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ON THE INF–SUP CONDITION OF MIXED FINITE ELEMENT FORMULATIONS FOR ACOUSTIC FLUIDS

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The objective of this paper is to present a study of the solvability, stability and optimal error bounds of certain mixed finite element formulations for acoustic fluids. An analytical proof of the stability and optimal error bounds of a set of three-field mixed finite element discretizations is given, and the interrelationship between the inf–sup condition, including the numerical inf–sup test, and the eigenvalue problem pertaining to the natural frequencies is discussed.

1. Introduction

In recent years, it has been recognized that mixed finite element formulations with elements satisfying the inf–sup condition can be used reliably in many engineering applications, e.g. the analyses of rubber-like material, elasto-plasticity, creep, and Stokes flow.[1] In this paper, we focus on the so-called acoustoelastic problems, which involve acoustic fluids and their interaction with elastic structures.

Following the discovery of nonzero frequency spurious modes associated with displacement-based acoustic fluid elements,[14] extensive research efforts have been devoted to the improvement of finite element formulations for both frequency and dynamic analyses of acoustic fluid-structure interaction problems.[2,3,15,19–21]

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In these research efforts, the origins of nonzero spurious modes have been identified and both the displacement/pressure ($\mathbf{u}/p$) and displacement–pressure–vorticity moment ($\mathbf{u}$-p-$\mathbf{\Lambda}$) formulations have been found to be reliable and accurate provided that finite element interpolations are selected according to the inf–sup condition and the boundary discretization satisfies the mass and momentum conservation.[2,19] Although the inf–sup condition for mixed formulations has been proposed some time ago, an analytical proof of whether the inf–sup condition is satisfied by a specific element or discretization can be very difficult.[1,7] In practice, the numerical inf–sup test proposed in Ref. 8 is valuable.

Without restricting the essence of our exposition in this paper, we only consider two-dimensional cases. The finite element formulations for three-dimensional cases can be directly constructed. Consider an open, bounded, convex domain $V \subset \mathcal{R}^2$ with a sufficiently smooth boundary $\partial V = S$, e.g. a $C^{1,1}$ or piecewise smooth boundary with no re-entrant corners. The components of the strain tensor $\boldsymbol{\varepsilon}$ and the deviatoric strain tensor $\boldsymbol{\varepsilon}'$ are defined as $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $\varepsilon'_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$, where $\mathbf{u}$ stands for the displacement vector. Define the Sobolev space $[H^1_{0,S_u}(V)]^2 = \{\mathbf{v}|\mathbf{v} \in [H^1(V)]^2, \mathbf{v}|_{S_u} = \mathbf{0}\}$, where S_u and S_f stand for the Dirichlet and Neumann boundaries with $S = S_f \cup S_u$ and $S_f \cap S_u = \emptyset$. The variational discrete problem of the $\mathbf{u}/p$ formulation for nearly incompressible media with the bulk modulus β and the shear modulus G can be expressed as,[1]

$$\min_{\mathbf{v}^h \in \mathcal{V}^h} \left\{ a(\mathbf{v}^h, \mathbf{v}^h) + \frac{\beta}{2}\int_V [P^h(\operatorname{div}\mathbf{v}^h)]^2 dV - \int_V \mathbf{f}^B \cdot \mathbf{v}^h dV - \int_{S_f} \mathbf{f}^{S_f} \cdot \mathbf{v}^h dS \right\}, \tag{1.1}$$

where $a(\mathbf{v}^h, \mathbf{v}^h) = G\int_V |\boldsymbol{\varepsilon}'(\mathbf{v}^h)|^2 dV$, and the projection operator P^h is defined by

$$\int_V [P^h(\operatorname{div}\mathbf{v}^h) - \operatorname{div}\mathbf{v}^h] q^h dV = 0\,, \quad \forall q^h \in \mathcal{Q}^h\,, \tag{1.2}$$

with $\mathcal{V}^h \subset [H^1_{0,S_u}(V)]^2$ and $\mathcal{Q}^h \subset L^2(V)$.

The variational forms (1.1) and (1.2) can also be rewritten as,

$$2G\int_V \boldsymbol{\varepsilon}'(\mathbf{u}^h) : \boldsymbol{\varepsilon}'(\mathbf{v}^h) dV - \int_V p^h \operatorname{div}\mathbf{v}^h dV$$
$$= \int_V \mathbf{f}^B \cdot \mathbf{v}^h dV + \int_{S_f} \mathbf{f}^{S_f} \cdot \mathbf{v}^h dS\,, \quad \forall \mathbf{v}^h \in \mathcal{V}^h\,, \tag{1.3}$$

$$\int_V \left(\frac{p^h}{\beta} + \operatorname{div}\mathbf{u}^h \right) q^h dV = 0\,, \quad \forall q^h \in \mathcal{Q}^h\,. \tag{1.4}$$

Using the standard finite element interpolation procedure with $\mathbf{u}^h = \mathbf{H}\mathbf{U}$, $p^h = \mathbf{H}_p\mathbf{P}$, $\varepsilon'_{ij}(\mathbf{u}^h) = \mathbf{B}\mathbf{U}$, and $\nabla \cdot \mathbf{u}^h = \bar{\mathbf{B}}\mathbf{U}$, where $\mathbf{H}$ and $\mathbf{H}_p$ are the interpolation matrices, and $\mathbf{U}$ and $\mathbf{P}$ are the solution vectors, respectively, Eqs. (1.3) and (1.4) can be written in the algebraic form as follows[1]:

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R} \\ \mathbf{0} \end{Bmatrix}, \tag{1.5}$$

where

$$\mathbf{K}_{uu} = \int_V 2G\mathbf{B}^{\mathrm{T}}\mathbf{B}dV, \qquad \mathbf{K}_{up} = -\int_V \bar{\mathbf{B}}^{\mathrm{T}}\mathbf{H}_p dV;$$

$$\mathbf{K}_{pp} = -\int_V \frac{1}{\beta}\mathbf{H}_p^{\mathrm{T}}\mathbf{H}_p dV, \qquad \mathbf{R} = \int_V \mathbf{H}^{\mathrm{T}}\mathbf{f}^B dV + \int_{S_f} \mathbf{H}^{S_f\mathrm{T}}\mathbf{f}^{S_f} dS;$$

with $\mathbf{K}_{pu} = \mathbf{K}_{up}^{\mathrm{T}}$, and $\mathbf{H}^{S_f}$ obtained from $\mathbf{H}$.

