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AN ARTIFICIAL BOUNDARY CONDITION FOR THE INCOMPRESSIBLE VISCOUS FLOWS IN A NO-SLIP CHANNEL*

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Abstract

The numerical simulation of the steady incompressible viscous flows in a no-slip channel is considered. A discrete artificial boundary condition on a given segmental artificial boundary is designed by the method of lines. Then the original problem is reduced to a boundary value problem of Navier-Stokes equations on a bounded domain. The numerical examples show that this artificial boundary condition is very effective and more accurate than Dirichlet and Neumann boundary conditions used in engineering literature.

1. Introduction

When computing the numerical solutions of viscous fluid flow problems in an unbounded domain, one often introduces artificial boundaries, and sets up an artificial bopundary condition on them; then the original problem is reduced to a problem on a bounded computational domain. In order to limit the computational cost, these boundaries must not be too far from the domain of interest. Therefore, the artificial boundary conditions must be good approximation to the "exact" boundary conditions (so that the solution of the problem in the bounded domain is equal to the solution in the original problem). Thus, the accuracy of the artificial boundary conditions and the computational cost are closely related. Designing artificial boundary conditions with high accuracy on a given artificial boundary has become an important and effective method for solving partial differential equations on an unbounded domain. In the last ten years, many authors have worked on this subject for various problems by different techniques. For details, refer to the works by Goldstein [1], Feng [2], Han [3,4,5], Hagstrom [6], Halpern [7] and Nataf [8] and the references there in.

The purpose of this paper is to design discrete artificial boundary conditions for the steady incompressible viscous flow in stream function vorticity formulation in the case when the domain is a no-slip channel. We use a direct method of lines [9] in the exterior domain and design discrete artificial boundary conditions at the segmental artificial

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boundaries. Then the original problem is reduced to a bounded computational domain. Furthermore, the numerical examples show that the artificial boundary condition given in this paper is very effective and more accurate than the Dirichlet and Neumann boundary conditions used in engineering literature.

2. Navier-Stokes Equations and Their Linearization

In this paper, we consider the numerical simulation of a steady incompressible viscous flow arounding a body (domain Ω_i) in a no-slipping channel defined by $\Re\times[0,L]$. Let u, v denote the components of the velocity in the x and y coordinate directions, and p denote the pressure. Then u, v and p satisfy the following Navier-Stokes (N-S) equations in domain $\Omega=\Re\times(0,L)\setminus\bar{\Omega}_i$:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}=\nu\triangle u, \tag{2.1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}=\nu\triangle v, \tag{2.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.3}$$

and boundary conditions

$$u|_{y=0,L}=v|_{y=0,L}=0,\quad -\infty<x<+\infty, \tag{2.4}$$

$$u|_{\partial\Omega_i}=v|_{\partial\Omega_i}=0 \tag{2.5}$$

$$u(x,y)\to u_\infty(y)=\alpha y(L-y),\quad v(x,y)\to v_\infty=0,\quad \text{when } x\to\pm\infty, \tag{2.6}$$

where $\nu>0$ is the kinematic viscosity, and $\alpha>0$ is a constant.

Introduce the stream function ψ and vorticity ω. Then

$$\frac{\partial\psi}{\partial y}=u,\quad \frac{\partial\psi}{\partial x}=-v, \tag{2.7}$$

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{2.8}$$

Then the problem (2.1)–(2.6) is reduced to the following problem:

$$\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}-\nu\triangle\omega=0,\quad \text{in } \Omega, \tag{2.9}$$

$$\triangle\psi+\omega=0,\quad \text{in } \Omega, \tag{2.10}$$

$$\psi|_{y=0}=0,\quad \psi|_{y=L}=\psi_L\equiv\int_0^L u_\infty(s)ds,\quad -\infty<x<+\infty, \tag{2.11}$$

$$\frac{\partial\psi}{\partial y}|_{y=0,L}=0,\quad -\infty<x<\infty, \tag{2.12}$$

$$\psi|_{\partial\Omega_i} = \text{constant}, \quad \frac{\partial\psi}{\partial n}|_{\partial\Omega_i} = 0, \tag{2.13}$$

$$\psi \to \psi_\infty(y) \equiv \int_0^y u_\infty(s)ds, \quad \omega \to \omega_\infty(y) \equiv -u'_\infty(y), \quad \text{when } x \to \pm\infty, \tag{2.14}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative.

