we obtain

$$\partial_t \psi(x, t_n) = \sum_{l=1}^{\infty} \widehat{\psi}_l'(t_n)\ \Phi_l(x), \quad \partial_t \phi(x, t_n) = \sum_{l=1}^{\infty} \widehat{\phi}_l'(t_n)\ \Phi_l(x), \tag{3.39}$$

where

$$\widehat{\psi}_l'(t_1) = -\beta_l \sin(\beta_l \tau)\,\widehat{(\psi^{(0)})}_l + \cos(\beta_l\tau)\,\widehat{(\psi^{(1)})}_l - \int_0^\tau \widehat{F}_l^0(s)\cos(\beta_l(\tau - s))\,ds,$$

$$\widehat{\phi}_l'(t_1) = -\mu_l \sin(\mu_l \tau)\,\widehat{(\phi^{(0)})}_l + \cos(\mu_l\tau)\,\widehat{(\phi^{(1)})}_l - \mu_l^2 \int_0^\tau \widehat{G}_l^0(s)\cos(\mu_l(\tau - s))\,ds.$$

$$\widehat{\psi}_l'(t_{n+1}) = \widehat{\psi}_l'(t_{n-1}) - 2\beta_l \sin(\beta_l\tau)\,\widehat{\psi}_l(t_n) - \int_0^\tau \widehat{(F^+)}_l^n(s)\cos(\beta_l(\tau - s))\,ds,$$

$$\widehat{\phi}_l'(t_{n+1}) = \widehat{\phi}_l'(t_{n-1}) - 2\mu_l \sin(\mu_l\tau)\,\widehat{\phi}_l(t_n) - \mu_l^2 \int_0^\tau \widehat{(G^+)}_l^n(s)\cos(\mu_l(\tau - s))\,ds.$$

The rest of the proof is similar to those procedures in Theorem 3.1 and we omit the details here for brevity. ∎

Remark 3.1. The numerical methods we have discussed in the previous sections for the KGZ system can be easily extended to two dimensions when $\Omega = [a, b] \times [c, d]$ is a rectangle and three dimensions when $\Omega = [a, b] \times [c, d] \times [e, f]$ is a box. Let h be the mesh size. Then the error bounds for the EWI-SP in Theorems 3.1 and 3.2 are still valid in two and three dimensions provided that we replace the operator L by $(-\Delta)^{-1/2}$.

4. Numerical results. In this section, numerical results are reported to illustrate our error estimates for the EWI-SP method and compare different time integrators with fixed $\varepsilon = \varepsilon_0 = \mathcal{O}(1)$ and $\gamma = \gamma_0 = \mathcal{O}(1)$. In addition, we also study and compare numerically the resolution (or ε-scalability) of different numerical methods for the KGZ system in the simultaneous high-plasma-frequency and subsonic limit regime, i.e., $(\varepsilon, \gamma) \to 0$ under $\varepsilon \lesssim \gamma$. In order to do so, we take $d = 1$ and $\gamma = 5\varepsilon$ in (1.1)–(1.2) and choose the initial conditions in (1.3) as

$$\psi^{(0)}(x) = \frac{\sin(x)\mathrm{e}^{0.5x^2}}{1 + \mathrm{e}^{x^2}},\ \psi^{(1)}(x) = \frac{2\mathrm{e}^{-x^2}}{\sqrt{\pi}\varepsilon^2},\ \phi^{(0)}(x) = \frac{2\cos(x)\mathrm{e}^{-0.5x^2}}{\sqrt{\pi}},\ \phi^{(1)}(x) = 0.$$

In our computation, we truncate the problem on a bounded interval $\Omega = [-16, 16]$, i.e., $a = -16$ and $b = 16$, which is large enough such that the homogeneous Dirichlet

TABLE 4.1
Spatial error analysis of EWI-SP: $\mathrm{e}_\varepsilon^{\tau,h}$ *at* $t = 1$ *with* $\tau = 10^{-6}$ *for different* ε *and mesh size* h.

