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Dimension reduction for anisotropic Bose–Einstein condensates in the strong interaction regime

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Abstract

We study the problem of dimension reduction for the three dimensional Gross–Pitaevskii equation (GPE) describing a Bose–Einstein condensate confined in a strongly anisotropic harmonic trap. Since the gas is assumed to be in a strong interaction regime, we have to analyse two combined singular limits: a semi-classical limit in the transport direction and the strong partial confinement limit in the transversal direction. We prove that both limits commute together and we provide convergence rates. The by-products of this work are approximated models in reduced dimension for the GPE, with *a priori* estimates of the approximation errors.

Keywords: Gross–Pitaevskii equation, dimension reduction, semi-classical limit, averaging, asymptotic analysis, highly oscillatory equations, nonlinear Schrödinger equations

Mathematics Subject Classification: 35Q55, 35Q40

1. Introduction and main results

In this paper, we study dimension reduction for the three-dimensional Gross–Pitaevskii equation (GPE) modelling Bose–Einstein condensation [1,10,13]. In contrast with the existing literature on this topic [7–9], we will *not* assume that the gas is in a weak interaction regime.

Based on the mean field approximation [15, 18, 19], the Bose–Einstein condensate is modelled by its wavefunction $\Psi := \Psi(t, \boldsymbol{x})$ satisfying the GPE written in physical variables as

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\Delta\Psi + V(\boldsymbol{x})\Psi + Ng|\Psi|^2\Psi, \tag{1.1}$$

where Δ is the Laplace operator, $V(\boldsymbol{x})$ denotes the trapping harmonic potential, $m > 0$ is the mass, $\hbar$ is the Planck constant, $g = \frac{4\pi\hbar^2 a_s}{m}$ describes the interaction between atoms in the condensate with the s-wave scattering length a_s and N denotes the number of particles in the condensate. The wave function is normalized according to

$$\int |\Psi(t, \boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} = 1.$$

1.1. Scaling assumptions

We assume that the harmonic potential is strongly anisotropic and confines particles from dimension $n + d \geqslant 2$ to dimension $n \geqslant 1$. In applications, we will have $n + d = 3$ and, either $n = 2$ for disc-shaped condensates, or $n = 1$ for cigar-shaped condensates. We shall denote $\boldsymbol{x} = (x, z)$, where $x \in \mathbb{R}^n$ denotes the variable in the confined direction(s) and $z \in \mathbb{R}^d$ denotes the variable in the transversal direction(s). The harmonic potential reads [3, 20, 21]

$$V(\boldsymbol{x}) = \frac{m}{2}\left(\omega_x^2|x|^2 + \omega_z^2|z|^2\right),$$

where $\omega_z \gg \omega_x$. We introduce the two dimensionless parameters

$$\varepsilon = \sqrt{\omega_x/\omega_z}, \qquad \beta = \frac{4\pi N|a_s|}{a_0},$$

where the harmonic oscillator length is defined by [3, 20, 21]

$$a_0 = \left(\frac{\hbar}{m\omega_x}\right)^{1/2}.$$

Let us rewrite the GPE (1.1) in dimensionless form. For that, we introduce the new variables $\tilde{t}$, $\tilde{x}$, $\tilde{z}$ and the associated unknown $\widetilde{\Psi}$ defined by [3, 20, 21]

$$\tilde{t} = \omega_x t, \qquad \tilde{x} = \frac{x}{a_0}, \qquad \tilde{z} = \frac{z}{a_0}, \qquad \widetilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{z}) = a_0^{(n+d)/2}\Psi(t, x, z).$$

The dimensionless GPE equation reads [3, 20, 21]

$$\mathrm{i}\partial_{\tilde{t}}\widetilde{\Psi} = -\frac{1}{2}\Delta\widetilde{\Psi} + \frac{1}{2}\left(|\tilde{x}|^2 + \frac{1}{\varepsilon^4}|\tilde{z}|^2\right)\widetilde{\Psi} + \beta\sigma|\widetilde{\Psi}|^2\widetilde{\Psi}, \tag{1.2}$$

where $\sigma = \mathrm{sign}\,(a_s) \in \{-1, 1\}$. In order to observe the condensate at the correct space scales, we will now proceed to a rescaling in x and z. Let us define

$$\alpha = \varepsilon^{2d/n}\beta^{-2/n}, \tag{1.3}$$

and set

$$t' = \tilde{t}, \qquad z' = \frac{\tilde{z}}{\varepsilon}, \qquad x' = \alpha^{1/2}\tilde{x},$$

which means that the typical length scales of the dimensionless variables are ε in the z-direction and $\alpha^{-1/2}$ in the x-direction. The wavefunction is rescaled as follows:

$$\Psi^{\varepsilon,\alpha}(t', x', z') := \varepsilon^{d/2}\alpha^{-n/4}\widetilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{z})\mathrm{e}^{\mathrm{i}\tilde{t}d/2\varepsilon^2}.$$

Notice that the L^2 norm of $\Psi^{\varepsilon,\alpha}$ is left invariant by this rescaling, so we still have

$$\int_{\mathbb{R}^{n+d}} |\Psi^{\varepsilon,\alpha}(t, x, z)|^2 \, \mathrm{d}x \, \mathrm{d}z = 1.$$

We end up with the following rescaled GPE (for simplicity we omit the primes on the variables):

$$\mathrm{i}\alpha\partial_t\Psi^{\varepsilon,\alpha} = \frac{\alpha}{\varepsilon^2}\mathcal{H}_z\Psi^{\varepsilon,\alpha} - \frac{\alpha^2}{2}\Delta_x\Psi^{\varepsilon,\alpha} + \frac{|x|^2}{2}\Psi^{\varepsilon,\alpha} + \alpha\sigma|\Psi^{\varepsilon,\alpha}|^2\Psi^{\varepsilon,\alpha}, \tag{1.4}$$

where the transversal Hamiltonian is

$$\mathcal{H}_z := -\frac{1}{2}\Delta_z + \frac{|z|^2}{2} - \frac{d}{2}$$

and the scaling assumptions are

$$\alpha \ll 1 \quad \text{and} \quad \varepsilon \ll 1.$$

The spectrum of $\mathcal{H}_z$ is the set of integers $\mathbb{N}$, its ground state (associated to the eigenvalue 0) is $\omega_0(z) = \pi^{-d/4}\mathrm{e}^{-|z|^2/2}$.

The dimension reduction of the GPE (1.2) from three dimensions (3D) to lower dimensions was studied formally in [5,6] and numerically in [4] for fixed β when $\varepsilon \to 0$. The mathematical rigorous justification for this dimension reduction was given in [2, 7–9] for $\alpha = 1$ and $\varepsilon \ll 1$ in (1.4) so that $\beta = \varepsilon^d \ll 1$ in (1.2), which corresponds to a weak interaction regime in the GPE (1.2). However, it is an open problem to justify mathematically the dimension reduction of the GPE (1.2) in the strong interaction regime, i.e. for fixed β when $\varepsilon \to 0$. The key difficulty is due to that the energy associated to the reduced GPE in lower dimensions is unbounded when $\varepsilon \to 0$ [3,5]. In this paper, we study the strong interaction regime by adapting a proper re-scaling. This amounts to considering simultaneously the strong confinement limit and the semi-classical limit for the solution $\Psi^{\varepsilon,\alpha}$ to (1.4) as $\varepsilon \to 0$ and $\alpha \to 0$. Note that $\beta = \varepsilon^d\alpha^{-n/2}$ may tend to every constants $\gamma \in \mathbb{R}^+$ and even to $+\infty$.

Our key mathematical assumption will be that the wavefunction $\Psi^{\varepsilon,\alpha}$ at time $t = 0$ is under the WKB form:

$$\Psi^{\varepsilon,\alpha}(0, x, z) = \Psi^{\alpha}_{\mathrm{init}}(x, z) := A_0(x, z)\mathrm{e}^{\mathrm{i}S_0(x)/\alpha}, \quad \forall (x, z) \in \mathbb{R}^{n+d}. \tag{1.5}$$

Here A_0 is a complex-valued function and S_0 is real-valued.

Remark 1.1. With respect to the small parameter α, equation (1.4) is in a semi-classical regime which is usually referred to as 'weakly nonlinear geometric optics', see [11]. The more singular regime

$$\mathrm{i}\alpha\partial_t\Psi = \frac{\alpha}{\varepsilon^2}\mathcal{H}_z\Psi - \frac{\alpha^2}{2}\Delta_x\Psi + \frac{|x|^2}{2}\Psi + \sigma|\Psi|^2\Psi, \tag{1.6}$$

similar to supercritical geometric optics, would correspond to the choice

$$\alpha = \left(\varepsilon^d\beta^{-1}\right)^{\frac{2}{n+2}},$$

instead of the choice $\alpha = (\varepsilon^d\beta^{-1})^{\frac{2}{n}}$ that we have made in (1.3). Hence, the difference between these two regimes lies in the assumption on the initial wavefunction: in the regime (1.4) studied here, the wavefunction is assumed to have a broader extension in the x direction than in the more singular regime (1.6).

1.2. Heuristics

In the section, we derive formally the limiting behaviour of the solution of (1.4). We have the choice to first let $\varepsilon \to 0$ (strong confinement limit), then $\alpha \to 0$ (semi-classical limit), or to exchange these two limits: first $\alpha \to 0$, then $\varepsilon \to 0$. Our main result, stated in the next section, will be that in fact both limits commute together: the limit is valid as ε and α converge *independently* to zero.