According to Refs. 1 and 7, two solvability conditions must be satisfied, i.e. (a) $(\mathbf{v}^h)^{\mathrm{T}}\mathbf{K}_{uu}\mathbf{v}^h > 0$, $\forall\, \mathbf{v}^h \in \ker(\mathbf{K}_{pu})$ and (b) $\ker(\mathbf{K}_{up}) = \{\mathbf{0}\}$. Obviously, a difficulty arises from the essential assumption for acoustic fluids that $G = 0$. However, as discussed in Ref. 20, when we consider the inertia force $-\rho\ddot{\mathbf{u}}$ in $\mathbf{f}^B$, the corresponding equation of motion can be expressed as

$$\begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{U}} \\ \ddot{\mathbf{P}} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R} \\ \mathbf{0} \end{Bmatrix}, \tag{1.6}$$

where ρ is the mass density, $\mathbf{n}$ is the unit normal vector (pointing outwards), $\mathbf{M}_{uu} = \int_V \rho\mathbf{H}^{\mathrm{T}}\mathbf{H}dV$, and $R = -\int_{S_f} \mathbf{H}_n^{S_f\mathrm{T}}\bar{p}dS$, with $\bar{p} = -\mathbf{f}^{S_f}\cdot\mathbf{n}$. As discussed in Ref. 19, assuming that the physical constant pressure mode arising from the boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ on S has been eliminated and there is no spurious zero frequency,[19] the solvability conditions are satisfied in a transient direct step-by-step solution, where at each time step, we have

$$\begin{bmatrix} \mathbf{K}_{uu}^* & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{R}} \\ \mathbf{0} \end{Bmatrix}, \tag{1.7}$$

where $\hat{\mathbf{R}}$ is the effective load vector, and $\mathbf{K}_{uu}^* = \hat{C}\mathbf{M}_{uu}$, with $\hat{C}$ a positive constant associated with the direct time integration scheme, e.g. $\hat{C} = \dfrac{4}{\Delta t^2}$ for the trapezoidal rule.

In addition, for the frequency analysis, we have the eigenvalue problem, $\mathbf{K}\boldsymbol{\phi} = \omega^2\mathbf{M}\boldsymbol{\phi}$, with $\mathbf{K} = -\mathbf{K}_{up}\mathbf{K}_{pp}^{-1}\mathbf{K}_{pu}$ and $\mathbf{M} = \mathbf{M}_{uu}$. Obviously, $\mathbf{M}$ is positive definite and the eigenvalue problem is well-posed. In fact, for n displacement unknowns and m pressure degrees of freedom, the number of zero frequencies is $n - m$.

Considering next the stability of a discretization scheme (refer to Refs. 1 and 7), the following two conditions must be satisfied,

$$\begin{aligned} &\text{Ellipticity}: \quad a(\mathbf{v}^h, \mathbf{v}^h) \geq C\|\mathbf{v}^h\|_1^2, \quad \forall\, \mathbf{v}^h \in \ker(\mathbf{K}_{pu}), \\ &\text{Inf–sup:} \quad \inf_{q^h\in\mathcal{Q}^h} \sup_{\mathbf{v}^h\in\mathcal{V}^h} \frac{\int_V q^h \boldsymbol{\nabla}\cdot\mathbf{v}^h dV}{\|\mathbf{v}^h\|_1\|q^h\|_0} \geq \beta_0 > 0, \end{aligned} \tag{1.8}$$

where C is a positive constant independent of the mesh size h, and β_0 is a positive constant independent of both h and β.

Note that when $G \neq 0$, it is obvious that the ellipticity condition is satisfied.[9] Therefore, considering an acoustic fluid, the ellipticity condition can always be satisfied by some modifications to the variational formulation as discussed in Refs. 1, 6 and 11, see Sec. 2. Of course, in practice, a very small shear modulus compared with β to represent the acoustic fluid can simply be used.

If these modifications are not employed, the loss of ellipticity introduces zero frequency modes which correspond to zero deviatoric strain energy and can be effectively removed from the eigenvalue solutions in engineering computations. Therefore, the key stability requirement is the inf–sup condition for the selection of displacement and pressure interpolations, which governs the convergence of the true physical nonzero frequency modes as confirmed in Refs. 19 and 20.

Furthermore, to reduce the number of zero eigenvalues, according to Refs. 2, 11 and 19, we can use, for acoustic fluids, the so-called displacement–pressure–vorticity moment ($\mathbf{u}$-p-$\mathbf{\Lambda}$) formulation. Assigning $\operatorname{rot}\lambda = (\partial\lambda/\partial x_2, -\partial\lambda/\partial x_1)^{\mathrm{T}}$ and $\operatorname{rot}\mathbf{v} = \partial v_2/\partial x_1 - \partial v_1/\partial x_2$ in association with $\operatorname{grad} p = (\partial p/\partial x_1, \partial p/\partial x_2)^{\mathrm{T}}$ and $\operatorname{div}\mathbf{v} = \partial v_1/\partial x_1 + \partial v_2/\partial x_2$, respectively, we can replace Eqs. (1.3) and (1.4) with

$$\int_V \lambda^h \operatorname{rot} \mathbf{v}^h dV - \int_V p^h \operatorname{div} \mathbf{v}^h dV$$

$$= \int_V \mathbf{f}^B \cdot \mathbf{v}^h dV - \int_{S_f} \bar{p}^h (\mathbf{v}^h \cdot \mathbf{n}) dS\,, \quad \forall \mathbf{v}^h \in \mathcal{V}^h\,, \tag{1.9}$$

$$\int_V \left(\frac{p^h}{\beta} + \operatorname{div}\mathbf{u}^h\right) q^h dV = 0\,, \quad \forall q^h \in \mathcal{Q}^h\,, \tag{1.10}$$

$$\int_V \left(\frac{\lambda^h}{\alpha} - \operatorname{rot}\mathbf{u}^h\right) \mu^h dV = 0\,, \quad \forall \mu^h \in \mathcal{P}^h\,, \tag{1.11}$$

where α, the constant associated with the irrotationality, is a very large number, and $\mathcal{P}^h = \mathcal{Q}^h$.

For the $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulation, let $\lambda^h = \mathbf{H}_\lambda \mathbf{\Lambda}$ and $\operatorname{rot}\mathbf{u}^h = \hat{\mathbf{B}}\mathbf{U}$, where $\mathbf{H}_\lambda$ is the interpolation matrix for the variable λ^h, and $\mathbf{\Lambda}$ is the solution vector for λ^h, we have the additional matrices

$$\mathbf{K}_{u\lambda} = \int_V \hat{\mathbf{B}}^{\mathrm{T}} \mathbf{H}_\lambda dV \quad \text{and} \quad \mathbf{K}_{\lambda\lambda} = -\int_V \frac{1}{\alpha} \mathbf{H}_\lambda^{\mathrm{T}} \mathbf{H}_\lambda dV\,. \tag{1.12}$$

Of course, the key benefit of replacing the $\mathbf{u}/p$ formulation with the $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulation is to reduce the number of zero frequencies to $n - m - k$, where k is the number of vorticity moment degrees of freedom.[2] In addition, in the study of stability and optimal error bounds, it is necessary to use a modified $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulation in order to fulfil the ellipticity condition by considering both $\operatorname{div}\mathbf{v}^h$ and $\operatorname{rot}\mathbf{v}^h$, as discussed in Ref. 11.

