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a Bose gas**

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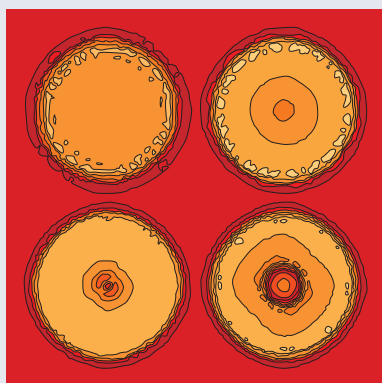
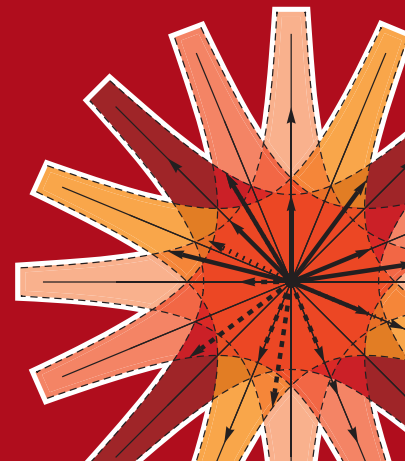
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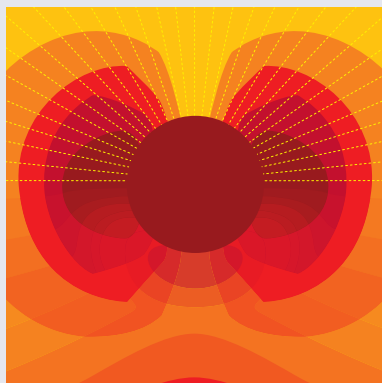
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Image: Ornamental multiplication of space-time figures of temperature transformation rules (adapted from T. S. Bíró and P. Ván 2010 *EPL* **89** 30001; artistic impression by Frédérique Swist).

Breathing oscillations of a trapped impurity in a Bose gas

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Abstract – Motivated by a recent experiment (CATANI J. *et al.*, *Phys. Rev. A*, **85** (2012) 023623) we study breathing oscillations in the width of a harmonically trapped impurity interacting with a separately trapped Bose gas. We provide an intuitive physical picture of such dynamics at zero temperature, using a time-dependent variational approach. The amplitudes of breathing oscillations are suppressed by self-trapping, due to interactions with the Bose gas. Further, exciting phonons in the Bose gas leads to damped oscillations and non-Markovian dynamics of the width of the impurity, the degree of which can be engineered through controllable parameters. Our results, supported by simulations, reproduce the main features of the dynamics observed by Catani *et al.* despite the temperature of that experiment. Moreover, we predict novel effects at lower temperatures due to self-trapping and the inhomogeneity of the trapped Bose gas.

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The ability to trap and cool atoms of different species to ultra-low temperatures has led to the realisation of various theoretical models in which the intriguing physics of binary mixtures of spin hyperfine states or different elements can be studied [1–5]. In particular, highly imbalanced mixtures have made it possible to investigate the dynamics, interactions and decoherence of single atoms, generally referred to as impurities, immersed in a background atomic gas [6–11]. As a prominent example, signatures of polaron effects, caused by impurity-induced density fluctuations of the background gas, have been investigated theoretically [12,13] and observed in experiments [14,15].

More recently, Catani *et al.* created a harmonically trapped impurity suspended in a separately trapped Bose gas [16]. They studied the dynamics of the system following a sudden lowering of the trap frequency of the impurity. Primarily, they observed breathing oscillations of the width σ of the impurity density distribution for various impurity-Bose gas interaction strengths. Several features of the experiment were amenable to interpretation in terms of a quantum Langevin equation in conjunction with a polaronic mass shift; however, we show that at even lower

temperatures a different model is required to fully describe the dynamics of the system.

In this letter, we develop a versatile analytical model to describe the breathing oscillations of an impurity in a Bose gas at zero temperature. At first, our analysis is based on a variational approach in the Gross-Pitaevskii (GP) regime. We show that the impurity density distribution, which we describe by a Gaussian, has a width obeying a Newtonian equation of motion for a fictitious particle with position σ . The potential governing this motion accounts for the quantum pressure of the impurity, the inhomogeneity of the trapped Bose gas and the localised distortion of this background gas induced by the impurity. The latter leads to a strong confinement of the impurity, known as self-trapping [17–20]. Subsequently, we extend our model by including excitations of the Bose gas in the form of Bogoliubov phonons. A variational ansatz, which describes the bosons as a product of coherent phonon states, allows us to track the evolution of the system, including the exchange of energy between the impurity and the phonon bath. The dynamics turns out to be non-Markovian because of the back-action of phonons created by the impurity, and we show that the timescale of memory effects can be varied by adjusting the trapping parameters, allowing for a comprehensive study

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of the transition between Markovian and non-Markovian dynamics.

