Numerical Methods for Problems in Unbounded Domains

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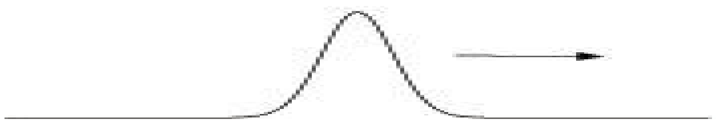
Outline

- Motivation
- Different approaches
- For model problems
- New `optimal' error estimates
- Extension of the results
- Application to Navier-Stokes equations
- Conclusion & Future challenges

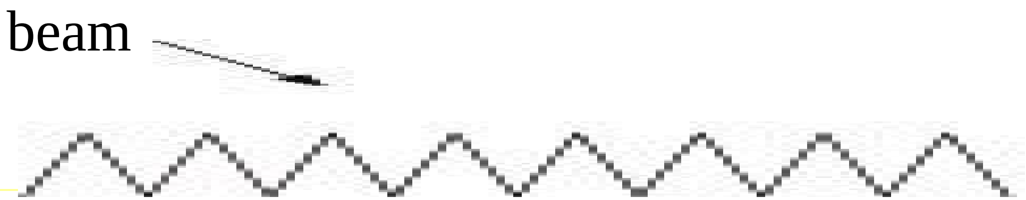
Motivation

- Problems in unbounded domains
  - – Potential flow

  - – Wave propagation & radiation

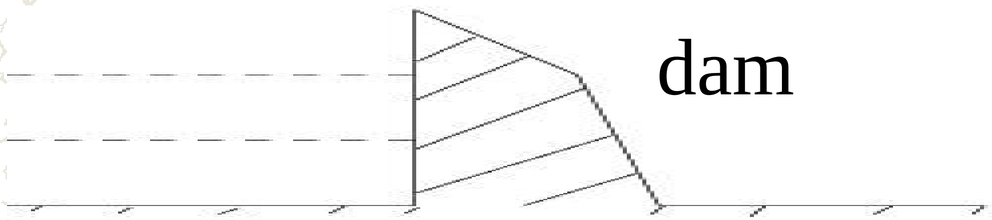


  - – Linear/nonlinear optics

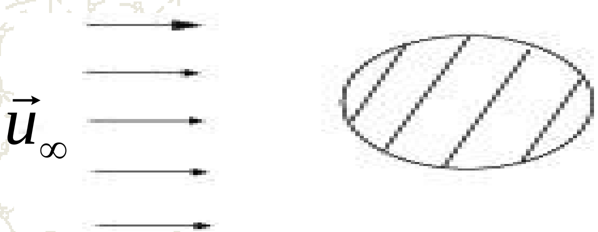


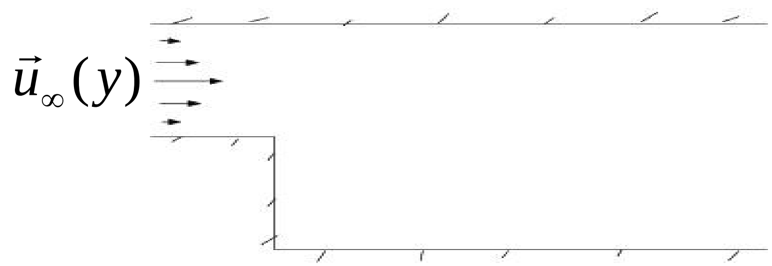
Motivation

– Coupling of structures with foundation



– Fluid flow around obstacle or in channel

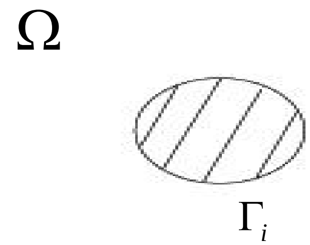




– Quantum physics & chemistry

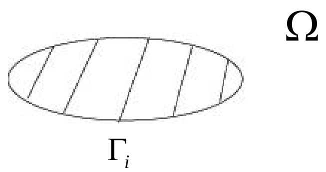
Motivation

- Numerical difficulties
  - Unboundedness of physical domain
  - Others

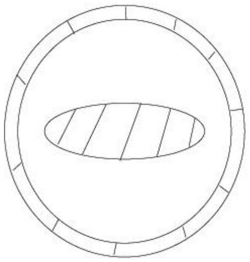


- Classical numerical methods
  - Finite element method (FEM)
  - Finite difference method (FDM)
  - Finite volume method (FVM)
- Linear/nonlinear system with infinite unknowns

Different Approaches

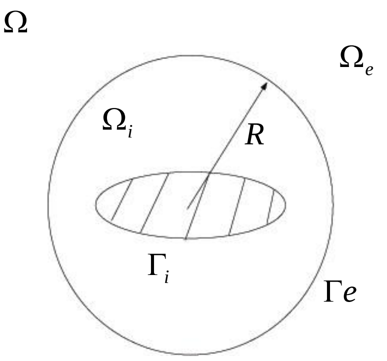


- Integral equation
  - Boundary element method (BEM): Feng, Yu, Du, …
  - Fast Multipole method (FMM): Roklin & Greengard, …
- Infinite element method: Xathis, Ying, Han, …
- Domain mapping
- Perfect matched layer (PML): Beranger
- FEM with two different types basis functions:
- Spectral method: Shen, Guo, …



Artificial Boundary Conditions

- Artificial boundary conditions (ABCs)
  - Introduce an artificial boundary Γ_e
  - Engineers use
    - Dirichlet or Neumann boundary condition on it
  - Better way:
    - Solve on Ω_e analytically
    - Design high-order ABC (DtN) on Γ_e based on transmission conditions, i.e. establish $\frac{\partial u}{\partial n}\Big|_{\Gamma_e} = K(u|_{\Gamma_e}) \quad K : H^{1/2}(\Gamma_e) \to H^{-1/2}(\Gamma_e)$
  - Reduce to Ω_i
  - Solve the reduced problem by a classical method
  - How to design high-order ABCs & do error analysis ???



Artificial Boundary Conditions

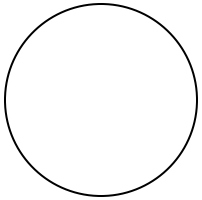
- 1&2D wave equation: Engquist & majda, 77' 3D case Teng, 03'
- Helmholtz equation in waveguides: Goldstein, 82'
- Elliptic equations: Bayliss, Gunzburger & Turkel, 82'
- Helmholtz equation (local ABC): Feng 84'
- Laplace & Navier system: Han & Wu, 85' & 92', Yu 85'
- Elliptic equations in a cylinder: Hagstrom & Keller, 86'
- Linear advection diffusion equation: Halpern, 86'
- Helmholtz equation (DtN): Givoli & Keller, 95'

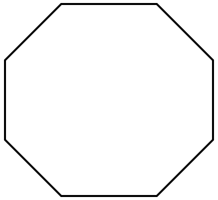
Artificial Boundary Conditions

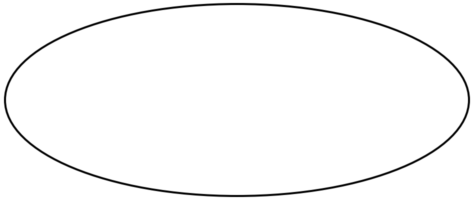
- Stokes system: Bao & Han, 97'
- Navier-Stokes equations: Halpern 89'; Bao, 95', 97', 00'
- Linear Schrodinger equation: Arnold, 99'; NLS Besse 02'
- Ginburg-Landau equation: Du & Wu, 99'
- New `optimal' error estimates: Bao & Han, 00', 03'
- Flow around a submerged body: Bao & Wen, 01'
- Shrodinger-Poisson: Ben Abdallah 98'
- Landau-Lipschitz: Bao & Wang,