(a) Strong confinement limit first, then semi-classical limit. Following [8], in order to analyse the strong partial confinement limit, it is convenient to begin by filtering out the fast oscillations at scale ε^2 induced by the transversal Hamiltonian. To this aim, we introduce the new unknown

$$\Phi^{\varepsilon,\alpha}(t,\cdot) = \mathrm{e}^{\mathrm{i}t\mathcal{H}_z/\varepsilon^2}\Psi^{\varepsilon,\alpha}(t,\cdot).$$

It satisfies the equation

$$\mathrm{i}\alpha\partial_t\Phi^{\varepsilon,\alpha} = -\frac{\alpha^2}{2}\Delta_x\Phi^{\varepsilon,\alpha} + \frac{|x|^2}{2}\Phi^{\varepsilon,\alpha} + \alpha F\left(\frac{t}{\varepsilon^2}, \Phi^{\varepsilon,\alpha}\right)$$

where the nonlinear function is defined by

$$F(\theta, \Phi) = \sigma \mathrm{e}^{\mathrm{i}\theta\mathcal{H}_z}\left(\left|\mathrm{e}^{-\mathrm{i}\theta\mathcal{H}_z}\Phi\right|^2 \mathrm{e}^{-\mathrm{i}\theta\mathcal{H}_z}\Phi\right). \tag{1.7}$$

A fundamental remark is that for all fixed Φ, the function $\theta \mapsto F(\theta, \Phi)$ is 2π-periodic, since the spectrum of $\mathcal{H}_z$ only contains integers. For any fixed $\alpha > 0$, Ben Abdallah *et al* [7, 8] proved by an averaging argument that we have $\Phi^{\varepsilon,\alpha} = \Phi^{0,\alpha} + \mathcal{O}(\varepsilon^2)$, where $\Phi^{0,\alpha}$ solves the averaged equation

$$\mathrm{i}\alpha\partial_t\Phi^{0,\alpha} = -\frac{\alpha^2}{2}\Delta_x\Phi^{0,\alpha} + \frac{|x|^2}{2}\Phi^{0,\alpha} + \alpha F_{\mathrm{av}}(\Phi^{0,\alpha}), \qquad \Phi^{0,\alpha}(t=0) = \Psi^{\alpha}_{\mathrm{init}}, \tag{1.8}$$

where F_{av} is the averaged vector field

$$F_{\mathrm{av}}(\Phi) = \frac{1}{2\pi}\int_0^{2\pi} F(\theta, \Phi)\,\mathrm{d}\theta. \tag{1.9}$$

Now we can proceed to the second limit $\alpha \to 0$. As we said in remark 1.1, (1.8) is written in the semi-classical regime of 'weakly nonlinear geometric optics', which can be studied by a WKB analysis. Here we are only interested in the limiting model, so in the first stage of the WKB expansion. Let us introduce the solution $S(t, x)$ of the eikonal equation

$$\partial_t S + \frac{|\nabla_x S|^2}{2} + \frac{|x|^2}{2} = 0, \qquad S(0, x) = S_0(x) \tag{1.10}$$

and, again, filter out the oscillatory phase of the wavefunction by setting

$$A^{0,\alpha} = \mathrm{e}^{-\mathrm{i}S(t,x)/\alpha}\,\Phi^{0,\alpha}.$$

This function $A^{0,\alpha}(t, x, z)$ satisfies

$$\partial_t A^{0,\alpha} + \nabla_x S\cdot\nabla_x A^{0,\alpha} + \frac{1}{2}A^{0,\alpha}\Delta_x S = \mathrm{i}\frac{\alpha}{2}\Delta_x A^{0,\alpha} - \mathrm{i}F_{\mathrm{av}}(A^{0,\alpha}), \tag{1.11}$$

with the initial data

$$A^{0,\alpha}(0, x, z) = A_0(x, z).$$

As long as the phase $S(t, x)$ remains smooth, i.e. before the formation of caustics in the eikonal equation, we expect to have $A^{0,\alpha} = A + \mathcal{O}(\alpha)$, where $A(t, x, z)$ solves the limiting transport equation

$$\partial_t A + \nabla_x S\cdot\nabla_x A + \frac{1}{2}A\Delta_x S = -\mathrm{i}F_{\mathrm{av}}(A), \qquad A(0, x, z) = A_0(x, z). \tag{1.12}$$

To summarize, the solution $\Psi^{\varepsilon,\alpha}$ of (1.4) is expected to behave as

$$\Psi^{\varepsilon,\alpha}(t, x, z) = \mathrm{e}^{-\mathrm{i}t\mathcal{H}_z/\varepsilon^2}\mathrm{e}^{\mathrm{i}S(t,x)/\alpha}A(t, x, z) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\alpha). \tag{1.13}$$

(b) Semi-classical limit first, then strong confinement limit. Coming back to the GPE (1.4), let us first proceed to the semi-classical limit $\alpha \to 0$. We define

$$A^{\varepsilon,\alpha} = e^{it\mathcal{H}_z/\varepsilon^2} e^{-iS(t,x)/\alpha}\, \Psi^{\varepsilon,\alpha}, \tag{1.14}$$

where $S(t, x)$ is still the solution of the eikonal equation (1.10). A direct computation shows that this function satisfies the equation

$$\partial_t A^{\varepsilon,\alpha} + \nabla_x S \cdot \nabla_x A^{\varepsilon,\alpha} + \frac{1}{2} A^{\varepsilon,\alpha} \Delta_x S = i\frac{\alpha}{2}\Delta_x A^{\varepsilon,\alpha} - iF\left(\frac{t}{\varepsilon^2}, A^{\varepsilon,\alpha}\right), \tag{1.15}$$
$$A^{\varepsilon,\alpha}(0, x, z) = A_0(x, z),$$

where F is still defined by (1.7). For all fixed ε, we can expect that, as $\alpha \to 0$, we have $A^{\varepsilon,\alpha} = A^{\varepsilon,0} + \mathcal{O}(\alpha)$, where $A^{\varepsilon,0}$ solves the equation

$$\partial_t A^{\varepsilon,0} + \nabla_x S \cdot \nabla_x A^{\varepsilon,0} + \frac{1}{2} A^{\varepsilon,0} \Delta_x S = -iF\left(\frac{t}{\varepsilon^2}, A^{\varepsilon,0}\right), \tag{1.16}$$
$$A^{\varepsilon,0}(0, x, z) = A_0(x, z).$$

The last step consists in letting $\varepsilon \to 0$ in this equation (strong confinement limit), which amounts to average out the oscillatory nonlinear term in (1.16). This step yields the limiting equation (1.12), and we have $A^{\varepsilon,0} = A + \mathcal{O}(\varepsilon^2)$.

Remark 1.2. A key point here in this analysis is that the nonlinearities F and F_{av} are gauge invariant i.e. for all $U \in L^2(\mathbb{R}^{n+d})$ and for all t, we have

$$F(t, Ue^{iS/\alpha}) = F(t, U)e^{iS/\alpha}, \qquad F_{av}(Ue^{iS/\alpha}) = F_{av}(U)e^{iS/\alpha}.$$

1.3. Main results

Our main contribution is to prove rigorously the limit of the coupled averaging and semi-classical limits as $\varepsilon \to 0$ and $\alpha \to 0$ independently and to prove the estimate (1.13). It is natural—and equivalent as long as the phase $S(t, x)$ is well defined and is smooth—to work with the function $A^{\varepsilon,\alpha}$ defined by (1.14).

1.3.1. Existence, uniqueness and uniform boundedness results. Let us make precise our functional framework. For wavefunctions, we will use the scale of Sobolev spaces adapted to quantum harmonic oscillators:

$$B^m(\mathbb{R}^{n+d}) := \{u \in H^m(\mathbb{R}^{n+d}) \quad \text{such that } \left(|x|^m + |z|^m\right) u \in L^2(\mathbb{R}^{n+d})\}$$

for $m \in \mathbb{N}$. For the phase S, we will use the space of subquadratic functions, defined by

$$\mathtt{SQ}_k(\mathbb{R}^n) = \{f \in \mathcal{C}^k(\mathbb{R}^n; \mathbb{R})) \quad \text{such that} \quad \partial_x^\kappa f \in L^\infty(\mathbb{R}^n), \ \text{for all } \ 2 \leqslant |\kappa| \leqslant k\},$$

where $k \in \mathbb{N}$, $k \geqslant 2$. In the following theorem, we give existence and uniqueness results for equations (1.10), (1.11), (1.12), (1.15) and (1.16) as well as uniform bounds on the solutions.

Theorem 1.3. *Let $A_0 \in B^m(\mathbb{R}^{n+d})$ and $S_0 \in \mathtt{SQ}_{s+1}(\mathbb{R}^n)$, where m and s are integers such that $m > \frac{n+d}{2}$ and $s \geqslant m + 2$. Then the following holds:*

(i) There exists $T > 0$ such that the eikonal equation (1.10) admits a unique solution $S \in \mathcal{C}([0, T]; \mathtt{SQ}_s(\mathbb{R}^n)) \cap \mathcal{C}^s([0, T] \times \mathbb{R}^n)$.