Our results reproduce the main features observed in the experiment of Catani *et al.* despite neglecting finite temperature effects, but also demonstrate that cooling the system further should introduce novel effects. We obtain an amplitude reduction of the breathing oscillation for large impurity-Bose gas interactions. In particular, we observe a sudden quench of the breathing amplitude once the impurity-Bose gas coupling substantially exceeds the boson-boson interaction strength. We find that the damping of breathing oscillations, due to the net dissipation of energy into the Bose gas, is highly non-Markovian, contrary to a treatment of the system at higher temperatures [16]. Moreover, our model explains the different behaviour of attractive and repulsive impurities as a consequence of the inhomogeneity of the trapped Bose gas.

Model. – The specific system we consider consists of a single impurity of mass m_a in a harmonic trap with frequency Ω_a , and identical bosons of mass m_b , trapped by a separate external potential $v_b(\mathbf{r})$, whose densities weakly interact with strength g . The impurity interacts with the Bose gas also via a density-density interaction of strength ηg , where the dimensionless parameter η controls the relative strengths of interactions. As such, the many-body Hamiltonian for the combined system is given by $\hat{H} = \hat{H}_a + \hat{H}_b + \hat{H}_{ab}$, where

$$\hat{H}_a = \int d\mathbf{r} \hat{\chi}^\dagger \left(-\frac{\hbar^2 \nabla^2}{2m_a} + \frac{m_a}{2} \Omega_a^2 r^2 \right) \hat{\chi}, \quad (1a)$$

$$\hat{H}_b = \int d\mathbf{r} \hat{\phi}^\dagger \left(-\frac{\hbar^2 \nabla^2}{2m_b} + v_b + \frac{g}{2} \hat{\phi}^\dagger \hat{\phi} \right) \hat{\phi}, \quad (1b)$$

$$\hat{H}_{ab} = \eta g \int d\mathbf{r} \hat{\chi}^\dagger \hat{\chi} \hat{\phi}^\dagger \hat{\phi}, \quad (1c)$$

describe the impurity, bosons and impurity-Bose gas coupling, respectively. Here, $\hat{\chi}(\mathbf{r}, t)$ and $\hat{\phi}(\mathbf{r}, t)$ are the impurity and boson field operators. The single impurity has wave function $\chi(\mathbf{r}, t)$.

Breathing oscillations in the GP regime. – In the GP approach, we replace $\hat{\phi}(\mathbf{r}, t)$ by the condensate mode of the Bose gas $\varphi(\mathbf{r}, t)$. For low temperatures and in a dilute regime it follows from eqs. (1) that the impurity and condensate wave functions evolve according to

$$i\hbar \frac{\partial \chi}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m_a} + \frac{m_a}{2} \Omega_a^2 r^2 + \eta g |\varphi|^2 \right) \chi, \quad (2a)$$

$$i\hbar \frac{\partial \varphi}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m_b} + v_b + \eta g |\chi|^2 + g |\varphi|^2 \right) \varphi, \quad (2b)$$

respectively, where φ is normalised to the number of particles N in the Bose gas [21].

We consider the condensate in the Thomas-Fermi regime, in which the kinetic energy of the condensate, represented by the first term on the right-hand side of eq. (2b), is negligible. If the impurity and Bose gas are

decoupled the condensate evolves as $\varphi_0(\mathbf{r}, t) = \sqrt{n_0(\mathbf{r})} e^{-i\mu_b t/\hbar}$, where the decoupled condensate density is given by $n_0(\mathbf{r}) = [\mu_b - v_b(\mathbf{r})]/g$ for $\mu_b > v_b(\mathbf{r})$ and zero otherwise, and μ_b is the chemical potential. Returning to the case of non-zero impurity-boson coupling, we assume a similar form for the time-evolution of the condensate $\varphi(\mathbf{r}, t) = \sqrt{n(\mathbf{r})} e^{-i\mu_b t/\hbar}$. It then follows from eq. (2b) that, for attractive (repulsive) interactions, the condensate density $n(\mathbf{r}) = n_0(\mathbf{r}) - \eta |\chi(\mathbf{r})|^2$ is enhanced (suppressed) from its decoupled value at the location of the impurity. The self-consistency of our assumption regarding the time evolution of the condensate requires $n_0(\mathbf{r}) \gg \eta |\chi(\mathbf{r})|^2$, *i.e.*, sufficiently weak impurity-boson coupling.