The objective of this paper is to present a comprehensive study of the solvability, stability, and optimal error bounds of mixed finite element formulations for acoustic fluids, including the interrelationship between the numerical inf–sup test and the eigenvalue problem pertaining to the natural frequencies of a coupled acoustoelastic system. In addition, we extend the technique proposed by Stenberg in Refs. 16–18 for the analytical proof of the inf–sup condition for Stokes flow to the study of the inf–sup condition for acoustic fluids and provide an analytical proof for the stability and optimal error bounds of a set of three-field mixed finite element discretizations.

We begin with preliminaries of the governing equations and relevant theorems in Sec. 2, and then review in Sec. 3 a technique useful for the analytical proof of the inf–sup condition. The interrelationship between the discrete eigenvalue problems for both the $\mathbf{u}/p$ and $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulations and the numerical inf–sup tests is discussed in Sec. 4. In Sec. 5, we use the theorems developed in Sec. 3 to prove analytically that the 9-3-1 and 9-4c-1 elements satisfy the inf–sup condition for the three-field mixed formulation, and in the concluding section, we reiterate the important findings.

2. Preliminaries

Following the examples in Refs. 2, 11 and 14, we consider the eigenvalue problem: Find ω, $\mathbf{u}$, p and λ such that

$$\rho\omega^2\mathbf{u} - \operatorname{grad} p - \operatorname{rot}\lambda = 0 \qquad \text{in } V\,, \tag{2.1}$$

$$\frac{p}{\beta} + \operatorname{div}\mathbf{u} = 0 \qquad \text{in } V\,, \tag{2.2}$$

$$\frac{\lambda}{\alpha} - \operatorname{rot}\,\mathbf{u} = 0 \qquad \text{in } V\,, \tag{2.3}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{on } S_u\,, \tag{2.4}$$

$$(k - m\omega^2)\mathbf{u}\cdot\mathbf{n} = \int_{S_f} p dS \qquad \text{on } S_f\,, \tag{2.5}$$

$$\lambda = 0 \qquad \text{on } S\,, \tag{2.6}$$

where k and m are the mass and stiffness of the piston connected with the acoustic fluid at the boundary S_f representing the elastic structure in the acoustoelastic problem, and the boundary condition (2.5) implies that $\mathbf{u}\cdot\mathbf{n}$ is constant along S_f. According to Ref. 11, let

$$\begin{aligned}
\mathcal{V} &= \{\mathbf{v}\in[H^1(V)]^2 | \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } S_u\,,\ \mathbf{v}\cdot\mathbf{n} \text{ is constant on } S_f\}\,,\\
\mathcal{V}_0 &= \{\mathbf{v}\in[H^1(V)]^2 | \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } S\}\,,\\
\mathcal{Q} &= \mathcal{P} = L^2(V)\,,\\
\mathcal{Q}_0 &= \left\{q\in\mathcal{Q}\,\middle|\, \int_V q dV = 0\right\}.
\end{aligned}$$

The corresponding variational form of (2.1)–(2.6) can be expressed as: Find $\omega \in \mathcal{R}$, $\mathbf{u} \in \mathcal{V}$, $p \in \mathcal{Q}$, and $\lambda \in \mathcal{P}$ such that

$$a(\mathbf{u},\mathbf{v})+\left(\frac{\gamma_1}{\beta}-1\right)b(\mathbf{v},p)-\left(\frac{\gamma_2}{\alpha}-1\right)c(\mathbf{v},\lambda)=\rho\omega^2(\mathbf{u},\mathbf{v})+m\omega^2 n(\mathbf{u},\mathbf{v}), \quad \forall \mathbf{v}\in\mathcal{V},$$

$$\frac{(p,q)}{\beta}+b(\mathbf{u},q)=0, \quad \forall q\in\mathcal{Q}, \tag{2.7}$$

$$\frac{\lambda,\mu}{\alpha}-c(\mathbf{u},\mu)=0, \quad \forall \mu\in\mathcal{P};$$

where γ_1 and γ_2 are positive constants such that $0<\gamma_1\le\beta$ and $0<\gamma_2\le\alpha$, $(\cdot,\cdot)$ is the usual inner product on $\mathcal{Q}\times\mathcal{Q}$, $\mathcal{P}\times\mathcal{P}$, or $\mathcal{V}\times\mathcal{V}$, and

$$\begin{aligned}
n(\mathbf{u},\mathbf{v}) &= (\mathbf{u}\cdot\mathbf{n})|_{S_f}(\mathbf{v}\cdot\mathbf{n})|_{S_f}, \quad \forall \mathbf{u},\mathbf{v}\in\mathcal{V},\\
a(\mathbf{u},\mathbf{v}) &= \gamma_1(\operatorname{div}\mathbf{u},\operatorname{div}\mathbf{v})+\gamma_2(\operatorname{rot}\mathbf{u},\operatorname{rot}\mathbf{v})+kn(\mathbf{u},\mathbf{v}), \quad \forall \mathbf{u},\mathbf{v}\in\mathcal{V},\\
b(\mathbf{v},q) &= (\operatorname{div}\mathbf{v},q), \quad \forall \mathbf{v}\in\mathcal{V}, q\in\mathcal{Q},\\
c(\mathbf{v},\mu) &= (\operatorname{rot}\mathbf{v},\mu), \quad \forall \mathbf{v}\in\mathcal{V}, \mu\in\mathcal{P}.
\end{aligned}$$

Note that in this formulation, the bilinear form $a(\mathbf{u},\mathbf{v})$ is coercive on $\mathcal{V}\times\mathcal{V}$ if the two artificial constants γ_1 and γ_2 are positive. Of key importance is that for the bilinear forms $b(\mathbf{v},q)$ and $c(\mathbf{v},\mu)$, we have the following lemma.

Lemma 2.1. *There exists a constant* C_0 *such that*

$$\sup_{\mathbf{v}\in\mathcal{V}_0\setminus\{\mathbf{0}\}}\frac{b(\mathbf{v},q)+c(\mathbf{v},\mu)}{\|\mathbf{v}\|_1}\ge C_0(\|q\|_0+\|\mu\|_0), \quad \forall q\in\mathcal{Q}_0, \mu\in\mathcal{P}. \tag{2.8}$$

Proof. For any $q\in\mathcal{Q}_0$, noting that the domain V is an open, bounded, convex domain with no re-entrant corners, there exists a unique $\phi\in H^2(V)$ satisfying (see Ref. 12 for details)

$$-\nabla^2\phi=q \quad \text{in } V, \tag{2.9}$$

$$\frac{\partial\phi}{\partial n}=0 \quad \text{on } S, \tag{2.10}$$

with $\int_V \phi dV=0$.

