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UNIFORM ERROR ESTIMATES OF A FINITE DIFFERENCE METHOD FOR THE KLEIN-GORDON-SCHRÖDINGER SYSTEM IN THE NONRELATIVISTIC AND MASSLESS LIMIT REGIMES

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Abstract. We establish a uniform error estimate of a finite difference method for the Klein-Gordon-Schrödinger (KGS) equations with two dimensionless parameters $0 < \gamma \leq 1$ and $0 < \varepsilon \leq 1$, which are the mass ratio and inversely proportional to the speed of light, respectively. In the simultaneously nonrelativistic and massless limit regimes, i.e., $\gamma \sim \varepsilon$ and $\varepsilon \to 0^+$, the KGS equations converge singularly to the Schrödinger-Yukawa (SY) equations. When $0 < \varepsilon \ll 1$, due to the perturbation of the wave operator and/or the incompatibility of the initial data, which is described by two parameters $\alpha \geq 0$ and $\beta \geq -1$, the solution of the KGS equations oscillates in time with $O(\varepsilon)$-wavelength, which requires harsh meshing strategy for classical numerical methods. We propose a uniformly accurate method based on two key points: (i) reformulating KGS system into an asymptotic consistent formulation, and (ii) applying an integral approximation of the oscillatory term. Using the energy method and the limiting equation via the SY equations with an oscillatory potential, we establish two independent error bounds at $O(h^2 + \tau^2/\varepsilon)$ and $O(h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*})$ with h mesh size, τ time step and $\alpha^* = \min\{1, \alpha, 1+\beta\}$. This implies that the method converges uniformly and optimally with quadratic convergence rate in space and uniformly in time at $O(\tau^{4/3})$ and $O(\tau^{1+\frac{\alpha^*}{2+\alpha^*}})$ for well-prepared ($\alpha^* = 1$) and ill-prepared ($0 \leq \alpha^* < 1$) initial data, respectively. Thus the ε-scalability of the method is $\tau = O(1)$ and $h = O(1)$ for $0 < \varepsilon \leq 1$, which is significantly better than classical methods. Numerical results are reported to confirm our error bounds. Finally, the method is applied to study the convergence rates of KGS equations to its limiting models in the simultaneously nonrelativistic and massless limit regimes.

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Key words and phrases. Klein-Gordon-Schrödinger equations, Schrödinger-Yukawa equations, nonrelativistic limit, massless limit, highly oscillatory, finite difference method, error estimates.

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1. **Introduction.** We consider the coupled Klein-Gordon-Schrödinger (KGS) equations which describe a system of conserved scalar nucleons interacting with neutral scalar mesons coupled through the Yukawa interaction [12, 27, 40]:

$$\begin{cases} i\hbar\partial_t\psi(\mathbf{x},t) + \dfrac{\hbar^2}{2m_1}\Delta\psi(\mathbf{x},t) + \eta\phi(\mathbf{x},t)\psi(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathbb{R}^d, \quad d=1,2,3, \\ \dfrac{1}{c^2}\partial_{tt}\phi(\mathbf{x},t) - \Delta\phi(\mathbf{x},t) + \dfrac{m_2^2c^2}{\hbar^2}\phi(\mathbf{x},t) - \eta|\psi(\mathbf{x},t)|^2 = 0, \quad t>0. \end{cases} \tag{1.1}$$

Here ψ represents a complex scalar nucleon field and ϕ is a real scalar meson field, $\hbar$ is the Planck constant, c is the speed of light, $m_1 > 0$ is the mass of a nucleon, $m_2 > 0$ is the mass of a meson and $\eta > 0$ is the coupling constant. The KGS system describes a classical model of the Yukawa interaction between conservative complex nucleon field and neutral meson in quantum field theory [40]. It is widely applied in many physical fields, such as many-body physics [11], nonlinear plasmas and complex geophysical flows [18], nonlinear optics and optical communications [33] and nonlinear quantum electrodynamics [31].

For scaling the KGS system (1.1), introduce

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{x_s}, \quad \tilde{t} = \frac{t}{t_s}, \quad \tilde{\psi}(\tilde{\mathbf{x}},\tilde{t}) = x_s^{d/2}\psi(\mathbf{x},t), \quad \tilde{\phi}(\tilde{\mathbf{x}},\tilde{t}) = \frac{\phi(\mathbf{x},t)}{\phi_s}, \tag{1.2}$$

where x_s, $t_s = \frac{2m_1x_s^2}{\hbar}$ and $\phi_s = \frac{\hbar x_s^{-d/2}}{\sqrt{2m_1}}$ are length unit, time unit and meson field unit, respectively, to be taken for the nondimensionalization of the KGS (1.1) via (1.2). Plugging (1.2) into (1.1) and removing all '$\sim$', we get the following dimensionless KGS system as

$$\begin{cases} i\partial_t\psi(\mathbf{x},t) + \Delta\psi(\mathbf{x},t) + \lambda\phi(\mathbf{x},t)\psi(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ \varepsilon^2\partial_{tt}\phi(\mathbf{x},t) - \Delta\phi(\mathbf{x},t) + \dfrac{\gamma^2}{\varepsilon^2}\phi(\mathbf{x},t) - \lambda|\psi(\mathbf{x},t)|^2 = 0, \end{cases} \tag{1.3}$$

where $\gamma := \frac{m_2}{2m_1}$ is the mass ratio, $\lambda = \frac{\eta\sqrt{2m_1}x_s^{2-d/2}}{\hbar}$ and $\varepsilon := \frac{v}{c} = \frac{\hbar}{2cm_1x_s}$ is the ratio of the wave speed $v = \frac{x_s}{t_s} = \frac{\hbar}{2m_1x_s}$ and the speed of light. For the KGS system (1.3), it is dispersive and time symmetric. Moreover, it conserves the *mass*

$$\mathcal{M}(t) = \|\psi(\cdot,t)\|^2 := \int_{\mathbb{R}^d}|\psi(\mathbf{x},t)|^2d\mathbf{x} \equiv \mathcal{M}(0), \quad t\geq 0, \tag{1.4}$$

and the *Hamiltonian*

$$\mathcal{H}(t) := \int_{\mathbb{R}^d}\left[\frac{1}{2}\left(\varepsilon^2|\partial_t\phi|^2 + |\nabla\phi|^2 + \frac{\gamma^2}{\varepsilon^2}|\phi|^2\right) + |\nabla\psi|^2 - \lambda\phi|\psi|^2\right]d\mathbf{x} \equiv \mathcal{H}(0). \tag{1.5}$$

If one sets the dimensionless length unit $x_s = \frac{\hbar}{2cm_1}$, then $\varepsilon = 1$, which corresponds to the classical regime. This choice of x_s is appropriate when the wave speed is at the same order of the speed of light. However, a different choice of x_s is more suitable when the wave speed is much smaller than the speed of light. We remark here that the choice of x_s determines the observation scale of time evolution of the system and decides which phenomena can be resolved by discretization on specified spatial/temporal grids and which phenomena is 'visible' by asymptotic analysis.

Different parameter regimes could be considered for the KGS system(1.3) which is displayed in Figure 1.1:

- Standard regime, i.e., $\varepsilon = 1$ and $\gamma = 1$ ($\Longleftrightarrow x_s = \frac{\hbar}{2cm_1}$ and $m_2 = 2m_1$): there were extensive analytical and numerical studies for the KGS equations

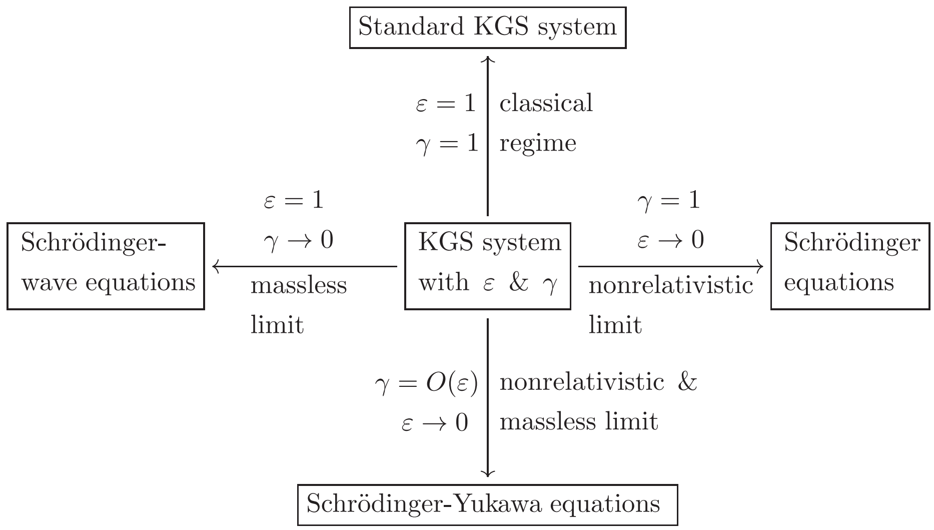

FIGURE 1.1. Diagram of different limits of the KGS system (1.3).

(1.3) with $\varepsilon = \gamma = 1$ in the last two decades. For the well-posedness, we refer to [12–15, 19]; for the attractors and asymptotic behavior of the system, we refer to [7, 16, 17, 25, 26, 29]; and for plane, solitary, and periodic wave solutions, we refer to [9, 21, 36] as well as the references therein. For the numerical part, many efficient numerical methods have been proposed for the KGS system, such as the finite difference method [30, 38, 41], the conservative spectral method [39], the time-splitting spectral method [5], trigonometric spectral method [20], the discrete-time orthogonal cubic spline collocation method [37], the Chebyshev pseudospectral multidomain method [10], and the symplectic and multi-symplectic methods [22–24].

- Massless limit regime, i.e., $\varepsilon = 1$ and $0 < \gamma \ll 1$ ($\Longleftrightarrow x_s = \frac{\hbar}{2cm_1}$ and $m_2 \ll m_1$): the KGS system (1.3) converges – regularly – to the Schrödinger-wave equations

$$\begin{cases} i\partial_t\psi(\mathbf{x},t) + \Delta\psi(\mathbf{x},t) + \lambda\phi(\mathbf{x},t)\psi(\mathbf{x},t) = 0, \\ \partial_{tt}\phi(\mathbf{x},t) - \Delta\phi(\mathbf{x},t) - \lambda|\psi(\mathbf{x},t)|^2 = 0, \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \end{cases} \tag{1.6}$$

with quadratic convergence rate in terms of γ. Any numerical methods for the KGS equations (1.3) in the standard regime can be applied in this regime.

- Nonrelativistic limit regime, i.e., $\gamma = 1$ and $0 < \varepsilon \ll 1$: by taking the ansatz

$$\phi(\mathbf{x},t) = e^{it/\varepsilon^2} z(\mathbf{x},t) + e^{-it/\varepsilon^2}\overline{z}(\mathbf{x},t) + o(\varepsilon), \quad \mathbf{x}\in\mathbb{R}^d, \quad t\ge 0, \tag{1.7}$$

where $\bar{z}$ denotes the complex conjugate of a complex-valued function z, the KGS (1.3) converges – singularly – to the Schrödinger equations [6], i.e., (ψ, z) satisfies either the Schrödinger equations with wave operator [6]

$$\begin{cases} i\partial_t\psi(\mathbf{x},t) + \Delta\psi(\mathbf{x},t) + \lambda\left(e^{it/\varepsilon^2} z(\mathbf{x},t) + e^{-it/\varepsilon^2}\overline{z}(\mathbf{x},t)\right)\psi(\mathbf{x},t) = 0, \\ 2i\partial_t z(\mathbf{x},t) + \varepsilon^2\partial_{tt} z(\mathbf{x},t) - \Delta z(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \end{cases} \tag{1.8}$$

or the Schrödinger equations [6]

$$\begin{cases} i\partial_t\psi(\mathbf{x},t)+\Delta\psi(\mathbf{x},t)+\lambda\left(e^{it/\varepsilon^2}z(\mathbf{x},t)+e^{-it/\varepsilon^2}\overline{z}(\mathbf{x},t)\right)\psi(\mathbf{x},t)=0, \\ 2i\partial_t z(\mathbf{x},t)-\Delta z(\mathbf{x},t)=0, \qquad \mathbf{x}\in\mathbb{R}^d, \qquad t>0. \end{cases} \tag{1.9}$$

In addition, a multiscale time integrator Fourier pseudospectral method was proposed in [6] and it was proved that the method converges in space and time with exponential and linear convergence rates, respectively, which are uniformly for $0<\varepsilon\le 1$.

- Simultaneously nonrelativistic and massless limit regimes, i.e., $\gamma\sim\varepsilon$ and $0<\varepsilon\ll 1$, the KGS system (1.3) converges – singularly – to the Schrödinger-Yukawa (SY) equations, which was rigourously analyzed in [2]. To our best knowledge, there is no rigorous numerical analysis for different numerical methods for the KGS system (1.3) in this regime, especially on how the error bound depends on the small parameter $\varepsilon\in(0,1]$.

In this paper we consider the KGS equations (1.3) in the simultaneously nonrelativistic and massless limit regimes, i.e., $0<\varepsilon\ll 1$ and $\gamma=\delta\varepsilon$ with $\delta>0$ a fixed constant which is independent of ε. For simplicity of notation, we choose $\delta=1$ and $\lambda=1$, in which case we denote the functions as $(\psi^\varepsilon,\phi^\varepsilon)$ in (1.3) and the system reads as

$$\begin{cases} i\partial_t\psi^\varepsilon(\mathbf{x},t)+\Delta\psi^\varepsilon(\mathbf{x},t)+\phi^\varepsilon(\mathbf{x},t)\psi^\varepsilon(\mathbf{x},t)=0, \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ \varepsilon^2\partial_{tt}\phi^\varepsilon(\mathbf{x},t)-\Delta\phi^\varepsilon(\mathbf{x},t)+\phi^\varepsilon(\mathbf{x},t)-|\psi^\varepsilon(\mathbf{x},t)|^2=0, \end{cases} \tag{1.10}$$

with initial data

$$\psi^\varepsilon(\mathbf{x},0)=\psi_0(\mathbf{x}), \quad \phi^\varepsilon(\mathbf{x},0)=\phi_0^\varepsilon(\mathbf{x}), \quad \partial_t\phi^\varepsilon(\mathbf{x},0)=\phi_1^\varepsilon(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d. \tag{1.11}$$

Similar to the properties of the Zakharov system [3, 28, 32], the solution of the KGS equations (1.10) propagates highly oscillatory waves at wavelength $O(\varepsilon)$ and $O(1)$ in time and space, respectively, and rapid outgoing initial layers at speed $O(1/\varepsilon)$ in space. This highly temporal oscillatory nature in the solution of the KGS equations (1.10) brings significant numerical difficulties, especially when $0<\varepsilon\ll 1$ [3, 28, 32]. For example, classical methods may request harsh meshing strategy (or ε-scalability) in order to get 'correct' oscillatory solutions when $\varepsilon\ll 1$ [8, 34]. Recently, we proposed and analyzed uniform accurate finite difference methods for the Zakharov system [3] and Klein-Gordon-Zakharov system [4] in the subsonic limit regime by adopting an asymptotic consistent formulation. The main aim of this paper is to propose and analyze a finite difference method for the KGS equations, which is uniformly accurate in both space and time for $0<\varepsilon\ll 1$. The key ingredients rely on (i) reformulating the KGS system into an asymptotic consistent formulation and (ii) using an integral approximation of the oscillatory term. Other techniques include the energy method, cut-off technique for treating the nonlinearity and the inverse inequalities to bound the numerical solution, and the limiting equation via a Schrödinger-Yukawa system with an oscillatory potential.

The rest of the paper is organized as follows. In Section 2, we recall the singular limit of the KGS system in the nonrelativistic and massless limit regimes and introduce an asymptotic consistent formulation for the KGS equations. In Section 3, we present a finite difference method and state our main results. Section 4 is devoted to the details of the error estimates. Numerical results are reported in Section 5 to confirm our error bounds. Finally some conclusions are drawn in Section 6. Throughout the paper, we adopt the standard Sobolev spaces as well as the corresponding norms and denote $A \lesssim B$ to represent that there exists a generic constant $C > 0$ independent of ε, τ, h, such that $|A| \le C\,B$.

2. Singular limit of the KGS equations in the nonrelativistic and massless limit regimes.

In this section, we recall the limit behavior of the KGS system (1.10) when $\varepsilon \to 0$ and give an asymptotic consistent formulation for (1.10).