Artificial Boundary Conditions

- Types of artificial boundary
  - Circle
  - Straight line
  - Segments
  - Polygonal line
  - Elliptic curve







Artificial Boundary Conditions

- Types of ABCs
  - Local: Dirichlet or Neumann

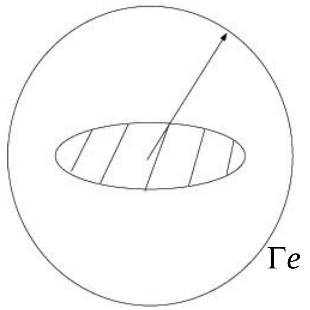
$$u\big|_{\Gamma_e} = u_\infty \quad \text{or} \quad \frac{\partial u}{\partial n}\Big|_{\Gamma_e} = 0$$

  - Nonlocal: DtN boundary condition

$$\frac{\partial u}{\partial n}\Big|_{\Gamma_e} = K\left(u\big|_{\Gamma_e}\right) \approx K_N\left(u\big|_{\Gamma_e}\right)$$

  - Discrete:

$$\begin{pmatrix} \frac{\partial u}{\partial n}\Big|_{\vec{x}=\vec{x}_1} \\ \cdots \\ \frac{\partial u}{\partial n}\Big|_{\vec{x}=\vec{x}_m} \end{pmatrix} = A \begin{pmatrix} u\big|_{\vec{x}=\vec{x}_1} \\ \cdots \\ u\big|_{\vec{x}=\vec{x}_m} \end{pmatrix}$$



Model Problem (I)

- 1D problem:

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{du(r)}{dr}\right)+\frac{m^2}{r^2}u(r)=f(r),\quad 0<r<\infty$$

$$u(0)=0,\qquad u\to 0\quad \text{when}\quad r\to\infty$$

  – Assume that: $m\neq 0,\qquad f(r)=0\quad \text{when}\quad r\geq r_0>0$

  – Artificial boundary: $r=R\geq r_0$

  – Exterior problem:

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{du(r)}{dr}\right)+\frac{m^2}{r^2}u(r)=0,\quad R<r<\infty$$

$$u(R)\ \text{given},\qquad u\to 0\quad \text{when}\quad r\to\infty$$

Model Problem (I)

- Exact solution: $u(r) = u(R)\,(R/r)^{|m|} \qquad r \ge R$
- Transmission conditions:

$$u(R^-) = u(R^+) \qquad u'(R^-) = u'(R^+)$$

- Exact boundary condition:

$$u'(R) = -|m|\,u(R)/R$$

- Reduced problem:

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{du(r)}{dr}\right) + \frac{m^2}{r^2}u(r) = f(r), \quad 0 < r < R$$

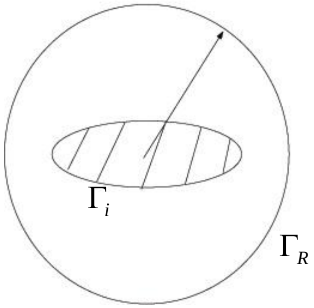
$$u(0) = 0, \qquad u'(R) = -|m|\,u(R)/R$$

Model Problem (II)

- 2D problem:

$$-\Delta u = f \qquad \text{in} \quad \Omega = Q^c$$

$$u\big|_{\Gamma_i} = g, \qquad u \quad \text{bounded when} \quad r = |\vec{x}| \to \infty$$



  - Assume: $\text{supp}(f) \cup \Gamma_i \subset B_{R_0}(\vec{0})$
  - Artificial boundary: $\Gamma_R = \{(R,\theta) \mid 0 \le \theta \le 2\pi\}$ with $R \ge R_0$
  - Exterior problem:

$$-\Delta u = 0, \qquad r > R$$

$$u(R,\theta) \text{ given}, \qquad u \quad \text{bounded when} \quad r \to \infty$$

Model Problem (II)

– Exact solution:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{R}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \qquad r \geq R$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} u(R,\theta) \cos n\theta \, d\theta, \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} u(R,\theta) \sin n\theta \, d\theta$$

– Transmission conditions:

$$u(R^-,\theta) = u(R^+,\theta) \qquad \frac{\partial u}{\partial r}(R^-,\theta) = \frac{\partial u}{\partial r}(R^+,\theta) \quad 0 \leq \theta \leq 2\pi$$

– Exact boundary condition:

$$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{\pi R} \sum_{n=1}^{\infty} n \int_0^{2\pi} u(R,\phi) \cos n(\theta-\phi) \, d\phi = -\frac{1}{\pi R} \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\partial u}{\partial \phi}(R,\phi) \sin n(\theta-\phi) \, d\phi$$

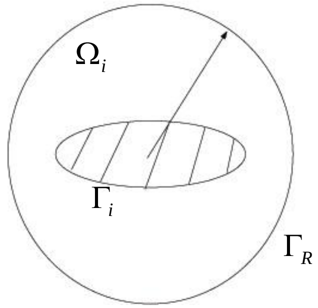
$$:= K(u(R,\phi)) = K(u\big|_{\Gamma_R}), \qquad K : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$$

Model Problem (II)

– Approximate ABCs:

$$\frac{\partial u}{\partial r}(R,\theta) \approx K_N(u|_{\Gamma_R}) := -\frac{1}{\pi R}\sum_{n=1}^{N} n \int_0^{2\pi} u(R,\theta)\cos n(\theta-\phi)\,d\phi \quad 0 \le \phi \le 2\pi \qquad N \ge 0$$

$$N = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial r}(R,\theta) = 0 \quad \text{(Neumann BC)}$$



– Reduced problem:

$$-\Delta u = f \qquad \text{in} \quad \Omega_i$$

$$u|_{\Gamma_i} = g, \qquad \frac{\partial u}{\partial r}(R,\theta) = K_N(u|_{\Gamma_R}) \quad 0 \le \theta \le 2\pi$$

Model Problem (II)

- Variational formulation:

$$V=\{v\in H^1(\Omega_i)\mid v|_{\Gamma_i}=0\}\quad\text{with norm}\quad \|v\|_V:=\|v\|_{1,2,\Omega_i}$$

$$A(u,v)=\int_{\Omega_i}\nabla u\bullet\nabla v\,d\vec{x}\qquad F(v)=\int_{\Omega_i} f\,v\,d\vec{x}$$

$$B(u,v)=\sum_{n=1}^{\infty} n[\int_0^{2\pi}u(R,\theta)\cos n\theta\,d\theta\int_0^{2\pi}v(R,\theta)\cos n\phi\,d\phi+\int_0^{2\pi}u(R,\theta)\sin n\theta\,d\theta\int_0^{2\pi}v(R,\theta)\sin n\phi\,d\phi]$$

$$B_N(u,v)=\sum_{n=1}^{N} n[\int_0^{2\pi}u(R,\theta)\cos n\theta\,d\theta\int_0^{2\pi}v(R,\theta)\cos n\phi\,d\phi+\int_0^{2\pi}u(R,\theta)\sin n\theta\,d\theta\int_0^{2\pi}v(R,\theta)\sin n\phi\,d\phi]$$