By elliptic regularity we have,

$$\|\phi\|_2\le\bar{C}\|q\|_0, \tag{2.11}$$

where the positive constant $\bar{C}$ only depends on the area of V. Analogously, for any $\mu\in\mathcal{P}$, there exists a unique $\psi\in H^2(V)$ satisfying (see Ref. 12 for details)

$$\nabla^2\psi=\mu \quad \text{in } V, \tag{2.12}$$

$$\psi=0 \quad \text{on } S, \tag{2.13}$$

and by elliptic regularity we have

$$\|\psi\|_2 \le \bar{C}\|\mu\|_0 \,. \tag{2.14}$$

Let $\mathbf{z} = -\mathrm{grad}\ \phi - \mathrm{rot}\ \psi$ with $\mathbf{z} \in [H^1(V)]^2$, we can derive,

$$\begin{aligned} \mathrm{div}\,\mathbf{z} &= -\nabla^2\phi = q \,, \\ \mathrm{rot}\,\mathbf{z} &= \ \ \nabla^2\psi = \mu \,, \end{aligned} \tag{2.15}$$

and

$$\mathbf{z}\cdot\mathbf{n}|_s = \left(-\frac{\partial\phi}{\partial n} - \frac{\partial\psi}{\partial\tau}\right)\Big|_S = 0 \,, \tag{2.16}$$

$$\|\mathbf{z}\|_1 \le \|\phi\|_2 + \|\psi\|_2 \le \bar{C}(\|q\|_0 + \|\mu\|_0) \,, \tag{2.17}$$

where $\boldsymbol{\tau}$ is the unit tangent vector.

Hence, from (2.17), using the Schwarz inequality, we obtain

$$\begin{aligned} \sup_{\mathbf{v}\in\mathcal{V}_0\backslash\{\mathbf{0}\}} \frac{b(\mathbf{v},q) + c(\mathbf{v},\mu)}{\|\mathbf{v}\|_1} &\ge \frac{\|q\|_0^2 + \|\mu\|_0^2}{\bar{C}(\|q\|_0 + \|\mu\|_0)} \\ &\ge C_0(\|q\|_0 + \|\mu\|_0) \,, \quad \forall q \in \mathcal{Q}_0, \mu \in \mathcal{P} \,, \end{aligned} \tag{2.18}$$

with $C_0 = 1/(2\bar{C})$. □

Let $\mathcal{V}^h$, $\mathcal{Q}^h$ and $\mathcal{P}^h$ be finite element subspaces of $\mathcal{V}$, $\mathcal{Q}$ and $\mathcal{P}$, respectively, the finite element approximation of the problem (2.7) becomes: Find $\omega^h \in \mathcal{R}$, $\mathbf{u}^h \in \mathcal{V}^h$, $p^h \in \mathcal{Q}^h$, and $\lambda^h \in \mathcal{P}^h$, such that

$$\begin{aligned} & a(\mathbf{u}^h,\mathbf{v}^h) + \left(\frac{\gamma_1}{\beta} - 1\right) b(\mathbf{v}^h,p^h) - \left(\frac{\gamma_2}{\alpha} - 1\right) c(\mathbf{v}^h,\lambda^h) \\ & \quad = \rho(\omega^h)^2(\mathbf{u}^h,\mathbf{v}^h) + m(\omega^h)^2 n(\mathbf{u}^h,\mathbf{v}^h) \,, \quad \forall \mathbf{v}^h \in \mathcal{V}^h \,, \\ & \quad \frac{(p^h,q^h)}{\beta} + b(\mathbf{u}^h,q^h) = 0 \,, \quad \forall q^h \in \mathcal{Q}^h \,, \\ & \quad \frac{(\lambda^h,\mu^h)}{\alpha} - c(\mathbf{u}^h,\mu^h) = 0 \,, \quad \forall \mu^h \in \mathcal{P}^h \,. \end{aligned} \tag{2.19}$$

Now, in order to have a good finite element method, the finite element spaces have to be chosen such that they inherit the property (2.8), i.e. they should satisfy,

$$\sup_{\mathbf{v}^h\in\mathcal{V}_0^h\backslash\{\mathbf{0}\}} \frac{b(\mathbf{v}^h,q^h) + c(\mathbf{v}^h,\mu^h)}{\|\mathbf{v}^h\|_1} \ge C_0(\|q^h\|_0 + \|\mu^h\|_0) \,, \quad \forall q^h \in \mathcal{Q}_0^h, \mu^h \in \mathcal{P}^h \,, \tag{2.20}$$

and the theory of mixed methods provides the following optimal error estimate (see Ref. 13 for detail):

$$|\omega - \omega^h| \le \tilde{C}\varepsilon_h^2 \,, \tag{2.21}$$

where $\tilde{C}$ is a positive constant independent of h and material properties, ω and ω^h are solutions of the problems (2.7) and (2.19), respectively; and

$$\varepsilon_h = \sup_{\|\mathbf{u}\|_1+\|p\|_0+\|\lambda\|_0=1} \inf_{\substack{(\mathbf{v}^h, q^h, \mu^h) \in \\ \mathcal{V}^h \times \mathcal{Q}^h \times \mathcal{P}^h}} \left(\|\mathbf{u}-\mathbf{v}^h\|_1 + \|p - q^h\|_0 + \|\lambda - \mu^h\|_0\right).$$

In the following section, we will use the technique proposed by Stenberg[16–18] to check the discrete inf–sup condition (2.20).

3. Inf–Sup Condition

In order to provide a precise discussion, we have to define our concepts properly. Considering (2.20), we use $\mathcal{V}_0^h$ instead of $\mathcal{V}^h$, and if the inf–sup condition in the form of (2.20) is satisfied, since $\mathcal{V}_0^h \subset \mathcal{V}^h$, the inf–sup condition for the problem (2.19) is also satisfied, provided that the constant pressure mode is eliminated, i.e. we are using $\mathcal{Q}_0^h$ instead of $\mathcal{Q}^h$. Therefore, for simplicity and clarity, we adopt $\mathcal{V}_0^h$ and $\mathcal{Q}_0^h$ in the rest of the paper.

Let $\mathcal{T}^h$ be a partition of V which consists of either triangular or convex quadrilateral elements. Naturally, the partition is assumed to satisfy the usual compatibility and regularity conditions.[1] Let us further assume that the finite element polynomial spaces $\mathcal{V}_0^h$, $\mathcal{Q}_0^h$, and $\mathcal{P}^h$ can be uniquely defined on $\mathcal{T}^h$ by using a reference element $\hat{T}$, i.e. the unit triangle, or square element within the reference polynomial spaces, $\hat{\mathcal{V}}$, $\hat{\mathcal{Q}}$ and $\hat{\mathcal{P}}$. For $T \in \mathcal{T}^h$, let $\mathbf{F}_T : \hat{T} \to T$ be an affine or bilinear mapping from $\hat{T}$ onto T, i.e. $\mathbf{x} = \mathbf{F}_T(\hat{\mathbf{x}})$. We define

$$\mathcal{V}_0^h = \{\mathbf{v}^h \in \mathcal{V}_0 | \mathbf{v}^h(\mathbf{x})|_T = \hat{\mathbf{v}}^h(\mathbf{F}_T^{-1}(\mathbf{x}))\,, \quad \hat{\mathbf{v}}^h \in \hat{\mathcal{V}}\,, \quad \mathbf{x} \in T\,, \quad \forall T \subset V\}\,,$$