$\mathrm{e}_\varepsilon^{\tau,h}$	$h_0 = 1$	$h_0/2$	$h_0/4$	$h_0/8$
$\varepsilon_0 = 0.4$	6.88E-2	5.20E-3	9.86E-7	6.10E-13
$\varepsilon_0/2^2$	2.58E-1	9.40E-3	2.42E-6	1.54E-12
$\varepsilon_0/2^4$	1.21E-1	7.70E-3	4.83E-6	1.62E-12
$\varepsilon_0/2^6$	2.14E-1	3.30E-3	6.65E-7	1.49E-12
$\varepsilon_0/2^8$	1.84E-1	1.20E-3	7.66E-7	1.34E-12
$\varepsilon_0/2^{10}$	1.92E-1	9.97E-4	7.44E-7	1.19E-12

TABLE 4.2
Temporal error analysis of EWI-SP: $\mathrm{e}_\varepsilon^{\tau,h}$ *at* $t = 1$ *with* $h = 1/8$ *for different* ε *and time step* τ.

$\mathrm{e}_\varepsilon^{\tau,h}$	$\tau_0 = 0.1$	$\tau_0/2^2$	$\tau_0/2^4$	$\tau_0/2^6$	$\tau_0/2^8$	$\tau_0/2^{10}$
$\varepsilon_0 = 0.4$	2.88E−2	1.70E−3	1.04E−4	6.47E−6	4.03E−7	3.65E−8
rate	—	2.05	2.00	2.00	2.00	1.80
$\varepsilon_0/2^1$	4.21E+0	2.01E−2	1.20E−3	7.45E−5	4.59E−6	2.22E−7
rate	—	3.85	2.03	2.00	2.01	2.10
$\varepsilon_0/2^2$	7.36E−1	1.36E+0	1.49E−2	8.95E−4	5.49E−5	2.57E−6
rate	—	-0.44	3.25	2.02	2.01	2.20
$\varepsilon_0/2^3$	1.07E+0	4.65E−1	4.09E−1	9.60E−3	5.69E−4	2.66E−5
rate	—	0.60	0.10	2.70	2.04	2.20
$\varepsilon_0/2^4$	8.75E−1	2.44E−1	3.49E−1	3.26E−1	7.30E−3	3.31E−4
rate	—	0.92	-0.26	0.05	2.73	2.20
$\varepsilon_0/2^5$	2.93E+0	1.59E+0	2.35E−1	4.22E−1	4.18E−1	6.90E−3
rate	—	0.44	1.38	-0.42	0.01	2.95

boundary condition (2.3) does not introduce significant aliasing errors relative to the whole space problem. In order to quantify and compare the numerical results by different methods, we introduce the following "error" function

$$e_\varepsilon^{\tau,h}(t_n) := \|\psi^n - \psi(\cdot,t_n)\|_h + \|D_x(\psi^n - \psi(\cdot,t_n))\|_h + \|\phi^n - \phi(\cdot,t_n)\|_h,$$

where $\|\cdot\|_h$ is the standard l^2-norm for a vector $U \in X_M$ and

$$\|U\|_h^2 = h\sum_{j=1}^{M-1}|U_j|^2, \qquad \|D_xU\|_h^2 = \frac{b-a}{2}\sum_{l=1}^{M-1}\mu_l^2|\widetilde{U}_l|^2.$$