  – With exact BCs: Find $u\in V_g$ s.t.

$$A(u,v)+B(u,v)=F(v),\qquad \forall v\in V$$

  – With approximate ABCs: Find $u_N\in V_g$ s.t.

$$A(u_N,v)+B_N(u_N,v)=F(v),\qquad \forall v\in V$$

Model Problem (II)

- Finite element approximation: Find $u_N^h \in V_g^h$ s.t.

$$A(u_N^h, v^h) + B_N(u_N^h, v^h) = F(v^h), \qquad \forall v^h \in V^h$$

- Properties of $A(u,v)$

$$|A(u,v)| \le M_1 \|u\|_V \|v\|_V \qquad M_2 \|v\|_V^2 \le A(v,v)$$

- Properties of $B(u,v)$ & $B_N(u,v)$

$$|B(u,v)| \le M_3 \|u\|_V \|v\|_V \qquad |B_N(u,v)| \le M_3 \|u\|_V \|v\|_V$$

$$0 \le B_N(v,v) \le B(v,v)$$

Model Problem (II)

- Existing error estimates (Han&Wu, 85', Yu, 85', Givoli & Keller 89')

$$\left\| u - u_N^h \right\|_V \le C(R,u)\left[h^p + \frac{1}{(N+1)^p} \right] \quad \text{or} \quad C(R,u)\left[h^p + \frac{1}{(N+1)^k}\left(\frac{R_0}{R}\right)^N \right]$$

- Deficiency
  - N=0: no convergence, but numerically gives
  - How does error depend on R?
- Find new error estimates depend on
  - h, N & R ?????

New `Optimal' Error Estimate

$$\left| u - u_N^h \right|_{1,\Omega_i} \le C_0 \left[h^p \, |u|_{p+1,\Omega_i} + \frac{1}{(N+1)^p} \left(\frac{R_0}{R} \right)^{N+1} |u|_{\Gamma_0}|_{p+1/2,\Gamma_0} \right]$$

- N=0, convergence linearly as $R \to \infty$
- Fixed N, $R >> R_0$
- $\text{Ꞗ} = \text{Ꞗ}^0$,convergence as $N \to \infty$
- Tradeoff between N and R
- In practice, $R \ge R_0, \quad N : 5 \sim 10$ (Bao & Han, SIAMNA 00')

New `Optimal' Error Estimate

Ideas (Bao & Han, SIAMNA, 00'):

- Use an equivalent norm on V: $\|v\|_* = [A(v,v)]^{1/2} = |v|_{1,2,\Omega_i}$

$$|A(u,v)| \le \|u\|_* \|v\|_* \qquad \|v\|_*^2 \le A(v,v)$$

- Analysis B(u,v) carefully

$$B_N(v,v) = \pi \sum_{n=1}^{N} n(c_n^2 + d_n^2) \le \pi \sum_{n=1}^{\infty} n(c_n^2 + d_n^2) = B(v,v) = \left| v|_{\Gamma_R} / \mathrm{R} \right|^2_{1/2,\Gamma_R} \le \|v\|_*^2$$

$$v(R,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta)$$

- Notice u satisfying Laplacian when $r > R_0$

Numerical example

- Poisson equation outside a disk with radius 0.5:
  - Choose f and g s.t. there exists exact solution
  - V^h : piecewise linear finite element subspace
- Test cases:
  - Mesh size h effect: $R = R_0 = 1, N = 101$
  - N effect: $R = R_0 = 1$
  - R effect: $R > R_0$ varies

next

h effect

Mesh	$h_0 = 0.31416$	$h_0/2$	$h_0/4$	$h_0/8$
$\max\lvert u - \mathbf{u}_N^h\rvert$	$1.306E-2$	$3.24E-3$	$8.097E-4$	$2.024E-4$
$\lVert u - \mathbf{u}_N^h\rVert_{0,\Omega_i}$	$4.175E-2$	$1.081E-2$	$2.728E-3$	$6.836E-4$
$\lvert u - \mathbf{u}_N^h\rvert_{1,\Omega_i}$	0.517	0.275	0.139	0.070

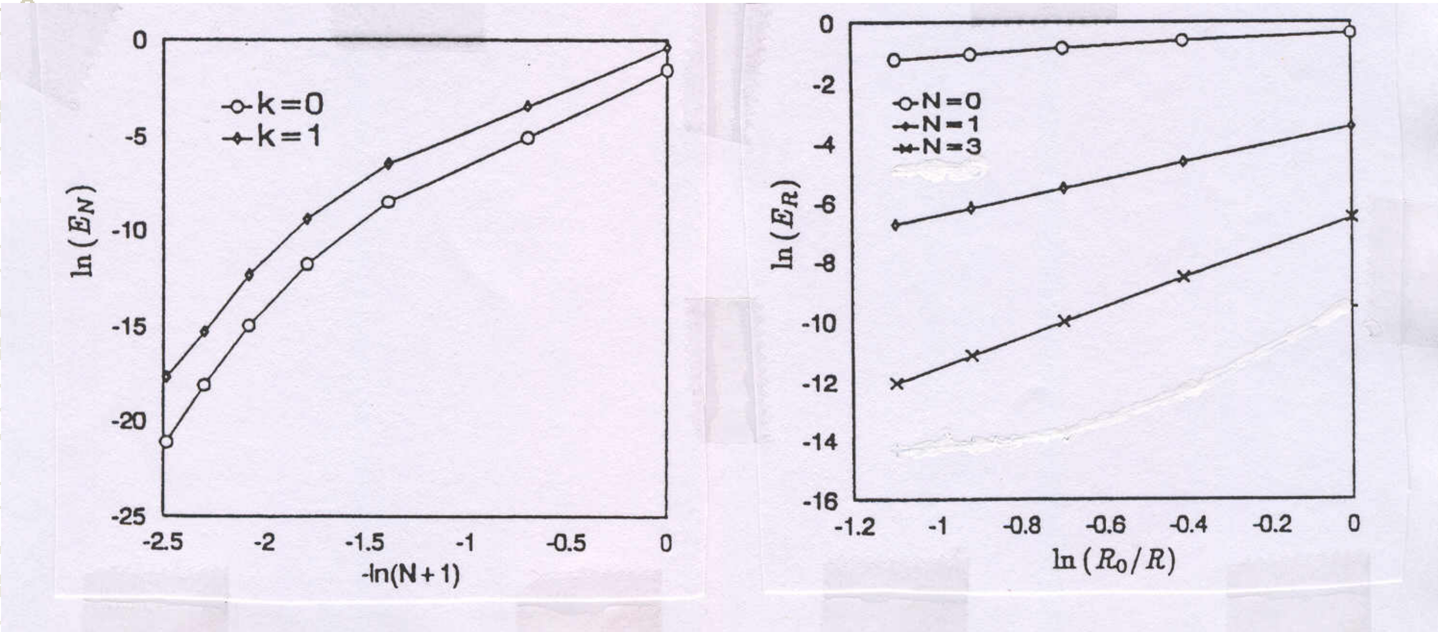
- Conclusion:

$$\lvert u - \mathbf{u}_N^h\rvert_{1,\Omega_i} = O(h)$$

back

$$\lVert u - \mathbf{u}_N^h\rVert_{0,\Omega_i} = O(h^2) \qquad \max\lvert u - \mathbf{u}_N^h\rvert = O(h^2)$$

h & N effect



- Conclusion:

$$\left|u_\infty^h - u_N^h\right|_{k,\Omega_i} = O\left(\frac{1}{(N+1)^p}\right), \quad R = R_0 \qquad \left|u_\infty^R - u_N^R\right|_{1,\Omega_i} = O\left(\frac{R_0}{R}\right)^{N+1}$$