2.1. Convergence of the KGS system to the Schrödinger-Yukawa equations.

Formally setting $\varepsilon \to 0$ in the KGS equations (1.10), one can get the following nonlinear Schrödinger-Yukawa (SY) system [2, 29]:

$$\begin{cases} i\partial_t\psi^0(\mathbf{x},t) + \Delta\psi^0(\mathbf{x},t) + \phi^0(\mathbf{x},t)\psi^0(\mathbf{x},t) = 0, \\ -\Delta\phi^0(\mathbf{x},t) + \phi^0(\mathbf{x},t) - |\psi^0(\mathbf{x},t)|^2 = 0, \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ \psi^0(\mathbf{x},0) = \psi_0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d. \end{cases} \tag{2.12}$$

It can be derived from (2.12) that $\phi^0(\mathbf{x},t)$ satisfies

$$\phi^0 := \phi^0(\mathbf{x},t) = (-\Delta + I)^{-1}|\psi^0(\mathbf{x},t)|^2, \quad \mathbf{x}\in\mathbb{R}^d, \quad t\ge 0, \tag{2.13}$$

where I is the identity operator. Setting $t = 0$ in (2.13), we get

$$\phi_0(\mathbf{x}) := \phi^0(\mathbf{x},0) = (-\Delta + I)^{-1}|\psi_0(\mathbf{x})|^2, \quad \mathbf{x}\in\mathbb{R}^d. \tag{2.14}$$

Multiplying the first equation in (2.12) by $\overline{\psi_0(\mathbf{x},t)}$ and subtracting from its conjugate, we obtain

$$\begin{aligned} \partial_t|\psi^0(\mathbf{x},t)|^2 &= -i\left[\psi^0(\mathbf{x},t)\Delta\overline{\psi^0(\mathbf{x},t)} - \overline{\psi^0(\mathbf{x},t)}\Delta\psi^0(\mathbf{x},t)\right] \\ &= -i\nabla\cdot\left[\psi^0(\mathbf{x},t)\nabla\overline{\psi^0(\mathbf{x},t)} - \overline{\psi^0(\mathbf{x},t)}\nabla\psi^0(\mathbf{x},t)\right] \\ &= -\nabla\cdot\mathbf{c}^0(\mathbf{x},t), \qquad \mathbf{x}\in\mathbb{R}^d, \quad t\ge 0, \end{aligned} \tag{2.15}$$

where $\overline{f}$ denotes the complex conjugate of f and $\mathbf{c}^0$ is the so-called current [2] for the SY equations (2.12) with its definition as

$$\mathbf{c}^0(\mathbf{x},t) = i\left[\psi^0(\mathbf{x},t)\nabla\overline{\psi^0(\mathbf{x},t)} - \overline{\psi^0(\mathbf{x},t)}\nabla\psi^0(\mathbf{x},t)\right]. \tag{2.16}$$

Differentiating (2.13) with respect to t, setting $t = 0$ and noticing (2.15), we get

$$\phi_1(\mathbf{x}) := \partial_t\phi^0(\mathbf{x},0) = (-\Delta + I)^{-1}\partial_t|\psi^0|^2(\mathbf{x},0) = -(-\Delta + I)^{-1}\nabla\cdot\mathbf{c}_0(\mathbf{x}), \tag{2.17}$$

where

$$\mathbf{c}_0(\mathbf{x}) := \mathbf{c}^0(\mathbf{x},0) = i\left[\psi_0(\mathbf{x})\nabla\overline{\psi_0(\mathbf{x})} - \overline{\psi_0(\mathbf{x})}\nabla\psi_0(\mathbf{x})\right], \quad \mathbf{x}\in\mathbb{R}^d.$$

Based on the above results, the initial data $(\psi_0, \phi_0^\varepsilon, \phi_1^\varepsilon)$ can be decomposed as

$$\phi_0^\varepsilon(\mathbf{x}) = \phi_0(\mathbf{x}) + \varepsilon^\alpha\omega_0(\mathbf{x}), \quad \phi_1^\varepsilon(\mathbf{x}) = \phi_1(\mathbf{x}) + \varepsilon^\beta\omega_1(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d, \tag{2.18}$$

where $\alpha \ge 0$ and $\beta \ge -1$ are parameters describing the incompatibility of the initial data of the KGS equations (1.10) with respect to that of the SY equations (2.12) in the nonrelativistic and massless limit regimes such that the Hamiltonian (1.5) is bounded, $\omega_0(\mathbf{x})$ and $\omega_1(\mathbf{x})$ are two given real functions independent of ε. Due to the perturbation of the wave operator '$\varepsilon^2\partial_{tt}$' or the inconsistency of the initial data, the solution of the KGS equations would display high oscillation in time at $O(\varepsilon)$-wavelength with amplitude at $O(\varepsilon^{\min\{2,\alpha,1+\beta\}})$, and propagate rapid outspreading initial layers at speed $O(1/\varepsilon)$ in space. To illustrate the temporal oscillation and rapid outgoing wave phenomena, Figure 2.2 shows the solutions $\phi^\varepsilon(x,1)$, $\phi^\varepsilon(1,t)$ of the KGS system (1.10) for $d=1$ and the initial data

$$\begin{aligned} &\psi_0(x) = e^{-\frac{x^2}{2}+i\frac{x}{2}}, \quad \omega_0(x) = f\left((x+18)/10\right) f\left((18-x)/9\right)\sin\left(2x+\pi/6\right), \\ &\omega_1(x) = f\left((x+10)/5\right) f\left((10-x)/5\right)\sin\left(x/2\right), \end{aligned} \tag{2.19}$$

with

$$f(x) = \chi_{[1,\infty)} + \chi_{(0,1)}\left(1+e^{\frac{1-2x}{x-x^2}}\right)^{-1}, \tag{2.20}$$

and χ being the characteristic function, $\alpha=\beta=0$ in (2.18) for different ε, which was obtained numerically by the exponential-wave-integrator and time-splitting sine pseudospectral method on a bounded interval $[-200,200]$ with homogenous Dirichlet boundary condition [5].

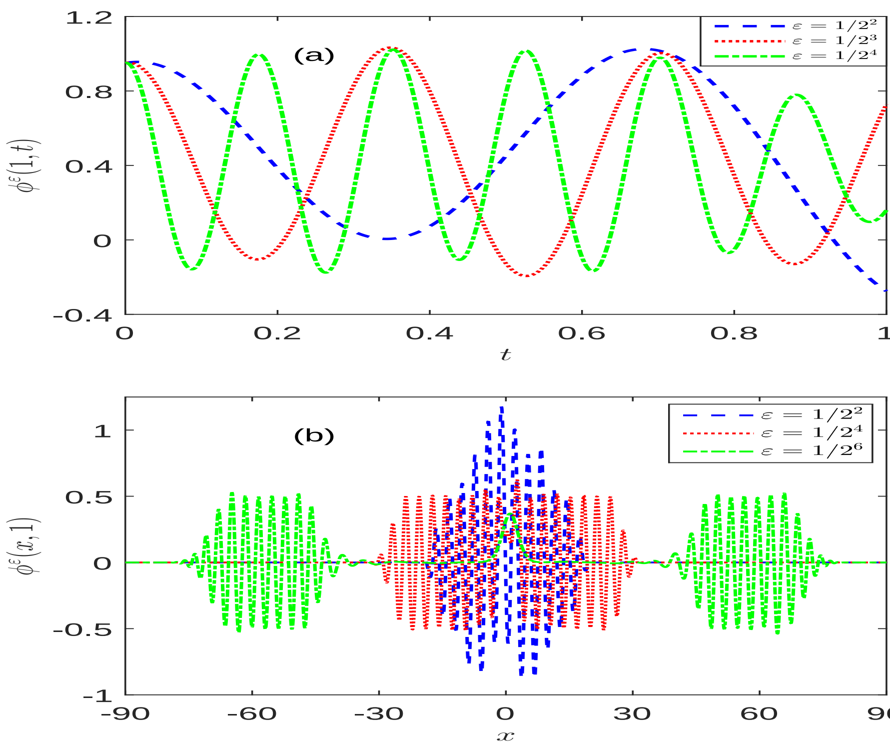

FIGURE 2.2. The temporal oscillation (a) and rapid outspreading wave in space (b) of the KGS system (1.10).

2.2. **An asymptotic consistent formulation.** Inspired by the analysis concerning on the convergence between the Zakharov system (ZS) and the limiting cubically Schrödinger equation [28] and the uniform method for solving the ZS in the subsonic limit regime [3], we introduce

$$\chi^\varepsilon(\mathbf{x},t) = \phi^\varepsilon(\mathbf{x},t) - \varphi^\varepsilon(\mathbf{x},t) - \omega^\varepsilon(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^d, \quad t\ge 0, \tag{2.21}$$

where $\varphi^\varepsilon(\mathbf{x},t) = (-\Delta+I)^{-1}|\psi^\varepsilon|^2(\mathbf{x},t)$, and $\omega^\varepsilon(\mathbf{x},t)$ represents the initial layer caused by the incompatibility of the initial data (2.18), which is the solution of the linear wave-type equation

$$\begin{cases} \varepsilon^2\partial_{tt}\omega^\varepsilon(\mathbf{x},t) - \Delta\omega^\varepsilon(\mathbf{x},t) + \omega^\varepsilon(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ \omega^\varepsilon(\mathbf{x},0) = \varepsilon^\alpha\omega_0(\mathbf{x}), \quad \partial_t\omega^\varepsilon(\mathbf{x},0) = \varepsilon^\beta\omega_1(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d. \end{cases} \tag{2.22}$$

Substituting (2.21) into the KGS equations (1.10), we can reformulate it into an asymptotic consistent formulation as

$$\begin{cases} i\partial_t\psi^\varepsilon(\mathbf{x},t) + \Delta\psi^\varepsilon(\mathbf{x},t) + \left[\varphi^\varepsilon(\mathbf{x},t) + \chi^\varepsilon(\mathbf{x},t) + \omega^\varepsilon(\mathbf{x},t)\right]\psi^\varepsilon(\mathbf{x},t) = 0, \\ \varepsilon^2\partial_{tt}\chi^\varepsilon(\mathbf{x},t) - \Delta\chi^\varepsilon(\mathbf{x},t) + \chi^\varepsilon(\mathbf{x},t) + \varepsilon^2\partial_{tt}\varphi^\varepsilon(\mathbf{x},t) = 0, \quad t>0, \\ -\Delta\varphi^\varepsilon(\mathbf{x},t) + \varphi^\varepsilon(\mathbf{x},t) - |\psi^\varepsilon(\mathbf{x},t)|^2 = 0, \quad \mathbf{x}\in\mathbb{R}^d, \\ \psi^\varepsilon(\mathbf{x},0) = \psi_0(\mathbf{x}), \quad \chi^\varepsilon(\mathbf{x},0) = 0, \quad \partial_t\chi^\varepsilon(\mathbf{x},0) = 0, \quad \mathbf{x}\in\mathbb{R}^d. \end{cases} \tag{2.23}$$

The advantage of this formulation is that the main oscillatory wave with amplitude at $O(1)$ in ϕ^ε, which is caused by the inconsistency of the initial data, is now removed by the initial layer ω^ε in (2.22), which is easy to solve separately. In the nonrelativistic and massless limit regimes, i.e., $\varepsilon\to 0^+$, formally we have $\psi^\varepsilon(\mathbf{x},t)\to\psi^0(\mathbf{x},t)$ and $\chi^\varepsilon(\mathbf{x},t)\to 0$, where $\psi^0(\mathbf{x},t)$ is the solution of the SY system (2.12). Moreover, as $\varepsilon\to 0^+$, formally we can also get $\psi^\varepsilon(\mathbf{x},t)\to\widetilde{\psi}^\varepsilon(\mathbf{x},t)$, where $\widetilde{\psi}^\varepsilon := \widetilde{\psi}^\varepsilon(\mathbf{x},t)$ is the solution of the Schrödinger-Yukawa equations with an oscillatory potential $\omega^\varepsilon(\mathbf{x},t)$ (SY-OP):

$$\begin{cases} i\partial_t\widetilde{\psi}^\varepsilon(\mathbf{x},t) + \Delta\widetilde{\psi}^\varepsilon(\mathbf{x},t) + \left[\widetilde{\varphi}^\varepsilon(\mathbf{x},t) + \omega^\varepsilon(\mathbf{x},t)\right]\widetilde{\psi}^\varepsilon(\mathbf{x},t) = 0, \\ -\Delta\widetilde{\varphi}^\varepsilon(\mathbf{x},t) + \widetilde{\varphi}^\varepsilon(\mathbf{x},t) - |\widetilde{\psi}^\varepsilon(\mathbf{x},t)|^2 = 0, \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ \widetilde{\psi}^\varepsilon(\mathbf{x},0) = \psi_0(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d. \end{cases} \tag{2.24}$$

Similar to the convergence of the Zakharov system to the nonlinear Schrödinger equation in the subsonic limit [28], we can obtain the following result concerning on the convergence from the KGS system (2.23) to the SY-OP (2.24)

$$\|\chi^\varepsilon\|_{H^1} + \|\chi^\varepsilon\|_{L^\infty} + \|\psi^\varepsilon(\cdot,t) - \widetilde{\psi}^\varepsilon(\cdot,t)\|_{H^1} \le C_T\varepsilon^2, \quad 0\le t\le T, \tag{2.25}$$

where $0<T<T^*$ with $T^*>0$ being the maximum common existence time of the solutions of the KGS system (2.23) and the SY-OP (2.24) and C_T is a positive constant independent of ε. To illustrate this, Figure 2.3 depicts the convergence behavior between the solutions of the KGS equations (2.23) and the SY-OP (2.24), where $e^\varepsilon_\chi(t) := \|\chi^\varepsilon(\cdot,t)\|_{H^1}$, $e^\varepsilon_\infty(t) := \|\chi^\varepsilon(\cdot,t)\|_{L^\infty}$ and $e^\varepsilon_\psi(t) := \|\psi^\varepsilon(\cdot,t) - \widetilde{\psi}^\varepsilon(\cdot,t)\|_{H^1}$ for different ε with the same initial data as in (2.19) for $d=1$, $\alpha=0$ and $\beta=-1$.

3. A finite difference method and main results.

In this section, we present a finite difference scheme for the reformulated KGS equations (2.23) and give its uniform error bounds.

3.1. A uniformly accurate finite difference method.

For simplicity of notation, we only present the numerical method for the KGS system on one space dimension, and extensions to higher dimensions are straightforward. Practically, similar to most works for computation of the Zakharov-type equations [3,30], (2.23) is truncated on a bounded domain $\Omega=(a,b)$ with homogeneous Dirichlet boundary condition:

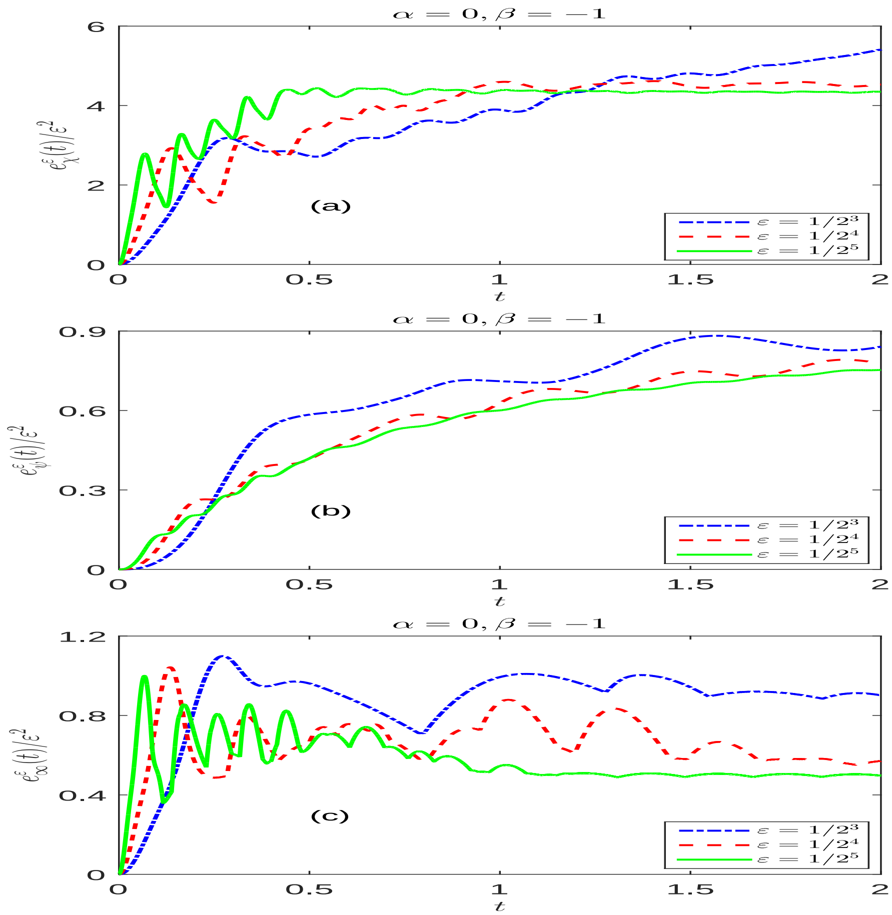

FIGURE 2.3. Time evolution of $e^\varepsilon_\chi(t)$, $e^\varepsilon_\psi(t)$ and $e^\varepsilon_\infty(t)$.

$$\begin{cases} i\partial_t\psi^\varepsilon(x,t) + \partial_{xx}\psi^\varepsilon(x,t) + [\varphi^\varepsilon(x,t) + \chi^\varepsilon(x,t) + \omega^\varepsilon(x,t)]\,\psi^\varepsilon(x,t) = 0, \\ \varepsilon^2\partial_{tt}\chi^\varepsilon(x,t) - \partial_{xx}\chi^\varepsilon(x,t) + \chi^\varepsilon(x,t) + \varepsilon^2\partial_{tt}\varphi^\varepsilon(x,t) = 0, \\ -\partial_{xx}\varphi^\varepsilon(x,t) + \varphi^\varepsilon(x,t) - |\psi^\varepsilon(x,t)|^2 = 0, \quad x \in \Omega, \quad t > 0, \\ \psi^\varepsilon(x,0) = \psi_0(x), \quad \chi^\varepsilon(x,0) = 0, \quad \partial_t\chi^\varepsilon(x,0) = 0, \quad x \in \overline{\Omega}, \\ \psi^\varepsilon(a,t) = \varphi^\varepsilon(a,t) = \chi^\varepsilon(a,t) = 0, \quad \psi^\varepsilon(b,t) = \varphi^\varepsilon(b,t) = \chi^\varepsilon(b,t) = 0, \end{cases} \tag{3.26}$$

where $\omega^\varepsilon(x,t)$ is defined as the solution of (2.22) with homogeneous Dirichlet boundary condition for $d = 1$,

$$\begin{cases} \varepsilon^2\partial_{tt}\omega^\varepsilon(x,t) - \partial_{xx}\omega^\varepsilon(x,t) + \omega^\varepsilon(x,t) = 0, \quad x \in \Omega, \quad t > 0, \\ \omega^\varepsilon(x,0) = \varepsilon^\alpha\omega_0(x), \quad \partial_t\omega^\varepsilon(x,0) = \varepsilon^\beta\omega_1(x), \quad x \in \overline{\Omega}, \\ \omega^\varepsilon(a,t) = \omega^\varepsilon(b,t) = 0, \quad t \geq 0. \end{cases} \tag{3.27}$$