(ii) There exists $\overline{T} \in (0, T]$ independent of ε and α such that the solutions $A^{\varepsilon,\alpha}$, $A^{0,\alpha}$, $A^{\varepsilon,0}$ and A of, respectively, (1.15), (1.11), (1.16) and (1.12), are uniquely defined in the space

$$C([0, \overline{T}]; B^m(\mathbb{R}^{n+d})) \cap C^1([0, \overline{T}]; B^{m-2}(\mathbb{R}^{n+d})).$$

(iii) For all $(\varepsilon, \alpha) \in (0, 1]^2$, the functions $A^{\varepsilon,\alpha}$, $A^{0,\alpha}$, $A^{\varepsilon,0}$, A are uniformly bounded in

$$C([0, \overline{T}]; B^m(\mathbb{R}^{n+d})) \cap C^1([0, \overline{T}]; B^{m-2}(\mathbb{R}^{n+d})).$$

1.3.2. Study of the limits $\alpha \to 0$ and $\varepsilon \to 0$. We are now able to study the behaviour of $A^{\varepsilon,\alpha}$ as $\varepsilon \to 0$ and $\alpha \to 0$.

Theorem 1.4. *Assume the hypothesis of theorem 1.3 true. Notice that, in particular, we have $m \geqslant 2$. Then, for all $(\varepsilon, \alpha) \in (0, 1]^2$, we have the following bounds:*

(i) Averaging results:

$$\|A^{\varepsilon,\alpha} - A^{0,\alpha}\|_{L^\infty([0,\overline{T}];\, B^{m-2}(\mathbb{R}^{n+d}))} \leqslant C\varepsilon^2 \tag{1.17}$$

and

$$\|A^{\varepsilon,0} - A\|_{L^\infty([0,\overline{T}];\, B^{m-2}(\mathbb{R}^{n+d}))} \leqslant C\varepsilon^2. \tag{1.18}$$

(ii) Semi-classical result:

$$\|A^{\varepsilon,\alpha} - A^{\varepsilon,0}\|_{L^\infty([0,\overline{T}];\, B^{m-2}(\mathbb{R}^{n+d}))} \leqslant C\alpha \tag{1.19}$$

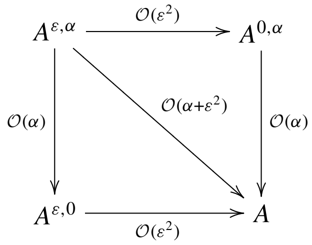
and

$$\|A^{0,\alpha} - A\|_{L^\infty([0,\overline{T}];\, B^{m-2}(\mathbb{R}^{n+d}))} \leqslant C\alpha. \tag{1.20}$$

(iii) Global result:

$$\|A^{\varepsilon,\alpha} - A\|_{L^\infty([0,\overline{T}];\, B^{m-2}(\mathbb{R}^{n+d}))} \leqslant C(\varepsilon^2 + \alpha). \tag{1.21}$$

The constant C here does not depend on α and ε. These estimates can be summarized in the following diagram:



Coming back to the original unknown, our theorem can be expressed in terms of GPEs.

Corollary 1.5. *Under the assumptions of theorem 1.3, the solution $\Psi^{\varepsilon,\alpha}$ of (1.4), (1.5) satisfies*

$$\left\| e^{-iS/\alpha} \left(\Psi^{\varepsilon,\alpha} - e^{-it\mathcal{H}_z/\varepsilon^2} \Phi^\alpha \right) \right\|_{L^\infty([0,\overline{T}];\, B^{m-2}(\mathbb{R}^{n+d}))} \leqslant C\varepsilon^2,$$

where Φ^α solves the GPE in reduced dimension (1.8). Moreover, we have

$$\left\| e^{-iS/\alpha} \Phi^\alpha - A \right\|_{L^\infty([0,\overline{T}];\, B^{m-2}(\mathbb{R}^{n+d}))} \leqslant C\alpha,$$

where A solves the transport equation (1.12). The constant C here is independent of $\varepsilon \in (0, 1]$ and $\alpha \in (0, 1]$.

Remark 1.6. In the weak interaction regime $\beta = \mathcal{O}(\varepsilon^d)$ (i.e. when α does not converge to zero), the previous corollary implies the result of Ben Abdallah *et al* [7, 8] with the same rate of convergence ε^2. In the strong interaction regime $\varepsilon \to 0$ with a fixed β, we have $\alpha(\varepsilon) = \mathcal{O}(\varepsilon^{2d/(3-d)})$ (in the physical case $n + d = 3$). Hence, denoting $\Psi^\varepsilon = \Psi^{\varepsilon,\alpha(\varepsilon)}$, our global error result (1.21) reads

$$\|e^{it\mathcal{H}_z/\varepsilon^2} e^{-iS(t,x)/\alpha}\, \Psi^\varepsilon - A\|_{L^\infty([0,\overline{T}];\, B^{m-2}(\mathbb{R}^{n+d}))} \leqslant C\varepsilon^d.$$

Remark 1.7. An interesting physical case corresponds to initial data polarized on the first eigenmode $\omega_0(z)$ of the confinement Hamiltonian $\mathcal{H}_z$. Assume that the Cauchy condition in (1.5) takes the form

$$\Psi^{\varepsilon,\alpha}(0,x,z) = \Psi^{\alpha}_{\mathrm{init}}(x,z) = a_0(x)\mathrm{e}^{\mathrm{i}S_0(x)/\alpha}\omega_0(z). \tag{1.22}$$

Then the solution $\Phi^{0,\alpha}$ of the GPE (1.8) in reduced dimension remains polarized on ω_0: we have

$$\Phi^{0,\alpha}(t,x,z) = \varphi^{\alpha}(t,x)\omega_0(z)$$

and $\varphi^{\alpha}(t,x)$ solves the equation

$$\mathrm{i}\alpha\partial_t\varphi^{\alpha} = -\frac{\alpha^2}{2}\Delta_x\varphi^{\alpha} + \frac{|x|^2}{2}\varphi^{\alpha} + \frac{\alpha\sigma}{(2\pi)^{d/2}}|\varphi^{\alpha}|^2\varphi^{\alpha}, \qquad \varphi^{\alpha}(0,x) = a_0(x)\mathrm{e}^{\mathrm{i}S_0(x)/\alpha}.$$

Moreover, the solution A of the limiting transport equation (1.12) takes the form

$$A(t,x,z) = a(t,x)\omega_0(z),$$

where $a(t,x)$ solves the equation

$$\partial_t a + \nabla_x S\cdot\nabla_x a + \frac{1}{2}a\Delta_x S = -\frac{\mathrm{i}\sigma}{(2\pi)^{d/2}}|a|^2 a, \qquad a(0,x) = a_0(x).$$

The sequel of this article is devoted to the proofs of our two theorems. In section 2.2, we prove theorem 1.3 and in section 3, we prove theorem 1.4.

2. Proof of theorem 1.3: well-posedness and uniform estimates

This section is devoted to the proof of theorem 1.3. We first prove the local in time well-posedness of the eikonal equation (proposition 2.2). Then we prove the local in time well-posedness of the four equations (1.15), (1.11), (1.16) and (1.12), as well as uniform bounds (proposition 2.8). Theorem 1.3 is then a direct consequence of these two propositions 2.2 and 2.8.

2.1. Solving the eikonal equation

We seek a solution of equation (1.10), where $S_0 \in \mathtt{SQ}_{s+1}(\mathbb{R}^n)$, for $s \geqslant 2$.

Example 2.1. *If* $S_0 = 0$, *the function defined by*

$$S : (t,x) \in (-\pi/2, \pi/2)\times\mathbb{R}^n \mapsto -\frac{1}{2}|x|^2\tan t$$

is the regular solution of equation (1.10). Let us remark that S *is not globally defined in time.*

Following [11], we use the method of characteristics to find a regular solution to (1.10). The characteristic equations associated with this Hamilton–Jacobi equation are

$$\begin{cases} \partial_t x(t,y) = \xi(t,y), & x(0,y) = y, \\ \partial_t \xi(t,y) = -x(t,y), & \xi(0,y) = \nabla_x S_0(y), \\ \partial_t z(t,y) = \dfrac{|\xi(t,y)|^2}{2} - \dfrac{|x(t,y)|^2}{2}, & z(0,y) = S_0(y) \end{cases}$$

(see for instance [16, section 3.2.5]).