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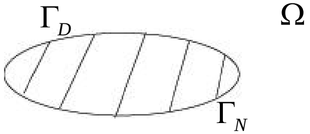
Extension of the Results



- Yukawa equation (Bao & Han, SIAMNA, 00'):

$$-\nabla\bullet(\kappa(\vec{x})\nabla u(\vec{x}))+\beta(\vec{x})u(\vec{x})=f(\vec{x}),\quad in\quad \Omega$$

$$u\Big|_{\Gamma_D}=g,\qquad \frac{\partial u}{\partial n}\Big|_{\Gamma_N}=k,\qquad u\to 0\quad \text{when}\quad r\to\infty$$

  - Assumption: $f(\vec{x})=0,\quad \kappa(\vec{x})=\kappa_0>0,\quad \beta(\vec{x})=\beta_0>0,\quad \text{when}\quad r=|\vec{x}|>R_0$
  - Exact & approximate ABCs:
  - Error bounds:

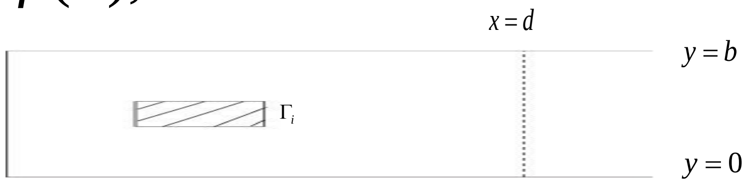
$$\left\|u-u_N^h\right\|_{1,\Omega_i}\le C_0\left[h^p+\frac{(1+\sqrt{\beta_0})K_{N+1}(R\sqrt{\beta_0})}{(N+1)^p K_{N+1}(R_0\sqrt{\beta_0})}\right]$$

Extension of the Results

- Problem in a semi-infinite strip (Bao & Han, SIAMNA,00’):

$$-\nabla \bullet (\kappa(\vec{x})\nabla u(\vec{x})) + \beta(\vec{x})u(\vec{x}) = f(\vec{x}), \quad in \quad \Omega$$

$+ \quad$ B.C.



- – Assumption:

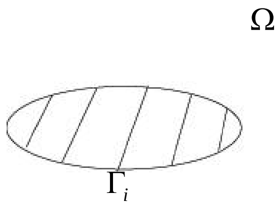
$$f(\vec{x}) = 0, \quad \kappa(\vec{x}) = \kappa_0 > 0, \quad \beta(\vec{x}) = \beta_0 > 0, \quad \text{when} \quad x > d_0$$

- – Exact & approximate ABCs:
- – Error bounds:

$$\left\| u - u_N^h \right\|_{1,\Omega_i} \le C_0 \left[h^p + \frac{1}{(N+1)^p} e^{-(d-d_0)\pi(n+1)/b} \right]$$

Extension of the Results

- Exterior Stokes Eqs. (Bao, IMANA, 03')



$$-\nu\,\Delta\vec{u} + \text{grad}\ p = \vec{f}, \qquad \text{div}\,\vec{u} = 0, \quad \text{in} \quad \Omega$$

$$\vec{u} = \vec{0} \qquad \text{on} \quad \Gamma_i$$

$$\vec{u} \quad \text{bounded} \qquad p \to 0 \qquad \text{when} \quad r \to \infty$$

- – Exact & approximate nonlocal/local ABCs:
- – Difficulty: Constant in inf-sup condition
- – Error bounds:

$$\left\|\vec{u} - \vec{u}_N^h\right\| + \left\|p - p_N^h\right\| \le C_0 \left[h^k + \frac{1}{(N+1)^k} \left(\frac{R_0}{R} \right)^{\max\{1, N-1\}} \right]$$

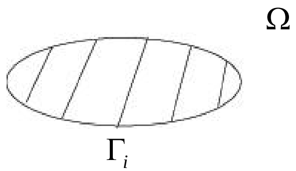
Extension of the Results

- Exterior linear elastic Eqs. (Bao & Han, Math. Comp. 01')

$$-\mu\,\Delta\vec{u} - (\lambda+\mu)\,\text{grad div}\,\vec{u} = \vec{f}, \quad \text{in} \quad \Omega$$

$$\vec{u} = \vec{0} \qquad \text{on} \quad \Gamma_i$$

$$\vec{u} \quad \text{bounded} \qquad \text{when} \quad r \to \infty$$



- Exact & approximate nonlocal/local ABCs:
- Difficulty: Constant in Korn inequality
- Error bounds:

$$\left\|\vec{u} - \vec{u}_N^h\right\| \le C_0\left[h^k + \frac{1}{(N+1)^k}\left(\frac{R_0}{R}\right)^{\max\{1,N-1\}}\right]$$

High-order Local ABCs

- Poisson Eq.

$$-\Delta u = f \qquad \text{in} \quad \Omega = Q^c$$

$$u\big|_{\Gamma_i} = g, \qquad u \quad \text{bounded when} \quad r = |\vec{x}| \to \infty$$

- Exact BC

$$\frac{\partial u}{\partial r}(R,\theta) = -\frac{1}{R}\sum_{n=1}^{\infty} n(a_n \cos n\theta + b_n \sin n\theta) = L\, u(R,\theta)$$

$$u(R,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

- Approximate $L \approx L_N$ s.t. correct for first N terms

$$L_N\, u(R,\theta) = -\frac{1}{R}\sum_{m=1}^{N}(-1)^m \alpha_m^{(N)} \frac{\partial^{2m}}{\partial\theta^{2m}} u(R,\theta) \Rightarrow \quad \sum_{m=1}^{N} n^{2m} \alpha_m^{(N)} = n,\ n = 1,\cdots,N$$

High-order Local ABCs

- N=1:

$$\frac{\partial u}{\partial r}(R,\theta) = \frac{1}{R}\frac{\partial^2 u}{\partial \theta^2}(R,\theta) \qquad 0 \le \theta \le 2\pi$$

- Finite element approximation:
- Error bounds: (Bao & Han, CMAME, 01')

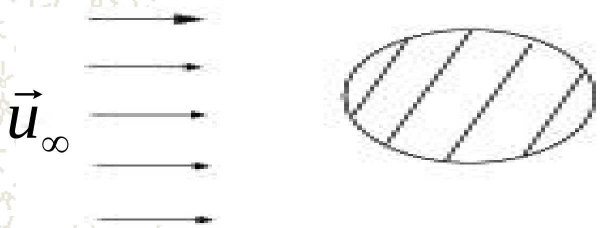
$$\left| u - \boldsymbol{u}_N^h \right|_{1,\Omega_i} \le C_{N,u}\left[h^p + \left(\frac{R_0}{R} \right)^{N+1} \right]$$