As a consequence of these approximations, eq. (2a) becomes the self-focussing non-linear Schrödinger equation

$$i\hbar \frac{\partial \chi}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m_a} + \eta g n_0 - \eta^2 g |\chi|^2 + \frac{m_a}{2} \Omega_a^2 r^2 \right) \chi. \quad (3)$$

We now specialise to the important case of a spherically symmetric system in d dimensions with the Bose gas trapped in the harmonic potential $v_b(r) = \frac{1}{2} m_b \Omega_b^2 r^2$ of frequency Ω_b . The chemical potential in this case is

$$\mu_b = \hbar \Omega_b \left[\frac{(2+d)\Gamma(1+d/2)}{2(2\pi)^{d/2}} \frac{Ng}{\hbar \Omega_b w_b^d} \right]^{2/(2+d)},$$

where $w_b = \sqrt{\hbar/m_b \Omega_b}$ is the width of the single particle ground state and $\Gamma(p) = \int_0^\infty dx e^{-x} x^{p-1}$ is the Gamma function.

We solve eq. (3) within the variational ansatz

$$\chi(r, \sigma, \gamma) = (\pi \sigma^2)^{-d/4} \exp(-r^2/2\sigma^2 - i\gamma r^2), \quad (4)$$

with time-dependent width σ and phase γ . This ansatz was shown to describe a Bose gas in a harmonic trap accurately (see [22] and references therein). We find equations of motion for the Gaussian parameters by extremising the action $S = \int dt d\mathbf{r} \mathcal{L}$ with respect to them. The Lagrangian density is

$$\mathcal{L} = \chi^* \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m_a} - \eta g n_0 + \frac{\eta^2}{2} g |\chi|^2 - \frac{m_a}{2} \Omega_a^2 r^2 \right] \chi.$$

As a result, $\gamma = -m_a \dot{\sigma}/2\hbar\sigma$ and the width of the impurity σ obeys the equation of motion

$$m_a \ddot{\sigma} = -\frac{\partial V(\sigma)}{\partial \sigma}, \quad (5)$$

with the potential $V = V_0 + V_{\text{st}} + V_{\text{inh}}$, where

$$V_0(\sigma) = \frac{\hbar^2}{2m_a \sigma^2} + \frac{m_a}{2} \Omega_a^2 \sigma^2, \quad (6a)$$

$$V_{\text{st}}(\sigma) = -\frac{\eta^2 g}{d(2\pi)^{d/2} \sigma^d}, \quad (6b)$$

$$V_{\text{inh}}(\sigma) = \eta \mu_b \left[\frac{2}{d} \hat{\Gamma} \left(\frac{d}{2}, \frac{R^2}{\sigma^2} \right) - \frac{\sigma^2}{R^2} \hat{\Gamma} \left(1 + \frac{d}{2}, \frac{R^2}{\sigma^2} \right) \right]. \quad (6c)$$