The two first lines form a closed system of equations which are called *Hamilton's equations*. The solution is unique, belongs to $\mathcal{C}^s(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^{2n})$ and is given by

$$\begin{pmatrix} x(t, y) \\ \xi(t, y) \end{pmatrix} = \begin{pmatrix} y\cos(t) + \nabla_x S_0(y)\sin(t) \\ -y\sin(t) + \nabla_x S_0(y)\cos(t) \end{pmatrix}.$$

Let us define the Jacobian determinant $J \in \mathcal{C}^{s-1}(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$ by

$$J_t(y) = \det \nabla_y x(t, y) = \det(I_n \cos(t) + \nabla^2_{xx} S_0(y)\sin(t)),$$

where I_n is the identity matrix of $\mathbb{R}^n$. Since S_0 is subquadratic, there exists $T > 0$ and $C > 0$ such that

$$\frac{1}{C} < J_t(y) < C, \quad |\partial_y^\kappa x(t, y)| \leqslant C \quad \text{for all} \quad t \in [0, T], \; y \in \mathbb{R}^n \quad \text{and} \quad |\kappa| = 1, 2.$$

By Schwartz's global inversion theorem [14, 23], $y \mapsto x(t, y)$ is a $\mathcal{C}^s$-diffeomorphism of $\mathbb{R}^n$. Let us denote by $y(t, \cdot)$ its inverse function so that S is defined for all $t \in [0, T]$ and $x \in \mathbb{R}^n$ by

$$S(t, x) = z(t, y(t, x)).$$

We obtain then the following proposition, see [16] for details:

Proposition 2.2. *If $S_0 \in \mathtt{SQ}_{s+1}(\mathbb{R}^n)$ with $s \geqslant 2$, there exists $T > 0$ such that the eikonal equation (1.10) admits a unique solution $S \in \mathcal{C}([0, T]; \mathtt{SQ}_s(\mathbb{R}^n)) \cap \mathcal{C}^s([0, T] \times \mathbb{R}^n)$.*

2.2. Well-posedness results and uniform estimates

Let us introduce the non-negative essentially self-adjoint operator $\Lambda_z^m := (1 + \mathcal{H}_z)^{m/2}$ on $L^2(\mathbb{R}^{n+d})$ whose domain is $\mathcal{D}(\Lambda_z^m) = L^2(\mathbb{R}^n) \otimes B_z^m(\mathbb{R}^d)$, where

$$B_z^m(\mathbb{R}^d) := \{u \in H^m(\mathbb{R}^d) : \quad |z|^m u \in L^2(\mathbb{R}^d)\}.$$

The space

$$B^m(\mathbb{R}^{n+d}) = \left\{u \in H^m(\mathbb{R}^{n+d}) : \quad (|x|^m + |z|^m)u \in L^2(\mathbb{R}^{n+d})\right\}$$

is endowed with the norm

$$\|u\|^2_{B^m} := \sum_{|\kappa| \leqslant m} \|\partial_x^\kappa u\|^2_{L^2} + \||x|^m u\|^2_{L^2} + \|\Lambda_z^m u\|^2_{L^2}.$$

We will use the L^2 real scalar product defined by

$$(u, v) = \mathrm{Re} \int_{\mathbb{R}^{n+d}} \overline{u(x, z)} v(x, z) \, \mathrm{d}x \, \mathrm{d}z$$

and we shall denote

$$(u, v)_{B^m} = \sum_{|\kappa| \leqslant m} \left(\partial_x^\kappa u, \partial_x^\kappa v\right) + \left(|x|^m u, |x|^m v\right) + \left(\Lambda_z^m u, \Lambda_z^m v\right).$$

Remark 2.3. Theorem VIII.33 of [22] ensures that if Λ is an essentially self-adjoint operator on the Hilbert space H_1 of domain $\mathcal{D}(\Lambda)$ and H_2 is another Hilbert space, then $\Lambda \otimes I$ is essentially self-adjoint on $H_1 \otimes H_2$ with domain $\mathcal{D}(\Lambda \otimes I) = \mathcal{D}(\Lambda) \otimes H_2$; here I is the identity of H_2.

Remark 2.4. It can be shown [17] that the following norms N_1^m and N_2^m defined for $u \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ by

$$N_1^m(u) = \|\Lambda_z^m u\|_{L^2(\mathbb{R}^d)},$$
$$N_2^m(u) = \|u\|_{H^m(\mathbb{R}^d)} + \||z|^m u\|_{L^2(\mathbb{R}^d)},$$

are equivalent. In the sequel, we will also make frequent use of the estimate

$$\| |z|^k \partial_z^\kappa u \|_{L^2} \leqslant C N_1^m(u), \quad \text{for all} \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^d) \quad \text{and} \quad k + |\kappa| \leqslant m.$$

Ben Abdallah *et al* generalized these results for a more general class of confining potential using Weyl–Hörmander calculus in [8].

As an immediate consequence, we get the following *tame estimate* for $m \geqslant 0$. Let $G : \mathbb{C} \to \mathbb{C}$ be a smooth function such that $G(0) = 0$, then for all $u \in B^m(\mathbb{R}^{n+d}) \cap L^\infty(\mathbb{R}^{n+d})$, we have $G(u) \in B^m(\mathbb{R}^{n+d}) \cap L^\infty(\mathbb{R}^{n+d})$ and

$$\|\Lambda_z^m G(u)\|_{L^2} \leqslant C\left(\|u\|_{L^\infty}\right) \|\Lambda_z^m u\|_{L^2}, \tag{2.1}$$

where $C : [0, \infty[\to [0, \infty[$ (see [8, proposition 2.5], [12, lemma 4.10.2] or [11, lemma 1.24]).

Remark 2.5. Assuming that $m > \frac{n+d}{2}$, we get that

$$B^m(\mathbb{R}^{n+d}) \hookrightarrow H^m(\mathbb{R}^{n+d}) \hookrightarrow L^\infty(\mathbb{R}^{n+d}),$$

and $B^m(\mathbb{R}^{n+d})$ is an algebra. Moreover, by using Leibniz's rule and Sobolev embeddings, one can prove easily that there exists C_m such that, for all $u \in B^m(\mathbb{R}^{n+d})$ and $v \in B^\ell(\mathbb{R}^{n+d})$ with $0 \leqslant \ell \leqslant m$,

$$\|uv\|_{B^\ell} \leqslant C\|u\|_{B^m}\|v\|_{B^\ell}. \tag{2.2}$$

The proof of uniform well-posedness for the four equations (1.15), (1.11), (1.16) and (1.12) will be based on the following lemma concerning a non-homogeneous linear equation (2.3) with a given source term R.

Lemma 2.6. *Let us assume that for some* $m \geqslant 2$, $s \geqslant m + 2$ *and* $T > 0$, *we have*

(i) $a_0 \in B^m(\mathbb{R}^{n+d})$,
(ii) $S \in \mathcal{C}([0, T]; \mathtt{SQ}_s(\mathbb{R}^n)) \cap \mathcal{C}^s([0, T] \times \mathbb{R}^n)$ *solves the eikonal equation (1.10),*
(iii) $R \in \mathcal{C}([0, T]; B^m(\mathbb{R}^{n+d}))$.

Then, for all $\alpha \in [0, 1]$, *there exists a unique solution* $a \in \mathcal{C}([0, T]; B^m(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0, T]; B^{m-2}(\mathbb{R}^{n+d}))$ *to the following equation:*

$$\partial_t a + \nabla_x S \cdot \nabla_x a + \frac{a}{2}\Delta_x S = \mathrm{i}\frac{\alpha}{2}\Delta_x a + R, \quad a(0, x, z) = a_0(x, z). \tag{2.3}$$

Moreover for all $t \in [0, T]$ *and* $0 \leqslant \ell \leqslant m$, *the function a satisfies the estimates*

$$\|a(t)\|_{B^\ell}^2 \leqslant \|a_0\|_{B^\ell}^2 + C\int_0^t \|a(s)\|_{B^\ell}^2\, \mathrm{d}s + \int_0^t (a(s), R(s))_{B^\ell}\, \mathrm{d}s \tag{2.4}$$

$$\leqslant \|a_0\|_{B^\ell}^2 + C\int_0^t \left(\|a(s)\|_{B^\ell}^2 + \|R(s)\|_{B^\ell}^2\right) \mathrm{d}s, \tag{2.5}$$

where C is a generic constant which depends only on m and on

$$\sup_{2 \leqslant |\kappa| \leqslant s} \|\partial_x^\kappa S\|_{L^\infty([0,T]\times\mathbb{R}^n)}.$$

Proof. We first prove the result for $0 < \alpha \leqslant 1$ and treat the case $\alpha = 0$ in a second step. Let us start with a few preliminary remarks. From assumption (ii), we deduce that $|\nabla_x S(t, x)| \leqslant C(1 + |x|)$ and that, for all $2 \leqslant k \leqslant m$, we have the equivalences

$$u \in \mathcal{C}([0, T]; B^k(\mathbb{R}^{n+d})) \iff u\mathrm{e}^{\mathrm{i}S/\alpha} \in \mathcal{C}([0, T]; B^k(\mathbb{R}^{n+d}))$$

and

$$u \in \mathcal{C}^1([0, T]; B^{k-2}(\mathbb{R}^{n+d})) \iff u\mathrm{e}^{\mathrm{i}S/\alpha} \in \mathcal{C}^1([0, T]; B^{k-2}(\mathbb{R}^{n+d})).$$

Moreover, a direct calculation using the fact that S solves the eikonal equation shows that a function $a \in \mathcal{C}([0,T]; B^2(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0,T]; L^2(\mathbb{R}^{n+d}))$ is a strong solution of (2.3) if, and only if $\Psi = a\mathrm{e}^{\mathrm{i}S/\alpha}$ is a strong solution of the non-homogeneous linear GPE

$$\mathrm{i}\alpha\partial_t\Psi = -\frac{\alpha^2}{2}\Delta_x\Psi + \frac{|x|^2}{2}\Psi + \mathrm{i}\alpha R\,\mathrm{e}^{\mathrm{i}S/\alpha}, \quad \Psi(0,x,z) = a_0(x,z)\mathrm{e}^{\mathrm{i}S_0(x)/\alpha}. \tag{2.6}$$

Note that $R\,\mathrm{e}^{\mathrm{i}S/\alpha} \in \mathcal{C}([0,T]; B^m(\mathbb{R}^{n+d}))$ and that $a_0\mathrm{e}^{\mathrm{i}S_0/\alpha} \in B^m$. Therefore, standard results on Schrödinger equations [12] give the existence and uniqueness of the strong solution $\Psi \in \mathcal{C}([0,T]; B^m(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0,T]; B^{m-2}(\mathbb{R}^{n+d}))$ to (2.6). This solution can be expressed in terms of the Duhamel formula

$$\Psi(t) = \mathrm{e}^{-\mathrm{i}t\mathcal{H}_x}\left(a_0\mathrm{e}^{\mathrm{i}S_0/\alpha}\right) + \int_0^t \mathrm{e}^{-\mathrm{i}(t-\tau)\mathcal{H}_x}\left(R(\tau)\mathrm{e}^{\mathrm{i}S(\tau)/\alpha}\right)\mathrm{d}\tau,$$

where $\mathcal{H}_x = -\frac{\alpha^2}{2}\Delta_x + \frac{|x|^2}{2}$. This proves the well-posedness of (2.3) for $0 < \alpha \leqslant 1$.