As $\varepsilon \to 0$, formally we have $\psi^\varepsilon(x,t) \to \widetilde{\psi}^\varepsilon(x,t)$ and $\chi^\varepsilon(x,t) \to 0$, where $\widetilde{\psi}^\varepsilon(x,t)$ is the solution of the SY-OP equations with homogeneous boundary condition

$$\begin{cases} i\partial_t\widetilde{\psi}^\varepsilon(x,t)+\partial_{xx}\widetilde{\psi}^\varepsilon(x,t)+\big[\widetilde{\varphi}^\varepsilon(x,t)+\omega^\varepsilon(x,t)\big]\widetilde{\psi}^\varepsilon(x,t)=0,\\ -\partial_{xx}\widetilde{\varphi}^\varepsilon(x,t)+\widetilde{\varphi}^\varepsilon(x,t)-|\widetilde{\psi}^\varepsilon(x,t)|^2=0,\quad x\in\Omega,\quad t>0,\\ \widetilde{\psi}^\varepsilon(x,0)=\psi_0(x),\quad x\in\overline{\Omega};\quad \widetilde{\psi}^\varepsilon(x,t)=\widetilde{\varphi}^\varepsilon(x,t)=0,\quad x\in\partial\Omega,\quad t\geq 0.\end{cases} \tag{3.28}$$

Choose a mesh size $h:=\Delta x=\frac{b-a}{M}$ with M being a positive integer and a time step $\tau:=\Delta t>0$. Denote the grid points and time steps as

$$x_j:=a+jh,\quad j=0,1,\cdots,M;\quad t_k:=k\tau,\quad k=0,1,2,\cdots.$$

Define the index sets $\mathcal{T}_M=\{j\mid j=1,2,\cdots,M-1\}$, $\mathcal{T}_M^0=\{j\mid j=0,1,\cdots,M\}$. Denote

$$X_M=\Big\{v=(v_0,v_1,\ldots,v_M)^T\ \mid v_0=v_M=0\Big\}\subseteq\mathbb{C}^{M+1},$$

equipped with inner products and norms defined as

$$(u,v)=h\sum_{j=1}^{M-1}u_j\overline{v_j},\quad \langle\delta_x^+u,\delta_x^+v\rangle=h\sum_{j=0}^{M-1}(\delta_x^+u_j)\,(\delta_x^+\overline{v_j}),\quad \|u\|_\infty=\sup_{j\in\mathcal{T}_M^0}|u_j|,$$
$$\|u\|^2=(u,u),\quad \|\delta_x^+u\|^2=\langle\delta_x^+u,\delta_x^+u\rangle,\quad \|u\|_{H^1}^2=\|u\|^2+\|\delta_x^+u\|^2.$$

Then we have for $u,\,v\in X_M$,

$$(-\delta_x^2u,v)=\langle\delta_x^+u,\delta_x^+v\rangle=(u,-\delta_x^2v). \tag{3.29}$$

Let $\psi_j^{\varepsilon,k}$, $\varphi_j^{\varepsilon,k}$ and $\chi_j^{\varepsilon,k}$ be the approximations of $\psi^\varepsilon(x_j,t_k)$, $\varphi^\varepsilon(x_j,t_k)$ and $\chi^\varepsilon(x_j,t_k)$, respectively, and denote $\psi^{\varepsilon,k}=(\psi_0^{\varepsilon,k},\cdots,\psi_M^{\varepsilon,k})^T$, $\varphi^{\varepsilon,k}=(\varphi_0^{\varepsilon,k},\cdots,\varphi_M^{\varepsilon,k})^T$, $\chi^{\varepsilon,k}=(\chi_0^{\varepsilon,k},\cdots,\chi_M^{\varepsilon,k})^T\in X_M$ as the numerical solution vectors at $t=t_k$. The finite difference operators are the standard notations as:

$$\delta_x^+E_j^k=\frac{E_{j+1}^k-E_j^k}{h},\quad \delta_t^+E_j^k=\frac{E_j^{k+1}-E_j^k}{\tau},\quad \delta_t^cE_j^k=\frac{E_j^{k+1}-E_j^{k-1}}{2\tau},$$
$$\delta_t^2E_j^k=\frac{E_j^{k+1}-2E_j^k+E_j^{k-1}}{\tau^2},\quad \delta_x^2E_j^k=\frac{E_{j+1}^k-2E_j^k+E_{j-1}^k}{h^2}.$$

To simplify notations, for a function $E(x,t)$, and a grid function $E^k\in X_M$ $(k\geq 0)$, we denote for $k\geq 1$

$$E(x,t_{[k]})=\frac{E(x,t_{k+1})+E(x,t_{k-1})}{2},\quad x\in\overline{\Omega};\quad E_j^{[k]}=\frac{E_j^{k+1}+E_j^{k-1}}{2},\quad j\in\mathcal{T}_M^0.$$

In this paper, we consider the finite difference discretization of (3.26) as following

$$i\delta_t^c\psi_j^{\varepsilon,k}+\big(\delta_x^2+\varphi_j^{\varepsilon,k}+\mu_j^{\varepsilon,k}+\chi_j^{\varepsilon,[k]}\big)\psi_j^{\varepsilon,[k]}=0, \tag{3.30a}$$
$$\varepsilon^2\delta_t^2\chi_j^{\varepsilon,k}+(1-\delta_x^2)\chi_j^{\varepsilon,[k]}+\varepsilon^2\delta_t^2\varphi_j^{\varepsilon,k}=0,\quad j\in\mathcal{T}_M,\quad k\geq 1, \tag{3.30b}$$
$$\varphi_j^{\varepsilon,k}-\delta_x^2\varphi_j^{\varepsilon,k}-|\psi_j^{\varepsilon,k}|^2=0,\quad j\in\mathcal{T}_M,\quad k\geq 0, \tag{3.30c}$$

where we apply an average of the oscillatory potential ω^ε over the interval $[t_{k-1},t_{k+1}]$

$$\mu_j^{\varepsilon,k}=\frac{1}{2\tau}\int_{t_{k-1}}^{t_{k+1}}\omega^\varepsilon(x_j,s)ds,\quad j\in\mathcal{T}_M,\quad k\geq 1. \tag{3.31}$$

Meanwhile, the initial condition is discretized as

$$\psi_j^{\varepsilon,0}=\psi_0(x_j),\quad \chi_j^{\varepsilon,0}=0,\quad j\in\mathcal{T}_M. \tag{3.32}$$

Choice of the first step value. By Taylor expansion, we get $\psi_j^{\varepsilon,1}$ as

$$\psi_j^{\varepsilon,1} = \psi_0(x_j) + \tau\psi_1(x_j) + \frac{\tau^2}{2}\psi_2(x_j), \quad \chi_j^{\varepsilon,1} = \frac{\tau^2}{2}\psi_3(x_j), \quad j \in \mathcal{T}_M, \tag{3.33}$$

where by (3.26),

$$\begin{aligned}
\psi_1(x) &:= \partial_t\psi^\varepsilon(x,0) = i\left[\psi_0''(x) + \phi_0^\varepsilon(x)\psi_0(x)\right], \\
\psi_2(x) &:= \partial_{tt}\psi^\varepsilon(x,0) = i\left[\psi_1''(x) + \phi_1^\varepsilon(x)\psi_0(x) + \phi_0^\varepsilon(x)\psi_1(x)\right], \\
\psi_3(x) &:= \partial_{tt}\chi^\varepsilon(x,0) = -\partial_{tt}\varphi^\varepsilon(x,0) = -(-\Delta+I)^{-1}\partial_{tt}|\psi^\varepsilon|^2(x,0) \\
&= 2(-\Delta+I)^{-1}\mathrm{Im}\left[\psi_0''(x)\overline{\psi_1(x)} + \psi_1''(x)\overline{\psi_0(x)}\right], \quad x \in \Omega.
\end{aligned}$$

Noticing (2.18), the above approximation for $\psi_j^{\varepsilon,1}$ implies $\max\limits_{0\le j\le N}|\psi_j^{\varepsilon,1}| = O(\tau^2\varepsilon^\beta)$ when $-1 \le \beta < 0$. In such case, in order to make sure $\psi^{\varepsilon,1}$ is uniformly bounded for $\varepsilon \in (0,1]$, τ has to be taken as $\tau \lesssim \varepsilon^{-\beta/2}$, which is too restrictive. To rescue this, we replace $\psi_2(x)$ above by a modified version [3]

$$\psi_2(x) = i\left[\psi_1''(x) + \left(\phi_1(x) + \frac{\varepsilon^{1+\beta}}{\tau}\sin\left(\frac{\tau}{\varepsilon}\right)\omega_1(x)\right)\psi_0(x) + \phi_0^\varepsilon(x)\psi_1(x)\right], \tag{3.34}$$

which yields the first step value with second order accuracy as

$$\begin{aligned}
\psi_j^{\varepsilon,1} = \psi_0(x_j) + \tau\psi_1(x_j) + \frac{i\tau^2}{2}\left[\psi_1''(x_j) + \phi_1(x_j)\psi_0(x_j) + \phi_0^\varepsilon(x_j)\psi_1(x_j)\right] \\
+ \frac{i\tau}{2}\varepsilon^{1+\beta}\sin\left(\frac{\tau}{\varepsilon}\right)\psi_0(x_j)\omega_1(x_j).
\end{aligned} \tag{3.35}$$

In practical computation, $\mu_j^{\varepsilon,k}$ in (3.31) can be obtained by solving the linear wave-type equation (3.27) via the sine pseudospectral discretization in space followed by integrating in time in phase space *exactly* [3] as

$$\begin{aligned}
\mu_j^{\varepsilon,k} &\approx \frac{1}{2\tau}\sum_{l=1}^{M-1}\sin\left(\frac{lj\pi}{M}\right)\int_{t_{k-1}}^{t_{k+1}}\left[\varepsilon^\alpha\widetilde{(\omega_0)}_l\cos(\theta_l u) + \frac{\varepsilon^\beta}{\theta_l}\widetilde{(\omega_1)}_l\sin(\theta_l u)\right]du \\
&= \sum_{l=1}^{M-1}\frac{1}{\tau\theta_l}\sin\left(\frac{lj\pi}{M}\right)\sin(\theta_l\tau)\left[\varepsilon^\alpha\widetilde{(\omega_0)}_l\cos(\theta_l t_k) + \frac{\varepsilon^\beta}{\theta_l}\widetilde{(\omega_1)}_l\sin(\theta_l t_k)\right],
\end{aligned}$$

where for $l \in \mathcal{T}_M$, $\theta_l = \frac{\sqrt{l^2\pi^2+(b-a)^2}}{\varepsilon(b-a)}$, and

$$\widetilde{(\omega_0)}_l = \frac{2}{M}\sum_{j=1}^{M-1}\omega_0(x_j)\sin\left(\frac{jl\pi}{M}\right), \quad \widetilde{(\omega_1)}_l = \frac{2}{M}\sum_{j=1}^{M-1}\omega_1(x_j)\sin\left(\frac{jl\pi}{M}\right).$$

3.2. **Main results.** Let $T^* > 0$ be the maximum common existence time for the solutions of the KGS system (3.26) and the SY-OP equations (3.28). Then for any fixed $0 < T < T^*$, according to the known results in [29,32], it is natural to assume that the solution $(\psi^\varepsilon, \varphi^\varepsilon, \chi^\varepsilon)$ of the KGS (3.26) and the solution $(\widetilde{\psi}^\varepsilon, \widetilde{\varphi}^\varepsilon)$ of the

SY-OP (3.28) are smooth enough over $\Omega_T := \Omega \times [0,T]$ and satisfy

$$
(A)\quad
\begin{aligned}
&\|\psi^\varepsilon\|_{W^{5,\infty}(\Omega)} + \|\psi_t^\varepsilon\|_{W^{1,\infty}(\Omega)} \lesssim 1, \ \ \|\psi_{tt}^\varepsilon\|_{W^{3,\infty}(\Omega)} \lesssim 1/\varepsilon, \ \ \|\partial_t^3\psi^\varepsilon\|_{L^\infty(\Omega)} \lesssim 1/\varepsilon^2,\\
&\|\chi^\varepsilon\|_{W^{4,\infty}(\Omega)} \lesssim \varepsilon^2, \quad \|\partial_t\chi^\varepsilon\|_{W^{4,\infty}(\Omega)} \lesssim \varepsilon, \quad \|\chi_{tt}^\varepsilon\|_{W^{2,\infty}(\Omega)} \lesssim 1,\\
&\|\partial_t^3\chi^\varepsilon\|_{W^{2,\infty}(\Omega)} \lesssim 1/\varepsilon, \quad \|\varphi^\varepsilon\|_{W^{4,\infty}(\Omega)} + \|\varphi_t^\varepsilon\|_{W^{4,\infty}(\Omega)} + \|\varphi_{tt}^\varepsilon\|_{W^{1,\infty}(\Omega)} \lesssim 1,\\
&\|\partial_t^4\chi^\varepsilon\|_{L^\infty(\Omega)} + \|\partial_t^4\varphi^\varepsilon\|_{L^\infty(\Omega)} \lesssim 1/\varepsilon^2, \quad \|\partial_t^5\chi^\varepsilon\|_{L^\infty(\Omega)} + \|\partial_t^5\varphi^\varepsilon\|_{L^\infty(\Omega)} \lesssim 1/\varepsilon^3,\\
&\|\widetilde{\psi}^\varepsilon\|_{W^{5,\infty}(\Omega)} + \|\widetilde{\psi}_t^\varepsilon\|_{W^{3,\infty}(\Omega)} + \|\widetilde{\varphi}^\varepsilon\|_{W^{4,\infty}(\Omega)} + \|\widetilde{\varphi}_t^\varepsilon\|_{W^{4,\infty}(\Omega)} \lesssim 1,\\
&\|\widetilde{\varphi}_{tt}^\varepsilon\|_{W^{1,\infty}(\Omega)} \lesssim 1, \quad \|\partial_t^3\widetilde{\varphi}^\varepsilon\|_{L^\infty(\Omega)} \lesssim 1/\varepsilon^{1-\alpha^*},
\end{aligned}
$$

where $\alpha^* = \min\{1, \alpha, 1+\beta\} \in [0,1]$. Moreover, we assume the initial data satisfies

$$
(B)\qquad \|\psi_0\|_{W^{5,\infty}(\Omega)} + \|\omega_0\|_{W^{3,\infty}(\Omega)} + \|\omega_1\|_{W^{3,\infty}(\Omega)} \lesssim 1.
$$

Then one can obtain

$$
\|\partial_s^m \omega^\varepsilon(\cdot,s)\|_{W^{3,\infty}(\Omega)} \lesssim \varepsilon^{\alpha^*-m}, \quad m = 0,1,2,3. \tag{3.36}
$$

Define the error functions $e_\psi^{\varepsilon,k}$, $e_\varphi^{\varepsilon,k}$ and $e_\chi^{\varepsilon,k} \in X_M$ for $0 \le k \le \frac{T}{\tau}$ as

$$
e_{\psi,j}^{\varepsilon,k} = \psi^\varepsilon(x_j,t_k) - \psi_j^{\varepsilon,k}, \quad e_{\chi,j}^{\varepsilon,k} = \chi^\varepsilon(x_j,t_k) - \chi_j^{\varepsilon,k}, \quad e_{\varphi,j}^{\varepsilon,k} = \varphi^\varepsilon(x_j,t_k) - \varphi_j^{\varepsilon,k}. \tag{3.37}
$$

Then we have the following error estimates for (3.30) with (3.31)-(3.35).