Take two constants $b < d$, such that $\bar{\Omega}_i \subset (b,d) \times (0,L)$. Then Ω is divided into three parts: Ω_b, Ω_T and Ω_d by the artificial boundaries Γ_b and Γ_d with

$$\begin{aligned}
&\Gamma_b = \{x = b, \ \ 0 \le y \le L\},\\
&\Gamma_d = \{x = d, \ \ 0 \le y \le L\},\\
&\Omega_b = \{(x,y), \ | -\infty < x < b, \ \ 0 < y < L\},\\
&\Omega_T = \{(x,y) \ \ | \, b < x < d, \ \ 0 < y < L\} \setminus \bar{\Omega}_i,\\
&\Omega_d = \{(x,y) \ \ | \, d < x < +\infty, \ \ 0 < y < L\}.
\end{aligned}$$

When $|b|$, d are sufficiently large, in the domain $\Omega_b \cup \Omega_d$ the flow is almost a Poiseulle flow. So the nonlinear N-S equations (2.9)–(2.10) can be linearized, namely, in domain Ω_d (and Ω_b) the solution ω and ψ of the problem (2.9)–(2.14) approximately satisfy the following problem:

$$\triangle\,\omega - u_\infty(y) Re \frac{\partial\omega}{\partial x} = 0, \quad \text{in } \ \Omega_d, \tag{2.15}$$

$$\triangle\,\psi + \omega = 0, \quad \text{in } \Omega_d, \tag{2.16}$$

$$\psi|_{y=0} = 0, \ \ \psi|_{y=L} = \psi_L, \ \ d \le x < +\infty, \tag{2.17}$$

$$\frac{\partial\psi}{\partial y}|_{y=0,L} = 0, \quad d \le x < +\infty, \tag{2.18}$$

$$\psi(x,y) \to \psi_\infty(y), \ \ \omega(x,y) \to \omega_\infty(y), \quad \text{when } x \to +\infty \tag{2.19}$$

where Re$=\frac{1}{\nu}$. Let

$$\tilde{\omega}(x,y) = \omega(x,y) - \omega_\infty(y), \quad \tilde{\psi}(x,y) = \psi(x,y) - \psi_\infty(y). \tag{2.20}$$

Since $\psi_\infty(y)$ is a polynomial of degree three, $\omega_\infty(y)$ is a polynomial of degree one and $\psi''_\infty(y) + \omega_\infty(y) = 0$, it is straightforward to check that $\tilde{\omega}$ and $\tilde{\psi}$ satisfy the equations (2.15)–(2.16) and the following boundary conditions:

$$\tilde{\psi}|_{y=0,L} = \frac{\partial\tilde{\psi}}{\partial y}|_{y=0,L} = 0, \quad d \le x < +\infty, \tag{2.21}$$

$$\tilde{\psi}(x,y) \to 0, \ \ \tilde{\omega}(x,y) \to 0, \ \ \text{when } x \to +\infty. \tag{2.22}$$

Since the boundary condition on the artificial boundary Γ_d is unknown, equations (2.15)–(2.16) with boundary conditions (2.21)–(2.22) are an incompletely posed problem, which can not be solved. Let

$$\tilde{\psi}|_{x=d} = \tilde{\psi}_d(y), \ \ \tilde{\omega}|_{x=d} = \tilde{\omega}_d(y), \ \ 0 \le y \le L. \tag{2.23}$$

For given functions $\tilde{\psi}_d(y)$ and $\tilde{\omega}_d(y)$ with $\tilde{\psi}_d(0) = \tilde{\psi}_d(L) = 0$ and $\frac{d\tilde{\psi}_d(y)}{dy}|_{y=0,L} = 0$, we discuss the numerical solution of equations (2.15)–(2.16) with boundary conditions (2.21)–(2.23), and design a discrete artificial boundary condition on the segment Γ_d for the problem (2.9)–(2.14).

3. An Artificial Boundary Condition

We now consider the semidiscretization approximations of the problem (2.15)–(2.16), (2.21)–(2.23). Let $\Delta y = \frac{L}{N}$ be the mesh size, where N is a positive integer. The domain Ω_d is divided into N strips, i.e. $\bar{\Omega}_d = \cup_{k=1}^{N} \bar{\Omega}_k$, where

$$\Omega_k = \{(x, y) \mid \ d < x < +\infty, \ \ (k-1)\Delta y = y_{k-1} < y < y_k = k\Delta y\}.$$