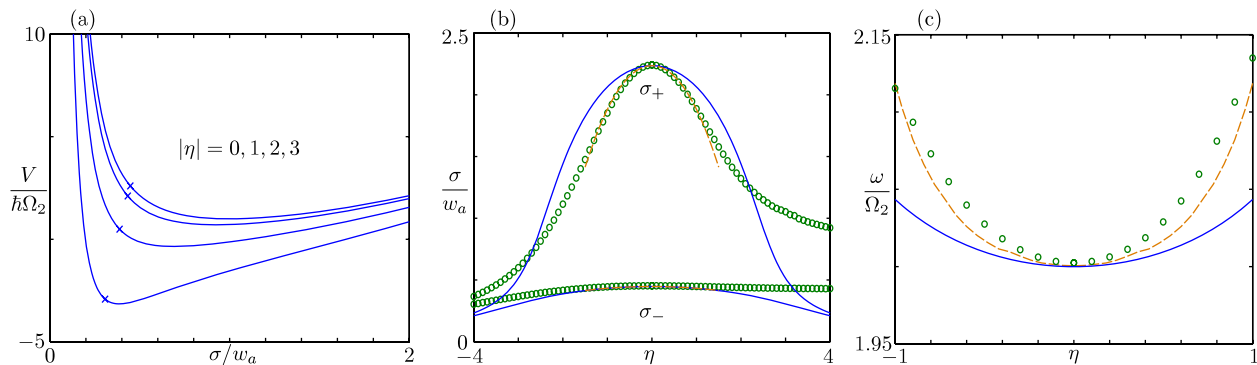


Fig. 1: (Colour on-line) One-dimensional homogeneous Bose gas. (a) The potential V governing the evolution of the width σ of an impurity trapped with frequency Ω_2 for various interactions strengths η . The crosses mark the width σ at the minimum of V for tight trapping with frequency $\Omega_1 = 5\Omega_2$. (b) The maximum σ_+ and minimum σ_- widths during breathing oscillations as a function of η . The analytical results of the GP approach predict that self-trapping strongly suppresses breathing oscillations (full blue lines). Additional suppression occurs in the Bogoliubov approach due to dissipation into phonon modes (dashed orange lines), matching the numerical results (green dots). (c) The frequency of the breathing oscillations ω as a function of η . The parameters are set to $m_b = 2m_a$, $n_0 = \sqrt{m_b\Omega_2/\hbar}$ and $g = \hbar\Omega_2\sqrt{\hbar/m_b\Omega_2}$.

Here, $R = \sqrt{2\mu_b/m_b\Omega_b^2}$ is the Thomas-Fermi radius of the condensate and $\tilde{\Gamma}(p, z) = [\Gamma(p)]^{-1} \int_0^z dx e^{-x} x^{p-1}$ is the normalised lower incomplete Gamma function.

The two terms in V_0 describe the quantum pressure and the harmonic trapping of the impurity. The contribution V_{st} is second order in η and represents self-trapping effects, which result from the deformation of the condensate due to the interaction with the impurity. The inhomogeneity in the condensate density due to its trapping gives rise to the potential V_{inh} that is first order in η . It has a simple form: The expression in the square brackets of eq. (6c) is a monotonically decreasing function in σ/R , taking the value $2/d$ at $\sigma/R = 0$ and asymptotically approaching zero as $\sigma/R \rightarrow \infty$. Its gradient is largest for widths $\sigma \sim R$.

We see that eq. (5), governing the evolution of the width σ , has just the form of Newton's second law for the motion of a fictitious particle with position σ and mass m_a in the potential $V(\sigma)$. This analogy allows us to understand the behaviour of the width σ as the impurity undergoes a procedure similar to that in the experiment by Catani *et al.* [16]. In this procedure the impurity initially feels a tight trap with frequency Ω_1 and the coupled impurity-Bose gas system is cooled to the ground state (although in the experiment such low temperatures were not achieved). At time $t = 0$ the trap frequency is reduced to Ω_2 and the subsequent evolution of the width for $t \geq 0$ is observed. In our model this corresponds to the width of the impurity sitting at the minimum of V with frequency $\Omega_a = \Omega_1$ for times $t < 0$. Then for $t \geq 0$ the width moves in the potential V with lower trapping frequency $\Omega_a = \Omega_2$.

Our approach, with $V = V_0$, exactly captures the correct evolution of the impurity wave function for zero impurity-boson coupling. The square of the width oscillates harmonically with frequency $2\Omega_2$ according to

$$\sigma^2(t) = \sigma_+^2 \sin^2(\Omega_2 t) + \sigma_-^2 \cos^2(\Omega_2 t),$$

where $\sigma_{\pm}^2 = w_a^2(\Omega_1/\Omega_2)^{\pm 1}$ and $w_a = \sqrt{\hbar/m_a\Omega_2}$ is the width of the harmonic-oscillator ground state. Note that this harmonic solution exists for σ^2 despite its non-linear equation of motion.

Homogeneous condensate. For non-zero impurity-boson coupling, we first consider the special case of a homogeneous condensate whose decoupled density n_0 is uniform, corresponding to the regime of a flat trap $R \gg \sigma$. In this case the potential V_{inh} is also uniform and can be neglected, leaving the potential V symmetric with respect to attractive and repulsive interactions.

The self-trapping V_{st} imparts a force on the width of the impurity towards smaller values of σ , shown in fig. 1(a), and this has several effects. First, the minimum of the potential V is shifted to smaller widths. Second, it shortens the distance between the minimum of V for $t < 0$ and its minimum for $t \geq 0$. Third, the curvature of potential V near its minimum is increased. These effects lead to an increased localisation of the impurity in the ground state at times $t < 0$. Further, at times $t \geq 0$ the amplitudes of the breathing oscillations are reduced, along with an increase in the frequency of the oscillations. This is shown in figs. 1(b) and (c), where we have plotted the maximum and minimum widths, and the frequency of the breathing oscillations, respectively. The frequencies ω are found from the best fitting curve of the form $\sigma^2(t)/w_a^2 = A \cos(\omega t) + \sigma_0^2$ to the oscillations, where A and σ_0 are fitting parameters.