Since there is no analytical solution for the problem, the "exact" solution is obtained by using the EWI-SP method with very small mesh size h and time step τ. Table 4.1 lists the spatial discretization errors $e_\varepsilon^{\tau,h}(t)$ at $t = 1$ for the EWI-SP method with a very small time step $\tau = 10^{-6}$, which guarantees that the temporal discretization error can be ignored, for different ε and mesh size h. The spatial discretization errors of ECFD-SP, SIFD-SP, and EXFD-SP behave similarly and we omit them here for brevity. Similarly, Tables 4.2, 4.3, and 4.4 show the temporal discretization error of EWI-SP, ECFD-SP, and EXFD-SP, respectively, with a very small mesh size $h = 1/8$, which guarantees that the spatial discretization error can be ignored, for different ε and time step τ. Again, the temporal discretization errors of SIFD-SP behaves similarly to ECFD-SP and we omit them here for brevity too. For comparison, Table 4.5 shows temporal discretization errors of different numerical methods. In addition, Figure 4.1 depicts time evolution of the solution $\psi(0,t)$ for different ε and

Table 4.3
Temporal error analysis of ECFD-SP: $\mathrm{e}_\varepsilon^{\tau,h}$ *at* $t=1$ *with* $h=1/8$ *for different* ε *and time step* τ.

$\mathrm{e}_\varepsilon^{\tau,h}$	$\tau_0=0.1$	$\tau_0/2^2$	$\tau_0/2^4$	$\tau_0/2^6$	$\tau_0/2^8$	$\tau_0/2^{10}$
$\varepsilon_0=0.4$	1.02E+0	7.78E−2	4.90E−3	3.10E−4	1.94E−5	1.22E−6
rate	—	1.86	1.99	1.99	2.00	1.99
$\varepsilon_0/2^1$	1.32E+0	2.80E+0	2.27E−1	1.41E−2	8.84E−4	5.53E−5
rate	—	-0.54	1.81	2.00	2.00	2.00
$\varepsilon_0/2^2$	2.40E+0	2.20E+0	1.87E+0	8.13E−1	5.30E−2	3.30E−3
rate	—	0.06	0.12	0.60	1.97	2.00
$\varepsilon_0/2^3$	2.58E+0	2.78E+0	1.53E+0	2.78E+0	2.65E+0	1.50E−1
rate	—	0.10	0.43	-0.43	0.03	2.07
$\varepsilon_0/2^4$	3.09E+0	2.06E+0	2.39E+0	2.27E+0	3.36E+0	2.49E+0
rate	—	0.29	-0.11	0.04	-0.28	0.22
$\varepsilon_0/2^5$	2.63E+0	2.38E+0	1.88E+0	2.04E+0	2.79E+0	1.53E+0
rate	—	0.07	0.17	-0.06	-0.22	0.43

Table 4.4
Temporal error analysis of EXFD-SP: $\mathrm{e}_\varepsilon^{\tau,h}$ *at* $t=1$ *with* $h=1/8$ *for different* ε *and time step* τ. *Here and in what follows, "*" means that the method is numerically unstable under the corresponding choice of* ε *and* τ.

$\mathrm{e}_\varepsilon^{\tau,h}$	$\tau_0=0.1$	$\tau_0/2^2$	$\tau_0/2^4$	$\tau_0/2^6$	$\tau_0/2^8$	$\tau_0/2^{10}$
$\varepsilon_0=0.4$	2.79E+3	1.58E−2	9.87E−4	6.18E−5	3.86E−6	2.51E−7
rate	—	8.72	2.00	2.00	2.00	1.97
$\varepsilon_0/2^1$	*	*	4.51E−2	2.80E−3	1.76E−4	1.10E−5
rate	—	*	*	2.00	2.00	2.00
$\varepsilon_0/2^2$	*	*	2.15E+0	1.70E−1	1.06E−2	6.63E−4
rate	—	*	*	1.83	2.00	2.00
$\varepsilon_0/2^3$	*	*	*	6.85E−1	4.52E−1	2.94E−2
rate	—	*	*	*	0.30	1.97
$\varepsilon_0/2^4$	*	*	*	*	2.32E+0	1.75E+0
rate	—	*	*	*	*	0.21
$\varepsilon_0/2^5$	*	*	*	*	*	2.16E+0
rate	—	*	*	*	*	*

Table 4.5
Comparison between different methods: $\mathrm{e}_\varepsilon^{\tau,h}$ *at* $t=1$ *with* $h=1/8$ *for different* ε *and time step* τ.