The following semidiscretization scheme is used to solve equations (2.15)–(2.16) with boundary conditions (2.21)–(2.23):

$$\frac{d^2\tilde{\omega}_k(x)}{dx^2} + \frac{\tilde{\omega}_{k+1}(x) - 2\tilde{\omega}_k(x) + \tilde{\omega}_{k-1}(x)}{\Delta y^2} - u_\infty(y_k) Re \frac{d\tilde{\omega}(x)}{dx} = 0, \tag{3.1}$$

$$\frac{d^2\tilde{\psi}_k(x)}{dx^2} + \frac{\tilde{\psi}_{k+1}(x) - 2\tilde{\psi}_k(x) + \tilde{\psi}_{k-1}(x)}{\Delta y^2} + \tilde{\omega}_k(x) = 0, \quad 1 \le k \le N-1, \tag{3.2}$$

with boundary conditions

$$\tilde{\psi}_0(x) = \tilde{\psi}_N(x) = 0, \ \ d \le x < +\infty, \tag{3.3}$$

$$\tilde{\omega}_0(x) = -\frac{1}{2}\tilde{\omega}_1(x) - \frac{3\tilde{\psi}_1(x)}{\Delta y^2}, \ \ \tilde{\omega}_N(x) = -\frac{1}{2}\tilde{\omega}_{N-1}(x) - \frac{3\tilde{\psi}_{N-1}(x)}{\Delta y^2}, \ \ d \le x < +\infty, \tag{3.4}$$

$$\tilde{\psi}_k(d) = \tilde{\psi}_d(y_k), \ \ \tilde{\omega}_k(d) = \tilde{\omega}_d(y_k), \ \ 1 \le k \le N-1, \tag{3.5}$$

$$\lim_{x\to+\infty} \tilde{\psi}_k(x) = \lim_{x\to+\infty} \tilde{\omega}_k(x) = 0, \ \ 1 \le k \le N-1. \tag{3.6}$$

Let

$$X_0 = [\tilde{\omega}_d(y_1), \ldots, \tilde{\omega}_d(y_{N-1}), \tilde{\psi}_d(y_1), \ldots, \tilde{\psi}_d(y_{N-1})]^T,$$

$$X(x) = [\tilde{\omega}_1(x), \ldots, \tilde{\omega}_{N-1}(x), \tilde{\psi}_1(x), \ldots, \tilde{\psi}_{N-1}(x)]^T.$$

Therefore, the problem (3.1)–(3.6) is equivalent to the following ordinary differential system with constant coefficients:

Find X such that

$$\ddot{X}(x) + A_0\dot{X}(x) + B_0X(x) = 0, \tag{3.7}$$

$$X(d) = X_0, \quad \lim_{x\to+\infty} X(x) = 0 \tag{3.8}$$

where B_0 and A_0 are $2(N-1)\times 2(N-1)$ matrices and

$$B_0 = \begin{pmatrix} -\frac{5\beta}{2} & \beta & 0 & \dots & 0 & -3\beta^2 & 0 & 0 & \dots & 0 \\ \beta & -2\beta & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \beta & -2\beta & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta & -\frac{5\beta}{2} & 0 & 0 & \dots & 0 & -3\beta^2 \\ 1 & 0 & 0 & \dots & 0 & -2\beta & \beta & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & \beta & -2\beta & \beta & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \beta & -2\beta & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & \beta & -2\beta \end{pmatrix},$$

$$A_0 = -Re\begin{pmatrix} C_0 & 0 \\ 0 & 0 \end{pmatrix},$$

with

$$C_0 = \begin{pmatrix} u_\infty(y_1) & 0 & \dots & 0 \\ 0 & u_\infty(y_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_\infty(y_{N-1}) \end{pmatrix},$$

and

$$\beta = \frac{1}{\Delta y^2}.$$

Let

$$Y(x) = \dot{X}(x), \quad W(x) = \begin{pmatrix} X(x) \\ Y(x) \end{pmatrix}.$$

Then we have

$$\dot{Y}(x) + A_0 Y(x) + B_0 X(x) = 0.$$

Therrefore the problem (3.7)–(3.8) is equivalent to the following boundary value problem of a system of ordinary differential equations:

$$\text{Find} \quad W \quad \text{such that}$$

$$\dot{W}(x) = D_0 W(x), \tag{3.9}$$

$$X(d) = X_0, \quad \lim_{x\to+\infty} X(x) = 0 \tag{3.10}$$

where

$$D_0 = \begin{pmatrix} 0 & I_{2N-2} \\ -B_0 & -A_0 \end{pmatrix},$$

and I_{2n-2} is a $(2N-2)\times(2N-2)$ unit matrix.

We now solve the problem (3.9)–(3.10) by a direct method. Suppose $e^{\lambda(x-d)}\varsigma$ is a nonzero solution of the system of ordinary differential equations in (3.9). Then we know that the constant λ and nonzero vector ς are a solution of the following standard eigenvalue problem:

$$(\lambda I_{4N-4} - D_0)\varsigma = 0. \tag{3.11}$$

From the boundary condition $\lim_{x\to+\infty} X(x) = 0$, the real part of λ must be negative. So to solve the problem (3.9)–(3.10) we need only to calculate the eigenvalues with negative relal parts and the corresponding eigenvectors of the eigenvalue problem (3.11). Fortunately, in our numerical examples, the computation shows that the problem (3.11) has just $2N-2$ eigenvalues with negative real parts, which is what we expect. Therefore, we assume λ_i $(1 \le i \le 2N-2)$ are the eigenvalues with negative real parts of the problem (3.11) and ς_i $(1 \le i \le 2N-2)$ are the corresponding eigenvectors. Particularly, we assume λ_i $(1 \le i \le 2r)$ are real eigenvalues, λ_i $(2r+1 \le i \le 2N-2)$ are complex eigenvalues with nonzero imaginary parts and $\lambda_{2l} = \bar{\lambda}_{2l-1}$ $(r+1 \le l \le N-1)$.