and

$$\mathcal{Q}_0^h = \{q^h \in \mathcal{Q}_0 | q^h(\mathbf{x})|_T = \hat{q}^h(\mathbf{F}_T^{-1}(\mathbf{x}))\,, \quad \hat{q}^h \in \hat{\mathcal{Q}}\,, \quad \mathbf{x} \in T\,, \quad \forall T \subset V\}\,,$$

$$\mathcal{P}^h = \{\mu^h \in \mathcal{P} | \mu^h(\mathbf{x})|_T = \hat{\mu}^h(\mathbf{F}_T^{-1}(\mathbf{x}))\,, \quad \hat{\mu}^h \in \hat{\mathcal{P}}\,, \quad \mathbf{x} \in T\,, \quad \forall T \subset V\}\,;$$

or

$$\mathcal{Q}_0^h = \{q^h \in \mathcal{Q}_0 \cap C(V) | q^h(\mathbf{x})|_T = \hat{q}^h(\mathbf{F}_T^{-1}(\mathbf{x}))\,, \quad \hat{q}^h \in \hat{\mathcal{Q}}\,, \quad \mathbf{x} \in T\,, \quad \forall T \subset V\}\,,$$

$$\mathcal{P}^h = \{\mu^h \in \mathcal{P} \cap C(V) | \mu^h(\mathbf{x})|_T = \hat{\mu}^h(\mathbf{F}_T^{-1}(\mathbf{x}))\,, \quad \hat{\mu}^h \in \hat{\mathcal{P}}\,, \quad \mathbf{x} \in T\,, \quad \forall T \subset V\}\,;$$

where $C(V)$ denotes the set of continuous functions on V, and both $\mathbf{x}$ and $\hat{\mathbf{x}}$ represent the position vectors in the two-dimensional domain considered in this paper and its reference domain, respectively.

The second choice for $\mathcal{Q}_0^h$ and $\mathcal{P}^h$ is also called the Taylor–Hood discretization. For $T \in \mathcal{T}^h$ we denote its characteristic diameter as h_T, its boundary as ∂T, its boundary unit normal vector as $\mathbf{n}_T$, its boundary unit tangent vector as $\boldsymbol{\tau}_T$, the set of its edges as $\ell(T)$, the characteristic length of its edges as h_E, and

$$\ell^h = \bigcup_{T \in \mathcal{T}^h} \ell(T) = \ell_V^h \cup \ell_S^h\,,$$

with

$$\ell_V^h = \{E \in \ell^h | E \cap V \neq \emptyset\},$$

$$\ell_S^h = \{E \in \ell^h | E \subset S\}.$$

In general, by a macroelement M we mean the union of one or more neighboring elements in $\mathcal{T}^h$. Analogously, a macroelement M is said to be equivalent to a macroelement in the reference domain $\hat{M}$ if there is a mapping $\mathbf{F}_M$: $\hat{M} \to M$ satisfying the following conditions:

(i) $\mathbf{F}_M$ is continuous and bijective;
(ii) If $\hat{M} = \bigcup_{j=1}^{m} \hat{T}_j$, where $\hat{T}_j$, $j = 1, \ldots, m$ are the elements of $\hat{M}$, then $T_j = \mathbf{F}_M(\hat{T}_j)$, $j = 1, \ldots, m$ are the elements of M;
(iii) $\mathbf{F}_M|_{\hat{T}_j} = \mathbf{F}_{T_j} \circ \mathbf{F}_{\hat{T}_j}^{-1}$, where $\mathbf{F}_{\hat{T}_j}$ and $\mathbf{F}_{T_j}$ are affine or bilinear mappings from the reference element $\hat{T}$ onto $\hat{T}_j$ and T_j, respectively, with $j = 1, \ldots, m$.

We denote the family of macroelements equivalent to M and $\hat{M}$ as $\mathcal{M}$ and $\hat{\mathcal{M}}$, respectively. Hence, for a given macroelement M, analogous to the definition of finite element spaces, and ℓ^h, ℓ_V^h and ℓ_S^h on V, we can define finite element spaces $\mathcal{V}_M^h$, $\mathcal{Q}_M^h$ and $\mathcal{P}_M^h$, along with ℓ_M^h, $\ell_{M,V}^h$ and $\ell_{M,S}^h$ on M. Notice here that $\mathcal{Q}_M^h$ corresponds to $\mathcal{Q}$ rather than $\mathcal{Q}_0$ and $\mathcal{V}_M^h$ corresponds to $\mathcal{V}_0$ rather than $\mathcal{V}$ for the domain occupied by a macroelement M.

Introduce the following norms in $\mathcal{Q}_0^h$ and $\mathcal{P}^h$,

$$\|q^h\|_{V,\mathcal{Q}}^2 = \sum_{T \in \mathcal{T}^h} h_T^2 \| \operatorname{grad} q^h\|_{0,T}^2 + \sum_{E \in \ell_V^h} h_E \int_E |[q^h]|^2 dS, \quad \forall q^h \in \mathcal{Q}_0^h, \quad (3.1)$$

$$\|\mu^h\|_{V,\mathcal{P}}^2 = \sum_{T \in \mathcal{T}^h} h_T^2 \| \operatorname{rot} \mu^h\|_{0,T}^2 + \sum_{E \in \ell_V^h} h_E \int_E |[\mu^h]|^2 dS + \sum_{E \in \ell_S^h} h_E \int_E |\mu^h|^2 dS, \quad \forall \mu^h \in \mathcal{P}^h; \quad (3.2)$$

where $[a]$ denotes the jump of the variable a across edges $E \in \ell_V^h$, and analogously the following semi-norms in $\mathcal{Q}_M^h$ and $\mathcal{P}_M^h$,

$$|q^h|_{M,\mathcal{Q}}^2 = \sum_{T \subset M} h_T^2 \| \operatorname{grad} q^h\|_{0,T}^2 + \sum_{E \in \ell_M^h} h_E \int_E |[q^h]|^2 dS, \quad (3.3)$$

$$|\mu^h|_{M,\mathcal{P}}^2 = \sum_{T \subset M} h_T^2 \| \operatorname{rot} \mu^h\|_{0,T}^2 + \sum_{E \in \ell_{M,V}^h} h_E \int_E |[\mu^h]|^2 dS + \sum_{E \in \ell_{M,S}^h} h_E \int_E |\mu^h|^2 dS. \quad (3.4)$$