Let us now prove the estimate (2.4). Applying ∂_x^κ to equation (2.3), where $|\kappa| \leqslant \ell$, yields

$$\partial_t\partial_x^\kappa a + \nabla_x S\cdot\nabla_x\partial_x^\kappa a - [\nabla_x S\cdot\nabla_x, \partial_x^\kappa]a + \partial_x^\kappa\left(\frac{a}{2}\Delta_x S\right) = \mathrm{i}\frac{\alpha}{2}\Delta_x\partial_x^\kappa a + \partial_x^\kappa R.$$

Take the L^2 real scalar product of this equation with $\partial_x^\kappa a$. Since $\mathrm{i}\Delta_x$ is skew-symmetric, we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\kappa a\|_{L^2}^2 + \left(\partial_x^\kappa a, \nabla_x S\cdot\nabla_x\partial_x^\kappa a\right) = \left(\partial_x^\kappa a, R_1 + R_2 + \partial_x^\kappa R\right),$$

where $R_1 = [\nabla_x S\cdot\nabla_x, \partial_x^\kappa]a$ and $R_2 = -\partial_x^\kappa(\frac{a}{2}\Delta_x S)$. We have by an integration by part that

$$\left|\left(\partial_x^\kappa a, \nabla_x S\cdot\nabla_x\partial_x^\kappa a\right)\right| = \frac{1}{2}\left|\left(\Delta_x S\,\partial_x^\kappa a, \partial_x^\kappa a\right)\right|.$$

We recall that $S \in \mathcal{C}([0,T], \mathtt{SQ}_s)$ with $s \geqslant m+2$, hence all the derivatives of $\Delta_x S$ up to the order m are bounded, so that

$$\left|\left(\partial_x^\kappa a, \nabla_x S\cdot\nabla_x\partial_x^\kappa a\right)\right| \leqslant C\|a\|_{B^\ell}^2$$

and

$$\|R_2\|_{L^2} \leqslant C\|a\|_{B^\ell}.$$

Let us now remark that the commutator $[\nabla_x S\cdot\nabla_x, \partial_x^\kappa]a$ is only composed of differential operators of order $\leqslant \ell$ multiplied by L^∞ functions, since S is subquadratic. Hence, we get

$$\|R_1\|_{L^2} \leqslant C\|a\|_{B^\ell}.$$

It comes finally

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\kappa a\|_{L^2}^2 \leqslant C\|a\|_{B^\ell}^2 + \left(\partial_x^\kappa a, \partial_x^\kappa R\right). \tag{2.7}$$

Applying now the operator Λ_z^ℓ to (2.3) yields (recall that S does not depend on z)

$$\partial_t\Lambda_z^\ell a + \nabla_x S\cdot\nabla_x\Lambda_z^\ell \text{ÃŠ}a + \Lambda_z^\ell\left(\frac{a}{2}\Delta_x S\right) = \mathrm{i}\frac{\alpha}{2}\Delta_x\Lambda_z^\ell a + \Lambda_z^\ell R.$$

Hence, taking the L^2 real scalar product with $\Lambda_z^\ell a$ gives, after integrations by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_z^\ell a\|_{L^2}^2 = \left(\Lambda_z^\ell a, \partial_x^\kappa R\right). \tag{2.8}$$

Let us finally apply the operator $|x|^\ell$ to (2.3). We get that

$$\partial_t \left(|x|^\ell a\right) + \nabla_x S \cdot \nabla_x (|x|^\ell a) + \frac{|x|^\ell a}{2} \Delta_x S = [\nabla_x S \cdot \nabla_x, |x|^\ell] a + \frac{\mathrm{i}\alpha}{2} \Delta_x \left(|x|^\ell a\right) + \frac{\mathrm{i}\alpha}{2} [|x|^\ell, \Delta_x] a + |x|^\ell R. \tag{2.9}$$

Since S is subquadratic, we have

$$\|[\nabla_x S \cdot \nabla_x, |x|^\ell] a\|_{L^2} = \ell \| |x|^{\ell-2} x \cdot \nabla_x S \, a\|_{L^2} \leqslant C \| \left(1 + |x|^\ell\right) a\|_{L^2} \leqslant C \|a\|_{B^\ell}$$

and we compute also

$$\|[|x|^\ell, \Delta_x] a\|_{L^2} = \|\Delta_x (|x|^\ell) a + 2\nabla_x (|x|^\ell) \cdot \nabla_x a\|_{L^2} \leqslant C \| |x|^{\ell-2} a\|_{L^2} + C \| |x|^{\ell-1} \nabla_x a\|_{L^2} \leqslant C \|a\|_{B^\ell}.$$

Taking the L^2 real scalar product of (2.9) with $|x|^\ell a$, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| |x|^\ell a\|_{L^2}^2 \leqslant C \|a\|_{B^\ell}^2 + \left(|x|^\ell a, |x|^\ell R\right). \tag{2.10}$$

Finally, from (2.7), (2.8) and (2.10), we deduce (2.4) for $0 < \alpha \leqslant 1$. From (2.4) and Cauchy–Schwarz, we obtain then the second estimate (2.5). Note that the above calculations are rigorous only if we know a better regularity for a, for instance $a \in \mathcal{C}([0, T]; B^{m+2}(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0, T]; B^m(\mathbb{R}^{n+d}))$. A standard regularization argument, that we skip here, enables to fully justify this proof.

Let us now prove the result in the case $\alpha = 0$. To this aim, we consider a regularized sequence a_0^δ, S^δ, R^δ, where $\delta > 0$ is a regularization parameter, such that

(i)' $a_0^\delta \in B^{m+2}(\mathbb{R}^{n+d})$ and $\|a_0^\delta - a_0\|_{B^m} \to 0$ as $\delta \to 0$,

(ii)' $S^\delta \in \mathcal{C}([0, T^\delta]; \mathtt{SQ}_{s+2}(\mathbb{R}^n)) \cap \mathcal{C}^{s+2}([0, T^\delta] \times \mathbb{R}^n)$ solves the eikonal equation (1.10), $\|S^\delta - S\|_{\mathcal{C}^s} \to 0$ as $\delta \to 0$ and $T^\delta \to T$,

(iii)' $R^\delta \in \mathcal{C}([0, T]; B^{m+2}(\mathbb{R}^{n+d}))$ and $\|R^\delta - R\|_{L^\infty([0,T]; B^m)} \to 0$ as $\delta \to 0$.

Note that, to construct S^δ, we need to regularize the associated initial data S_0, which may make the existence time T depend on δ.

We consider a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of positive numbers converging to 0 and denote by $(a_n^\delta)_{n\in\mathbb{N}}$ the sequence of solutions of

$$\partial_t a_n^\delta + \nabla_x S^\delta \cdot \nabla_x a_n^\delta + \frac{a_n^\delta}{2} \Delta_x S^\delta = \mathrm{i} \frac{\alpha_n}{2} \Delta_x a_n^\delta + R^\delta, \quad a_n^\delta(0, x, z) = a_0^\delta(x, z).$$

In a first step, we consider $\delta > 0$ as fixed. From (2.5) and Gronwall's lemma, we infer

$$\max_{0\leqslant t\leqslant T^\delta} \left\|a_n^\delta(t)\right\|_{B^{m+2}}^2 \leqslant \left(\left\|a_0^\delta\right\|_{B^{m+2}}^2 + C T^\delta \|R^\delta\|_{L^\infty([0,T^\delta]; B^{m+2})}^2\right) \mathrm{e}^{CT^\delta},$$

so this sequence $(a_n^\delta)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}([0, T^\delta]; B^{m+2}(\mathbb{R}^{n+d}))$. Moreover, we have

$$\partial_t (a_p^\delta - a_q^\delta) + \nabla_x S^\delta \cdot \nabla_x (a_p^\delta - a_q^\delta) + \frac{(a_p^\delta - a_q^\delta)}{2} \Delta_x S^\delta = \mathrm{i} \frac{\alpha_q}{2} \Delta_x (a_p^\delta - a_q^\delta) + \mathrm{i} \frac{\alpha_p - \alpha_q}{2} \Delta_x a_p^\delta. \tag{2.11}$$

Applying again (2.5) with R replaced by $\mathrm{i}\frac{\alpha_p - \alpha_q}{2} \Delta a_p^\delta \in \mathcal{C}([0, T^\delta]; B^m(\mathbb{R}^{n+d}))$ and $a_0 = 0$ gives

$$\|a_p^\delta(t) - a_q^\delta(t)\|_{B^m}^2 \leqslant C_\delta (\alpha_p - \alpha_q)^2 + C_\delta \int_0^t \left(\|a_p^\delta(s) - a_q^\delta(s)\|_{B^m}^2\right) \mathrm{d}s$$

and Gronwall's lemma implies that

$$\max_{0\leqslant t\leqslant T^\delta} \|a_p^\delta(t) - a_q^\delta(t)\|_{B^m}^2 \leqslant C_\delta(\alpha_p - \alpha_q)^2.$$