Thus

$$W(x) = \sum_{j=1}^{2r} a_j e^{\lambda_j(x-d)}\varsigma_j + \sum_{j=r+1}^{N-1} [\frac{a_{2j-1}}{2}(e^{\lambda_{2j}(x-d)}\varsigma_{2j} + e^{\bar{\lambda}_{2j}(x-d)}\bar{\varsigma}_{2j})$$
$$+ \frac{a_{2j}}{2i}(e^{\lambda_{2j}(x-d)}\varsigma_{2j} - e^{\bar{\lambda}_{2j}(x-d)}\bar{\varsigma}_{2j}) \tag{3.12}$$

satisfies the system of ordinary differential equation (3.7) and the boundary condition $\lim_{x\to+\infty} X(x) = 0$ for any constants $a_1, a_2, \dots, a_{2N-2}$.

Letting $x = d$ in the equality (3.12), we have

$$W(d) = \sum_{j=1}^{2r} a_j\varsigma_j + \sum_{j=r+1}^{N-1} [a_{2j-1}\mathrm{Re}\varsigma_{2j} + a_{2j}\mathrm{Im}\varsigma_{2j}], \tag{3.13}$$

where $\mathrm{Re}\varsigma$ and $\mathrm{Im}\varsigma$ denote the real and imaginary part of ς. Introduce matrices

$$D = [\varsigma_1, \dots, \varsigma_{2r},\ \mathrm{Re}\ \varsigma_{2r+2},\ \mathrm{Im}\varsigma_{2r+2}, \dots,\ \mathrm{Re}\ \varsigma_{2N-2},\ \mathrm{Im}\ \varsigma_{2N-2}] = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},$$
$$a = [a_1, a_2, \dots, a_{2N-2}]^T,$$

where D_1, D_2 are both $(2N-2)\times(2N-2)$ matrices.

From (3.13), we have

$$X(d) = D_1 a, \quad \dot{X}(d) = Y(d) = D_2 a.$$

Then, we obtain

$$\dot{X}(d) = D_2 D_1^{-1} X(d). \tag{3.14}$$

Let

$$Z(x) = [\omega_1(x), \omega_2(x), \dots, \omega_{N-1}(x), \psi_1(x), \psi_2(x), \dots, \psi_{N-1}(x)]^T.$$

Thus,

$$Z(d) = X(d) + c, \quad \dot{Z}(d) = \dot{X}(d), \tag{3.15}$$

where

$$c = [\omega_\infty(y_1), \ldots, \omega_\infty(y_{N-1}), \psi_\infty(y_1), \ldots, \psi_\infty(y_{N-1})]^T.$$

Substituting (3.15) into (3.14), we obtain the following discrete artificial boundary condition on artificial boundary Γ_d:

$$\dot{Z}(d) = HZ(d) + s, \tag{3.16}$$

with

$$H = D_2 D_1^{-1}, \quad s = -D_2 D_1^{-1} c.$$

In a similar way, we can get the artificial boundary conditions on the boundary Γ_b.

Suppose that $\{l_k(y),\ k = 0,\ 1,\ \ldots,\ N\}$ is a basic set of interpolating functions on Γ_d, for example $\{l_k(y), k = 0, 1, \ldots, N\}$ is a Lagrange interpolating polynomial. Then we have

$$\frac{\partial \omega(d,y)}{\partial x} = \sum_{k=0}^{N} l_k(y) \frac{\partial \omega(d,y_k)}{\partial x} = \sum_{k=1}^{N-1} l_k(y) \frac{\partial \omega(d,y_k)}{\partial x}, \tag{3.17}$$

$$\frac{\partial \psi(d,y)}{\partial x} = \sum_{k=0}^{N} l_k(y) \frac{\partial \psi(d,y_k)}{\partial x} = \sum_{k=1}^{N-1} l_k(y) \frac{\partial \psi(d,y_k)}{\partial x}. \tag{3.18}$$