Condensate in a harmonic trap. In the general case the decoupled condensate density n_0 depends on the radial position r and the potential V is augmented by V_{inh} . The total potential V resulting from a trapped condensate is plotted in fig. 2(a). Since the contribution of V_{inh} is first order in η the symmetry between attractive and repulsive impurity-boson coupling is broken.

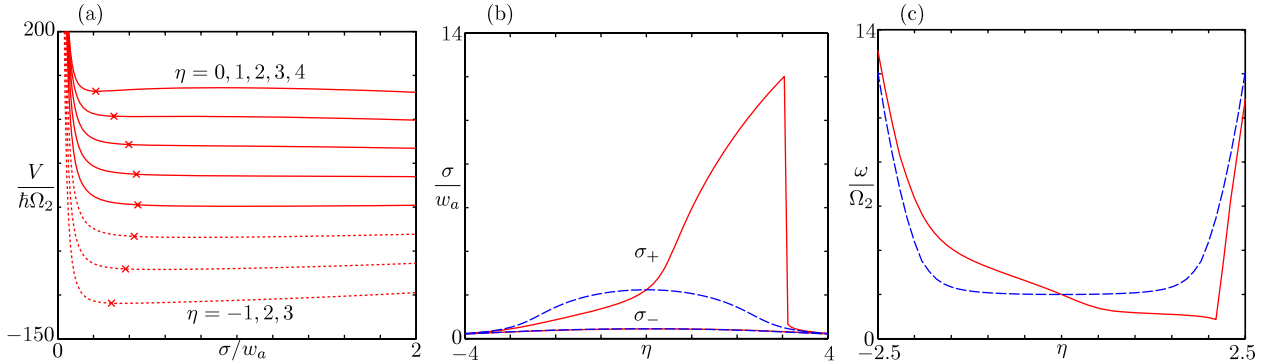


Fig. 2: (Colour on-line) One-dimensional Bose gas in a harmonic trap. (a) The potential V governing the evolution of the width σ of an impurity trapped with frequency Ω_2 for positive (full lines) and negative (dotted lines) interactions strengths η . The crosses mark the width σ at the minimum of V for tight trapping with frequency $\Omega_1 = 5\Omega_2$. (b) The maximum σ_+ and minimum σ_- widths during breathing oscillations as a function of η for a trapped (full red lines) and homogeneous (dashed blue lines) Bose gas. For a trapped Bose gas, breathing oscillations increase in amplitude with η and at a critical value, here $\eta \approx 3.1$, undergo a sudden quench. (c) The frequency of the breathing oscillations ω as a function of η . The parameters are set to $m_b = 2m_a$, $N = 200$, $\Omega_b = \Omega_2/\sqrt{2}$, and $g = \hbar\Omega_2\sqrt{\hbar/m_b\Omega_2}$.

For repulsive coupling, the force imparted by V_{inh} acts to increase the width of the impurity, opposing self-trapping. Specifically, in the regime $\eta \lesssim 1$ the inhomogeneity of the Bose gas flattens the potential V near its minimum. In contrast, for sufficiently large η the self-trapping term dominates and the potential develops a narrow trough at a small width, as for the homogeneous condensate. If interactions are attractive, V_{inh} represents a force on the width of the impurity towards smaller values of σ and enhances the self-trapping seen for the homogeneous condensate.

The effects of V_{inh} on the dynamics of the impurity following a sudden decrease in Ω_a are captured in figs. 2(b) and (c). For weak repulsive interactions $\eta \lesssim 1$, the effect of the flattening of the potential is to increase the amplitude of oscillations and decrease the frequency. For strong repulsive interactions $\eta \gg 1$ the creation of a second narrow minimum in the potential V results in small-amplitude and high-frequency oscillations. Our results predict a sudden transition between these two regimes, causing a sharp drop in the amplitude of oscillations and an increase in frequency. For attractive impurities, the effect of V_{inh} is to enhance the effects seen for a homogeneous condensate.