Furthermore, we define,

$$\mathcal{V}^h_{0,M} = \{\mathbf{v}^h \in \mathcal{V}^h_M | \mathbf{v}^h|_{\partial M\setminus S} = \mathbf{0}\,, \quad \mathbf{v}^h \cdot \mathbf{n}|_{\partial M \cap S} = 0\}\,,$$

$$\mathcal{N}_M = \{(q^h, \mu^h) \in \mathcal{Q}^h_M \times \mathcal{P}^h_M | (\text{div } \mathbf{v}^h, q^h)_M + (\text{rot } \mathbf{v}^h, \mu^h)_M = 0\,, \quad \forall \mathbf{v}^h \in \mathcal{V}^h_{0,M}\}\,,$$

where

$$(\text{div } \mathbf{v}^h, q^h)_M + (\text{rot } \mathbf{v}^h, \mu^h)_M$$
$$= \sum_{T\subset M} \int_T (q^h \text{ div } \mathbf{v}^h + \mu^h \text{ rot } \mathbf{v}^h) dV\,, \quad \forall \mathbf{v}^h \in \mathcal{V}^h_{0,M}, q^h \in \mathcal{Q}^h_M, \mu^h \in \mathcal{P}^h_M\,. \quad (3.5)$$

Then we have the following lemma:

Lemma 3.1. *There exist two positive constants C_a and C_b independent of h and material properties such that*

$$\sup_{\mathbf{v}^h \in \mathcal{V}^h_0 \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)}{\|\mathbf{v}^h\|_1} \geq C_a(\|q^h\|_0 + \|\mu^h\|_0) - C_b(\|q^h\|_{V,\mathcal{Q}} + \|\mu^h\|_{V,\mathcal{P}})\,,$$
$$\forall q^h \in \mathcal{Q}^h_0, \mu^h \in \mathcal{P}^h\,. \quad (3.6)$$

Proof. Let $(q^h, \mu^h) \in \mathcal{Q}^h_0 \times \mathcal{P}^h \subset \mathcal{Q}_0 \times \mathcal{P}$ be arbitrary, noting that the domain V is an open, bounded, convex domain with no re-entrant corners, then, from the inequality (2.8), there exists $\mathbf{v} \in \mathcal{V}_0$, such that

$$b(\mathbf{v}, q^h) + c(\mathbf{v}, \mu^h) \geq C_0(\|q^h\|_0 + \|\mu^h\|_0)^2 \quad (3.7)$$

and

$$\|\mathbf{v}\|_1 \leq \|q^h\|_0 + \|\mu^h\|_0\,. \quad (3.8)$$

We now interpolate $\mathbf{v}$ with $\mathbf{v}^h \in \mathcal{V}^h_0$ defined by the technique of Clemént[10] such that we have the following error estimates,

$$\sum_{T\in\mathcal{T}^h} h_T^{-2} \|\mathbf{v} - \mathbf{v}^h\|^2_{0,T} + \sum_{E\in\ell^h} h_E^{-1} \int_E |\mathbf{v} - \mathbf{v}^h|^2 dS \leq C_1 |\mathbf{v}|^2_1 \quad (3.9)$$

and

$$\|\mathbf{v}^h\|_1 \leq C_2 \|\mathbf{v}\|_1\,, \quad (3.10)$$

with C_1 and C_2 independent of h and material properties.

Then, using integration by parts, we obtain

$$\begin{aligned} b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h) &= b(\mathbf{v}^h - \mathbf{v}, q^h) + c(\mathbf{v}^h - \mathbf{v}, \mu^h) + b(\mathbf{v}, q^h) + c(\mathbf{v}, \mu^h) \\ &\geq b(\mathbf{v}^h - \mathbf{v}, q^h) + c(\mathbf{v}^h - \mathbf{v}, \mu^h) + C_0(\|q^h\|_0 + \|\mu^h\|_0)^2 \\ &= \sum_{T\in\mathcal{T}^h} (\mathbf{v} - \mathbf{v}^h, \text{ grad } q^h - \text{rot } \mu^h) + C_0(\|q^h\|_0 + \|\mu^h\|_0)^2 \end{aligned}$$

$$+ \sum_{E\in\ell_V^h} \int_E [(\mathbf{v}^h - \mathbf{v})\cdot \mathbf{n}_T([q^h]) + (\mathbf{v}^h - \mathbf{v})\cdot \boldsymbol{\tau}_T([\mu^h])]dS$$

$$+ \sum_{E\in\ell_S^h} \int_E (\mathbf{v}^h - \mathbf{v})\cdot \boldsymbol{\tau}_T \mu^h dS\,,$$

and furthermore, using the Hölder and Schwarz inequalities, (3.8), the fact that $|\mathbf{v}|_1^2 \le \|\mathbf{v}\|_1^2$, and the following inequality based on (3.8) and (3.10),

$$\|\mathbf{v}^h\|_1 \le C_2\|\mathbf{v}\|_1 \le C_2(\|q^h\|_0 + \|\mu^h\|_0)\,, \tag{3.11}$$

we have

$$\begin{aligned}
&b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)\\
&\ge -\left(\sum_{T\in\mathcal{T}^h} h_T^{-2}\|\mathbf{v}-\mathbf{v}^h\|_{0,T}^2 + \sum_{E\in\ell^h} h_E^{-1}\int_E |\mathbf{v}-\mathbf{v}^h|^2 dS\right)^{1/2}\\
&\quad\times(\|q^h\|_{V,\mathcal{Q}} + \|\mu^h\|_{V,\mathcal{P}}) + C_0(\|q^h\|_0 + \|\mu^h\|_0)^2\\
&\ge -\sqrt{C_1}|\mathbf{v}|_1(\|q^h\|_{V,\mathcal{Q}} + \|\mu^h\|_{V,\mathcal{P}}) + C_0(\|q^h\|_0 + \|\mu^h\|_0)^2\\
&\ge -\sqrt{C_1}(\|q^h\|_0 + \|\mu^h\|_0)(\|q^h\|_{V,\mathcal{Q}} + \|\mu^h\|_{V,\mathcal{P}}) + C_0(\|q^h\|_0 + \|\mu^h\|_0)^2\\
&= [C_0(\|q^h\|_0 + \|\mu^h\|_0) - \sqrt{C_1}(\|q^h\|_{V,\mathcal{Q}} + \|\mu^h\|_{V,\mathcal{P}})](\|q^h\|_0 + \|\mu^h\|_0)\\
&\ge [C_a(\|q^h\|_0 + \|\mu^h\|_0) - C_b(\|q^h\|_{V,\mathcal{Q}} + \|\mu^h\|_{V,\mathcal{P}})]\|\mathbf{v}^h\|_1\,,
\end{aligned} \tag{3.12}$$

with $C_a = C_0/C_2$ and $C_b = \sqrt{C_1}/C_2$. □

Lemma 3.2. *Suppose that for every $M \in \mathcal{M}$, the space $\mathcal{N}_M$, consisting of functions that are constant vectors on M, is two-dimensional if $\partial M \cap S = \emptyset$ or one-dimensional if $\partial M \cap S = \emptyset$. Then there exists a positive constant $\beta_{\mathcal{M}}$ such that the condition*

$$\sup_{\mathbf{v}^h\in\mathcal{V}_{0,M}^h\setminus\{\mathbf{0}\}} \frac{b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)}{\|\mathbf{v}^h\|_1} \ge \beta_{\mathcal{M}}(|q^h|_{M,\mathcal{Q}} + |\mu^h|_{M,\mathcal{P}})\,,$$

$$\forall q^h \in \mathcal{Q}_M^h, \mu^h \in \mathcal{P}_M^h\,, \tag{3.13}$$

holds for every $M \in \mathcal{M}$.