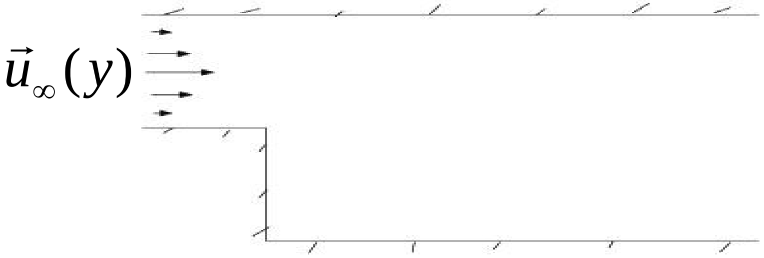
For Navier-Stokes Eqs. (Bao, JCP,95',97',00')

$$(\vec{u}\bullet\nabla)\vec{u}+\nabla p=\frac{1}{\mathrm{Re}}\Delta\vec{u} \qquad \text{in} \quad R^2\setminus\Omega_i$$

$$\nabla\bullet\vec{u}=0$$

- Two types exterior flows: around obstacles & in channel





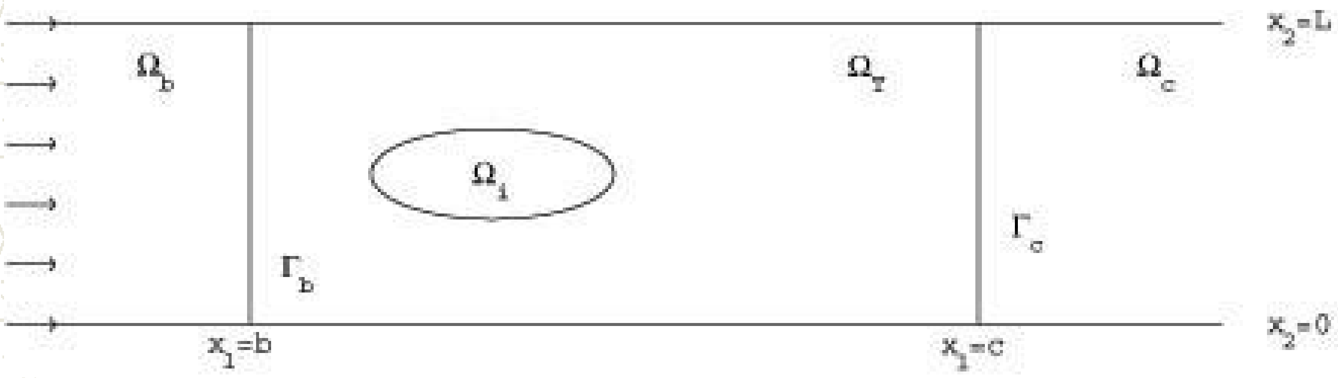
Ideas

- Introduce two lines $x_2 = 0 \,\&\, x_2 = L$ and set

$$u_2\Big|_{x_2=0,L} = \sigma_{12}\Big|_{x_2=0,L} = \frac{1}{\mathrm{Re}}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\Big|_{x_2=0,L} = 0 \qquad -\infty < x_1 < \infty$$

- Introduce a segment $x_1 = b$ and set

$$\vec{u}(\vec{x})\Big|_{x_1=b} = \vec{u}_\infty \qquad 0 \le x_2 \le L$$



Ideas

- Introduce a segment $x_1 = c$ and design ABCs
  - Linearize NSEs on Ω_c by Oseen Eq.
  - Solve Oseen Eq. on Γ_c analytically by given $\vec{u}(c, x_2)$
  - Use transmission conditions

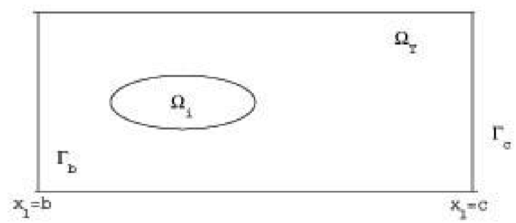
$$\vec{u}(c^-, x_2) = \vec{u}(c^+, x_2) \qquad \sigma_n(c^-, x_2) = \sigma_n(c^+, x_2)$$

  - Design ABCs on Ω_c

$$\sigma_n(c, x_2) = T(\vec{u}\big|_{x_1=c}, p\big|_{x_1=c}) \approx T_N(\vec{u}(c, x_2), p(c, x_2)) \quad 0 \le x_2 \le L$$

Ideas

- Reduction



$$(\vec{u}\bullet\nabla)\vec{u}+\nabla p=\frac{1}{\mathrm{Re}}\Delta\vec{u},\qquad \nabla\bullet\vec{u}=0\qquad \text{in}\quad \Omega_T$$

$$\vec{u}\Big|_{\partial\Omega_i}=\vec{0},\qquad \frac{\partial u_1}{\partial x_2}\Big|_{x_2=0,L}=u_2\Big|_{x_2=0,L}=0,\quad b\le x_1\le c$$

$$\vec{u}(b,x_2)=\vec{u}_\infty,\qquad \sigma_n(c,x_2)=T_N(\vec{u}(c,x_2),p(c,x_2))\quad 0\le x_2\le L.$$

- Solve the reduced problem

Well-posedness

- Variational formulation:

$$A(\vec{u}^N,\vec{v}) + A_1(\vec{u}^N,\vec{u}^N,\vec{v}) + A_2^N(\vec{u}^N,\vec{v}) + B(\vec{v},p^N) = F(\vec{v}) \quad \forall \vec{v} \in V \equiv [H_*^1(\Omega_T)]^2$$

$$B(u^N,q) = 0 \qquad \forall q \in W \equiv L^2(\Omega_T)$$

with

$$A(\vec{u},\vec{v}) = \frac{2}{\mathrm{Re}}\int_{\Omega_T}\sum_{i,j=1}^{2}\varepsilon_{ij}(\vec{u})\varepsilon_{ij}(\vec{v})d\vec{x} = \frac{1}{2\,\mathrm{Re}}\int_{\Omega_T}\sum_{i,j=1}^{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)d\vec{x}$$

$$A_1(\vec{u},\vec{v},\vec{w}) = \frac{1}{2}\int_{\Omega_T}[((\vec{u}\bullet\nabla)\vec{v})\bullet\vec{w} - ((\vec{u}\bullet\nabla)\vec{w})\bullet\vec{v}]\,d\vec{x}$$

$$B(\vec{v},q) = -\int_{\Omega_T} q\nabla\bullet\vec{v}\;d\vec{x} \qquad F(v) = \frac{a^2}{2}\int_0^L v_1(c,x_2)\,dx_2$$

$$A_2^N(\vec{u},\vec{v}) = \sum_{m=1}^{N}[\cdots\int_0^L u_1(c,x_2)\cos\frac{m\pi x_2}{L}dx_2\int_0^L v_1(c,x_2)\cos\frac{m\pi x_2}{L}dx_2 + \cdots]$$

Well-posedness

- Well-posedness:
  - There exists solution of the reduced problem
  - When Re is not too big, uniqueness
- Error estimates for N-S Eqs.

$$\left\|\vec{u}-\vec{u}^{N}\right\|+\left\|p-p^{N}\right\|\leq\frac{C}{(N+1)^{3/2}}\left\|\vec{u}\right\|_{2,\Gamma_c}$$