Damping and memory effects. – Up to now we have not included a mechanism for the transfer of energy between the impurity and the Bose gas. We now show, for a homogeneous Bose gas, that the oscillations of the impurity create Bogoliubov phonons and thereby lead to the exchange of energy with the Bose gas. Working in a volume \mathcal{V} with periodic boundaries, we expand $\hat{\phi} = \varphi_0 + \delta\hat{\phi}$. The decoupled mode φ_0 only describes the Bose gas in the ground state if $\eta = 0$; therefore, in this picture, interactions with the impurity generate excitations even in the ground state of the system. Terms in \hat{H} higher than second order in either the deformation $\delta\hat{\phi}$ or the coupling parameter η are neglected [12]. Then, expressing

the deformation $\delta\hat{\phi} = \sum_{\mathbf{q}} [u_{\mathbf{q}}(\mathbf{r})\hat{b}_{\mathbf{q}} + v_{\mathbf{q}}^*(\mathbf{r})\hat{b}_{\mathbf{q}}^\dagger]$ in terms of Bogoliubov modes $\hat{b}_{\mathbf{q}}$ with momenta $\hbar\mathbf{q}$, the Hamiltonian of the Bose gas reads [23]

$$\hat{H}_b = E_0 + \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}}\hat{b}_{\mathbf{q}}^\dagger\hat{b}_{\mathbf{q}}.$$

Here, E_0 is the energy of the mode φ_0 , $u_{\mathbf{q}}$ and $v_{\mathbf{q}}$ solve the Bogoliubov-de Gennes equations, $\hbar\omega_{\mathbf{q}} = \sqrt{\epsilon_{\mathbf{q}}(\epsilon_{\mathbf{q}} + 2gn_0)}$ are the phonon energies, and $\epsilon_{\mathbf{q}} = \hbar^2q^2/2m_b$ are the free particle energies [21]. Under the same approximations the interaction Hamiltonian simplifies to

$$\hat{H}_{ab} = \eta gn_0 + \eta g \sum_{\mathbf{q} \neq 0} (\hat{b}_{\mathbf{q}}^\dagger + \hat{b}_{\mathbf{q}}) f_{\mathbf{q}}, \quad (7)$$

with the coefficients

$$f_{\mathbf{q}} = \sqrt{\frac{n_0\epsilon_{\mathbf{q}}}{\mathcal{V}\hbar\omega_{\mathbf{q}}}} \int d\mathbf{r} |\chi(\mathbf{r}, t)|^2 e^{i\mathbf{q}\cdot\mathbf{r}}.$$

The first and the second terms in eq. (7) represent the interaction of the impurity with the condensate mode φ_0 and the Bogoliubov phonons $\hat{b}_{\mathbf{q}}$, respectively. Moreover, the latter describes the creation of Bogoliubov phonons due to a classical driving force $f_{\mathbf{q}}$. In our case, $f_{\mathbf{q}}$ depends parametrically on the oscillating width σ and thus the driving force is approximately periodic.

We solve for both the ground state and the evolution of the total system variationally within the ansatz $|\Psi\rangle = |\sigma, \gamma\rangle \otimes \{|\alpha_{\mathbf{q}}\rangle\}$. For the impurity, $|\sigma, \gamma\rangle$ corresponds to the Gaussian ansatz in eq. (4). The Bose gas is restricted to a product of coherent states $\{|\alpha_{\mathbf{q}}\rangle\} = \bigotimes_{\mathbf{q}} e^{-|\alpha_{\mathbf{q}}|^2/2} e^{\alpha_{\mathbf{q}}\hat{b}_{\mathbf{q}}^\dagger}|0\rangle$. This is known to describe bosonic modes coupled to a classical field such as a density and in particular Bogoliubov modes coupled to an impurity [12].

We derive equations of motion for the total system by extremising the action $S = \int dt L$ with respect to the ansatz parameters, where $L = \langle\Psi|(i\hbar\partial_t - \hat{H})|\Psi\rangle$ is