Theorem 3.1. *Under the assumptions (A)-(B), there exist $h_0, \tau_0 > 0$ sufficiently small and independent of $0 < \varepsilon \le 1$ such that, when $0 < h \le h_0$ and $0 < \tau \le \tau_0$, the following two error estimates of the scheme (3.30) with (3.31)-(3.35) hold*

$$
\|e_\psi^{\varepsilon,k}\|_{H^1} + \|e_\chi^{\varepsilon,k}\|_{H^1} + \|e_\varphi^{\varepsilon,k}\|_{H^1} \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \qquad 0 \le k \le \frac{T}{\tau}, \quad 0 < \varepsilon \le 1, \tag{3.38}
$$

$$
\|e_\psi^{\varepsilon,k}\|_{H^1} + \|e_\chi^{\varepsilon,k}\|_{H^1} + \|e_\varphi^{\varepsilon,k}\|_{H^1} \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}. \tag{3.39}
$$

Thus by taking the minimum among the two error bounds for $\varepsilon \in (0,1]$, we obtain a uniform error estimate for $\alpha \ge 0$ and $\beta \ge -1$,

$$
\begin{aligned}
\|e_\psi^{\varepsilon,k}\|_{H^1} + \|e_\chi^{\varepsilon,k}\|_{H^1} + \|e_\varphi^{\varepsilon,k}\|_{H^1} &\lesssim h^2 + \max_{0<\varepsilon\le 1} \min\left\{\tau^2 + \varepsilon^{\alpha^*}(\tau+\varepsilon), \frac{\tau^2}{\varepsilon}\right\}\\
&\lesssim h^2 + \tau^{1+\frac{\alpha^*}{2+\alpha^*}}.
\end{aligned} \tag{3.40}
$$

4. **Error estimates.** In order to prove Theorem 3.1, we will use the energy method to obtain one error bound (3.38) and use the limiting equation SY-OP (3.28) to get the other one (3.39). To deal with the nonlinearity and to overcome the difficulty that there is no a priori bound for the numerical solution, we use the cut-off technique which has been widely used in the literatures [8, 35], i.e., the nonlinearity is firstly truncated by a global Lipschitz function with compact support and then the error bound can be achieved if the exact solution is bounded and the numerical solution is close to the exact solution under some conditions on the mesh size and time step. Specifically, choose a smooth function $\rho(s) \in C^\infty(\mathbb{R})$ such that

$$
\rho(s) = \begin{cases} 1, & |s| \le 1,\\ \in [0,1], & |s| \le 2,\\ 0, & |s| \ge 2, \end{cases}
$$

and by assumption (A) we can set $M_0 > 0$ as

$$M_0 = \max\left\{\sup_{\varepsilon\in(0,1]}\|\psi^\varepsilon\|_{L^\infty(\Omega_T)}, \sup_{\varepsilon\in(0,1]}\|\widetilde{\psi}^\varepsilon\|_{L^\infty(\Omega_T)}\right\}.$$

For $s \geq 0$, $y_1, y_2 \in \mathbb{C}$, define $\rho_{_B}(s) = s\,\rho\left(\frac{s}{B}\right)$, with $B = (M_0+1)^2$, and

$$g(y_1, y_2) = \frac{y_1+y_2}{2}\int_0^1 \rho'_{_B}(s|y_1|^2 + (1-s)|y_2|^2)ds.$$

Then $\rho_{_B}(s)$ is globally Lipschitz and

$$|\rho_{_B}(s_1) - \rho_{_B}(s_2)| \lesssim |\sqrt{s_1} - \sqrt{s_2}|, \quad \forall s_1, s_2 \geq 0. \tag{4.41}$$

Set $\widehat{\psi}^{\varepsilon,k} = \psi^{\varepsilon,k}$, $\widehat{\chi}^{\varepsilon,k} = \chi^{\varepsilon,k}$, $k = 0, 1$, and define $\widehat{\psi}^{\varepsilon,k}$, $\widehat{\chi}^{\varepsilon,k} \in X_M$ as following

$$\begin{aligned}
&i\delta_t^c\widehat{\psi}_j^{\varepsilon,k} + (\delta_x^2 + \mu_j^{\varepsilon,k})\widehat{\psi}_j^{\varepsilon,[k]} + (\widehat{\varphi}_j^{\varepsilon,k} + \widehat{\chi}_j^{\varepsilon,[k]})g(\widehat{\psi}_j^{\varepsilon,k+1}, \widehat{\psi}_j^{\varepsilon,k-1}) = 0,\\
&\varepsilon^2\delta_t^2\widehat{\chi}_j^{\varepsilon,k} + \left(1-\delta_x^2\right)\widehat{\chi}_j^{\varepsilon,[k]} + \varepsilon^2\delta_t^2\widehat{\varphi}_j^{\varepsilon,k} = 0, \quad j \in \mathcal{T}_M, \quad k \geq 1,\\
&\widehat{\varphi}_j^{\varepsilon,k} - \delta_x^2\widehat{\varphi}_j^{\varepsilon,k} - \rho_B(|\widehat{\psi}_j^{\varepsilon,k}|^2) = 0, \quad j \in \mathcal{T}_M, \quad k \geq 0.
\end{aligned} \tag{4.42}$$

Here $(\widehat{\psi}^{\varepsilon,k}, \widehat{\varphi}^{\varepsilon,k}, \widehat{\chi}^{\varepsilon,k})$ can be viewed as another approximation of $(\psi^\varepsilon, \varphi^\varepsilon, \chi^\varepsilon)|_{t=t_k}$. Define the error function $\widehat{e}_{\psi,j}^{\varepsilon,k}$, $\widehat{e}_{\varphi,j}^{\varepsilon,k}$, $\widehat{e}_{\chi,j}^{\varepsilon,k} \in X_M$ for $k \geq 0$ as

$$\widehat{e}_{\psi,j}^{\varepsilon,k} = \psi^\varepsilon(x_j, t_k) - \widehat{\psi}_j^{\varepsilon,k}, \quad \widehat{e}_{\varphi,j}^{\varepsilon,k} = \varphi^\varepsilon(x_j, t_k) - \widehat{\varphi}_j^{\varepsilon,k}, \quad \widehat{e}_{\chi,j}^{\varepsilon,k} = \chi^\varepsilon(x_j, t_k) - \widehat{\chi}_j^{\varepsilon,k},$$

and the local truncation error $\widehat{\xi}_j^{\varepsilon,k}$, $\widehat{\eta}_j^{\varepsilon,k}$, $\widehat{\zeta}_j^{\varepsilon,k} \in X_M$ as

$$\begin{aligned}
&\widehat{\xi}_j^{\varepsilon,k} = i\delta_t^c\psi^\varepsilon(x_j, t_k) + \left[\delta_x^2 + \mu_j^{\varepsilon,k} + \varphi^\varepsilon(x_j, t_k) + \chi^\varepsilon(x_j, t_{[k]})\right]\psi^\varepsilon(x_j, t_{[k]}),\\
&\widehat{\eta}_j^{\varepsilon,k} = \varepsilon^2\delta_t^2\chi^\varepsilon(x_j, t_k) + (1-\delta_x^2)\chi^\varepsilon(x_j, t_{[k]}) + \varepsilon^2\delta_t^2\varphi^\varepsilon(x_j, t_k), \quad k \geq 1,\\
&\widehat{\zeta}_j^{\varepsilon,k} = \varphi^\varepsilon(x_j, t_k) - \delta_x^2\varphi^\varepsilon(x_j, t_k) - |\psi^\varepsilon(x_j, t_k)|^2, \quad j \in \mathcal{T}_M, \quad k \geq 0.
\end{aligned} \tag{4.43}$$

For the local truncation, we have the following error bounds.

Lemma 4.1 (Local truncation error)**.** *Under the assumption (A), we have*

$$\begin{aligned}
&\|\widehat{\xi}^{\varepsilon,k}\|_{H^1} \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \quad 1 \leq k < \frac{T}{\tau}; \quad \|\widehat{\zeta}^{\varepsilon,k}\| \lesssim h^2, \quad 0 \leq k \leq \frac{T}{\tau};\\
&\|\delta_t^+\widehat{\zeta}^{\varepsilon,k}\| \lesssim h^2, \quad 0 \leq k < \frac{T}{\tau}; \quad \|\delta_t^c\widehat{\zeta}^{\varepsilon,k}\| \lesssim h^2, \quad 1 \leq k < \frac{T}{\tau};\\
&\|\widehat{\eta}^{\varepsilon,k}\| \lesssim h^2 + \tau^2, \quad 1 \leq k < \frac{T}{\tau}; \quad \|\delta_t^c\widehat{\eta}^{\varepsilon,k}\| \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \quad 2 \leq k < \frac{T}{\tau} - 1.
\end{aligned}$$

Proof. By Taylor expansion, we have

$$\begin{aligned}
&\left(\delta_x^2 + \mu_j^{\varepsilon,k} + \varphi^\varepsilon(x_j, t_k) + \chi^\varepsilon(x_j, t_{[k]})\right)\psi^\varepsilon(x_j, t_{[k]})\\
&= \psi_{xx}^\varepsilon(x_j, t_k) + \left(\varphi^\varepsilon(x_j, t_k) + \mu_j^{\varepsilon,k} + \chi^\varepsilon(x_j, t_k)\right)\psi^\varepsilon(x_j, t_k)\\
&\quad + \frac{h^2}{12}\int_{-1}^1 (1-|s|)^3\left(\partial_x^4\psi^\varepsilon(x_j + sh, t_k + \tau) + \partial_x^4\psi^\varepsilon(x_j + sh, t_k - \tau)\right)ds\\
&\quad + \frac{\tau^2}{2}\int_{-1}^1 (1-|s|)\Big(\psi_{xxtt}^\varepsilon(x_j, t_k + s\tau) + \psi^\varepsilon(x_j, t_k)\chi_{tt}^\varepsilon(x_j, t_k + s\tau)\Big)ds\\
&\quad + \frac{\tau^2}{2}\left(\varphi^\varepsilon(x_j, t_k) + \mu_j^{\varepsilon,k} + \chi^\varepsilon(x_j, t_{[k]})\right)\int_{-1}^1 (1-|s|)\psi_{tt}^\varepsilon(x_j, t_k + s\tau)ds.
\end{aligned}$$

By (3.26) and using Taylor expansion, we get for $j \in \mathcal{T}_M$ and $1 \le k \le \frac{T}{\tau} - 1$,

$$
\begin{aligned}
i\delta_t^c \psi^\varepsilon(x_j, t_k) &= \frac{i}{2\tau}\int_{t_{k-1}}^{t_{k+1}} \partial_t \psi^\varepsilon(x_j, s) ds \\
&= (-\psi_{xx}^\varepsilon - \psi^\varepsilon\varphi^\varepsilon - \psi^\varepsilon\chi^\varepsilon)|_{(x_j,t_k)} - \frac{1}{2\tau}\int_{t_{k-1}}^{t_{k+1}} \psi^\varepsilon(x_j, s)\omega^\varepsilon(x_j, s) ds \\
&\quad - \frac{\tau^2}{4}\int_{-1}^{1} (1-|s|)^2 \partial_{tt}(\psi_{xx}^\varepsilon + \psi^\varepsilon\varphi^\varepsilon + \psi^\varepsilon\chi^\varepsilon)(x_j, t_k + s\tau) ds.
\end{aligned}
$$

Note that by (3.31), we have

$$
\begin{aligned}
&\mu_j^{\varepsilon,k}\psi^\varepsilon(x_j, t_k) - \frac{1}{2\tau}\int_{t_{k-1}}^{t_{k+1}} \psi^\varepsilon(x_j, s)\omega^\varepsilon(x_j, s) ds \\
&= \frac{\tau^2}{2}\psi_t^\varepsilon(x_j, t_k)\int_0^1 s\int_{-s}^{s} \omega_t^\varepsilon(x_j, t_k + \theta\tau) d\theta ds \\
&\quad + \frac{\tau^2}{2}\int_{-1}^{1}\int_0^s (s-\theta)\,\omega^\varepsilon(x_j, t_k + s\tau)\psi_{tt}^\varepsilon(x_j, t_k + \theta\tau) d\theta ds.
\end{aligned}
$$

Accordingly, by the assumption (A) and (3.36), we can conclude that

$$
\begin{aligned}
|\widehat{\xi}_j^{\varepsilon,k}| &\lesssim h^2\|\partial_x^4\psi^\varepsilon\|_{L^\infty} + \tau^2\Big[\|\psi_{xxtt}^\varepsilon\|_{L^\infty} + \|\psi_{tt}^\varepsilon\|_{L^\infty}(\|\omega^\varepsilon\|_{L^\infty} + \|\chi^\varepsilon\|_{L^\infty} + \|\varphi^\varepsilon\|_{L^\infty}) \\
&\quad + \|\psi_t^\varepsilon\|_{L^\infty}(\|\omega_t^\varepsilon\|_{L^\infty} + \|\chi_t^\varepsilon\|_{L^\infty} + \|\varphi_t^\varepsilon\|_{L^\infty}) + \|\psi^\varepsilon\|_{L^\infty}(\|\chi_{tt}^\varepsilon\|_{L^\infty} + \|\varphi_{tt}^\varepsilon\|_{L^\infty}\Big] \\
&\lesssim h^2 + \frac{\tau^2}{\varepsilon}, \qquad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.
\end{aligned}
$$

Applying δ_x^+ to $\widehat{\xi}^{\varepsilon,k}$ and using the same approach, we can get $|\delta_x^+\widehat{\xi}_j^{\varepsilon,k}| \lesssim h^2 + \frac{\tau^2}{\varepsilon}$. Similarly, we obtain

$$
\begin{aligned}
\widehat{\eta}_j^{\varepsilon,k} &= \frac{\varepsilon^2\tau^2}{6}\int_{-1}^{1} (1-|s|)^3\left(\partial_t^4\chi^\varepsilon(x_j, t_k + s\tau) + \partial_t^4\varphi^\varepsilon(x_j, t_k + s\tau)\right) ds \\
&\quad + \frac{\tau^2}{2}\int_{-1}^{1} (1-|s|)\left(\chi_{tt}^\varepsilon(x_j, t_k + s\tau) - \chi_{xxtt}^\varepsilon(x_j, t_k + s\tau)\right) ds \\
&\quad - \frac{h^2}{12}\int_{-1}^{1} (1-|s|)^3\left(\partial_x^4\chi^\varepsilon(x_j + sh, t_k + \tau) + \partial_x^4\chi^\varepsilon(x_j + sh, t_k - \tau)\right) ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
|\widehat{\eta}_j^{\varepsilon,k}| &\lesssim h^2\|\partial_x^4\chi^\varepsilon\|_{L^\infty} + \tau^2(\|\chi_{tt}^\varepsilon\|_{L^\infty} + \|\chi_{xxtt}^\varepsilon\|_{L^\infty} + \varepsilon^2\|\partial_t^4\chi^\varepsilon\|_{L^\infty} + \varepsilon^2\|\partial_t^4\varphi^\varepsilon\|_{L^\infty}) \\
&\lesssim \varepsilon^2 h^2 + \tau^2, \qquad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.
\end{aligned}
$$

Applying δ_t^c to $\widehat{\eta}_j^{\varepsilon,k}$, we have

$$
\begin{aligned}
&|\delta_t^c\widehat{\eta}_j^{\varepsilon,k}| \\
&\lesssim h^2\|\partial_x^4\chi_t^\varepsilon\|_{L^\infty} + \tau^2(\|\partial_t^3\chi^\varepsilon\|_{L^\infty} + \|\partial_t^3\chi_{xx}^\varepsilon\|_{L^\infty} + \varepsilon^2(\|\partial_t^5\chi^\varepsilon\|_{L^\infty} + \|\partial_t^5\varphi^\varepsilon\|_{L^\infty})) \\
&\lesssim \varepsilon h^2 + \frac{\tau^2}{\varepsilon}, \qquad j \in \mathcal{T}_M, \quad 2 \le k \le \frac{T}{\tau} - 2.
\end{aligned}
$$

Finally, it can be easily deduced that

$$\widehat{\zeta}_j^{\varepsilon,k} = \frac{h^2}{6}\int_{-1}^{1}(1-|s|)^3\partial_x^4\varphi^\varepsilon(x_j+sh,t_k)ds,$$

which gives that

$$|\widehat{\zeta}_j^{\varepsilon,k}| \lesssim h^2\|\partial_x^4\varphi^\varepsilon\|_{L^\infty} \lesssim h^2, \quad |\delta_t^+\widehat{\zeta}_j^{\varepsilon,k}| + |\delta_t^c\widehat{\zeta}_j^{\varepsilon,k}| \lesssim h^2\|\partial_x^4\varphi_t^\varepsilon\|_{L^\infty} \lesssim h^2.$$

Thus the proof is completed. □

For the initial step, we have the following estimates.

Lemma 4.2 (Error bounds for $k=1$)**.** *Under the assumption (A), the first step errors of the discretization* (3.33) *satisfy*

$$\widehat{e}_\psi^{\varepsilon,0} = \widehat{e}_\chi^{\varepsilon,0} = \mathbf{0}, \quad \|\widehat{e}_\psi^{\varepsilon,1}\|_{H^1} + \|\delta_t^+\widehat{e}_\chi^{\varepsilon,0}\| \lesssim \frac{\tau^2}{\varepsilon}, \quad \|\delta_t^+\widehat{e}_\psi^{\varepsilon,0}\| \lesssim \frac{\tau^2}{\varepsilon^2}; \quad \|\widehat{e}_\chi^{\varepsilon,1}\|_{H^1} \lesssim \frac{\tau^3}{\varepsilon}.$$

Proof. By the definition of $\widehat{\psi}_j^{\varepsilon,1}$ (3.35), and noticing $\beta \geq -1$, we obtain

$$\begin{aligned}|\widehat{e}_{\psi,j}^{\varepsilon,1}| &\leq \tau^2\Big[\Big|\int_0^1(1-s)\psi_{tt}^\varepsilon(x_j,s\tau)ds - \frac{\psi_{tt}^\varepsilon(x_j,0)}{2}\Big| + \frac{\varepsilon^\beta}{2}\big|\psi_0(x_j)\omega_1(x_j)\big|\Big|1-\frac{\sin\left(\frac{\tau}{\varepsilon}\right)}{\tau/\varepsilon}\Big|\Big]\\ &\lesssim \tau^2\left(\|\psi_{tt}^\varepsilon\|_{L^\infty} + \varepsilon^\beta\|\psi_0\|_{L^\infty}\|\omega_1\|_{L^\infty}\right) \lesssim \frac{\tau^2}{\varepsilon}.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}|\widehat{e}_{\psi,j}^{\varepsilon,1}| &\leq \frac{\tau^3}{2}\left[\left|\int_0^1(1-s)^2\psi_{ttt}^\varepsilon(x_j,s\tau)ds\right| + \frac{|\psi_0(x_j)\omega_1(x_j)|}{\varepsilon^{1-\beta}}\left|\int_0^1(1-s)\sin\left(\frac{\tau s}{\varepsilon}\right)ds\right|\right]\\ &\lesssim \tau^3\left(\|\psi_{ttt}^\varepsilon\|_{L^\infty} + \varepsilon^{-2}\|\psi_0\|_{L^\infty}\|\omega_1\|_{L^\infty}\right) \lesssim \frac{\tau^3}{\varepsilon^2},\end{aligned}$$

which implies that $|\delta_t^+\widehat{e}_{\psi,j}^{\varepsilon,0}| \lesssim \frac{\tau^2}{\varepsilon^2}$. It follows from (3.33) and assumption (A) that

$$|\widehat{e}_{\chi,j}^{\varepsilon,1}| = \frac{\tau^3}{2}\left|\int_0^1(1-s)^2\chi_{ttt}^\varepsilon(x_j,s\tau)ds\right| \lesssim \tau^3\|\chi_{ttt}^\varepsilon\|_{L^\infty} \lesssim \frac{\tau^3}{\varepsilon}.$$

Recalling that $\widehat{e}_{\chi,j}^{\varepsilon,0} = 0$, we can get that $|\delta_t^+\widehat{e}_{\chi,j}^{\varepsilon,0}| \lesssim \frac{\tau^2}{\varepsilon}$. Similarly, we can get $|\delta_x^+\widehat{e}_{\psi,j}^{\varepsilon,1}| \lesssim \frac{\tau^2}{\varepsilon}$, $|\delta_x^+\widehat{e}_{\chi,j}^{\varepsilon,1}| \lesssim \frac{\tau^3}{\varepsilon}$, which completes the proof. □

Proof of Theorem 3.1. The proof is divided into three main steps.