The last equalities in (3.17) and (3.18) are followed by

$$\frac{\partial \omega(d,y_0)}{\partial x} = \frac{\partial \omega(d,y_N)}{\partial x} = 0, \quad \frac{\partial \psi(d,y_0)}{\partial x} = \frac{\partial \psi(d,y_N)}{\partial x} = 0.$$

Thus we have

$$\begin{aligned}
\begin{pmatrix} \frac{\partial \omega(d,y)}{\partial x} \\ \frac{\partial \psi(d,y)}{\partial x} \end{pmatrix} &= \begin{pmatrix} \sum_{k=1}^{N-1} l_k(y) \frac{\partial \omega(d,y_k)}{\partial x} \\ \sum_{k=1}^{N-1} l_k(y) \frac{\partial \psi(d,y_k)}{\partial x} \end{pmatrix} \\
&= L_i \begin{pmatrix} \frac{\partial \omega(d,y_1)}{\partial x} & \cdots & \frac{\partial \omega(d,y_{N-1})}{\partial x} & \frac{\partial \psi(d,y_1)}{\partial x} & \cdots & \frac{\partial \psi(d,y_{N-1})}{\partial x} \end{pmatrix}^T \\
&= L_i(HZ(d) + s) \equiv \Pi_h^d(\omega, \psi),
\end{aligned} \tag{3.19}$$

where

$$L_i = \begin{pmatrix} l_1(y) & \ldots & l_{N-1}(y), & 0 & \ldots & 0 \\ 0 & \ldots & 0 & l_1(y) & \ldots & l_{N-1}(y) \end{pmatrix}.$$

Then on the domain Ω_T the original problem (2.9)–(2.14) can be approximated by the following problem:

$$\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} - \nu \triangle \omega = 0, \quad \text{in } \Omega_T, \tag{3.20}$$

$$\triangle \psi + \omega = 0, \quad \text{in } \Omega_T, \tag{3.21}$$

$$\psi|_{y=0} = \frac{\partial \psi}{\partial y}|_{y=0,L} = 0, \quad \psi|_{y=L} = \psi_L, \quad b \le x \le d \tag{3.22}$$

$$\psi|_{\partial\Omega_i} = \text{constant}, \quad \frac{\partial\psi}{\partial n}|_{\partial\Omega_i} = 0, \tag{3.23}$$

$$\psi|_{\Gamma_b} = \psi_\infty(y), \quad \omega|_{\Gamma_b} = 0, \tag{3.24}$$

$$(\frac{\partial\omega}{\partial x}, \frac{\partial\psi}{\partial x})^T|_{\Gamma_d} = \Pi_h^d(\omega, \psi). \tag{3.25}$$

4. Numerical Implementation and Example

In this section we now consider the numerical solution of the original problem (2.9)–(2.14) on the given computational domain Ω_T. This steady state solution is computed as the limit in time of the unsteady N-S equations, which are discretized by an ADI method[12]. For each example, we use the same inflow boundary conditions

$$\psi(b, y) = \psi_\infty(y), \quad \omega(b, y) = 0; \quad 0 \le y \le L, \tag{4.1}$$

at the artificial boundary Γ_b. To test the artificial boundary comnditions, we made three types of computation using diferent types of outflow boundary conditions at artificial boundary Γ_d in each example.

Type I. Dirichlet boundary condition

$$\psi(d, y) = \psi_\infty(y), \quad \omega(d, y) = 0, \quad 0 \le y \le L.$$

Type II. Neumann boundary condition

$$\frac{\partial\psi}{\partial x}(d, y) = 0, \quad \frac{\partial\omega}{\partial x}(d, y) = 0, \quad 0 \le y \le L.$$

Type III. Discrete artificial boundary condition (3.16) or (3.19).

In the example, the results are compared with an "exact solution". This solution is obtained by using an outflow boundary very far from the obstacle or step, at which is a prescribed Neumann boundary condition. To be precise, the distance between the inflow boundary and the outflow boundary for the "exact solution" is twice the distance in our example.

Example 1. Computed flow in a horizontal channel of finite width with a rectangular cylinder obstruction. The obstruction is defined by the domain

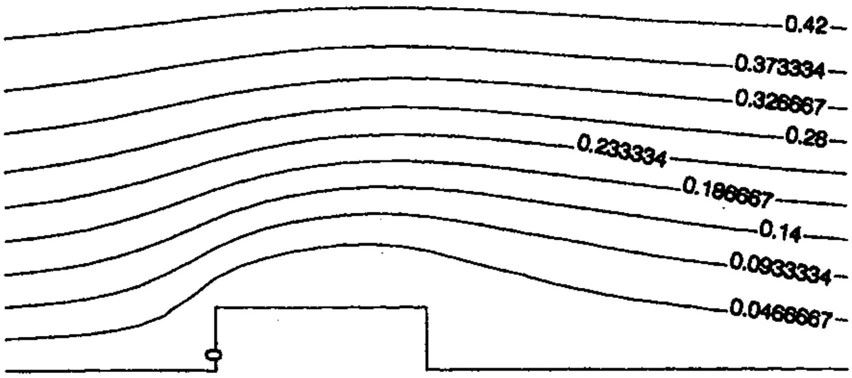
$$\Omega_i = \Big\{(x, y) \mid 0.8 < x < 1.2, \ \frac{3}{7}L < y < \frac{4}{7}L\Big\}.$$