Hence, $(a_n^\delta)_{n\in\mathbb{N}}$ is a Cauchy sequence of $\mathcal{C}([0,T^\delta]; B^m(\mathbb{R}^{n+d}))$. Inserting this information in (2.11) yields that it is also a Cauchy sequence of $\mathcal{C}^1([0,T^\delta]; B^{m-2}(\mathbb{R}^{n+d}))$. Therefore, as $n\to+\infty$, this sequence converges to a function

$$a^\delta \in \mathcal{C}([0,T^\delta]; B^m(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0,T^\delta]; B^{m-2}(\mathbb{R}^{n+d})),$$

which solves

$$\partial_t a^\delta + \nabla_x S^\delta \cdot \nabla_x a^\delta + \frac{a^\delta}{2}\Delta_x S^\delta = R^\delta, \quad a^\delta(0,x,z) = a_0^\delta(x,z). \tag{2.12}$$

Let us now proceed to the limit $\delta\to 0$. Using (2.5) for (2.12) (remark that the above proof of this estimate is valid also for $\alpha = 0$) enables to show that a^δ is a Cauchy sequence in $\mathcal{C}([0,T]; B^m(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0,T]; B^{m-2}(\mathbb{R}^{n+d}))$ and converges to a function a which satisfies (2.3) with $\alpha = 0$. The estimates (2.4) and (2.5) are also valid for this function a. Remark that the uniqueness of the solution a also stems from the estimate (2.5) (written for the difference between two solutions) and Gronwall's lemma. □

In order to prove the uniform well-posedness of the four nonlinear equations (1.15), (1.11), (1.12) and (1.16), we will need the following Lipschitz estimates for $g: u\mapsto |u|^2u$, $F(\theta,\cdot)$ defined by (1.7) and $F_{\rm av}$ defined by (1.9).

Lemma 2.7. *For all $m > \frac{n+d}{2}$ and $M>0$, there is a non-decreasing function $M\mapsto C_m(M) > 0$ such that, for all $0\leqslant \ell\leqslant m$,*

$$\begin{aligned}
\|g(u)-g(v)\|_{B^\ell} &\leqslant C_m(M)\|u-v\|_{B^\ell},\\
\|F_{\rm av}(u)-F_{\rm av}(v)\|_{B^\ell} &\leqslant C_m(M)\|u-v\|_{B^\ell},\\
\|F(\theta,u)-F(\theta,v)\|_{B^\ell} &\leqslant C_m(M)\|u-v\|_{B^\ell},
\end{aligned}$$

for all $u,v\in B^m(\mathbb{R}^{n+d})$ satisfying $\|u\|_{B^m}\leqslant M$, $\|v\|_{B^m}\leqslant M$ and for all $\theta\in\mathbb{R}$.

Proof. The first inequality is a direct consequence of (2.2) and

$$g(u)-g(v) = (|u|^2+\overline{u}v)(u-v) + v^2(\overline{u}-\overline{v}).$$

Using that $e^{-i\theta\mathcal{H}_z}$ is an isometry on B^ℓ and on B^m, we get that

$$\begin{aligned}
\|F(\theta,u)-F(\theta,v)\|_{B^\ell} &= \left\|g\left(e^{-i\theta\mathcal{H}_z}u\right) - g\left(e^{-i\theta\mathcal{H}_z}v\right)\right\|_{B^\ell}\\
&\leqslant C_m(M)\left\|e^{-i\theta\mathcal{H}_z}u - e^{-i\theta\mathcal{H}_z}v\right\|_{B^\ell}\\
&\leqslant C_m(M)\|u-v\|_{B^\ell}
\end{aligned}$$

and

$$\begin{aligned}
\|F_{\rm av}(u)-F_{\rm av}(v)\|_{B^\ell} &= \left\|\frac{1}{2\pi}\int_0^{2\pi}(F(\theta,u)-F(\theta,v))\,d\theta\right\|_{B^\ell}\\
&\leqslant \frac{1}{2\pi}\int_0^{2\pi}\|F(\theta,u)-F(\theta,v)\|_{B^\ell}\,d\theta\\
&\leqslant C_m(M)\|u-v\|_{B^\ell}.
\end{aligned}$$

□

The main result of this section is the following proposition.

Proposition 2.8. *Let* $(\varepsilon, \alpha) \in (0, 1]^2$ *and* $M > 0$*. Let* $m > \frac{n+d}{2}$ *and* $s \geqslant m + 2$*. Let* $S_0 \in \mathtt{SQ}_{s+1}(\mathbb{R}^n)$ *and* S *be the corresponding solution of the eikonal equation, given by proposition 2.2. Then there exist* $\overline{T} \in (0, T]$ *which depends only on* M *and*

$$\sup_{2\leqslant|\kappa|\leqslant s} \|\partial_x^\kappa S\|_{L^\infty([0,T]\times\mathbb{R}^n)}$$

such that, for all $A_0 \in B^m(\mathbb{R}^{n+d})$ *satisfying* $\|A_0\|_{B^m(\mathbb{R}^{n+d})} \leqslant M$*,*

(i) there is a unique solution $A^{\varepsilon,\alpha} \in \mathcal{C}([0, \overline{T}]; B^m(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0, \overline{T}]; B^{m-2}(\mathbb{R}^{n+d}))$ *to equation (1.15),*

(ii) there is a unique solution $A^{0,\alpha} \in \mathcal{C}([0, \overline{T}]; B^m(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0, \overline{T}]; B^{m-2}(\mathbb{R}^{n+d}))$ *to equation (1.11),*

(iii) there is a unique solution $A^{\varepsilon,0} \in \mathcal{C}([0, \overline{T}]; B^m(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0, \overline{T}]; B^{m-2}(\mathbb{R}^{n+d}))$ *to equation (1.16),*

(iv) there is a unique solution $A \in \mathcal{C}([0, \overline{T}]; B^m(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0, \overline{T}]; B^{m-2}(\mathbb{R}^{n+d}))$ *to equation (1.12).*

Moreover, we have

$$\|A^{\varepsilon,\alpha}\|_{L^\infty([0,\overline{T}];B^m)},\ \|A^{0,\alpha}\|_{L^\infty([0,\overline{T}];B^m)},\ \|A^{\varepsilon,0}\|_{L^\infty([0,\overline{T}];B^m)},\ \|A\|_{L^\infty([0,\overline{T}];B^m)} \leqslant 2M,$$

and the $L^\infty([0, \overline{T}]; B^{m-2})$ *norms of* $\partial_t A^{\varepsilon,\alpha}$*,* $\partial_t A^{0,\alpha}$*,* $\partial_t A^{\varepsilon,0}$ *and* $\partial_t A$ *are uniformly bounded with respect to* (ε, α)*.*

Proof. This proposition can be proved by iterative schemes. Let us only write the proof of item (i), the other items can be proved similarly. We denote by a^0 the function defined for all $t \in [0, T]$ by $a^0(t) = A_0$. Then, for all $k \in \mathbb{N}$, a^{k+1} is defined as the solution of the following equation:

$$\partial_t a^{k+1} + \nabla_x S \cdot \nabla_x a^{k+1} + \frac{a^{k+1}}{2}\Delta_x S = \mathrm{i}\frac{\alpha}{2}\Delta_x a^{k+1} - \mathrm{i}F\left(\frac{t}{\varepsilon^2}, a^k\right),$$

satisfying

$$a^{k+1}(0, x, z) = A_0(x, z).$$

From lemmas 2.6 and 2.7, we deduce that the sequence $(a^k)_{k\in\mathbb{N}}$ is well-defined in $\mathcal{C}([0, T]; B^m(\mathbb{R}^{n+d})) \cap \mathcal{C}^1([0, T]; B^{m-2}(\mathbb{R}^{n+d}))$ and that

$$\begin{aligned}\left\|a^{k+1}(t)\right\|_{B^m}^2 &\leqslant \|A_0\|_{B^m}^2 + C_0\int_0^t \left(\left\|a^{k+1}(s)\right\|_{B^m}^2 + \left\|F\left(\frac{s}{\varepsilon^2}, a^k\right)\right\|_{B^m}^2\right)\mathrm{d}s\\ &\leqslant \|A_0\|_{B^m}^2 + C_0\int_0^t \left(\left\|a^{k+1}(s)\right\|_{B^m}^2 + C_m(\|a^k\|_{B^m})^2\|a^k\|_{B^m}^2\right)\mathrm{d}s,\end{aligned}$$

where we used that $F(\theta, 0) = 0$. Let us prove by induction that, for

$$\overline{T} = \min\left(T, \frac{\log 2}{C_0}, \sup\left\{t > 0 :\ tC_0C_m(2M)^2\mathrm{e}^{C_0t} \leqslant \frac{1}{2}\right\}\right),$$

we have

$$\max_{0\leqslant t\leqslant\overline{T}} \|a^k(t)\|_{B^m} \leqslant 2M$$

for all $k \in \mathbb{N}$. This property is clearly true for $k = 0$. Assume that this condition is satisfied for $k \in \mathbb{N}$. By Gronwall's lemma, we obtain that this property is also true for $k + 1$, since

$$\left\|a^{k+1}(t)\right\|_{B^m}^2 \leqslant \mathrm{e}^{C_0t}M^2 + tC_0C_m(2M)^2\mathrm{e}^{C_0t}4M^2 \leqslant 4M^2 \quad \text{for} \quad 0 \leqslant t \leqslant \overline{T}.$$