Step 1 (Establish (3.38)-type error estimate for $\widehat{e}_\psi^{\varepsilon,k}$, $\widehat{e}_\chi^{\varepsilon,k}$, $\widehat{e}_\varphi^{\varepsilon,k}$). Subtracting (4.42) from (4.43), we have the error equations for $j \in \mathcal{T}_M$,

$$i\delta_t^c\widehat{e}_{\psi,j}^{\varepsilon,k} + (\delta_x^2 + \mu_j^{\varepsilon,k})\widehat{e}_{\psi,j}^{\varepsilon,[k]} + \widehat{r}_j^{\varepsilon,k} = \widehat{\xi}_j^{\varepsilon,k}, \tag{4.44a}$$

$$\varepsilon^2\delta_t^2\widehat{e}_{\chi,j}^{\varepsilon,k} + (1-\delta_x^2)\widehat{e}_{\chi,j}^{\varepsilon,[k]} + \varepsilon^2\delta_t^2\widehat{e}_{\varphi,j}^{\varepsilon,k} = \widehat{\eta}_j^{\varepsilon,k}, \quad 1 \leq k \leq \frac{T}{\tau}-1, \tag{4.44b}$$

$$\widehat{e}_{\varphi,j}^{\varepsilon,k} - \delta_x^2\widehat{e}_{\varphi,j}^{\varepsilon,k} - \widehat{p}_j^{\varepsilon,k} = \widehat{\zeta}_j^{\varepsilon,k}, \quad 0 \leq k \leq \frac{T}{\tau}, \tag{4.44c}$$

where $\widehat{p}_j^{\varepsilon,k} = |\psi^\varepsilon(x_j,t_k)|^2 - \rho_B(|\widehat{\psi}_j^{\varepsilon,k}|^2)$, and

$$\widehat{r}_j^{\varepsilon,k} = \left(\varphi^\varepsilon(x_j,t_k) + \chi^\varepsilon(x_j,t_{[k]})\right)\psi^\varepsilon(x_j,t_{[k]}) - \left(\widehat{\varphi}_j^{\varepsilon,k} + \widehat{\chi}_j^{\varepsilon,[k]}\right)g(\widehat{\psi}_j^{\varepsilon,k+1},\widehat{\psi}_j^{\varepsilon,k-1}).$$

By the property of ρ_B (cf. (4.41)), one can easily get that

$$|\widehat{p}_j^{\varepsilon,k}| = |\rho_B(|\psi^\varepsilon(x_j,t_k)|^2) - \rho_B(|\widehat{\psi}_j^{\varepsilon,k}|^2)| \lesssim |\widehat{e}_{\psi,j}^{\varepsilon,k}|, \ \ j \in \mathcal{T}_M, \ \ 0 \leq k \leq \frac{T}{\tau}. \tag{4.45}$$

By the definition of $g(\cdot,\cdot)$, and noticing that

$$\psi^\varepsilon(x_j,t_{[k]})=g\big(\psi^\varepsilon(x_j,t_{k+1}),\psi^\varepsilon(x_j,t_{k-1})\big),$$

it is known from [8] that for $j\in\mathcal{T}_M$, $1\le k\le\frac{T}{\tau}-1$,

$$\begin{aligned}&\left|g(\widehat{\psi}_j^{\varepsilon,k},\widehat{\psi}_j^{\varepsilon,k-1})\right|\lesssim 1,\quad \left|\psi^\varepsilon(x_j,t_{[k]})-g(\widehat{\psi}_j^{\varepsilon,k},\widehat{\psi}_j^{\varepsilon,k-1})\right|\lesssim|\widehat{e}_{\psi,j}^{\varepsilon,k}|+|\widehat{e}_{\psi,j}^{\varepsilon,k-1}|,\\&\left|\delta_x^+\big(\psi^\varepsilon(x_j,t_{[k]})-g(\widehat{\psi}_j^{\varepsilon,k+1},\widehat{\psi}_j^{\varepsilon,k-1})\big)\right|\lesssim\sum_{l=k\pm1}(|\widehat{e}_{\psi,j}^{\varepsilon,l}|+|\widehat{e}_{\psi,j+1}^{\varepsilon,l}|+|\delta_x^+\widehat{e}_{\psi,j}^{\varepsilon,l}|).\end{aligned}\tag{4.46}$$

Hence

$$|\widehat{r}_j^{\varepsilon,k}|\lesssim|\widehat{e}_{\psi,j}^{\varepsilon,k+1}|+|\widehat{e}_{\psi,j}^{\varepsilon,k-1}|+|\widehat{e}_{\varphi,j}^{\varepsilon,k}|+|\widehat{e}_{\chi,j}^{\varepsilon,k+1}|+|\widehat{e}_{\chi,j}^{\varepsilon,k-1}|.\tag{4.47}$$

Multiplying both sides of (4.44a) by $4\tau\overline{\widehat{e}_{\psi,j}^{\varepsilon,[k]}}$, summing together for $j\in\mathcal{T}_M$ and taking the imaginary parts, we obtain for $1\le k\le T/\tau$,

$$\|\widehat{e}_\psi^{\varepsilon,k+1}\|^2-\|\widehat{e}_\psi^{\varepsilon,k-1}\|^2=2\tau\,\mathrm{Im}(\widehat{\xi}^{\varepsilon,k}-\widehat{r}^{\varepsilon,k},\widehat{e}_\psi^{\varepsilon,k+1}+\widehat{e}_\psi^{\varepsilon,k-1}).\tag{4.48}$$

Multiplying both sides of (4.44a) by $4\tau\overline{\delta_t^c\widehat{e}_{\psi,j}^{\varepsilon,k}}$, summing together for $j\in\mathcal{T}_M$ and taking the real parts, we obtain for $1\le k\le T/\tau$,

$$\|\delta_x^+\widehat{e}_\psi^{\varepsilon,k+1}\|^2-\|\delta_x^+\widehat{e}_\psi^{\varepsilon,k-1}\|^2=2\,\mathrm{Re}\big(\widehat{r}^{\varepsilon,k}-\widehat{\xi}^{\varepsilon,k}+\mu^{\varepsilon,k}\widehat{e}_\psi^{\varepsilon,[k]},\widehat{e}_\psi^{\varepsilon,k+1}-\widehat{e}_\psi^{\varepsilon,k-1}\big).\tag{4.49}$$

Multiplying (4.44b) by $2\tau\,\delta_t^c(\widehat{e}_{\chi,j}^{\varepsilon,k}+\widehat{e}_{\varphi,j}^{\varepsilon,k})$, summing for $j\in\mathcal{T}_M$, we have

$$\begin{aligned}\varepsilon^2(\|\delta_t^+(\widehat{e}_\chi^{\varepsilon,k}+\widehat{e}_\varphi^{\varepsilon,k})\|^2-\|\delta_t^+(\widehat{e}_\chi^{\varepsilon,k-1}+\widehat{e}_\varphi^{\varepsilon,k-1})\|^2)+\frac12(\|\widehat{e}_\chi^{\varepsilon,k+1}\|_{H^1}^2-\|\widehat{e}_\chi^{\varepsilon,k-1}\|_{H^1}^2)\\+2\tau(\widehat{e}_\chi^{\varepsilon,[k]},\delta_t^c\widehat{p}^{\varepsilon,k}+\delta_t^c\widehat{\zeta}^{\varepsilon,k})=2\tau(\widehat{\eta}^{\varepsilon,k},\delta_t^c\widehat{e}_\chi^{\varepsilon,k}+\delta_t^c\widehat{e}_\varphi^{\varepsilon,k}),\end{aligned}\tag{4.50}$$

where we used (3.29) and (4.44c). Multiplying (4.44c) by $\widehat{e}_{\varphi,j}^{\varepsilon,k}$, summing together for $j\in\mathcal{T}_M$, we obtain

$$\|\widehat{e}_\varphi^{\varepsilon,k}\|_{H^1}^2=(\widehat{p}^{\varepsilon,k}+\widehat{\zeta}^{\varepsilon,k},\widehat{e}_\varphi^{\varepsilon,k}),$$

which together with Cauchy inequality and (4.45) gives that there exists $C_1>0$ such that

$$\|\widehat{e}_\varphi^{\varepsilon,k}\|_{H^1}^2\le C_1(\|\widehat{e}_\psi^{\varepsilon,k}\|^2+\|\widehat{\zeta}^{\varepsilon,k}\|^2).\tag{4.51}$$

Introduce a discrete 'energy' for $0\le k\le T/\tau-1$ by

$$\begin{aligned}\widehat{\mathcal{A}}^{\varepsilon,k}=C_1(\|\widehat{e}_\psi^{\varepsilon,k}\|^2+\|\widehat{e}_\psi^{\varepsilon,k+1}\|^2)+\|\delta_x^+\widehat{e}_\psi^{\varepsilon,k}\|^2+\|\delta_x^+\widehat{e}_\psi^{\varepsilon,k+1}\|^2+\frac12\|\widehat{e}_\chi^{\varepsilon,k}\|_{H^1}^2\\+\frac12\|\widehat{e}_\chi^{\varepsilon,k+1}\|_{H^1}^2+\varepsilon^2\|\delta_t^+(\widehat{e}_\chi^{\varepsilon,k}+\widehat{e}_\varphi^{\varepsilon,k})\|^2.\end{aligned}\tag{4.52}$$

Combining C_1*(4.48)+(4.49)+(4.50), we get for $1\le k\le\frac{T}{\tau}-1$

$$\begin{aligned}&\widehat{\mathcal{A}}^{\varepsilon,k}-\widehat{\mathcal{A}}^{\varepsilon,k-1}\\&=2C_1\tau\,\mathrm{Im}\left(\widehat{\xi}^{\varepsilon,k}-\widehat{r}^{\varepsilon,k},\widehat{e}_\psi^{\varepsilon,k+1}+\widehat{e}_\psi^{\varepsilon,k-1}\right)-2\tau\left(\widehat{e}_\chi^{\varepsilon,[k]},\delta_t^c\widehat{p}^{\varepsilon,k}+\delta_t^c\widehat{\zeta}^{\varepsilon,k}\right)\\&\quad+2\,\mathrm{Re}\left(\widehat{r}^{\varepsilon,k}-\widehat{\xi}^{\varepsilon,k}+\mu^{\varepsilon,k}\widehat{e}_\psi^{\varepsilon,[k]},2\tau\delta_t^c\widehat{e}_\psi^{\varepsilon,k}\right)+2\tau\left(\widehat{\eta}^{\varepsilon,k},\delta_t^c\widehat{e}_\chi^{\varepsilon,k}+\delta_t^c\widehat{e}_\varphi^{\varepsilon,k}\right).\end{aligned}\tag{4.53}$$

Now we estimate the terms in (4.53) respectively. It follows from (4.47) and (4.51) that

$$\begin{aligned}\left|\mathrm{Im}\left(\widehat{\xi}^{\varepsilon,k}-\widehat{r}^{\varepsilon,k},\widehat{e}_\psi^{\varepsilon,k+1}+\widehat{e}_\psi^{\varepsilon,k-1}\right)\right|&\lesssim\|\widehat{\xi}^{\varepsilon,k}\|^2+\|\widehat{r}^{\varepsilon,k}\|^2+\|\widehat{e}_\psi^{\varepsilon,k+1}\|^2+\|\widehat{e}_\psi^{\varepsilon,k-1}\|^2\\&\lesssim\|\widehat{\xi}^{\varepsilon,k}\|^2+\|\widehat{\zeta}^{\varepsilon,k}\|^2+\widehat{\mathcal{A}}^{\varepsilon,k}+\widehat{\mathcal{A}}^{\varepsilon,k-1}.\end{aligned}\tag{4.54}$$

In view of (4.44a), (4.47), (4.51) and (3.36), and using Cauchy inequality, we find

$$\begin{aligned}
&\left|2\,\mathrm{Re}(\mu^{\varepsilon,k}\widehat{e}_{\psi}^{\varepsilon,[k]}-\widehat{\xi}^{\varepsilon,k},2\tau\delta_t^c\widehat{e}_{\psi}^{\varepsilon,k})\right| = 4\tau\left|\mathrm{Im}\big(\mu^{\varepsilon,k}\widehat{e}_{\psi}^{\varepsilon,[k]}-\widehat{\xi}^{\varepsilon,k},\delta_x^2\widehat{e}_{\psi}^{\varepsilon,[k]}+\widehat{r}^{\varepsilon,k}\big)\right|\\
&\quad\lesssim \tau\left(1+\|\mu^{\varepsilon,k}\|_\infty+\|\delta_x^+\mu^{\varepsilon,k}\|_\infty\right)\left(\|\widehat{\xi}^{\varepsilon,k}\|_{H^1}^2+\|\widehat{\zeta}^{\varepsilon,k}\|^2+\widehat{\mathcal{A}}^{\varepsilon,k}+\widehat{\mathcal{A}}^{\varepsilon,k-1}\right)\\
&\quad\lesssim \tau\left(\|\widehat{\xi}^{\varepsilon,k}\|_{H^1}^2+\|\widehat{\zeta}^{\varepsilon,k}\|^2+\widehat{\mathcal{A}}^{\varepsilon,k}+\widehat{\mathcal{A}}^{\varepsilon,k-1}\right),\quad 1\le k\le\frac{T}{\tau}-1. \qquad (4.55)
\end{aligned}$$

We rewrite $\widehat{r}_j^{\varepsilon,k}$ as $\widehat{r}_j^{\varepsilon,k}=\widehat{q}_{1,j}^{\varepsilon,k}+\widehat{q}_{2,j}^{\varepsilon,k}$ with

$$\begin{aligned}
\widehat{q}_{1,j}^{\varepsilon,k}&=\big(\varphi^\varepsilon(x_j,t_k)+\chi^\varepsilon(x_j,t_{[k]})\big)\big(\psi^\varepsilon(x_j,t_{[k]})-g(\widehat{\psi}_j^{\varepsilon,k+1},\widehat{\psi}_j^{\varepsilon,k-1})\big),\\
\widehat{q}_{2,j}^{\varepsilon,k}&=g(\widehat{\psi}_j^{\varepsilon,k+1},\widehat{\psi}_j^{\varepsilon,k-1})\big(\widehat{e}_{\varphi,j}^{\varepsilon,k}+\widehat{e}_{\chi,j}^{\varepsilon,[k]}\big),\qquad j\in\mathcal{T}_M.
\end{aligned}$$

Applying assumption (A), (4.44a), (4.46) and (4.51), we obtain

$$\begin{aligned}
&\left|2\,\mathrm{Re}(\widehat{q}_1^{\varepsilon,k},2\tau\delta_t^c\widehat{e}_{\psi}^{\varepsilon,k})\right| = 4\tau\left|\mathrm{Im}\left(\widehat{q}_1^{\varepsilon,k},(\delta_x^2+\mu^{\varepsilon,k})\widehat{e}_{\psi}^{\varepsilon,[k]}+\widehat{r}^{\varepsilon,k}-\widehat{\xi}^{\varepsilon,k}\right)\right|\\
&\quad\lesssim \tau(1+\|\mu^{\varepsilon,k}\|_\infty)\left(\|\widehat{q}_1^{\varepsilon,k}\|_{H^1}^2+\|\widehat{r}^{\varepsilon,k}\|^2+\|\widehat{\xi}^{\varepsilon,k}\|^2+\widehat{\mathcal{A}}^{\varepsilon,k}+\widehat{\mathcal{A}}^{\varepsilon,k-1}\right)\\
&\quad\lesssim \tau\left(\|\widehat{\xi}^{\varepsilon,k}\|^2+\|\widehat{\zeta}^{\varepsilon,k}\|^2+\widehat{\mathcal{A}}^{\varepsilon,k}+\widehat{\mathcal{A}}^{\varepsilon,k-1}\right),\qquad 1\le k\le\frac{T}{\tau}-1. \qquad (4.56)
\end{aligned}$$