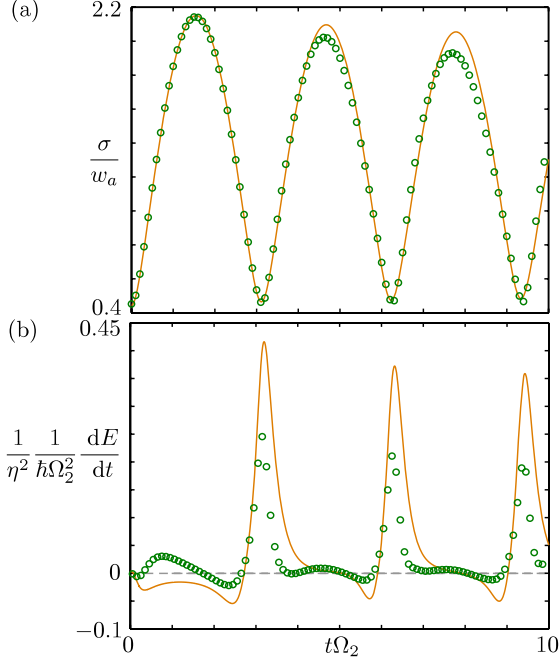


Fig. 3: (Colour on-line) Damping of oscillations due to loss of energy, for weak coupling. (a) Damped oscillations of the width σ of an impurity in a trap with frequency Ω_2 obtained analytically (full orange line) and numerically (green dots). The trap frequency for $t < 0$ is $\Omega_1 = 5\Omega_2$. (b) The rate of energy exchange \dot{E} between the impurity and the Bose gas. Positive (negative) values correspond to energy gained (lost) by the Bose gas. The parameters are set to $d=1$, $\eta=0.5$, $m_b=2m_a$, $n_0 = \sqrt{m_b\Omega_2/\hbar}$ and $g = \hbar\Omega_2\sqrt{\hbar/m_b\Omega_2}$. The dashed line marks $\dot{E} = 0$.

the Lagrangian. Working in the thermodynamic limit, where $\mathcal{V}^{-1}\sum_{\mathbf{q}} \rightarrow 1/(2\pi)^d \int d\mathbf{q}$, we evaluate the integrals analytically using the small-momentum approximation to the Bogoliubov dispersion relation $\omega_{\mathbf{q}} = cq$, with the speed of sound given by $c = \sqrt{gn_0/m_b}$. This approximation is accurate in the regime $q \ll m_b c/\hbar$. We found no qualitative differences when proceeding numerically without making this approximation.

Assuming the system is in the ground state at $t < 0$, applying the methods of the preceding paragraphs results in a Newtonian equation of motion describing the width of the impurity σ moving in the potential V_0 augmented by a time-dependent force F_{phon} . This force represents the interaction of the impurity with the phonon bath and, for $d=1$, is given by

$$F_{\text{phon}}(t) = -\frac{K\eta^2 g^2 n_0 \sigma(t)}{m_b c^2} \left\{ \frac{[\Sigma^2(t,0) - 2c^2 t^2] e^{-\frac{c^2 t^2}{\Sigma^2(t,0)}}}{\Sigma^5(t,0)} + 2c \int_0^t dt' \frac{c(t-t') [3\Sigma^2(t,t') - 2c^2(t-t')^2] e^{-\frac{c^2(t-t')^2}{\Sigma^2(t,t')}}}{\Sigma^7(t,t')} \right\}, \quad (8)$$

with compound width $\Sigma(t,t') = [\sigma^2(t) + \sigma^2(t')]^{1/2}$ and constant $K = 2\sqrt{\pi}/(4\pi)^{1/2}\Gamma(3/2)$. The first and second

terms in eq. (8) correspond to the interaction of the impurity with phonons present in the initial state and those created during the evolution, respectively. These interactions introduce non-Markovian effects with a memory that decays exponentially on a timescale σ/c . As a consequence, an experimentalist can, in principle, control the degree of non-Markovianity by altering g , n_0 and w_a .

Due to the form of F_{phon} , the equation of motion for the width of the impurity σ is an integro-differential equation, which we solve using an iterative procedure: Initially, we evaluate F_{phon} assuming a trajectory for σ and integrate the equations of motion. Subsequently, we use the resulting trajectory to evaluate F_{phon} in the next iteration. This procedure converges after a few iterations to a self-consistent evolution for σ . The evolution of the width of the impurity after a sudden decrease in Ω_a is plotted in fig. 3(a). We find that the breathing oscillations are damped out due to interactions with phonons. In figs. 1(b) and (c) we plot the maximum and minimum widths, and frequencies as functions of coupling strength η . Frequencies of oscillations ω are found from the best fitting curve $\sigma^2(t)/w_a^2 = Ae^{-\Lambda t}\cos(\omega t) + Ct + \sigma_0^2$ to our data, where A , B , C , Λ and σ_0 are fitting parameters. Dissipation before the first peak leads to a stronger suppression of maximum width than found when considering self-trapping only. We find that dissipation also results in a greater dependence of frequency on coupling.

To confirm our analytical approach we numerically solve the coupled time-dependent GP equations (2), with a description of the numerics left to the appendix. For small $|\eta|$, we find good agreement on the frequency of oscillations and the width at the first maximum, while the analytics underestimates the heights of later peaks, as exemplified in fig. 3(a). This leads to a close match of the numerically and analytically calculated maximum and minimum widths, and frequencies, as shown in figs. 1(b) and (c). Also in these two figures, the numerics show that the symmetry between attractive and repulsive interactions is broken for large coupling; an attractive impurity can infinitely enhance the condensate density at the origin, while an infinitely repulsive impurity only depletes the density of the Bose liquid at its location to zero [19].