$\mathrm{e}_\varepsilon^{\tau,h}$	$(\varepsilon_0=0.4,\tau_0=0.1)$	$(\varepsilon_0/2,\tau_0/2^2)$	$(\varepsilon_0/2^2,\tau_0/2^4)$	$(\varepsilon_0/2^3,\tau_0/2^6)$
EWI-SP	2.88E−2	2.01E−2	1.49E−2	9.60E−3
ECFD-SP	1.02E+0	2.80E+0	1.87E+0	2.78E+0
SIFD-SP	9.44E−1	2.77E+0	1.83E+0	2.76E+0
EXFD-SP	2.79E+3	*	2.15E+0	6.85E−1
$\mathrm{e}_\varepsilon^{\tau,h}$	$(\varepsilon_0=0.4,\tau_0=0.025)$	$(\varepsilon_0/2,\tau_0/2^2)$	$(\varepsilon_0/2^2,\tau_0/2^4)$	$(\varepsilon_0/2^3,\tau_0/2^6)$
EWI-SP	1.70E−3	1.20E−3	8.95E−4	5.69E−4
ECFD-SP	7.78E−2	2.27E−1	8.13E−1	2.65E+0
SIFD-SP	7.21E−2	2.21E−1	8.10E−1	2.65E+0
EXFD-SP	1.58E−2	4.51E−2	1.70E−1	4.52E−1

the error $|\psi(0,t)-\psi^n(0)|$, where $\psi^n(x)$ is obtained from the numerical solution ψ^n by interpolation, at $x=0$ with a very small mesh size $h=1/8$ for different ε and time step τ for the EWI-SP and ECFD-SP methods under different meshing strategies or ε-scalability.

