MA5250 Homework 2

(Due date: 6:00pm, March 21, 2014 (Friday))

1. Consider the Poisson equation in 1d:

$$u_{xx} = f(x), \qquad a < x < b, \tag{1}$$

with boundary condition

$$u(a) = \alpha, \qquad u(b) = \beta. \tag{2}$$

a). Write down a second order centered difference scheme for the problem on a uniform mesh.
b). Express the difference equations in linear system form.
c). What is the best algorithm to solve the linear system.
d). Derive a fast Poisson solver for solving the linear system.
e). Design a 4th-order compact scheme for the problem.

2. Consider the difference equations:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{\nu}{2}\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}\right]$$
$$+\frac{a}{2}\left[\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h} + \frac{u_{j+1}^n - u_{j-1}^n}{2h}\right], \quad 1 \le j \le M-1, \tag{3}$$
$$\tag{4}$$
$$u_0^{n+1} = u_M^{n+1} = 0, \tag{5}$$
$$u_j^0 = u_0(x_j), \qquad j = 0, 1, \cdots, M; \tag{6}$$

where ν and a are constants, and k is time step, h is mesh size.
a). Find the accuracy of the method to the equation $u_t = \nu\, u_{xx} + a\, u_x$.
b). Find the stability condition
c). Write down the difference equation in linear system form.

3. Consider

$$a\, u_x + \nu\, u_{xx} = 0, \qquad a < x < b, \tag{7}$$
$$u(a) = 0, \qquad u(b) = 1, \tag{8}$$

where $\nu > 0$ and a are two constants.
a). Find the exact solution.
b). Discuss the boundary layer effect when $\nu \to 0^+$.

4. Consider the Stokes equations:

$$\vec{\mathbf{u}}_t + \nabla p = \nu \triangle \vec{\mathbf{u}}, \qquad \text{in} \quad \Omega, \tag{9}$$
$$\nabla \cdot \vec{\mathbf{u}} = 0. \tag{10}$$

a). Obtain the Poisson equation for pressure p from this system:

$$\triangle p = 0. \tag{11}$$

b). Write down the projection method for this system.

c). Obtain the equation for pressure p from the discretization of the projection method:

$$(I - k\nu\triangle) \triangle p^{n+1} = 0; \tag{12}$$

where k is time step and p^{n+1} is the approximation of p at time $t = t_{n+1}$.

d). What conclusion can you get from a) and c)?

5. Consider incompressible viscous flows in vorticity-streamfunction formulation

$$\omega_t + (\vec{\mathbf{u}} \cdot \nabla)\, \omega = \nu \triangle \omega, \tag{13}$$

$$\triangle\psi = -\omega, \qquad \text{in} \quad \Omega, \tag{14}$$

$$u = \psi_y, \qquad v = -\psi_x, \tag{15}$$

$$\psi = 0, \qquad \frac{\partial \psi}{\partial \vec{n}} = 0, \qquad \text{on} \quad \partial\Omega; \tag{16}$$

where $\vec{n}$ is outward normal vector. Derive the vorticity boundary condition via using forward difference for $\dfrac{\partial \psi}{\partial \vec{n}}$ and 2nd centered difference for the equations.

6. Consider the Poisson equation

$$\triangle u = f, \qquad \text{in} \quad \Omega = [a, b] \times [c, d],$$

with boundary condition

$$u = u_b(x, y), \qquad \text{on} \quad \Gamma = \partial\Omega.$$

Choose $h_x = \dfrac{b-a}{M}$, $h_y = \dfrac{d-c}{N}$ be the mesh sizes and (x_i, y_j) with $x_i = a + i\ h_x$, $y_j = c + j\ h_y$, $i = 0, 1, \cdots, M$, $j = 0, 1, \cdots, N$ be grid points.

a). Write down the second order central difference discretization for the above problem.

b). Develop a code to implement the fast Poisson solver for the above problem.

c). Apply your code to solve the following problem

$$\triangle u = -5\sin(2x)\sin y + 4, \qquad \text{in} \quad \Omega = [0, 2\pi] \times [0, 2\pi],$$

$$u = x^2 + y^2, \qquad \text{on} \quad \partial\Omega.$$

Draw the contour plot and surface plot of your numerical solution under $h_x = h_y = \frac{\pi}{64}$. The exact solution of the problem is

$$u(x, y) = x^2 + y^2 + \sin(2x)\sin y.$$

Test the second order convergence rate by compute $\frac{\|u - u_h\|}{h^2}$ for $h = \frac{\pi}{16}, \frac{\pi}{32}, \frac{\pi}{64}$ and $\frac{\pi}{128}$.

d). Repeat a) and b) by replacing the Dirichlet boundary condition with Neumann boundary condition

$$\frac{\partial u}{\partial \vec{n}} = u_n(x, y), \qquad \text{on} \quad \Gamma = \partial\Omega.$$

Apply your code to solve the following problem

$$\begin{aligned}
&\triangle u = -5\cos x\cos(2y), && \text{in} \quad \Omega = [0, 2\pi] \times [0, 2\pi], \\
&\frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, 2\pi) = x, && 0 \le x \le 2\pi, \\
&\frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(2\pi, y) = y, && 0 \le y \le 2\pi.
\end{aligned}$$

Draw the contour plot and surface plot of your numerical solution under $h_x = h_y = \frac{\pi}{64}$.

P.S. You can download fast Sine and Cosine transform from http://www.netlib.org/cgi-bin/search.pi via searching for sint or cost.

7. Consider the incompressible viscous flow:

$$\begin{aligned}
&u_t + u\, u_x + v\, u_y + p_x = \nu\ \triangle u, \\
&v_t + u\, v_x + v\, v_y + p_y = \nu\ \triangle v, \\
&u_x + v_y = 0,
\end{aligned}$$

a). Write down the first-order (backward Euler) projection method.
b). Write down fully discretization with central difference for spatial discretization.
c). Develop a code to implement your discretization with a fast Poisson solver.
d). Use your code to simulate the time-dependent flow with $\nu = 10^{-2}$. Display the velocity field at different times.
e). Use your code to find steady state solution for $\nu = 1,\ 10^{-1},\ 10^{-2},\ 10^{-3}$ and 10^{-4}. Draw the velocity fields of the steady state solutions.

8. Consider the incompressible viscous flow:

$$\begin{aligned}
&\omega_t + u\, \omega_x + v\, \omega_y = \nu\ \triangle \omega, \\
&\triangle \psi = -\omega, \\
&u = \psi_y, \qquad v = -\psi_x,
\end{aligned}$$

a). Develop a code to implement a numerical method for this problem.
b). Apply your code to simulate the time dependent flow for $\nu = 10^{-2}$.
c). Apply your code to find the steady state solution for $\nu = 10^{-1}$, 10^{-2} and 10^{-3}.