We also calculate the energy deposited into the Bose gas $E(t) = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} (|\alpha_{\mathbf{q}}(t)|^2 - |\alpha_{\mathbf{q}}(0)|^2)$. The rate of energy lost by the impurity is

$$\frac{dE(t)}{dt} = \frac{K\eta^2 g^2 n_0}{m_b c} \left\{ -\frac{cte^{-c^2 t^2/\Sigma^2(t,0)}}{\Sigma^3(t,0)} + c \int_0^t dt' \frac{[\Sigma^2(t,t') - 2c^2(t-t')^2] e^{-c^2(t-t')^2/\Sigma^2(t,t')}}{\Sigma^5(t,t')} \right\},$$

which again is divided into two parts, representing interactions with phonons created at $t < 0$ and $t \geq 0$, respectively. In fig. 3(b) we plot this dissipation rate during the evolution shown in fig. 3(a). We find that most

dissipation occurs when the width of the impurity is near its minimum value and that \dot{E} can take negative values (the impurity can gain energy). It may, at first, appear at odds with the Landau criterion that dissipation is largest when the impurity density distribution is stationary. However, such arguments do not apply here, as in [24], since the impurity is not simply moving at a constant velocity. The form of \dot{E} also differs markedly from that predicted by modelling the impurity as a damped harmonic oscillator, as done in [16], whereby dissipation is due to a friction force. The qualitative features of our analytic results are confirmed by numerical simulations, as shown in fig. 3(b). Quantitative deviations arise from two approximations: We have neglected high-order phonon effects and the impurity density does not always have a Gaussian form.

Discussion and conclusion. – Our results reproduce the main features of the experiment by Catani *et al.* [16] even though we assumed zero temperature. For example, our model naturally explains the decoupled frequency of oscillations and ratio of maximum to minimum width, along with the basic aspects of damping and amplitude suppression arising from impurity-boson interactions. However, our zero-temperature calculations predict a much richer dependence of amplitude on coupling, including a sudden quench, resulting from the interplay of self-trapping and inhomogeneity induced by the Bose gas trapping. We also predict a dependence of oscillation frequency on the impurity-Bose gas coupling, indicating that the lack of a dependence observed by Catani *et al.* is a finite temperature effect. Our intuitive model explains differences between repulsive and attractive impurities, and highlights the way in which the non-Markovianity of the system arises and how it may be controlled.

In general, we expect the zero-temperature approximation to be valid if the thermal energy of the initial state $k_B T$ is small compared to the relevant energy scales of the system at $t < 0$. In the experiment by Catani *et al.*, prior to the equilibration the thermal energy of the Bose gas and the impurity were 350 nK and 50 nK, respectively. We find that the typical energy scales of our system, $\hbar\Omega_1$ and the initial energy of the Bose gas, can be set to 125 nK while retaining large oscillations $\Omega_1/\Omega_2 \sim 10$ and the impurity width on the scale of micrometers. However, mechanisms governing the evolution of the impurity at zero temperature, such as self-trapping, might still be relevant up to the temperatures realised in the experiment.

Our flexible analytical framework allows for several natural extensions, *e.g.*, impurities interacting directly and via the Bose gas, in the same trap or displaced. Future work will analyse many-impurity effects in such systems.

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Appendix on numerical simulations. – We solve the coupled time-dependent GP equations (2) for a homogeneous Bose gas numerically using time-splitting methods. We assume for numerical ease that the condensate is enclosed in an infinite spherical box with a radius \mathcal{R} that is large compared to the width σ and the healing length of the condensate. More precisely, we choose $\mathcal{R} = 40w_a$ and solve for $\varphi(r)$ and $\chi(r)$ on a spatial grid of 10^4 points using a time step $10^{-3}\Omega_2^{-1}$. The ground states are calculated using the normalised gradient flow [25–27] and the time-evolved states are calculated by applying a method combining the time-splitting technique [26,28], to decouple the non-linearity, with the second-order finite difference and Crank-Nicolson methods, to discretise the spatial and temporal derivatives, respectively [29]. To calculate the width of the impurity σ , we find the best fitting parameters to $|\chi(r)| = B \exp(-r^2/2\sigma^2)$. Maximum and minimum widths are calculated from this data.

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