Now, for all $k \in \mathbb{N}^*$, we get by lemmas 2.6 and 2.7 that

$$|a^{k+1}(t) - a^k(t)\|_{B^m}^2 \leqslant C_0 \int_0^t \left(\|a^{k+1}(s) - a^k(s)\|_{B^m}^2 + C_m(2M)^2\|a^k(s) - a^{k-1}(s)\|_{B^m}^2\right) \mathrm{d}s$$

and Gronwall's lemma ensures that

$$\|a^{k+1}(t) - a^k(t)\|_{B^m}^2 \leqslant C_0 C_m(2M)^2 \int_0^t \left(\mathrm{e}^{C_0(t-s)}\|a^k(s) - a^{k-1}(s)\|_{B^m}^2\right) \mathrm{d}s.$$

Then we obtain for $t \in [0, \overline{T}]$ that

$$\|a^{k+1} - a^k\|_{L^\infty([0,\overline{T}];B^m)}^2 \leqslant \overline{T} C_0 C_m (2M)^2 \mathrm{e}^{C_0\overline{T}} \|a^k - a^{k-1}\|_{L^\infty([0,\overline{T}];B^m)}^2 \leqslant 2^{-k}\|a^1 - a^0\|_{L^\infty([0,\overline{T}];B^m)}^2.$$

Hence the series $(a^{k+1} - a^k)_{k\in\mathbb{N}}$ converges in $\mathcal{C}([0, \overline{T}]; B^m(\mathbb{R}^{n+d}))$ so that $(a^k)_{k\in\mathbb{N}}$ converges to a solution $A^{\varepsilon,\alpha}$ of equation (1.15). Let us remark that $A^{\varepsilon,\alpha}$ satisfies the uniform estimate

$$\|A^{\varepsilon,\alpha}\|_{L^\infty([0,\overline{T}];B^m)} \leqslant 2M.$$

Inserting this estimate into (1.15) and using that S is subquadratic yields a uniform estimate of $\|\partial_t A^{\varepsilon,\alpha}\|_{L^\infty([0,\overline{T}];B^{m-2})}$. The uniqueness property follows also from Gronwall's lemma and from lemma 2.7. □

3. Proof of theorem 1.4: the limits $\alpha \longrightarrow 0$ and $\varepsilon \longrightarrow 0$

This section is devoted to the proof of theorem 1.4.

Strong confinement limits: proof of (1.17) and (1.18). Let us introduce the function

$$\begin{aligned} \mathscr{F} : \mathbb{R} \times B^m(\mathbb{R}^{n+d}) &\longrightarrow B^m(\mathbb{R}^{n+d}) \\ (\theta, u) &\longmapsto \textstyle\int_0^\theta (F(s,u) - F_{\mathrm{av}}(u))\,\mathrm{d}s, \end{aligned}$$

which satisfies the following properties for every $u \in B^m(\mathbb{R}^{n+d})$:

(a) $\theta \mapsto \mathscr{F}(\theta, u)$ is a 2π-periodic function, since $\theta \mapsto F(\theta, u)$ is 2π-periodic and F_{av} is its average,

(b) if $\|u\|_{B^m} \leqslant M$ then $\|\mathscr{F}(\theta, u)\|_{B^m} \leqslant 4\pi C_m(M)M$ for all $\theta \in \mathbb{R}$, where $C_m(\cdot)$ was defined in lemma 2.7.

Using the relation

$$\varepsilon^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathscr{F}(t/\varepsilon^2, u(t))\right) = \left(F(t/\varepsilon^2, u(t)) - F_{\mathrm{av}}(u(t))\right) + \varepsilon^2 D_u\mathscr{F}(t/\varepsilon^2, u(t))(\partial_s u(t))$$

and equations (1.11) and (1.15) (or their versions with $\alpha = 0$, i.e. (1.12) and (1.16)), we obtain for all $\alpha \in [0, 1]$ and $\varepsilon \in (0, 1]$,

$$\begin{aligned} \left(\partial_t + \nabla_x S\cdot\nabla_x + \frac{\Delta_x S}{2} - \frac{\mathrm{i}\alpha}{2}\Delta_x\right)\left(A^{\varepsilon,\alpha} - A^{0,\alpha}\right) &= -\mathrm{i}\left(F_{\mathrm{av}}(A^{\varepsilon,\alpha}) - F_{\mathrm{av}}(A^{0,\alpha})\right) \\ &\quad -\mathrm{i}\varepsilon^2\partial_t\mathscr{F}(t/\varepsilon^2, A^{\varepsilon,\alpha}) + \mathrm{i}\varepsilon^2 D_u\mathscr{F}(t/\varepsilon^2, A^{\varepsilon,\alpha})(\partial_t A^{\varepsilon,\alpha}). \end{aligned} \quad (3.1)$$

Hence, estimate (2.4) of lemma 2.6, with $\ell = m - 2$, ensures that

$$\|A^{\varepsilon,\alpha}(t) - A^{0,\alpha}(t)\|_{B^{m-2}}^2 \leqslant C \int_0^t \|A^{\varepsilon,\alpha}(s) - A^{0,\alpha}(s)\|_{B^{m-2}}^2 \mathrm{d}s + I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_0^t \left(\|F_{\rm av}(A^{\varepsilon,\alpha}) - F_{\rm av}(A^{0,\alpha})\|_{B^{m-2}}^2 + \varepsilon^4 \|D_u\mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha})(\partial_t A^{\varepsilon,\alpha})\|_{B^{m-2}}^2 \right) \mathrm{d}s,$$

$$I_2 = \sum_{|\kappa|\leqslant m-2} \varepsilon^2 \int_0^t \left(\partial_x^\kappa \left(A^{\varepsilon,\alpha} - A^{0,\alpha}\right), -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s,$$

$$I_3 = \varepsilon^2 \int_0^t \left(\Lambda_z^{m-2} \left(A^{\varepsilon,\alpha} - A^{0,\alpha}\right), -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s}\Lambda_z^{m-2} \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s,$$

$$I_4 = \varepsilon^2 \int_0^t \left(|x|^{2(m-2)} \left(A^{\varepsilon,\alpha} - A^{0,\alpha}\right), -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}s} \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s.$$

Let us remark that according to theorem 1.3 (iii), the sequences $(A^{\varepsilon,\alpha})_{\varepsilon,\alpha}$ and $(\partial_t A^{\varepsilon,\alpha})_{\varepsilon,\alpha}$ are uniformly bounded, respectively in $L^\infty([0,\overline{T}]; B^m(\mathbb{R}^{n+d}))$ and in $L^\infty([0,\overline{T}]; B^{m-2}(\mathbb{R}^{n+d}))$. Moreover, since $m > \frac{n+d}{2}$, (2.2) ensures that, for all $\theta \in [0, 2\pi]$,

$$\|D_u\mathscr{F}(\theta, A^{\varepsilon,\alpha})(\partial_t A^{\varepsilon,\alpha})\|_{B^{m-2}} = \left\| \int_0^\theta \left(D_u F(s, A^{\varepsilon,\alpha}) - D_u F_{\rm av}(A^{\varepsilon,\alpha})\right)(\partial_t A^{\varepsilon,\alpha})\,\mathrm{d}s \right\|_{B^{m-2}}$$

$$\leqslant \int_0^\theta \left\| 2|\mathrm{e}^{-\mathrm{i}s\mathcal{H}_z} A^{\varepsilon,\alpha}|^2 \mathrm{e}^{-\mathrm{i}s\mathcal{H}_z}\partial_t A^{\varepsilon,\alpha} + (\mathrm{e}^{-\mathrm{i}s\mathcal{H}_z} A^{\varepsilon,\alpha})^2 \mathrm{e}^{\mathrm{i}s\mathcal{H}_z}\overline{\partial_t A^{\varepsilon,\alpha}} \right\|_{B^{m-2}} \mathrm{d}s$$

$$+ \frac{\theta}{2\pi}\int_0^{2\pi} \|2|\mathrm{e}^{-\mathrm{i}s\mathcal{H}_z} A^{\varepsilon,\alpha}|^2 \mathrm{e}^{-\mathrm{i}s\mathcal{H}_z}\partial_t A^{\varepsilon,\alpha} + (\mathrm{e}^{-\mathrm{i}s\mathcal{H}_z} A^{\varepsilon,\alpha})^2 \mathrm{e}^{\mathrm{i}s\mathcal{H}_z}\overline{\partial_t A^{\varepsilon,\alpha}}\|_{B^{m-2}}\,\mathrm{d}s$$

satisfies

$$\sup_{s\in[0,\overline{T}]}\sup_{\varepsilon,\alpha} \|D_u\mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}(s))(\partial_t A^{\varepsilon,\alpha}(s))\|_{B^{m-2}} \leqslant C. \tag{3.2}$$

Using inequality (3.2) and lemma 2.7, we obtain that

$$I_1 \leqslant C\int_0^t \|A^{\varepsilon,\alpha}(s) - A^{0,\alpha}(s)\|_{B^{m-2}}^2\,\mathrm{d}s + C\varepsilon^4.$$