Moreover, in view of (4.44c), we get

$$\begin{aligned}
&2\,\mathrm{Re}\left(\widehat{q}_2^{\varepsilon,k},2\tau\delta_t^c\widehat{e}_{\psi}^{\varepsilon,k}\right)\\
&\quad=2\,\mathrm{Re}\left(g(\widehat{\psi}^{\varepsilon,k+1},\widehat{\psi}^{\varepsilon,k-1})(\widehat{e}_{\varphi}^{\varepsilon,k}+\widehat{e}_{\chi}^{\varepsilon,[k]}),\psi^\varepsilon(\cdot,t_{k+1})-\psi^\varepsilon(\cdot,t_{k-1})\right)\\
&\qquad-\left(\widehat{e}_{\varphi}^{\varepsilon,k}+\widehat{e}_{\chi}^{\varepsilon,[k]},\rho_{_B}(|\widehat{\psi}^{\varepsilon,k+1}|^2)-\rho_{_B}(|\widehat{\psi}^{\varepsilon,k-1}|^2)\right)\\
&\quad=\widehat{q}^{\varepsilon,k}+\left(\widehat{e}_{\varphi}^{\varepsilon,k}+\widehat{e}_{\chi}^{\varepsilon,[k]},\widehat{p}^{\varepsilon,k+1}-\widehat{p}^{\varepsilon,k-1}\right)\\
&\quad=\widehat{q}^{\varepsilon,k}+2\tau\left(\widehat{e}_{\chi}^{\varepsilon,[k]},\delta_t^c\widehat{p}^{\varepsilon,k}\right)+2\tau\left(\widehat{e}_{\varphi}^{\varepsilon,k},(1-\delta_x^2)\delta_t^c\widehat{e}_{\varphi}^{\varepsilon,k}-\delta_t^c\widehat{\zeta}^{\varepsilon,k}\right)\\
&\quad=\widehat{q}^{\varepsilon,k}+2\tau\left(\widehat{e}_{\chi}^{\varepsilon,[k]},\delta_t^c\widehat{p}^{\varepsilon,k}\right)+\big((\widehat{e}_{\varphi}^{\varepsilon,k+1},\widehat{e}_{\varphi}^{\varepsilon,k})-(\widehat{e}_{\varphi}^{\varepsilon,k},\widehat{e}_{\varphi}^{\varepsilon,k-1})\big)\\
&\quad+\big((\delta_x^+\widehat{e}_{\varphi}^{\varepsilon,k+1},\delta_x^+\widehat{e}_{\varphi}^{\varepsilon,k})-(\delta_x^+\widehat{e}_{\varphi}^{\varepsilon,k},\delta_x^+\widehat{e}_{\varphi}^{\varepsilon,k-1})\big)-2\tau(\widehat{e}_{\varphi}^{\varepsilon,k},\delta_t^c\widehat{\zeta}^{\varepsilon,k}), \qquad (4.57)
\end{aligned}$$

where

$$\widehat{q}^{\varepsilon,k}=2\,\mathrm{Re}\big((g(\widehat{\psi}^{\varepsilon,k+1},\widehat{\psi}^{\varepsilon,k-1})-\psi^\varepsilon(\cdot,t_{[k]}))(\widehat{e}_{\varphi}^{\varepsilon,k}+\widehat{e}_{\chi}^{\varepsilon,[k]}),\psi^\varepsilon(\cdot,t_{k+1})-\psi^\varepsilon(\cdot,t_{k-1})\big).$$

By Assumption (A), (4.46) and (4.51), we have

$$|\widehat{q}^{\varepsilon,k}|\lesssim\tau\|\partial_t\psi^\varepsilon\|_{L^\infty}(\widehat{\mathcal{A}}^{\varepsilon,k}+\widehat{\mathcal{A}}^{\varepsilon,k-1}+\|\widehat{\zeta}^{\varepsilon,k}\|^2)\lesssim\tau(\widehat{\mathcal{A}}^{\varepsilon,k}+\widehat{\mathcal{A}}^{\varepsilon,k-1}+\|\widehat{\zeta}^{\varepsilon,k}\|^2). \quad (4.58)$$

Noticing

$$(\widehat{e}_{\chi}^{\varepsilon,[k]},\delta_t^c\widehat{\zeta}^{\varepsilon,k})\lesssim\widehat{\mathcal{A}}^{\varepsilon,k}+\widehat{\mathcal{A}}^{\varepsilon,k-1}+\|\delta_t^c\widehat{\zeta}^{\varepsilon,k}\|^2,\quad(\widehat{e}_{\varphi}^{\varepsilon,k},\delta_t^c\widehat{\zeta}^{\varepsilon,k})\lesssim\widehat{\mathcal{A}}^{\varepsilon,k}+\|\widehat{\zeta}^{\varepsilon,k}\|^2+\|\delta_t^c\widehat{\zeta}^{\varepsilon,k}\|^2,$$

and combining (4.53)-(4.58), we can get

$$\begin{aligned}
&\widehat{\mathcal{A}}^{\varepsilon,k}-\widehat{\mathcal{A}}^{\varepsilon,k-1}-\big((\delta_x^+\widehat{e}_{\varphi}^{\varepsilon,k+1},\delta_x^+\widehat{e}_{\varphi}^{\varepsilon,k})-(\delta_x^+\widehat{e}_{\varphi}^{\varepsilon,k},\delta_x^+\widehat{e}_{\varphi}^{\varepsilon,k-1})\big)\\
&\quad-\big((\widehat{e}_{\varphi}^{\varepsilon,k+1},\widehat{e}_{\varphi}^{\varepsilon,k})-(\widehat{e}_{\varphi}^{\varepsilon,k},\widehat{e}_{\varphi}^{\varepsilon,k-1})\big)-2\tau(\widehat{\eta}^{\varepsilon,k},\delta_t^c\widehat{e}_{\chi}^{\varepsilon,k}+\delta_t^c\widehat{e}_{\varphi}^{\varepsilon,k})\\
&\lesssim\tau\big(\|\widehat{\xi}^{\varepsilon,k}\|_{H^1}^2+\|\widehat{\zeta}^{\varepsilon,k}\|^2+\|\delta_t^c\widehat{\zeta}^{\varepsilon,k}\|^2+\widehat{\mathcal{A}}^{\varepsilon,k}+\widehat{\mathcal{A}}^{\varepsilon,k-1}\big). \qquad (4.59)
\end{aligned}$$

In addition,

$$\sum_{l=1}^{k} 2\tau\left(\widehat{\eta}^{\varepsilon,l}, \delta_t^c \widehat{e}_\chi^{\varepsilon,l} + \delta_t^c \widehat{e}_\varphi^{\varepsilon,l}\right) = \sum_{l=k}^{k+1} (\widehat{\eta}^{\varepsilon,l-1}, \widehat{e}_\chi^{\varepsilon,l} + \widehat{e}_\varphi^{\varepsilon,l}) - \sum_{l=0}^{1} (\widehat{\eta}^{\varepsilon,l+1}, \widehat{e}_\chi^{\varepsilon,l} + \widehat{e}_\varphi^{\varepsilon,l})$$
$$- 2\tau \sum_{l=2}^{k-1} (\delta_t^c \widehat{\eta}^{\varepsilon,l}, \widehat{e}_\chi^{\varepsilon,l} + \widehat{e}_\varphi^{\varepsilon,l}), \tag{4.60}$$

with

$$\left|(\delta_t^c \widehat{\eta}^{\varepsilon,l}, \widehat{e}_\chi^{\varepsilon,l} + \widehat{e}_\varphi^{\varepsilon,l})\right| \lesssim \widehat{\mathcal{A}}^{\varepsilon,l} + \|\delta_t^c \widehat{\eta}^{\varepsilon,l}\|^2 + \|\widehat{\zeta}^{\varepsilon,l}\|^2.$$

Summing (4.59) from 1 to k, we obtain that

$$\widehat{\mathcal{A}}^{\varepsilon,k} - \widehat{\mathcal{A}}^{\varepsilon,0} - (\delta_x^+ \widehat{e}_\varphi^{\varepsilon,k+1}, \delta_x^+ \widehat{e}_\varphi^{\varepsilon,k}) - (\widehat{e}_\varphi^{\varepsilon,k+1}, \widehat{e}_\varphi^{\varepsilon,k}) + (\delta_x^+ \widehat{e}_\varphi^{\varepsilon,1}, \delta_x^+ \widehat{e}_\varphi^{\varepsilon,0}) + (\widehat{e}_\varphi^{\varepsilon,1}, \widehat{e}_\varphi^{\varepsilon,0})$$
$$- \sum_{l=k}^{k+1} (\widehat{\eta}^{\varepsilon,l-1}, \widehat{e}_\chi^{\varepsilon,l} + \widehat{e}_\varphi^{\varepsilon,l}) + \sum_{l=0}^{1} (\widehat{\eta}^{\varepsilon,l+1}, \widehat{e}_\chi^{\varepsilon,l} + \widehat{e}_\varphi^{\varepsilon,l})$$
$$\lesssim \tau \sum_{l=0}^{k} \widehat{\mathcal{A}}^{\varepsilon,l} + \tau \sum_{l=2}^{k-1} \|\delta_t^c \widehat{\eta}^{\varepsilon,l}\|^2 + \tau \sum_{l=1}^{k} \left(\|\widehat{\xi}^{\varepsilon,l}\|_{H^1}^2 + \|\widehat{\zeta}^{\varepsilon,l}\|^2 + \|\delta_t^c \widehat{\zeta}^{\varepsilon,l}\|^2\right). \tag{4.61}$$

Noticing that by Cauchy inequality and (4.51), we have

$$\left|(\delta_x^+ \widehat{e}_\varphi^{\varepsilon,k+1}, \delta_x^+ \widehat{e}_\varphi^{\varepsilon,k}) + (\widehat{e}_\varphi^{\varepsilon,k+1}, \widehat{e}_\varphi^{\varepsilon,k})\right| \le \frac{\widehat{\mathcal{A}}^{\varepsilon,k}}{2} + \frac{C_1}{2}(\|\widehat{\zeta}^{\varepsilon,k}\|^2 + \|\widehat{\zeta}^{\varepsilon,k+1}\|^2),$$
$$\Big|\sum_{l=k}^{k+1} (\widehat{\eta}^{\varepsilon,l-1}, \widehat{e}_\varphi^{\varepsilon,l} + \widehat{e}_\chi^{\varepsilon,l})\Big| \le 3(\|\widehat{\eta}^{\varepsilon,k-1}\|^2 + \|\widehat{\eta}^{\varepsilon,k}\|^2) + \frac{C_1}{4}\left(\|\widehat{e}_\psi^{\varepsilon,k}\|^2 + \|\widehat{e}_\psi^{\varepsilon,k+1}\|^2\right)$$
$$+ \frac{C_1}{4}\left(\|\widehat{\zeta}^{\varepsilon,k}\|^2 + \|\widehat{\zeta}^{\varepsilon,k+1}\|^2\right) + \frac{1}{8}\left(\|\widehat{e}_\chi^{\varepsilon,k}\|^2 + \|\widehat{e}_\chi^{\varepsilon,k+1}\|^2\right)$$
$$\le \frac{\widehat{\mathcal{A}}^{\varepsilon,k}}{4} + 3(\|\widehat{\eta}^{\varepsilon,k-1}\|^2 + \|\widehat{\eta}^{\varepsilon,k}\|^2) + \frac{C_1}{4}\left(\|\widehat{\zeta}^{\varepsilon,k}\|^2 + \|\widehat{\zeta}^{\varepsilon,k+1}\|^2\right).$$

Thus it can be deduced that

$$\widehat{\mathcal{A}}^{\varepsilon,k} - \widehat{\mathcal{A}}^{\varepsilon,0} - (\delta_x^+ \widehat{e}_\varphi^{\varepsilon,k+1}, \delta_x^+ \widehat{e}_\varphi^{\varepsilon,k}) - (\widehat{e}_\varphi^{\varepsilon,k+1}, \widehat{e}_\varphi^{\varepsilon,k}) - \sum_{l=k}^{k+1} (\widehat{\eta}^{\varepsilon,l-1}, \widehat{e}_\chi^{\varepsilon,l} + \widehat{e}_\varphi^{\varepsilon,l})$$
$$\ge \frac{1}{4}\widehat{\mathcal{A}}^{\varepsilon,k} - \widehat{\mathcal{A}}^{\varepsilon,0} - \frac{3C_1}{4}(\|\widehat{\zeta}^{\varepsilon,k}\|^2 + \|\widehat{\zeta}^{\varepsilon,k+1}\|^2) - 3(\|\widehat{\eta}^{\varepsilon,k}\|^2 + \|\widehat{\eta}^{\varepsilon,k-1}\|^2),$$

which together with (4.61) yields that

$$\widehat{\mathcal{A}}^{\varepsilon,k} \lesssim \widehat{\mathcal{A}}^{\varepsilon,0} + \|\widehat{\eta}^{\varepsilon,1}\|^2 + \|\widehat{\eta}^{\varepsilon,2}\|^2 + \sum_{l=k}^{k+1} \left(\|\widehat{\eta}^{\varepsilon,l-1}\|^2 + \|\widehat{\zeta}^{\varepsilon,l}\|^2\right) + \tau \sum_{l=1}^{k} \widehat{\mathcal{A}}^{\varepsilon,l}$$
$$+ \tau \sum_{l=2}^{k-1} \|\delta_t^c \widehat{\eta}^{\varepsilon,l}\|^2 + \tau \sum_{l=1}^{k} \left(\|\widehat{\xi}^{\varepsilon,l}\|_{H^1}^2 + \|\widehat{\zeta}^{\varepsilon,l}\|^2 + \|\delta_t^c \widehat{\zeta}^{\varepsilon,l}\|^2\right). \tag{4.62}$$

To estimate $\widehat{\mathcal{A}}^{\varepsilon,0}$, applying δ_t^+ to (4.44c) for $k = 0$ followed by multiplying both sides by $\delta_t^+ \widehat{e}_\varphi^{\varepsilon,0}$ and summing together for $j \in \mathcal{T}_M$, we get

$$\|\delta_t^+ \widehat{e}_\varphi^{\varepsilon,0}\|_{H^1}^2 = (\delta_t^+ \widehat{e}_\varphi^{\varepsilon,0}, \delta_t^+ \widehat{p}^{\varepsilon,0} + \delta_t^+ \widehat{\zeta}^{\varepsilon,0}).$$

Recalling $\widehat{p}^{\varepsilon,0}=0$, applying the Cauchy inequality, (4.45) and Lemmas 4.1, 4.2, one can obtain

$$\|\delta_t^+\widehat{e}_{\varphi}^{\varepsilon,0}\|^2 \lesssim \|\delta_t^+\widehat{p}^{\varepsilon,0}\|^2+\|\delta_t^+\widehat{\zeta}^{\varepsilon,0}\|^2 \lesssim \|\delta_t^+\widehat{e}_{\psi}^{\varepsilon,0}\|^2+\|\delta_t^+\widehat{\zeta}^{\varepsilon,0}\|^2 \lesssim \left(h^2+\tau^2/\varepsilon^2\right)^2,$$

which together with Lemma 4.2 derives that $\widehat{\mathcal{A}}^{\varepsilon,0}\lesssim \left(h^2+\tau^2/\varepsilon\right)^2$. Applying Lemma 4.1, it can be concluded that there exists $\tau_1>0$ such that when $\tau\le\tau_1$, we have

$$\widehat{\mathcal{A}}^{\varepsilon,k}\lesssim \left(h^2+\tau^2/\varepsilon\right)^2+\tau\sum_{l=1}^{k-1}\widehat{\mathcal{A}}^{\varepsilon,l}.$$

Using discrete Gronwall inequality, for sufficiently small τ, we can get that

$$\widehat{\mathcal{A}}^{\varepsilon,k}\lesssim \left(h^2+\tau^2/\varepsilon\right)^2, \quad 0\le k\le \frac{T}{\tau}-1,$$

which yields (3.38) for $\widehat{e}_{\psi}^{\varepsilon,k}$, $\widehat{e}_{\chi}^{\varepsilon,k}$, $\widehat{e}_{\varphi}^{\varepsilon,k}$ by recalling (4.52), i.e.,

$$\|e_{\psi}^{\varepsilon,k}\|_{H^1}+\|e_{\chi}^{\varepsilon,k}\|_{H^1}+\|e_{\varphi}^{\varepsilon,k}\|_{H^1}\lesssim h^2+\frac{\tau^2}{\varepsilon}, \quad 0\le k\le\frac{T}{\tau}, \quad 0<\varepsilon\le 1. \tag{4.63}$$

Step 2 (Establish (3.39)-type error bound for $\widehat{e}_{\psi}^{\varepsilon,k}$, $\widehat{e}_{\chi}^{\varepsilon,k}$, $\widehat{e}_{\varphi}^{\varepsilon,k}$). Define another set of error functions $\widetilde{e}_{\psi}^{\varepsilon,k}$, $\widetilde{e}_{\chi}^{\varepsilon,k}$, $\widetilde{e}_{\varphi}^{\varepsilon,k}\in X_M$ for $0\le k\le\frac{T}{\tau}$ as

$$\widetilde{e}_{\psi,j}^{\varepsilon,k}=\widetilde{\psi}^{\varepsilon}(x_j,t_k)-\widehat{\psi}_j^{\varepsilon,k}, \quad \widetilde{e}_{\chi,j}^{\varepsilon,k}=-\widehat{\chi}_j^{\varepsilon,k}, \quad \widetilde{e}_{\varphi,j}^{\varepsilon,k}=\widetilde{\varphi}^{\varepsilon}(x_j,t_k)-\widehat{\varphi}_j^{\varepsilon,k}, \quad j\in\mathcal{T}_M,$$

where $(\widetilde{\psi}^{\varepsilon},\widetilde{\varphi}^{\varepsilon})$ is the solution of the SY-OP (3.28), and their corresponding local truncation errors $\widetilde{\xi}^{\varepsilon,k}$, $\widetilde{\eta}^{\varepsilon,k}$, $\widetilde{\zeta}^{\varepsilon,k}\in X_M$ are defined as

$$\begin{aligned}
&\widetilde{\xi}_j^{\varepsilon,k}=i\delta_t^c\widetilde{\psi}^{\varepsilon}(x_j,t_k)+\left[\delta_x^2+\mu_j^{\varepsilon,k}+\widetilde{\varphi}^{\varepsilon}(x_j,t_k)\right]\widetilde{\psi}^{\varepsilon}(x_j,t_{[k]}),\\
&\widetilde{\eta}_j^{\varepsilon,k}=\varepsilon^2\delta_t^2\widetilde{\varphi}^{\varepsilon}(x_j,t_k), \quad j\in\mathcal{T}_M, \quad k\ge 1,\\
&\widetilde{\zeta}_j^{\varepsilon,k}=\widetilde{\varphi}^{\varepsilon}(x_j,t_k)-\delta_x^2\widetilde{\varphi}^{\varepsilon}(x_j,t_k)-|\widetilde{\psi}^{\varepsilon}(x_j,t_k)|^2, \quad j\in\mathcal{T}_M, \quad k\ge 0.
\end{aligned} \tag{4.64}$$