The bounded computational domain Ω_T is given by

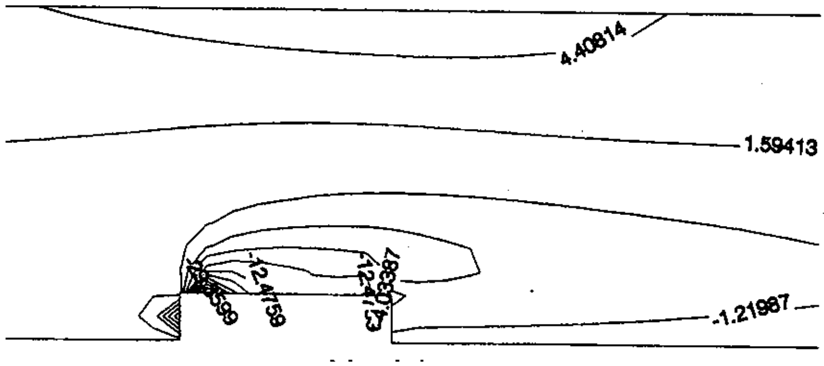
$$\Omega_T = \{(x, y) \mid b < x < d, \quad 0 < y < L\} \setminus \bar{\Omega}_i;$$

and $u_\infty(y) = \frac{4u_\infty}{L^2} y(L - y)$; hence $\psi_\infty(y)$ and $\omega_\infty(y)$ are given by

$$\omega_\infty(y) = \frac{4u_\infty}{L^2}(2y - L), \quad \psi_\infty(y) = \frac{4u_\infty}{L^2} y^2 (\frac{L}{2} - \frac{y}{3}).$$



Stream function



Vorticity

Fig.1

The constants b, d, L, u_∞ and Re are given by $b = 0$, $d = 2.8$, $L = 1.4$, $u_\infty = 1.0$ and $Re = 100$.

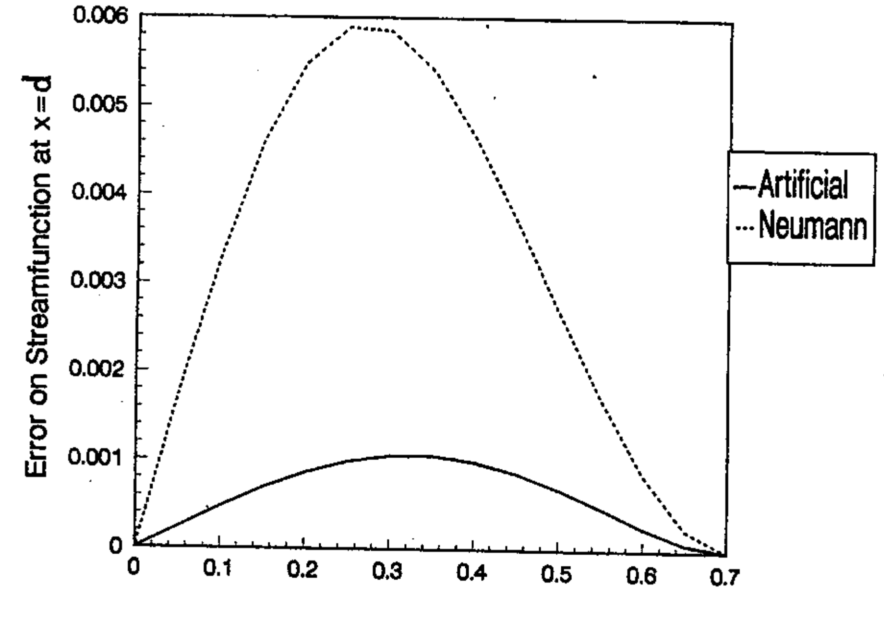
Let (ω_E, ψ_E) denote the "exact solution" (Fig.1) and $(\omega_i, \psi_i)(i = I, II, III)$ denote the numerical solutions corresponding to the boundary conditions of type I, II and III on artificial boundary Γ_d. The error $\omega_E - \omega_i$, $\psi_E - \psi_i$ on the boundary Γ_d are given in Fig.2. Let

$$\mathrm{err}(f_E - \tilde{f}_i) = \sqrt{\sum_{j=0}^{N} (f_E(d, y_j) - \tilde{f}_i(d, y_j))^2}$$

then the errors $\mathrm{err}(\omega_E - \omega_i)$, $\mathrm{err}(\psi_E - \psi_i)$ are given in Table 1.