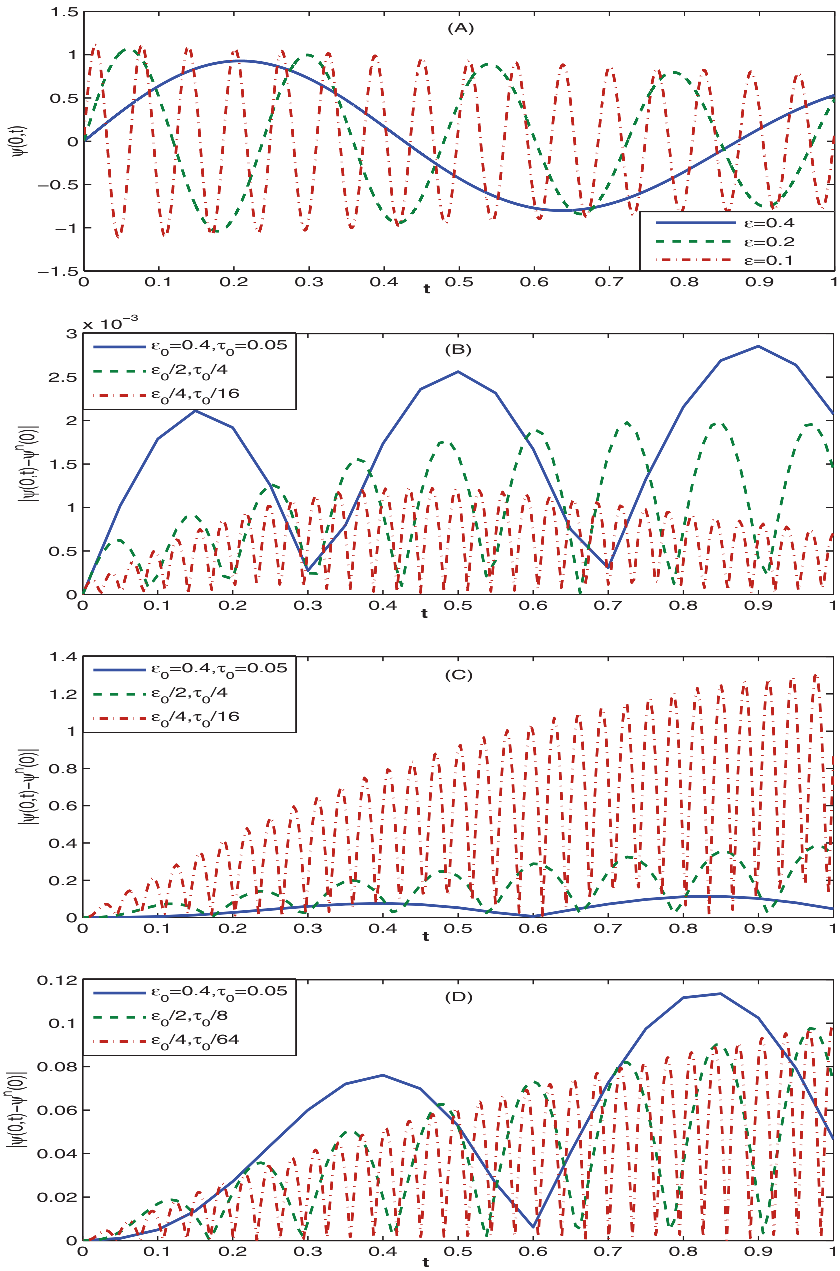

FIG. 4.1. *Time evolution of the solution* $\psi(0,t)$ *for different* ε (A), *and the error* $|\psi(0,t)-\psi^n(0)|$ *with a very small mesh size* $h = 1/8$ *for different* ε *and time step* τ *for EWI-SP under the meshing strategy (or* ε*-scalability)* $\tau = \mathcal{O}(\varepsilon^2)$ (B) *and for ECFD-SP under meshing strategies* $\tau = \mathcal{O}(\varepsilon^2)$ (C) *and* $\tau = \mathcal{O}(\varepsilon^3)$ (D).

From Tables 4.1–4.5 and Figure 4.1 as well as additional results not shown here for brevity, we can draw the following conclusions.

(i) When $0 < \varepsilon \ll 1$ and $0 < \gamma \ll 1$ satisfying $\varepsilon \lesssim \gamma$, the solution of the KGS system propagates waves with wavelength at $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon^2)$ in space and time, respectively (cf. Figure 4.1(A)).

(ii) For any fixed $0 < \varepsilon \leq 1$ and $0 < \gamma \leq 1$, the spatial discretization errors for all the methods EWI-SP, ECFD-SP, SIFD-SP, and EXFD-SP are spectrally accurate, which is independent of ε and τ (cf. Table 4.1).

(iii) For any fixed $\varepsilon = \varepsilon_0 = \mathcal{O}(1)$ and $\gamma = \gamma_0 = \mathcal{O}(1)$, the temporal discretization errors for all the methods EWI-SP, ECFD-SP, SIFD-SP, and EXFD-SP are second-order provided the time step τ is small enough (cf. Tables 4.2, 4.3, and 4.4, each row above the diagonal). In addition, for fixed τ, the error from EWI-SP is much smaller than those from ECFD-SP, SIFD-SP, and EXFD-SP, especially when ε and/or γ are very small (cf. Table 4.5).

(iv) In the simultaneous high-plasma-frequency and subsonic limit regime, i.e., $0 < \varepsilon \ll 1$ and $0 \leq \gamma \ll 1$ satisfying $\varepsilon \lesssim \gamma$, in order to obtain "correct" approximations of the KGZ system, the ε-scalability or meshing strategy for the EWI-SP method is (cf. Tables 4.1 and 4.2 and Figure 4.1(B))

$$\tau = \mathcal{O}(\varepsilon^2), \qquad h = \mathcal{O}(1)\text{-independent of } \varepsilon; \tag{4.1}$$

and, respectively, for the ECFD-SP, SIFD-SP, and EXFD-SP it is (cf. Tables 4.3–4.4 and Figure 4.1(D))

$$\tau = \mathcal{O}(\varepsilon^3), \qquad h = \mathcal{O}(1)\text{-independent of } \varepsilon. \tag{4.2}$$

These clearly demonstrate that the EWI-SP method has much better resolution than the ECFD-SP, SIFD-SP, and EXFD-SP methods in this regime. In fact, similar conclusions for these methods have been observed numerically and analyzed analytically when they are applied for the KG equation in the nonrelativistic limit regime [5].