Proof. Consider a fixed $M \in \mathcal{M}$. Define the constant β_M as

$$\begin{aligned}
\beta_M &= \inf_{\substack{(q^h,\mu^h)\in\mathcal{Q}_M^h\times\mathcal{P}_M^h\\ |q^h|_{M,\mathcal{Q}}^2 + |\mu^h|_{M,\mathcal{P}}^2 = 1}} \sup_{\mathbf{v}^h\in\mathcal{V}_{0,M}^h\setminus\{\mathbf{0}\}} \frac{b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)}{\|\mathbf{v}^h\|_1}\\
&= \inf_{\substack{(q^h,\mu^h)\in\mathcal{Q}_M^h\times\mathcal{P}_M^h\\ |q^h|_{M,\mathcal{Q}}^2 + |\mu^h|_{M,\mathcal{P}}^2 = 1}} \sup_{\substack{\mathbf{v}^h\in\mathcal{V}_{0,M}^h\setminus\{\mathbf{0}\}\\ \|\mathbf{v}^h\|_1 = 1}} [b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)]\,.
\end{aligned} \tag{3.14}$$

Since $\mathcal{N}_M$, consisting of functions that are constant vectors on M, is two-dimensional if $\partial M \cap S = \emptyset$ one-dimensional if $\partial M \cap S \neq \emptyset$, and $\mathcal{Q}^h_M$, $\mathcal{P}^h_M$, and $\mathcal{V}^h_{0,M}$ are finite-dimensional, it follows that $\beta_M > 0$.

Let us now prove that there is a constant $\beta_{\mathcal{M}}$ such that $\beta_M \geq \beta_{\mathcal{M}} > 0$ for every $M \in \mathcal{M}$. Let $\hat{\mathbf{x}}^1, \ldots, \hat{\mathbf{x}}^d$ be the vertices of the elements in $\hat{M}$. Every $M \in \mathcal{M}$ is now uniquely defined by its vertices $\mathbf{x}^i = \mathbf{F}_M(\hat{\mathbf{x}}^i)$, with $i = 1 \ldots, d$, and so we may write $\beta_M = \beta(\mathbf{x}^1, \ldots, \mathbf{x}^d)$. We will now consider the vertices as a point $\mathbf{y} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^d)$ in $\mathcal{R}^{2d}$, and $\beta_M = \beta(\mathbf{y})$ as a function of $\mathbf{y}$. Let $h_M = \max_{T \subset M} h_T$. We may assume that $h_M = 1$ and that $\mathbf{x}^1$ coincides with the origin in $\mathcal{R}^2$. Since the general case can be handled by a scaling argument using the mapping $\mathbf{G}(\mathbf{x}) = h_M^{-1}(\mathbf{x} - \mathbf{x}^1)$, where $\mathbf{x}^1$ is chosen as the origin, all vertices $\mathbf{x}^2, \ldots, \mathbf{x}^d$ lie within a given distance from the origin. Furthermore, every $T \subset M$ has a diameter less than or equal to unity and the triangulation $\mathcal{T}^h$ is regular. This means that the point $\mathbf{y}$ belongs to a compact set, denoted by $\mathcal{D}$, in $\mathcal{R}^{2d}$. It can easily be proved that the function β is continuous (see Ref. 16 for a similar argument), and since $\beta(\mathbf{y}) > 0$ for every $\mathbf{y} \in \mathcal{D}$, we conclude that there is a constant $\beta_{\mathcal{M}} > 0$, independent of h and material properties, such that $\beta(\mathbf{y}) \geq \beta_{\mathcal{M}}$ for every $\mathbf{y} \in \mathcal{D}$. Thus,

$$\inf_{\substack{(q^h,\mu^h) \in \mathcal{Q}^h_M \times \mathcal{P}^h_M \\ |q^h|^2_{M,\mathcal{Q}} + |\mu^h|^2_{M,\mathcal{P}} = 1}} \sup_{\substack{\mathbf{v}^h \in \mathcal{V}^h_{0,M}\backslash\{0\} \\ \|\mathbf{v}^h\|_1 = 1}} [b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)] = \beta_M \geq \beta_{\mathcal{M}} > 0, \ \forall M \in \mathcal{M}, \tag{3.15}$$

and the desired inequality (3.13) follows from (3.15) directly. □

Theorem 3.1. *Suppose that there is a fixed set of equivalent classes of macroelements $\mathcal{M}_i$, with $i = 1, \ldots, N$, a positive integer L, and a macroelement partition $\mathcal{M}^h$ for V, such that*

(M1) *For each $M \in \mathcal{M}_i$, with $i = 1, \ldots, N$, the space $\mathcal{N}_M$, two-dimensional if $\partial M \cap S = \emptyset$ or one-dimensional if $\partial M \cap S \neq \emptyset$, consists of functions that are constant on M.*
(M2) *Each $M \in \mathcal{M}^h$ belongs to one of the classes $\mathcal{M}_i$, with $i = 1, \ldots, N$.*
(M3) *Each $T \in \mathcal{T}^h$ is contained in at least one and not more than L macroelements of $\mathcal{M}^h$.*
(M4) *Each $E \in \ell^h_V$ is contained in the interior of at least one and not more than L macroelements of $\mathcal{M}^h$.*
(M5) *Each $E \in \ell^h_S$ is contained on the boundary of at least one and not more than L macroelements of $\mathcal{M}^h$.*

Then the inf–sup condition (2.20) holds.

Proof. Let $(q^h, \mu^h) \in \mathcal{Q}^h_0 \times \mathcal{P}^h$ be arbitrary. From Lemma 3.2 and (M1), we know that for every $M \in \mathcal{M}^h$ there exists $\mathbf{v}^h_M \in \mathcal{V}^h_{0,M}$, extending it with 0 outside of M, such that

$$b(\mathbf{v}_M^h, q^h) + c(\mathbf{v}_M^h, \mu^h) = \sum_{T\subset M} \int_T (q^h \text{ div } \mathbf{v}_M^h + \mu^h \text{ rot } \mathbf{v}_M^h) dV$$

$$\geq C_3(|q^h|_{M,\mathcal{Q}} + |\mu^h|_{M,\mathcal{P}})^2 , \tag{3.16}$$

$$\|\mathbf{v}_M^h\|_1 = \|\mathbf{v}_M^h\|_{1,V} = \|\mathbf{v}_M^h\|_{1,M} \leq |q^h|_{M,\mathcal{Q}} + |\mu^h|_{M,\mathcal{P}} , \tag{3.17}$$

where C_3 is a constant independent of M, h and material properties. □

Let us define

$$\mathbf{v}^h = \sum_{M\in\mathcal{M}^h} \mathbf{v}_M^h , \tag{3.18}$$

and from (M3)–(M5), noting (3.16), we have that

$$b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h) = \sum_{M\in\mathcal{M}^h} [b(\mathbf{v}_M^h, q^h) + c(\mathbf{v}_M^h, \mu^h)]$$

$$\geq \sum_{M\in\mathcal{M}^h} C_3(|q^h|_{M,\mathcal{Q}} + |\mu^h|_{M,\mathcal{P}})^2$$

$$\geq C_3(\|q^h\|_{V,\mathcal{Q}}^2 + \|\mu^h\|_{V,\mathcal{P}}^2) . \tag{3.19}$$