Similar to the proof in [3], we can get the local truncation error as

$$\begin{aligned}
&\|\widetilde{\xi}^{\varepsilon,k}\|_{H^1}\lesssim h^2+\tau^2+\tau\varepsilon^{\alpha^*}, \quad \|\widetilde{\eta}^{\varepsilon,k}\|\lesssim\varepsilon^2, \quad 1\le k<\frac{T}{\tau};\\
&\|\delta_t^c\widetilde{\eta}^{\varepsilon,k}\|\lesssim\varepsilon^{1+\alpha^*}, \quad 2\le k\le\frac{T}{\tau}-2; \quad \|\widetilde{\zeta}^{\varepsilon,k}\|+\|\delta_t^+\widetilde{\zeta}^{\varepsilon,k}\|+\|\delta_t^c\widetilde{\zeta}^{\varepsilon,k}\|\lesssim h^2,
\end{aligned} \tag{4.65}$$

and the error bounds at the first step as

$$\widetilde{e}_{\psi}^{\varepsilon,0}=\widetilde{e}_{\chi}^{\varepsilon,0}=\mathbf{0}, \quad \|\widetilde{e}_{\psi}^{\varepsilon,1}\|_{H^1}+\|\widetilde{e}_{\chi}^{\varepsilon,1}\|_{H^1}\lesssim\tau^2+\tau\varepsilon^{\alpha^*}, \quad \|\delta_t^+\widetilde{e}_{\psi}^{\varepsilon,0}\|+\|\delta_t^+\widetilde{e}_{\chi}^{\varepsilon,0}\|\lesssim\tau+\varepsilon^{\alpha^*}. \tag{4.66}$$

Subtracting (4.42) from (4.64), we obtain the error equations

$$\begin{aligned}
&i\delta_t^c\widetilde{e}_{\psi,j}^{\varepsilon,k}+(\delta_x^2+\mu_j^{\varepsilon,k})\widetilde{e}_{\psi,j}^{\varepsilon,[k]}+\widetilde{r}_j^{\varepsilon,k}=\widetilde{\xi}_j^{\varepsilon,k},\\
&\varepsilon^2\delta_t^2\widetilde{e}_{\chi,j}^{\varepsilon,k}+(1-\delta_x^2)\widetilde{e}_{\chi,j}^{\varepsilon,[k]}+\varepsilon^2\delta_t^2\widetilde{e}_{\varphi,j}^{\varepsilon,k}=\widetilde{\eta}_j^{\varepsilon,k}, \quad 1\le k<T/\tau,\\
&\widetilde{e}_{\varphi,j}^{\varepsilon,k}-\delta_x^2\widetilde{e}_{\varphi,j}^{\varepsilon,k}-\widetilde{p}_j^{\varepsilon,k}=\widetilde{\zeta}_j^{\varepsilon,k}, \quad 0\le k\le T/\tau,
\end{aligned} \tag{4.67}$$

where $\widetilde{r}^k\in X_M$ and $\widetilde{p}^k\in X_M$ are defined as

$$\widetilde{r}_j^{\varepsilon,k}=\widetilde{\varphi}^{\varepsilon}(x_j,t_k)\widetilde{\psi}^{\varepsilon}(x_j,t_{[k]})-\widehat{\varphi}_j^{\varepsilon,k}g(\widehat{\psi}_j^{\varepsilon,k+1},\widehat{\psi}_j^{\varepsilon,k-1}), \;\; \widetilde{p}_j^{\varepsilon,k}=|\widetilde{\psi}^{\varepsilon}(x_j,t_k)|^2-\rho_B(|\widehat{\psi}_j^{\varepsilon,k}|^2).$$

Define another discrete energy for $0 \le k \le \frac{T}{\tau} - 1$ by

$$\widetilde{\mathcal{A}}^{\varepsilon,k} = C_1(\|\widetilde{e}_\psi^{\varepsilon,k}\|^2 + \|\widetilde{e}_\psi^{\varepsilon,k+1}\|^2) + \|\delta_x^+ \widetilde{e}_\psi^{\varepsilon,k+1}\|^2 + \|\delta_x^+ \widetilde{e}_\psi^{\varepsilon,k}\|^2 + \frac{1}{2}\|\widetilde{e}_\chi^{\varepsilon,k}\|_{H^1}^2 + \frac{1}{2}\|\widetilde{e}_\chi^{\varepsilon,k+1}\|_{H^1}^2 + \varepsilon^2\|\delta_t^+(\widetilde{e}_\chi^{\varepsilon,k} + \widetilde{e}_\varphi^{\varepsilon,k})\|^2. \tag{4.68}$$

Applying the same approach as in **Step 1**, there exists $\tau_2 > 0$ sufficiently small and independent of $0 < \varepsilon \le 1$ such that when $0 < \tau \le \tau_2$,

$$\widetilde{\mathcal{A}}^{\varepsilon,k} \lesssim \widetilde{\mathcal{A}}^{\varepsilon,0} + \|\widetilde{\eta}^{\varepsilon,1}\|^2 + \|\widetilde{\eta}^{\varepsilon,2}\|^2 + \sum_{l=k}^{k+1}\left(\|\widetilde{\eta}^{\varepsilon,l-1}\|^2 + \|\widetilde{\zeta}^{\varepsilon,l}\|^2\right) + \tau\sum_{l=1}^{k}\widetilde{\mathcal{A}}^{\varepsilon,l} + \tau\sum_{l=2}^{k-1}\|\delta_t^c\widetilde{\eta}^{\varepsilon,l}\|^2 + \tau\sum_{l=1}^{k}\left(\|\widetilde{\xi}^{\varepsilon,l}\|_{H^1}^2 + \|\widetilde{\zeta}^{\varepsilon,l}\|^2 + \|\delta_t^c\widetilde{\zeta}^{\varepsilon,l}\|^2\right).$$

Moreover, by the Cauchy inequality, (4.65) and (4.66), we have

$$\|\delta_t^+\widetilde{e}_\varphi^{\varepsilon,0}\|^2 \lesssim \|\delta_t^+\widetilde{p}^{\varepsilon,0}\|^2 + \|\delta_t^+\widetilde{\zeta}^{\varepsilon,0}\|^2 \lesssim \|\delta_t^+\widetilde{e}_\psi^{\varepsilon,0}\|^2 + \|\delta_t^+\widetilde{\zeta}^{\varepsilon,0}\|^2 \lesssim (h^2 + \tau + \varepsilon^{\alpha^*})^2,$$

which together with (4.66) yields that $\widetilde{\mathcal{A}}^{\varepsilon,0} \lesssim (\tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*})^2$. Applying (4.65), we have

$$\widetilde{\mathcal{A}}^{\varepsilon,k} \lesssim \left(h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}\right)^2 + \tau\sum_{l=1}^{k-1}\widetilde{\mathcal{A}}^{\varepsilon,l}, \quad 1 \le k \le \frac{T}{\tau} - 1.$$

Using the discrete Gronwall inequality, when $0 < \tau \le \tau_2$, one has

$$\widetilde{\mathcal{A}}^{\varepsilon,k} \lesssim \left(h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}\right)^2, \quad 1 \le k \le \frac{T}{\tau} - 1.$$

which immediately gives $\|\widetilde{e}_\psi^{\varepsilon,k}\|_{H^1} + \|\widetilde{e}_\chi^{\varepsilon,k}\|_{H^1} + \|\widetilde{e}_\varphi^{\varepsilon,k}\|_{H^1} \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}$. Similar to (4.51), using (2.25), we can get the inequality

$$\|\varphi^\varepsilon - \widetilde{\varphi}^\varepsilon\|_{H^1} \lesssim \|\psi^\varepsilon - \widetilde{\psi}^\varepsilon\|_{L^2} \lesssim \varepsilon^2.$$

Then (3.39) for $\widehat{e}_\psi^{\varepsilon,k}$, $\widehat{e}_\chi^{\varepsilon,k}$, $\widehat{e}_\varphi^{\varepsilon,k}$ can be established via the triangle inequality, i.e.,

$$\|\widehat{e}_\psi^{\varepsilon,k}\|_{H^1} + \|\widehat{e}_\chi^{\varepsilon,k}\|_{H^1} + \|\widehat{e}_\varphi^{\varepsilon,k}\|_{H^1} \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}. \tag{4.69}$$

Step 3 (Obtain ε-uniform estimate (3.40)). Combining (4.63) and (4.69), when $\tau \le \min\{\tau_1, \tau_2\}$, it is established that

$$\|\widehat{e}_\psi^{\varepsilon,k}\|_{H^1} + \|\widehat{e}_\chi^{\varepsilon,k}\|_{H^1} + \|\widehat{e}_\varphi^{\varepsilon,k}\|_{H^1} \lesssim h^2 + \max_{0<\varepsilon\le 1}\min\left\{\tau^2 + \varepsilon^{\alpha^*}(\tau + \varepsilon), \frac{\tau^2}{\varepsilon}\right\} \lesssim h^2 + \tau^{1+\frac{\alpha^*}{2+\alpha^*}}.$$

This, together with the inverse inequality [35], implies

$$\|\widehat{\psi}^{\varepsilon,k}\|_\infty - \|\psi^\varepsilon(\cdot, t_k)\|_\infty \le \|\widehat{e}_\psi^{\varepsilon,k}\|_\infty \lesssim \|\widehat{e}^{\varepsilon,k}\|_{H^1} \lesssim h^2 + \tau^{1+\frac{\alpha^*}{2+\alpha^*}}, \quad 0 \le k \le \frac{T}{\tau}.$$

Thus, there exist $h_0 > 0$ and $\tau_3 > 0$ sufficiently small and independent of $0 < \varepsilon \le 1$ such that when $0 < h \le h_0$ and $0 < \tau \le \tau_3$,

$$\|\widehat{\psi}^{\varepsilon,k}\|_\infty \le 1 + \|\psi^\varepsilon(\cdot, t_k)\|_\infty \le 1 + M_0, \quad 0 \le k \le \frac{T}{\tau}.$$

Taking $\tau_0 = \min\{\tau_1, \tau_2, \tau_3\}$, when $0 < h \le h_0$ and $0 < \tau \le \tau_0$, the numerical method (4.42) collapses to (3.30), i.e.,

$$\widehat{\psi}_j^{\varepsilon,k} = \psi_j^{\varepsilon,k}, \quad \widehat{\varphi}_j^{\varepsilon,k} = \varphi_j^{\varepsilon,k}, \quad \widehat{\chi}_j^{\varepsilon,k} = \chi_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M^0, \quad 0 \le k \le \frac{T}{\tau}.$$

Thus the proof is completed. □

Remark 4.1. The error bounds in Theorem 3.1 are still valid in high dimensions, e.g., $d = 2, 3$, provided that an additional condition on the time step τ is added

$$\tau = o\left(C_d(h)^{1-\frac{\alpha^*}{2+2\alpha^*}}\right), \qquad \text{with} \qquad C_d(h) \sim \begin{cases} \dfrac{1}{|\ln h|}, & d = 2, \\ h^{1/2}, & d = 3. \end{cases}$$

The reason is due to the discrete Sobolev inequality [8,35]: $\|\psi_h\|_\infty \le \frac{1}{C_d(h)}\|\psi_h\|_{H^1}$, where ψ_h is a mesh function over Ω with homogeneous Dirichlet boundary condition.

5. Numerical results.

In this section, we present numerical results for the KGS equations (1.10) by our proposed finite difference method. Furthermore, we apply the method to numerically study convergence rates of the KGS equations to its limiting models (2.12) and (2.24) in the nonrelativistic and massless limit regimes. In order to do so, we take $d = 1$ in (1.10) and the initial condition is set as (2.19).

5.1. Accuracy test.

We mainly consider two types of initial data:

Case I. well-prepared initial data, i.e., $\alpha = 1$ and $\beta = 0$;

Case II. ill-prepared initial data, i.e., $\alpha = 0$ and $\beta = -1$.

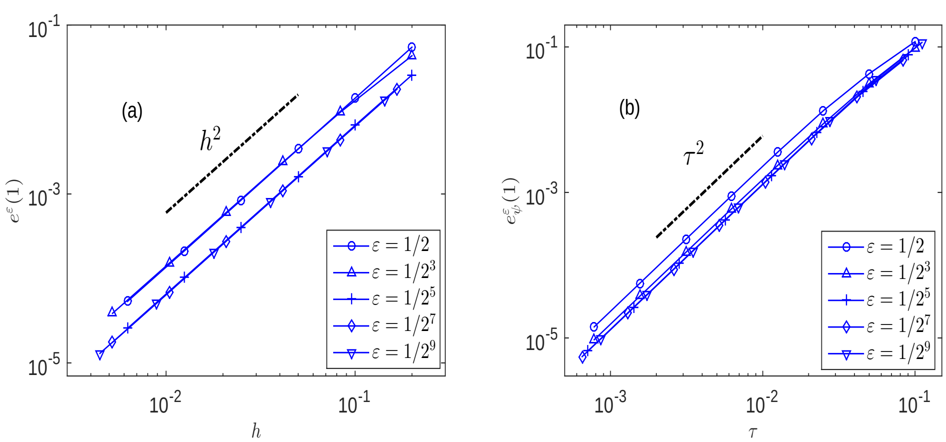

FIGURE 5.4. Spatial errors for Case II (a) and temporal errors of ψ^ε for Case I (b).

Practically, the problem is truncated on an interval $\Omega_\varepsilon = \left[-30 - \frac{1}{\varepsilon}, 30 + \frac{1}{\varepsilon}\right]$, which is large enough such that the truncation error of (3.26) to the original whole space problem (2.23) can be ignorable due to the homogeneous Dirichlet boundary condition. Due to the rapid outspreading waves with wave speed $O\left(\frac{1}{\varepsilon}\right)$ (cf. Figure 2.2(b)) and the homogeneous Dirichlet boundary condition truncated at the boundary, the computational domain Ω_ε has to be chosen as ε-dependent. The computational ε-dependent domain can be fixed as ε-independent if one applies other

TABLE 5.1. Temporal errors of ϕ^ε for Case I initial data.

$e^\varepsilon_\phi(1)$	$\tau_0=0.1$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$	$\tau_0/2^7$
$\varepsilon=1/2$	2.15E-2	5.48E-3	1.39E-3	3.49E-4	8.75E-5	2.19E-5	5.48E-6	1.38E-6
rate	-	1.97	1.98	1.99	2.00	2.00	2.00	1.99
$\varepsilon=1/2^2$	4.72E-2	1.57E-2	4.19E-3	1.07E-3	2.68E-4	6.72E-5	1.68E-5	4.21E-6
rate	-	1.59	1.91	1.97	1.99	2.00	2.00	2.00
$\varepsilon=1/2^3$	2.38E-2	1.36E-2	4.60E-3	1.24E-3	3.15E-4	7.92E-5	1.98E-5	4.96E-6
rate	-	0.81	1.56	1.89	1.97	1.99	2.00	2.00
$\varepsilon=1/2^4$	2.19E-2	8.12E-3	4.79E-3	2.16E-3	6.21E-4	1.59E-4	3.99E-5	1.00E-5
rate	-	1.43	0.76	1.15	1.80	1.97	1.99	2.00
$\varepsilon=1/2^5$	2.45E-2	5.22E-3	1.83E-3	1.37E-3	9.03E-4	3.11E-4	8.20E-5	2.07E-5
rate	-	2.23	1.51	0.42	0.60	1.54	1.92	1.99
$\varepsilon=1/2^6$	2.57E-2	7.93E-3	1.70E-3	4.97E-4	3.25E-4	3.06E-4	1.44E-4	4.16E-5
rate	-	1.70	2.22	1.77	0.61	0.09	1.09	1.79
$\varepsilon=1/2^7$	2.61E-2	6.58E-3	1.90E-3	4.20E-4	1.25E-4	7.70E-5	8.50E-5	5.75E-5
rate	-	1.99	1.79	2.18	1.75	0.69	-0.14	0.57
$\varepsilon=1/2^8$	2.62E-2	6.26E-3	1.75E-3	3.85E-4	1.05E-4	3.12E-5	1.97E-5	1.94E-5
rate	-	2.07	1.84	2.19	1.87	1.75	0.67	0.02
$\varepsilon=1/2^9$	2.62E-2	6.21E-3	1.54E-3	5.00E-4	1.06E-4	2.63E-5	7.80E-6	4.92E-6
rate	-	2.08	2.01	1.62	2.24	2.01	1.75	0.67
$\varepsilon=1/2^{10}$	2.62E-2	6.19E-3	1.50E-3	3.97E-4	1.17E-4	2.62E-5	6.58E-6	1.95E-6
rate	-	2.08	2.04	1.92	1.76	2.16	2.00	1.75

appropriate boundary conditions, e.g., absorbing boundary condition, or transparent boundary condition, or perfected matched layer for the wave-type equations in (3.26) and (3.27) during the truncation (cf. [3]).