Let us study the three remaining terms I_2, I_3 and I_4. By an integration by parts, we get that

$$I_2 = \sum_{|\kappa|\leqslant m-2} \varepsilon^2 \left(\partial_x^\kappa \left(A^{\varepsilon,\alpha}(t) - A^{0,\alpha}(t)\right), -\mathrm{i}\partial_x^\kappa \mathscr{F}(t/\varepsilon^2, A^{\varepsilon,\alpha}) \right) + \sum_{|\kappa|\leqslant m-2} J^\kappa$$

$$\leqslant C\varepsilon^4 + \frac{1}{4}\|A^{\varepsilon,\alpha}(t) - A^{0,\alpha}(t)\|_{B^{m-2}}^2 + \sum_{|\kappa|\leqslant m-2} J^\kappa,$$

where

$$J^\kappa = -\varepsilon^2 \int_0^t \left(\partial_x^\kappa \partial_t \left(A^{\varepsilon,\alpha} - A^{0,\alpha}\right), -\mathrm{i}\partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s.$$

Using again equation (3.1), we get that $J^\kappa = J_1^\kappa + J_2^\kappa + J_3^\kappa + J_4^\kappa + J_5^\kappa$, where

$$J_1^\kappa = \varepsilon^2 \int_0^t \left(\partial_x^\kappa \left(\nabla_x S\cdot\nabla_x\right)\left(A^{\varepsilon,\alpha} - A^{0,\alpha}\right), -\mathrm{i}\partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha})\,\mathrm{d}s \right),$$

$$J_2^\kappa = \varepsilon^2 \int_0^t \left(\partial_x^\kappa \left(\frac{\Delta_x S}{2} - \frac{\mathrm{i}\alpha}{2}\Delta_x\right)\left(A^{\varepsilon,\alpha} - A^{0,\alpha}\right), -\mathrm{i}\partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s,$$

$$J_3^\kappa = -\varepsilon^2 \int_0^t \left(\partial_x^\kappa \left(F_{\rm av}(A^{\varepsilon,\alpha}) - F_{\rm av}(A^{0,\alpha})\right), \partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s,$$

$$J_4^\kappa = \varepsilon^4 \int_0^t \left(\partial_x^\kappa D_u\mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha})(\partial_t A^{\varepsilon,\alpha}), \partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s,$$

$$J_5^\kappa = -\varepsilon^4 \int_0^t \left(\partial_x^\kappa \frac{\mathrm{d}}{\mathrm{d}s}\mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}), \partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s.$$

Using an integration by parts and the fact that the commutator $[\partial_x^\kappa, \nabla_x S \cdot \nabla_x]$ is an operator of order $m-2$, we get that

$$\begin{aligned}
J_1^\kappa &= \varepsilon^2 \int_0^t \left([\partial_x^\kappa, \nabla_x S \cdot \nabla_x] \left(A^{\varepsilon,\alpha} - A^{0,\alpha} \right), -\mathrm{i}\partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s \\
&\quad + \varepsilon^2 \int_0^t \left((\nabla_x S \cdot \nabla_x)\, \partial_x^\kappa \left(A^{\varepsilon,\alpha} - A^{0,\alpha} \right), -\mathrm{i}\partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s \\
&\leqslant C\varepsilon^2 \int_0^t \|A^{\varepsilon,\alpha} - A^{0,\alpha}\|_{B^{m-2}} \|\mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha})\|_{B^{m-2}} \, \mathrm{d}s \\
&\quad - \varepsilon^2 \int_0^t \left(\partial_x^\kappa \left(A^{\varepsilon,\alpha} - A^{0,\alpha} \right), -\mathrm{i}\nabla_x \cdot \left(\nabla_x S \partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \right) \mathrm{d}s \\
&\leqslant C\varepsilon^2 \int_0^t \|A^{\varepsilon,\alpha} - A^{0,\alpha}\|_{B^{m-2}} \|\mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha})\|_{B^{m}} \, \mathrm{d}s \\
&\leqslant C\varepsilon^4 + \int_0^t \|A^{\varepsilon,\alpha} - A^{0,\alpha}\|_{B^{m-2}}^2 \, \mathrm{d}s.
\end{aligned}$$

Since the operator $\frac{\alpha}{2}\Delta_x$ is symmetric, we obtain

$$\begin{aligned}
J_2^\kappa &\leqslant C\varepsilon^2 \int_0^t \|A^{\varepsilon,\alpha} - A^{0,\alpha}\|_{B^{m-2}} \|\mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha})\|_{B^{m-2}} \, \mathrm{d}s \\
&\quad + \varepsilon^2 \int_0^t \left(\partial_x^\kappa \left(A^{\varepsilon,\alpha} - A^{0,\alpha} \right), \partial_x^\kappa \frac{\alpha \Delta_x}{2} \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right) \mathrm{d}s \\
&\leqslant C\varepsilon^2 \int_0^t \|A^{\varepsilon,\alpha} - A^{0,\alpha}\|_{B^{m-2}} \|\mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha})\|_{B^{m}} \, \mathrm{d}s \\
&\leqslant C\varepsilon^4 + \int_0^t \|A^{\varepsilon,\alpha} - A^{0,\alpha}\|_{B^{m-2}}^2 \, \mathrm{d}s.
\end{aligned}$$

From the Lipschitz estimates of lemma 2.7, we deduce also

$$J_3^\kappa \leqslant C\varepsilon^4 + \int_0^t \|A^{\varepsilon,\alpha} - A^{0,\alpha}\|_{B^{m-2}}^2 \, \mathrm{d}s$$

and inequality (3.2) implies $J_4^\kappa \leqslant C\varepsilon^4$. Finally, we obtain that

$$J_5^\kappa = -\frac{\varepsilon^4}{2} \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \left\| \partial_x^\kappa \mathscr{F}(s/\varepsilon^2, A^{\varepsilon,\alpha}) \right\|_{L^2}^2 \mathrm{d}s = -\frac{\varepsilon^4}{2} \left\| \partial_x^\kappa \mathscr{F}(t/\varepsilon^2, A^{\varepsilon,\alpha}) \right\|_{L^2}^2 \leqslant 0$$

and, finally,

$$I_2 \leqslant C\varepsilon^4 + C \int_0^t \|A^{\varepsilon,\alpha}(s) - A^{0,\alpha}(s)\|_{B^{m-2}}^2 \, \mathrm{d}s + \frac{1}{4} \|A^{\varepsilon,\alpha}(t) - A^{0,\alpha}(t)\|_{B^{m-2}}^2.$$

The same arguments hold for the two remaining terms I_3 and I_4. Hence, we obtain

$$\|A^{\varepsilon,\alpha}(t) - A^{0,\alpha}(t)\|_{B^{m-2}}^2 \leqslant C \int_0^t \|A^{\varepsilon,\alpha}(s) - A^{0,\alpha}(s)\|_{B^{m-2}}^2 \, \mathrm{d}s + C\varepsilon^4$$

so that, by Gronwall's lemma, we get (1.17),

$$\|A^{\varepsilon,\alpha} - A^{0,\alpha}\|_{L^\infty([0,\overline{T}]; B^{m-2})} \leqslant C\varepsilon^2 \quad \text{for all } \alpha \in (0, 1],$$

and (1.18),

$$\|A^{\varepsilon,0} - A\|_{L^\infty([0,\overline{T}]; B^{m-2})} \leqslant C\varepsilon^2.$$

The semi-classical limits: proof of (1.19) and (1.20). The error estimates (1.19) and (1.20) are simple consequences of the uniform bounds given by theorem 1.3 (iii). We have indeed

$$\partial_t (A^{\varepsilon,\alpha} - A^{\varepsilon,0}) + \nabla_x S \cdot \nabla_x (A^{\varepsilon,\alpha} - A^{\varepsilon,0}) + \frac{\Delta_x S}{2}(A^{\varepsilon,\alpha} - A^{\varepsilon,0})$$
$$= \mathrm{i}\frac{\alpha}{2}\Delta_x (A^{\varepsilon,\alpha} - A^{\varepsilon,0}) - \mathrm{i}(F(s/\varepsilon^2, A^{\varepsilon,\alpha}) - F(s/\varepsilon^2, A^{\varepsilon,0})) - \mathrm{i}\frac{\alpha}{2}\Delta_x A^{\varepsilon,0}$$

so that, by lemmas 2.6 and 2.7 and by the uniform bound for $\|A^{\varepsilon,0}\|_{L^\infty([0,\overline{T}];B^m)}$,

$$\|A^{\varepsilon,\alpha}(t) - A^{0,\alpha}(t)\|^2_{B^{m-2}} \leqslant C\int_0^t \|A^{\varepsilon,\alpha}(s) - A^{\varepsilon,0}(s)\|^2_{B^{m-2}}\,\mathrm{d}s$$
$$+ \int_0^t \left(\|F(s/\varepsilon^2, A^{\varepsilon,\alpha}(s)) - F(s/\varepsilon^2, A^{0,\alpha}(s))\|^2_{B^{m-2}}\,\mathrm{d}s + \frac{\alpha}{2}\|A^{\varepsilon,0}(s)\|^2_{B^m}\right)\mathrm{d}s$$
$$\leqslant C\alpha + C\int_0^t \|A^{\varepsilon,\alpha}(s) - A^{\varepsilon,0}(s)\|^2_{B^{m-2}}\,\mathrm{d}s$$

and by Gronwall's lemma

$$\|A^{\varepsilon,\alpha} - A^{0,\alpha}\|^2_{C([0,\overline{T}];B^{m-2})} \leqslant C\alpha.$$

The proof of (1.19) is complete. The same proof holds for (1.20), replacing the function F by F_{av}. The proof of theorem 1.4 is complete.

□

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