Table 1

errors	i =I	i =II	i =III
$\mathrm{err}(\omega_E - \omega_i)$	2.5867	0.3423	0.0989
$\mathrm{err}(\psi_E - \psi_i)$	0.0754	0.0144	0.0026



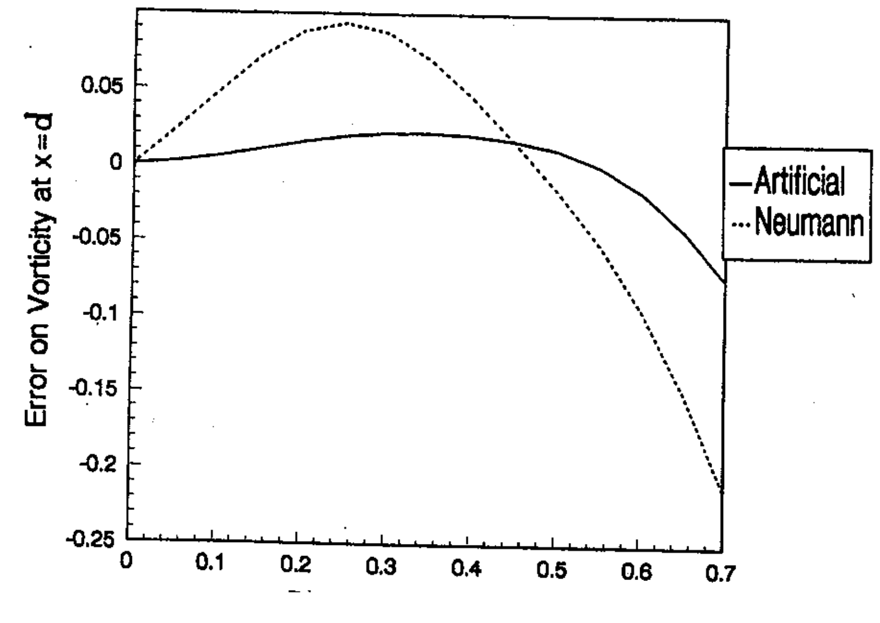
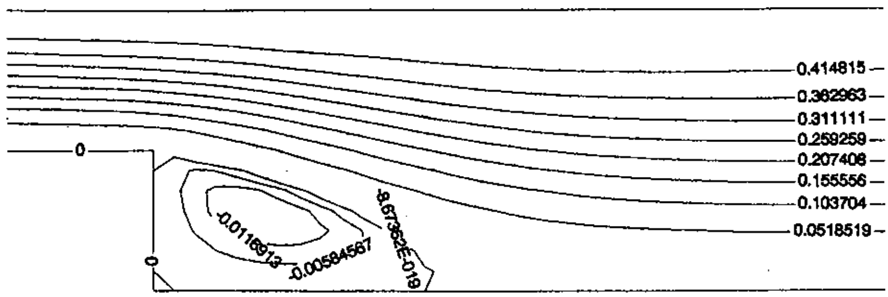


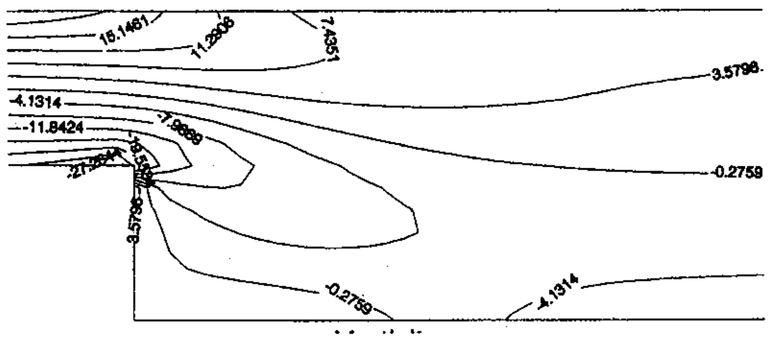
Fig.2

Example 2 Backward-facing step flow. The bounded computational domain given by

$$\Omega_T = \{(x,y) \mid b < x \le b + \frac{L}{2},\ \frac{L}{2} < y < L;\ \ b + \frac{L}{2} < x < d,\ 0 < y < L\}.$$



Stream function



Vorticity
Fig.3

The inflow condition at boundary $\Gamma_b = \{(x,y)|\ x = b,\ \frac{L}{2} \le y \le L\}$ is given by

$$\omega(b,y) = \frac{16u_\infty}{L^2}(4y-3L), \quad \psi(b,y) = \frac{8u_\infty}{3L^2}(y-\frac{L}{2})^2(5L-4y).$$