To quantify the numerical errors, we introduce the following error functions

$$e^\varepsilon_\psi(t_k) := \frac{\|e^{\varepsilon,k}_\psi\|_{H^1}}{\|\psi^\varepsilon(\cdot,t_k)\|_{H^1}},\quad e^\varepsilon_\phi(t_k) := \frac{\|\phi^\varepsilon(\cdot,t_k)-\phi^{\varepsilon,k}\|_{H^1}}{\|\phi^\varepsilon(\cdot,t_k)\|_{H^1}},\quad e^\varepsilon(t_k) = e^\varepsilon_\psi(t_k)+e^\varepsilon_\phi(t_k),$$

where $e^{\varepsilon,k}_{\psi,j} = \psi^\varepsilon(x_j,t_k) - \psi^{\varepsilon,k}_j$ and $\phi^{\varepsilon,k}_j = \varphi^{\varepsilon,k}_j + \chi^{\varepsilon,k}_j + \omega^\varepsilon(x_j,t_k)$ for $j\in\mathcal{T}_M$. The "exact" solution is obtained by the phase space analytical solver & time splitting spectral method [5] with very small mesh size $h = 1/32$ and time step $\tau = 10^{-6}$. The errors are displayed at $t = 1$. For spatial error analysis, we set the time step $\tau = 10^{-5}$ such that the temporal error can be neglected; for temporal error analysis, the mesh size h is set as $h = 2.5\times 10^{-4}$ such that the spatial error can be ignored.

Figure 5.4(a) depicts the spatial errors for Case II initial data with different mesh size h and $0 < \varepsilon \le 1$. It clearly demonstrates that our proposed finite difference method is second order accurate in space, which is uniformly for all $\varepsilon \in (0, 1]$. The results for other initial data are analogous, e.g., different $\alpha \ge 0$ and $\beta \ge -1$ and thus are omitted for brevity.

Figure 5.4(b), Tables 5.1 and 5.2 present the temporal errors for Case I and II initial data for different time step τ and $0 < \varepsilon \le 1$, respectively. Figure 5.4(b) shows the temporal errors of ψ^ε for Case I initial data, which suggests that the method is uniformly second order accurate for the nucleon field ψ^ε with well-prepared initial data. While for the messon field ϕ^ε, the upper and lower triangle parts of Table 5.1 suggest the error at $O(\tau^2/\varepsilon)$ and $O(\tau^2+\varepsilon^2)$, respectively. There is a resonance

TABLE 5.2. Temporal errors for Case II initial data.

$e^{\varepsilon}_{\psi}(1)$	$\tau_0=0.1$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$	$\tau_0/2^7$
$\varepsilon=1/2$	1.85E-1	7.00E-2	2.19E-2	5.89E-3	1.50E-3	3.76E-4	9.41E-5	2.36E-5
rate	-	1.40	1.67	1.90	1.98	1.99	2.00	2.00
$\varepsilon=1/2^2$	3.64E-1	1.99E-1	6.66E-2	1.75E-2	4.40E-3	1.10E-3	2.76E-4	6.90E-5
rate	-	0.87	1.58	1.93	1.99	2.00	2.00	2.00
$\varepsilon=1/2^3$	1.31E-1	5.94E-2	3.36E-2	1.62E-2	4.95E-3	1.28E-3	3.23E-4	8.09E-5
rate	-	1.14	0.82	1.05	1.71	1.95	1.99	2.00
$\varepsilon=1/2^4$	1.46E-1	4.21E-2	1.12E-2	2.91E-3	7.34E-4	1.84E-4	4.59E-5	1.15E-5
rate	-	1.79	1.91	1.95	1.99	2.00	2.00	2.00
$\varepsilon=1/2^5$	1.05E-1	4.15E-2	1.09E-2	2.62E-3	6.38E-4	1.57E-4	3.90E-5	9.75E-6
rate	-	1.35	1.93	2.06	2.04	2.02	2.01	2.00
$\varepsilon=1/2^6$	1.00E-1	3.14E-2	9.05E-3	3.02E-3	6.81E-4	1.60E-4	3.86E-5	9.53E-6
rate	-	1.67	1.79	1.58	2.15	2.09	2.05	2.02
$\varepsilon=1/2^7$	1.01E-1	3.30E-2	8.75E-3	2.88E-3	9.29E-4	1.93E-4	4.23E-5	9.88E-6
rate	-	1.61	1.92	1.61	1.63	2.27	2.19	2.10
$\varepsilon=1/2^8$	1.00E-1	3.30E-2	9.80E-3	2.59E-3	1.16E-3	3.30E-4	6.16E-5	1.21E-5
rate	-	1.61	1.75	1.92	1.17	1.81	2.42	2.35
$\varepsilon=1/2^9$	1.01E-1	3.31E-2	9.84E-3	3.05E-3	8.71E-4	5.22E-4	1.36E-4	2.31E-5
rate	-	1.61	1.75	1.69	1.81	0.74	1.94	2.55
$\varepsilon=1/2^{10}$	1.01E-1	3.34E-2	9.96E-3	3.11E-3	1.08E-3	3.41E-4	2.50E-4	6.16E-5
rate	-	1.59	1.75	1.68	1.52	1.67	0.45	2.02

$e^{\varepsilon}_{\phi}(1)$	$\tau_0=0.1$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$	$\tau_0/2^7$
$\varepsilon=1/2$	1.71E-2	4.30E-3	1.09E-3	2.74E-4	6.88E-5	1.72E-5	4.31E-6	1.08E-6
rate	-	1.99	1.98	1.99	2.00	2.00	2.00	1.99
$\varepsilon=1/2^2$	2.76E-2	9.96E-3	2.63E-3	6.69E-4	1.68E-4	4.21E-5	1.05E-5	2.64E-6
rate	-	1.47	1.92	1.97	1.99	2.00	2.00	2.00
$\varepsilon=1/2^3$	9.75E-3	8.65E-3	3.62E-3	1.05E-3	2.71E-4	6.83E-5	1.71E-5	4.28E-6
rate	-	0.17	1.26	1.79	1.95	1.99	2.00	2.00
$\varepsilon=1/2^4$	6.62E-3	2.61E-3	2.72E-3	1.58E-3	4.52E-4	1.15E-4	2.90E-5	7.25E-6
rate	-	1.34	-0.06	0.78	1.81	1.97	1.99	2.00
$\varepsilon=1/2^5$	3.24E-3	1.64E-3	7.12E-4	6.54E-4	7.69E-4	2.66E-4	6.90E-5	1.73E-5
rate	-	0.98	1.20	0.12	-0.23	1.53	1.94	1.99
$\varepsilon=1/2^6$	3.47E-3	1.17E-3	6.10E-4	2.23E-4	1.75E-4	1.47E-4	1.38E-4	3.84E-5
rate	-	1.57	0.94	1.45	0.35	0.26	0.09	1.84
$\varepsilon=1/2^7$	3.51E-3	1.12E-3	3.07E-4	2.75E-4	8.62E-5	4.14E-5	4.63E-5	5.33E-5
rate	-	1.65	1.86	0.16	1.67	1.06	-0.16	-0.20
$\varepsilon=1/2^8$	3.53E-3	1.01E-3	3.85E-4	1.19E-4	1.32E-4	3.88E-5	1.21E-5	1.19E-5
rate	-	1.80	1.39	1.70	-0.15	1.77	1.69	0.02
$\varepsilon=1/2^9$	3.56E-3	9.95E-4	3.38E-4	1.45E-4	5.13E-5	6.48E-5	1.86E-5	3.96E-6
rate	-	1.84	1.56	1.22	1.50	-0.34	1.80	2.23
$\varepsilon=1/2^{10}$	3.57E-3	1.01E-3	3.29E-4	1.34E-4	5.64E-5	2.39E-5	3.22E-5	9.21E-6
rate	-	1.82	1.62	1.30	1.24	1.24	-0.43	1.81

regime when $\tau \sim \varepsilon^{3/2}$ where the convergence rate degenerates to 4/3 (cf. Table 5.3). For Case II initial data, the resonance regime is $\tau \sim \varepsilon$, where the convergence

TABLE 5.3. Temporal error analysis at time $t = 1$ in the resonance regions for different ε and τ.

Case I $\tau = O(\varepsilon^{3/2})$	$\varepsilon_0 = 1/2$ $\tau_0 = 0.1$	$\varepsilon_0/2^2$ $\tau_0/2^3$	$\varepsilon_0/2^4$ $\tau_0/2^6$	$\varepsilon_0/2^6$ $\tau_0/2^9$
$e^{\varepsilon}_{\phi}(1)$	2.15E-2	1.24E-3	8.20E-5	5.18E-6
rate in time	-	4.12/3	3.92/3	3.98/3

Case II $\tau = O(\varepsilon)$	$\varepsilon_0 = 1/2^2$ $\tau_0 = 0.1/2^2$	$\varepsilon_0/2$ $\tau_0/2$	$\varepsilon_0/2^2$ $\tau_0/2^2$	$\varepsilon_0/2^3$ $\tau_0/2^3$	$\varepsilon_0/2^4$ $\tau_0/2^4$	$\varepsilon_0/2^5$ $\tau_0/2^5$
$e^{\varepsilon}_{\phi}(1)$	2.63E-3	1.05E-3	4.52E-4	2.66E-4	1.38E-4	5.33E-5
rate in time	-	1.32	1.21	0.76	0.95	1.37

rate is downgraded to the first order (cf. Table 5.3). Numerical results suggest that our analysis is sharp for ϕ^{ε}.

5.2. **Convergence rates of the KGS system to its limiting models when $\varepsilon \to 0$.** Let $(\psi^{\varepsilon}, \phi^{\varepsilon})$ be the solution of the KGS equations (1.10) with initial data (2.19) which is obtained numerically by the proposed finite difference method on a bounded interval Ω_{ε} for initial data (2.19) with a very fine mesh $h = 10^{-2}$ and a time step $\tau = 10^{-4}$. Let ψ^0 and $\widetilde{\psi}^{\varepsilon}$ be the solutions of the SY equations (2.12) and SY-OP equations (2.24), respectively. Denote $\widehat{\phi}^{\varepsilon}(x,t) = (-\Delta+I)^{-1}|\psi^0|^2(x,t)+\omega^{\varepsilon}(x,t)$ and $\widetilde{\phi}^{\varepsilon}(x,t) = (-\Delta + I)^{-1}|\widetilde{\psi}^{\varepsilon}|^2(x,t) + \omega^{\varepsilon}(x,t)$ with $\omega^{\varepsilon}(x,t)$ being the solution of the linear equation (2.22). Define the error functions as

$$\eta^{\varepsilon}_{\text{S}}(t) := \|\psi^{\varepsilon}(\cdot,t) - \psi^0(\cdot,t)\|_{H^1} + \|\phi^{\varepsilon}(\cdot,t) - \widehat{\phi}^{\varepsilon}(\cdot,t)\|_{H^1},$$
$$\eta^{\varepsilon}_{\text{SO}}(t) := \|\psi^{\varepsilon}(\cdot,t) - \widetilde{\psi}^{\varepsilon}(\cdot,t)\|_{H^1} + \|\phi^{\varepsilon}(\cdot,t) - \widetilde{\phi}^{\varepsilon}(\cdot,t)\|_{H^1}.$$

Figure 5.5 (a), (b), (c) plot the errors between the solutions of the KGS system (1.10) and the SY equations (2.12) with compatible initial data, i.e., $\omega_0(x) \equiv 0$ and $\omega_1(x) \equiv 0$ in (2.18), well-prepared initial data, i.e., $\alpha = 1$, $\beta = 0$, and ill-prepared initial data, i.e., $\alpha = 0$, $\beta = -1$ for different $\varepsilon > 0$; Figure 5.6 depicts the errors between the solutions of the KGS equations (1.10) and the SY-OP system (2.24) for ill-prepared initial data. The results for other initial data are similar and thus are omitted here for brevity.

From Figures 5.5-5.6, we can draw the following conclusions:

(i) The solution ψ^{ε} of the KGS equations (1.10) converges to that of the SY equations (2.12) ψ^0 and ϕ^{ε} converges to $\widehat{\phi}^{\varepsilon}$ when $\varepsilon \to 0^+$. In addition, we have the following convergence rates

$$\|\psi^{\varepsilon}(\cdot,t) - \psi^0(\cdot,t)\|_{H^1} + \|\phi^{\varepsilon}(\cdot,t) - \widehat{\phi}^{\varepsilon}(\cdot,t)\|_{H^1} \le C_1\,\varepsilon^{1+\alpha^*}, \tag{5.70}$$

where C_1 is a positive constant independent of $\varepsilon \in (0,1]$.

(ii) The solution ψ^{ε} of the KGS system (1.10) converges to $\widetilde{\psi}^{\varepsilon}$ of the SY-OP equations (2.24) and ϕ converges to $\widetilde{\phi}^{\varepsilon}$ with the following quadratic convergence rate for any kind of initial data

$$\|\psi^{\varepsilon}(\cdot,t) - \widetilde{\psi}^{\varepsilon}(\cdot,t)\|_{H^1} + \|\phi^{\varepsilon}(\cdot,t) - \widetilde{\phi}^{\varepsilon}(\cdot,t)\|_{H^1} \le C_2\,\varepsilon^2, \tag{5.71}$$

where $C_2 > 0$ is a constant independent of ε. Based on the above results, we can see that the SY-OP (2.24) is a more accurate limiting model to approximate the

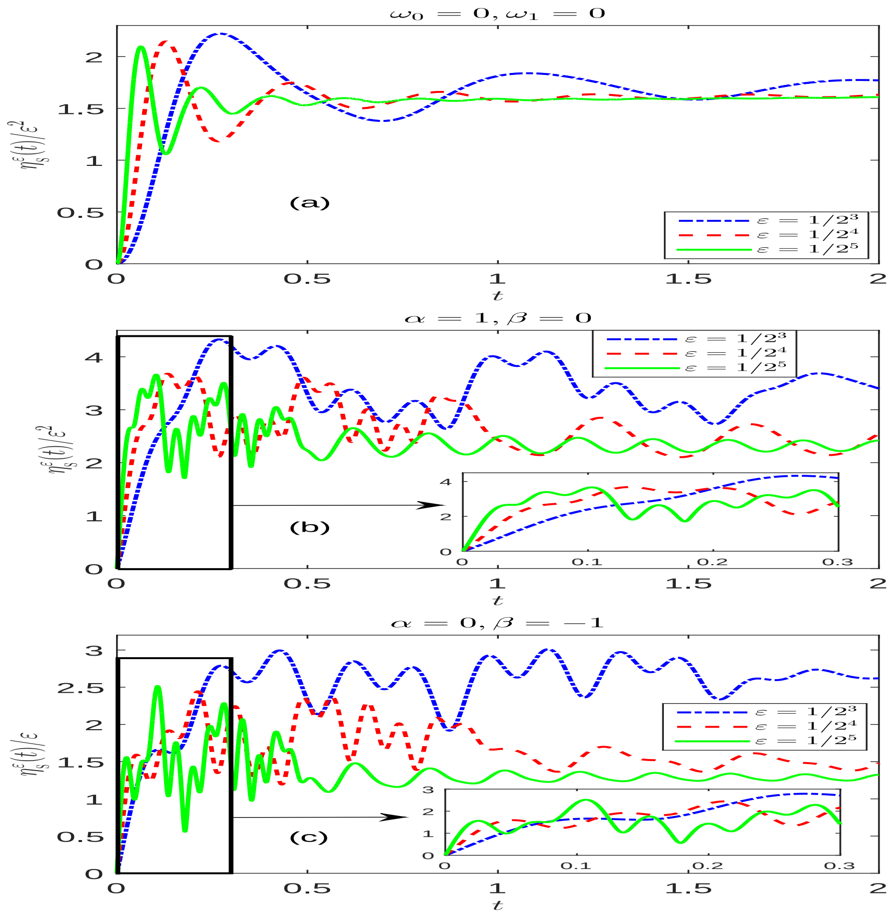

Figure 5.5. Convergence behavior between the KGS equations (1.10) and the SY equations (2.12) for different initial data.

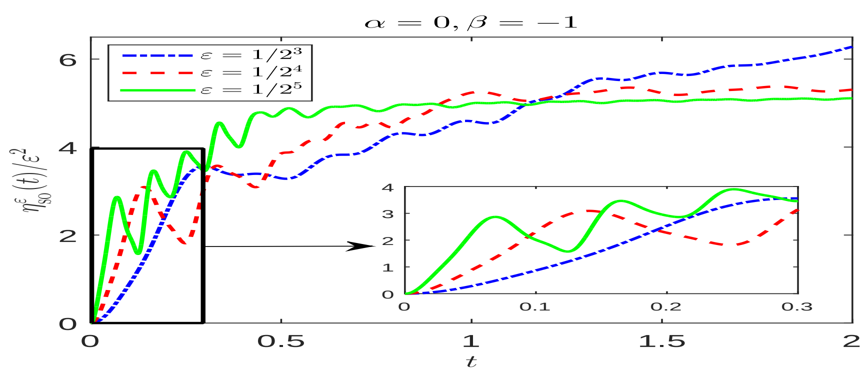

Figure 5.6. Convergence behavior between the KGS equations (1.10) and the SY-OP (2.24) with ill-prepared initial data, i.e., $\alpha = 0$, $\beta = -1$.

KGS equations in the simultaneously nonrelativistic and massless limit regimes, compared to the SY equations (2.12), especially for ill-prepared initial data.

6. **Conclusion.**

We presented a uniformly accurate finite difference method for the Klein-Gordon-Schrödinger (KGS) equations in the nonrelativistic and massless limit regimes — parameterized by a dimensionless parameter $0 < \varepsilon \leq 1$ — which is inversely proportional to the speed of light. When $0 < \varepsilon \ll 1$, the solution of KGS equations propagates highly oscillatory waves in time and rapid outspreading waves in space. Our method was designed by reformulating KGS system into an asymptotic consistent formulation and applying an integral approximation for the oscillating term. By using the energy method and the limiting model, we established two independent error bounds, which depend explicitly on the mesh size h, time step τ and the parameter $0 < \varepsilon \leq 1$. From the two error bounds, a uniform error estimate was obtained, which is uniformly accurate at second order in space and at least first order in time. Numerical experiments suggest that the error bounds are sharp. By adopting our numerical method, we observed that the Schrödinger-Yukawa system with an oscillatory potential approximates the KGS system quadratically in the nonrelativistic and massless limit regimes.

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