(v) The memory costs of all the methods EWI-SP, ECFD-SP, SIFD-SP, and EXFD-SP are similar, and at each time step, the computational costs for EWI-SP, SIFD-SP, and EXFD-SP are similar, where ECFD-SP is much more expensive. In addition, EWI-SP, ECFD-SP, and SIFD-SP are stable in computation for all $0 < \varepsilon \ll 1$, $0 < \gamma \ll 1$, $0 < h \lesssim 1$, and $0 < \tau \lesssim 1$, while EXFD-SP has a very severe numerical stability constraint when ε and/or γ are small (cf. Table 4.4). Specifically, ECFD-SP conserves the energy in the discretized level and the fluctuation of the energy in the EWI-SP method is bounded for a very long time and decreases quadratically when the time step τ decreases, i.e., the EWI-SP method conserves the energy essentially for the KGZ system in practical computation [12, 18, 19, 21, 22].

(vi) Based on these observations, when $\varepsilon = \mathcal{O}(1)$ and $\gamma = \mathcal{O}(1)$, all the numerical methods perform similarly. Thus any method can be used, e.g., EXFD-SP and/or EWI-SP have been used for solving the KGZ system due to their small computational cost. However, when $0 < \varepsilon \ll 1$ and/or $0 < \gamma \ll 1$, EWI-SP performs much better than ECFD-SP, SIFD-SP, and EXFD-SP in terms of stability, computational cost, and ε-scalability. Thus, in general, we suggest adopting the EWI-SP method for solving the KGZ system numerically, especially in the simultaneous high-plasma-frequency and subsonic limit regime, i.e., $0 < \varepsilon \ll 1$ and $0 \leq \gamma \ll 1$.

5. Conclusions. We proposed and analyzed an EWI-SP method for solving the KGZ system involving two dimensionless parameters $0 < \varepsilon \leq 1$ and $0 < \gamma \leq 1$ which are inversely proportional to the plasma frequency and the sound speed, respectively. The EWI-SP method is based on the application of sine pseudospectral discretization for spatial derivatives and a Gautschi-type EWI for temporal derivatives. It is fully explicit and thus efficient in practical computation, symmetric in time, of second-order accuracy in time, and spectral-order accuracy in space. In the $\mathcal{O}(1)$-wave regime, i.e., $\varepsilon = \varepsilon_0 = \mathcal{O}(1)$ and $\gamma = \gamma_0 = \mathcal{O}(1)$, we established rigorously optimal error bounds for the EWI-SP method in the energy space $H^1 \times L^2$. In the simultaneous high-plasma-frequency and subsonic limit regime, i.e., $(\varepsilon, \gamma) \to 0$ under $\varepsilon \lesssim \gamma$, we demonstrated numerically that the ε-scalability of the EWI-SP method is time step $\tau = \mathcal{O}(\varepsilon^2)$ and mesh size $h = \mathcal{O}(1)$-independent of ε. This ε-scalability is in general much better than those of the finite difference time integrators, which are usually at $\tau = \mathcal{O}(\varepsilon^3)$. In addition, we also observed numerically that the EWI-SP method has the property of near conservation of the total energy over a long time when it is applied to discretize the KGZ system. Finally, we remark that we are working on a rigorous error bound which depends explicitly on the mesh size h, time step τ, and ε for the EWI-SP method in the simultaneous high-plasma-frequency and subsonic limit regime, i.e., $(\varepsilon, \gamma) \to 0$ under $\varepsilon \lesssim \gamma$.

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