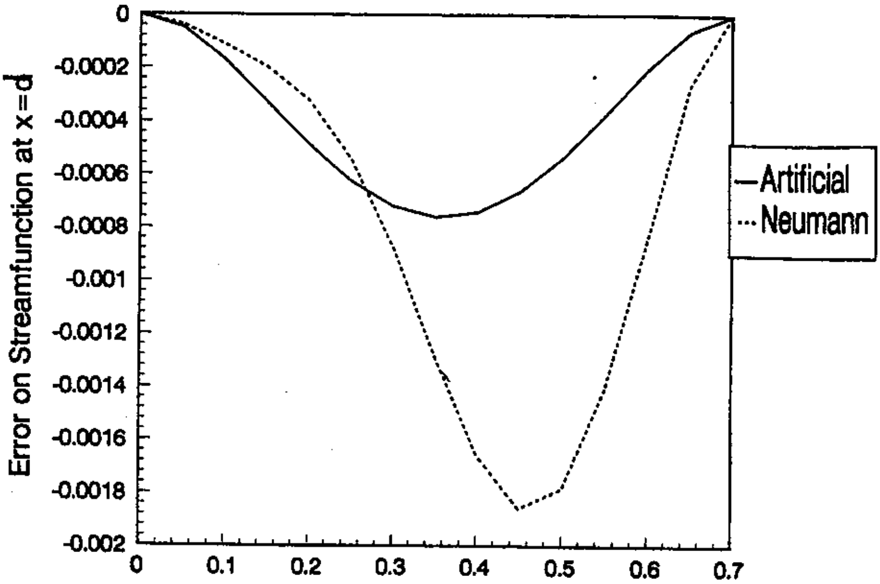
Then $u_\infty(y)$, $\psi_\infty(y)$ and $\omega_\infty(y)$ are the same as in example 1.

We take $b = 0$, $d = 2.1$, $L = 0.7$, $u_\infty = 1.0$ and $Re = 100$. The comparison of the "exact solution" $(\omega_E,\ \psi_E)$ (Fig.3) and the numerical solutions $(\omega_i,\ \psi_i)(i = II, III)$ on the boundary Γ_d are given in Fig.4. Table 2 shows the errors $\mathrm{err}(\omega_E - \omega_i)$ and $\mathrm{err}(\psi_E - \psi_i)$.

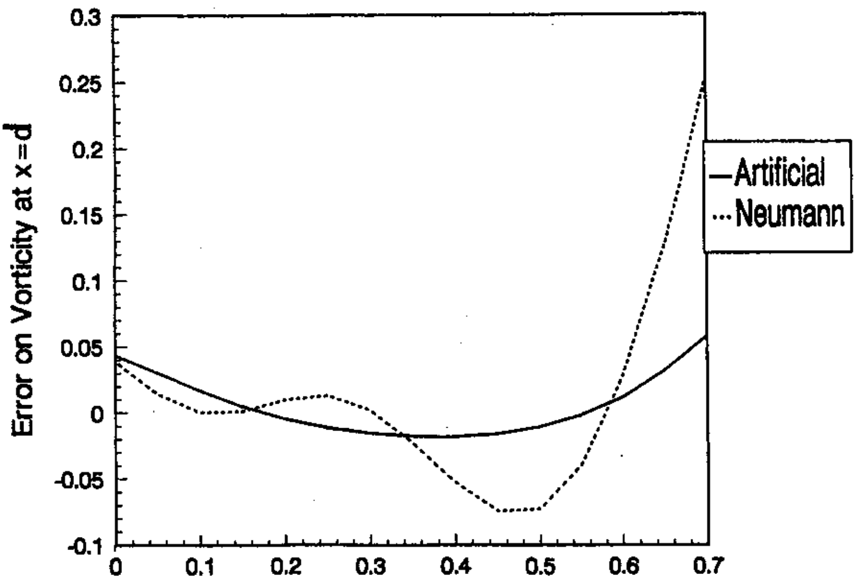
Table 2

errors	i =I	i =II	i =III
$\mathrm{err}(\omega_E - \omega_i)$	0.9969	0.3152	0.0951
$\mathrm{err}(\psi_E - \psi_i)$	0.0126	0.0038	0.0018

The examples show that the Dirichlet boundary condition is very inaccurate (Table 1 and Table 2), and the artificial boundary condition presented in this paper is more accurate than the Neumann boundary condition (Fig.2 and Fig.4). And is saves computing time, since for a given accuracy is is possible to use a smaller computational domain.



Distance from axis: y



Distance from axis: y

Fig.4

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