- Error estimates for Oseen Eqs.

$$\left\|\vec{u}-\vec{u}^{N}\right\|+\left\|p-p^{N}\right\|\leq\frac{C\,e^{-(N+1)(c-c_0)}}{(N+1)^{3/2}}\left\|\vec{u}\right\|_{2,\Gamma_{c0}}$$

Finite Element Approximation

- FEM approximation:

$$A(\vec{u}_h^N, \vec{v}_h) + A_1(\vec{u}_h^N, \vec{u}_h^N, \vec{v}_h) + A_2^N(\vec{u}_h^N, \vec{v}_h) + B(\vec{v}_h, p_h^N) = F(\vec{v}_h) \quad \forall \vec{v}_h \in V_h$$

$$B(u_h^N, q_h) = 0 \qquad \forall q_h \in W_h$$

- Error estimates for N-S Eqs.

$$\left\| \vec{u} - \vec{u}_h^N \right\| + \left\| p - p_h^N \right\| \le C_1\, h^m \left[\left| \vec{u} \right|_{m+1,\Omega_T} + \left| p \right|_{m,\Omega_T} \right] + \frac{C_2}{(N+1)^{3/2}} \left\| \vec{u} \right\|_{2,\Gamma_c}$$

- Error estimates for Oseen Eqs.

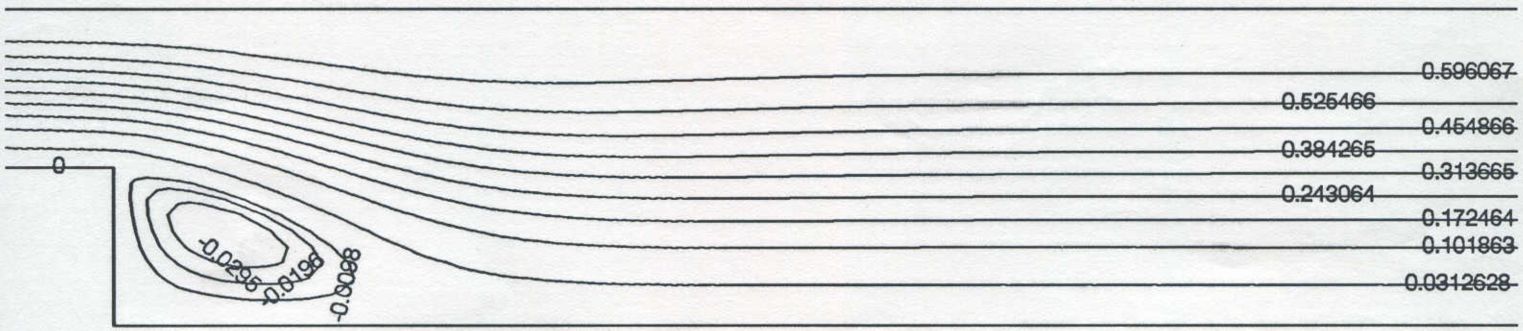
$$\left\| \vec{u} - \vec{u}_h^N \right\| + \left\| p - p_h^N \right\| \le C_1\, h^m \left[\left| \vec{u} \right|_{m+1,\Omega_T} + \left| p \right|_{m,\Omega_T} \right] + \frac{C_2\, e^{-(N+1)(c-c_0)}}{(N+1)^{3/2}} \left\| \vec{u} \right\|_{2,\Gamma_{c0}}$$

Examples

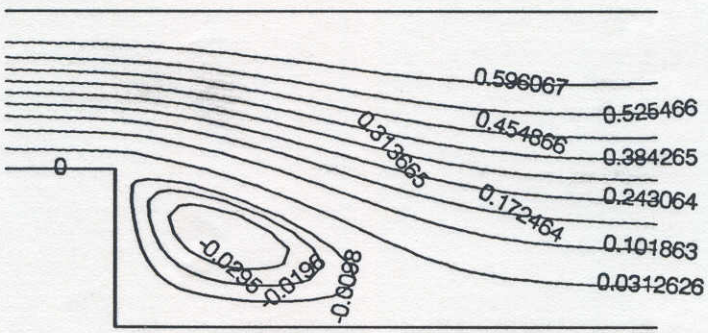
- Backward-facing step flow:
  - Streamfuction & vorticity
- Flow around rectangle cylinder :
  - Velocity field & near obstacle
- Flow around circular cylinder :
  - Velocity field & near obstacle

next

Flow in Channel



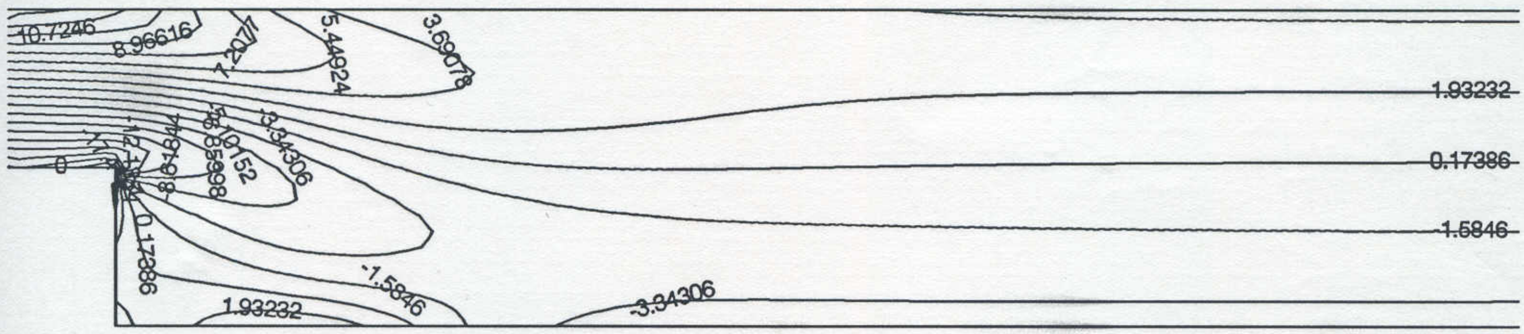
Streamfunction ("Exact solution")



Streamfunction (Artificial)

back

Flow in Channel



Vorticity ("Exact solution")

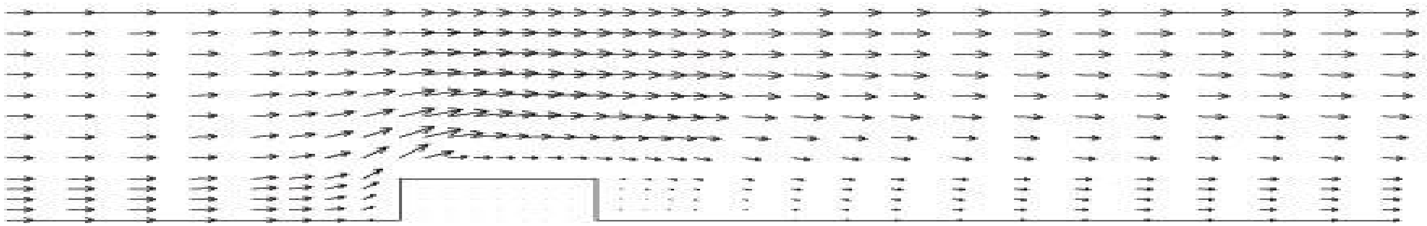
Vorticity (Artificial)

Vorticity (Dirichlet)

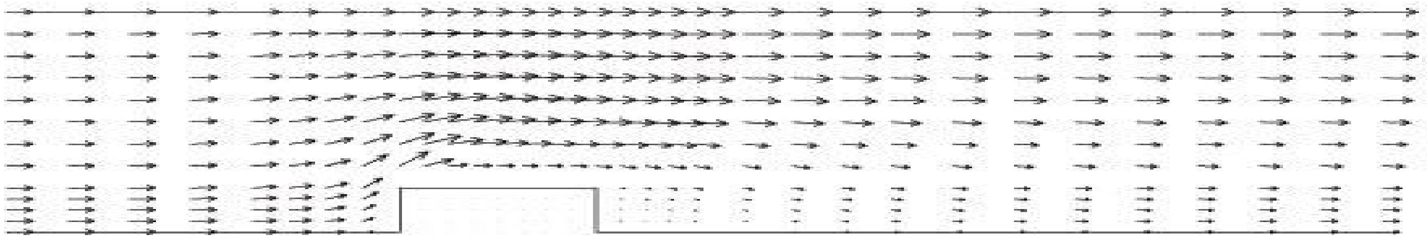
back

Flow around cylinder

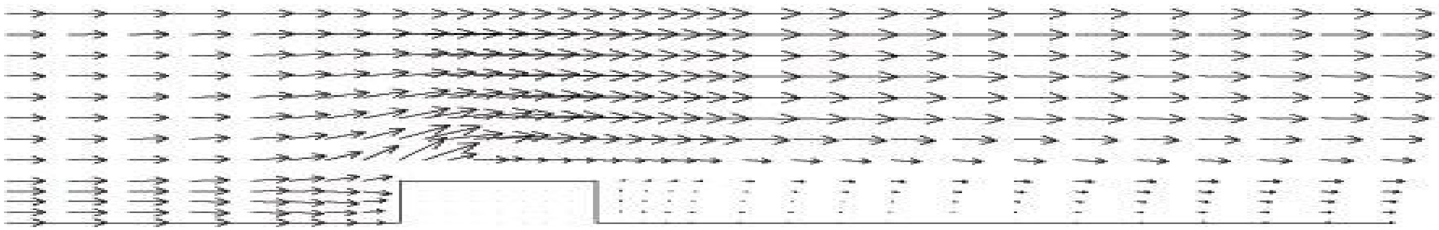
- Re=100



- Re=200



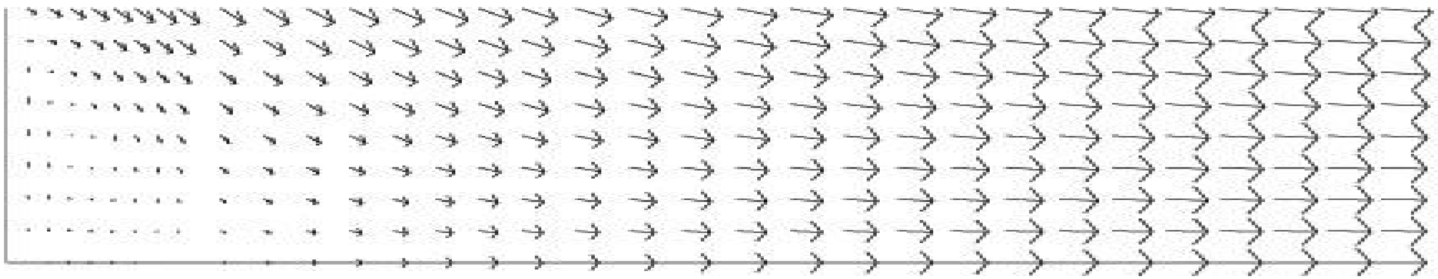
- Re=400



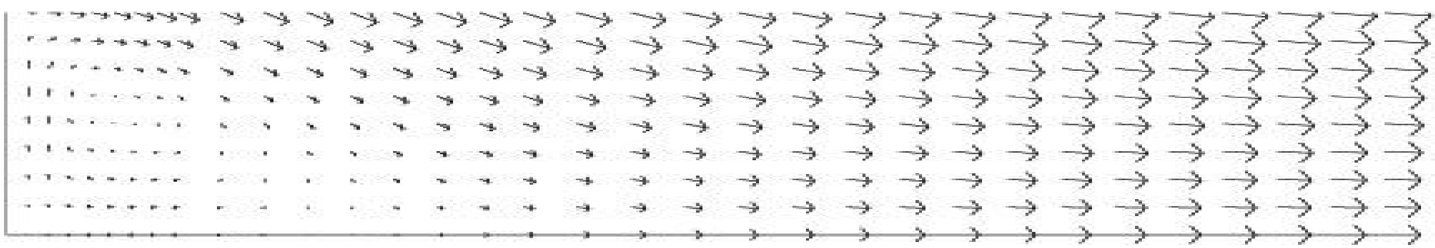
back

Flow around cylinder

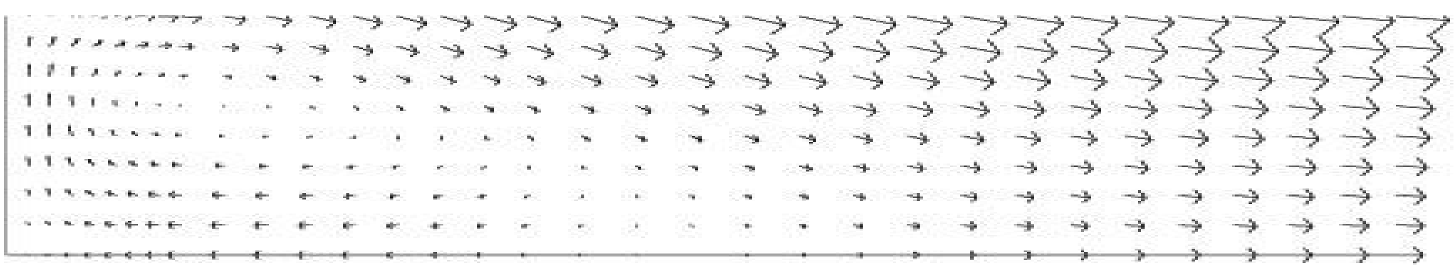
- Re=100



- Re=200



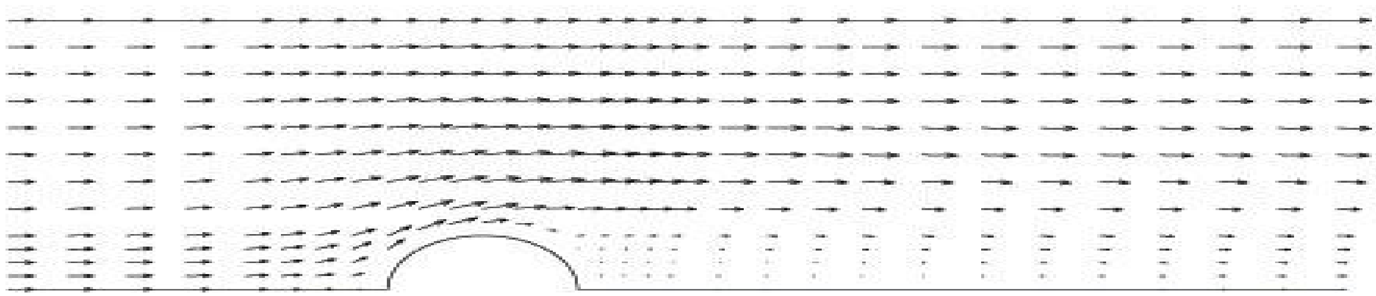
- Re=400



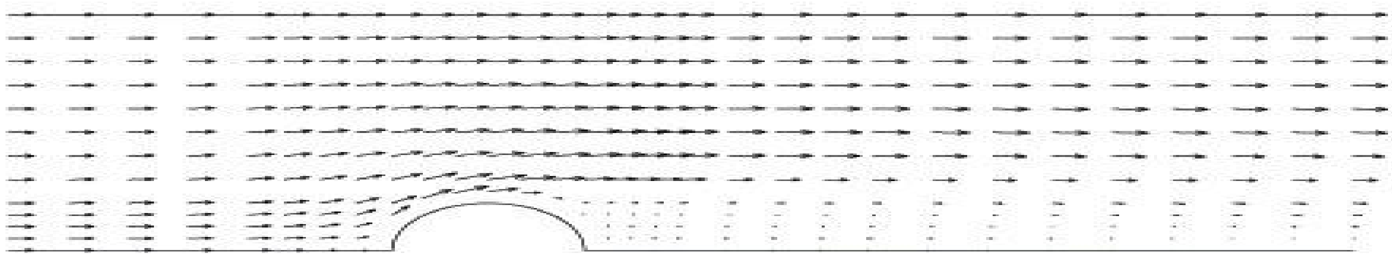
back

Flow around cylinder

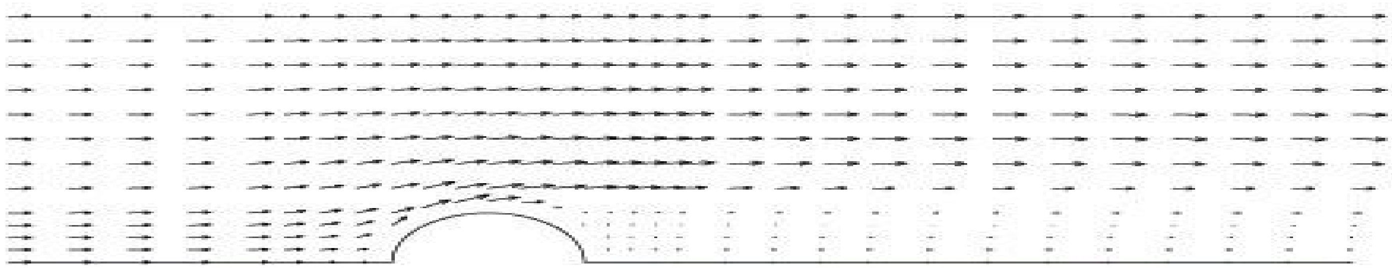
- Re=100



- Re=200



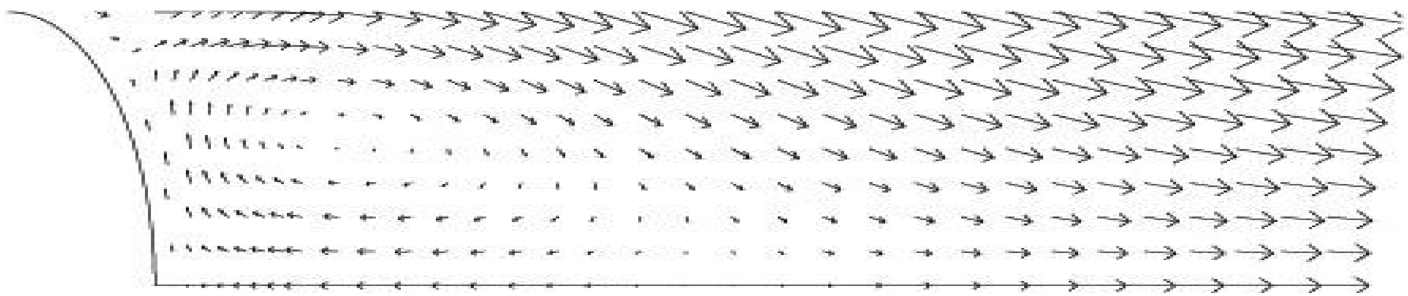
- Re=400



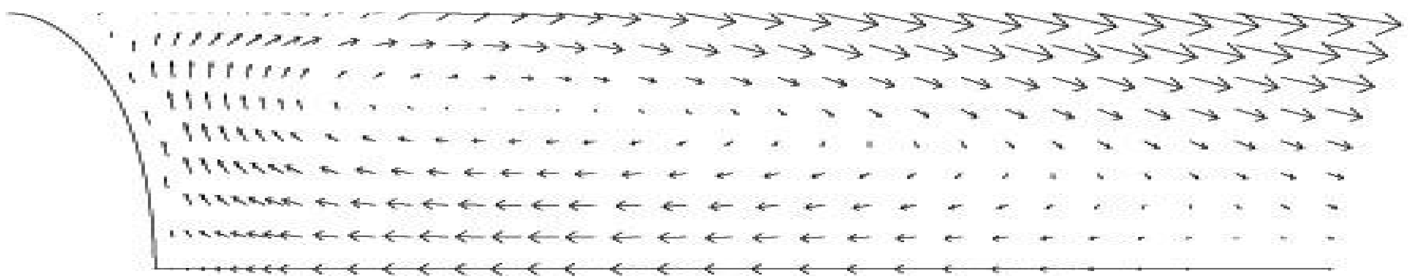
back

Flow around cylinder

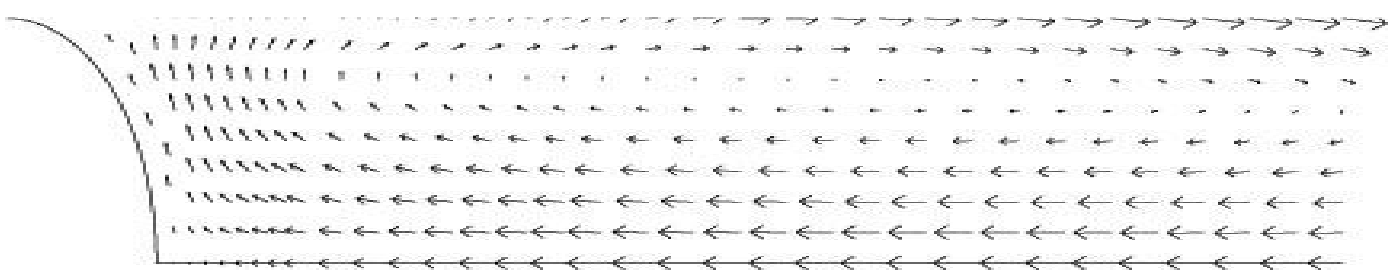
- Re=100



- Re=200



- Re=400



back

Conclusions & Future challenges

- Conclusions:
  - New `optimal' error estimates
  - New high-order local B.C.
  - Application to N-S Eqs.

- Future challenges
  - 3D problems
  - Nonlinear problems
  - Coupling system