Furthermore from (M4) and (M5), we know that each element $T \in \mathcal{T}^h$ is contained in at most L macroelements. This gives, using Schwarz's inequality and (3.17)

$$\|\mathbf{v}^h\|_1^2 = \left\| \sum_{M\in\mathcal{M}^h} \mathbf{v}_M^h \right\|_1^2 \leq L \sum_{M\in\mathcal{M}^h} \|\mathbf{v}_M^h\|_1^2$$

$$\leq L \sum_{M\in\mathcal{M}^h} (|q^h|_{M,\mathcal{Q}} + |\mu^h|_{M,\mathcal{P}})^2 \leq 2L^2(\|q^h\|_{V,\mathcal{Q}}^2 + \|\mu^h\|_{V,\mathcal{P}}^2) . \tag{3.20}$$

Therefore we obtain, using Schwarz's inequality

$$\sup_{\mathbf{w}^h\in\mathcal{V}_0^h\setminus\{\mathbf{0}\}} \frac{b(\mathbf{w}^h, q^h) + c(\mathbf{w}^h, \mu^h)}{\|\mathbf{w}^h\|_1}$$

$$\geq \frac{b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)}{\|\mathbf{v}^h\|_1} \geq \frac{\sqrt{2}C_3}{2L}(\|q^h\|_{V,\mathcal{Q}}^2 + \|\mu^h\|_{V,\mathcal{P}}^2)^{1/2}$$

$$\geq C_4(\|q^h\|_{V,\mathcal{Q}} + \|\mu^h\|_{V,\mathcal{P}}) , \quad \forall q^h \in \mathcal{Q}_0^h, \mu^h \in \mathcal{P}^h , \tag{3.21}$$

with $C_4 = C_3/(2L)$.

Combining Lemma 3.1 and the above inequality, we have that

$$\begin{aligned}
&\sup_{\mathbf{v}^h \in \mathcal{V}_0^h \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)}{\|\mathbf{v}^h\|_1} \\
&= \xi \sup_{\mathbf{v}^h \in \mathcal{V}_0^h \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)}{\|\mathbf{v}^h\|_1} + (1-\xi) \sup_{\mathbf{v}^h \in \mathcal{V}_0^h \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)}{\|\mathbf{v}^h\|_1} \\
&\geq \xi [C_a(\|q^h\|_0 + \|\mu^h\|_0) - C_b(\|q^h\|_{V,\mathcal{Q}} + \|\mu^h\|_{V,\mathcal{P}})] \\
&\quad + C_4(1-\xi)(\|q^h\|_{V,\mathcal{Q}} + \|\mu^h\|_{V,\mathcal{P}}) \\
&\geq \frac{C_a C_4}{C_b + C_4}(\|q^h\|_0 + \|\mu^h\|_0), \quad \forall q^h \in \mathcal{Q}_0^h, \mu^h \in \mathcal{P}, \qquad (3.22)
\end{aligned}$$

where we choose $\xi = C_4/(C_b + C_4) > 0$. □

4. Numerical Inf–Sup Tests

As defined in Sec. 1, the mass matrix $\mathbf{M}_{uu}$ corresponds to the L^2-norm $\|\cdot\|_0$ on $\mathcal{V}_0^h$, i.e. $\|\mathbf{u}^h\|_0^2 = \frac{1}{\rho}\mathbf{U}^{\mathrm{T}}\mathbf{M}_{uu}\mathbf{U}$, and the matrices $\mathbf{K}_{pp}$ and $\mathbf{K}_{\lambda\lambda}$ correspond to the 0-norm $\|\cdot\|_0$ on $\mathcal{Q}_0^h$ and $\mathcal{P}^h$, respectively. For clarity, we introduce two matrices $\tilde{\mathbf{K}}_{pp} = -\beta\mathbf{K}_{pp}$ and $\tilde{\mathbf{K}}_{\lambda\lambda} = -\alpha\mathbf{K}_{\lambda\lambda}$, such that $\|p^h\|_0^2 = \mathbf{P}^{\mathrm{T}}\tilde{\mathbf{K}}_{pp}\mathbf{P}$ and $\|\lambda^h\|_0^2 = \mathbf{\Lambda}^{\mathrm{T}}\tilde{\mathbf{K}}_{\lambda\lambda}\mathbf{\Lambda}$. Moreover, we define a matrix $\mathbf{S}_{uu}$ to represent the 1-norm $\|\cdot\|_1$ on $\mathcal{V}_0^h$, i.e. $\|\mathbf{u}^h\|_1^2 = \mathbf{U}^{\mathrm{T}}\mathbf{S}_{uu}\mathbf{U}$. In addition, we know that $\mathbf{K}_{up}$ and $\mathbf{K}_{u\lambda}$ are related to the bilinear forms $b(\mathbf{u}^h, p^h)$ and $c(\mathbf{u}^h, \lambda^h)$, i.e.

$$b(\mathbf{u}^h, p^h) = -\mathbf{U}^{\mathrm{T}}\mathbf{K}_{up}\mathbf{P} \text{ and } c(\mathbf{u}^h, \lambda^h) = \mathbf{U}^{\mathrm{T}}\mathbf{K}_{u\lambda}\mathbf{\Lambda}.$$

In the frequency analysis of acoustic fluids, as discussed in Refs. 2 and 19, for the $\mathbf{u}/p$ formulation, we need to solve the following eigenvalue problem for the nonzero eigenvalues,

$$\mathbf{K}_a\mathbf{U} = \omega_a^2\mathbf{M}_{uu}\mathbf{U}, \qquad (4.1)$$

with $\mathbf{K}_a = -\mathbf{K}_{up}\mathbf{K}_{pp}^{-1}\mathbf{K}_{pu}$. On the other hand, for the $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulation, we need to solve the following eigenvalue problem for the nonzero eigenvalues,

$$\mathbf{K}_b\mathbf{U} = \omega_b^2\mathbf{M}_{uu}\mathbf{U}, \qquad (4.2)$$

with $\mathbf{K}_b = -\mathbf{K}_{up}\mathbf{K}_{pp}^{-1}\mathbf{K}_{pu} - \mathbf{K}_{u\lambda}\mathbf{K}_{\lambda\lambda}^{-1}\mathbf{K}_{\lambda u}$ and $\mathbf{K}_{\lambda u} = \mathbf{K}_{u\lambda}^{\mathrm{T}}$.

As discussed in detail in Ref. 19, some of the nonzero eigenvalues of (4.2), if $\alpha \gg \beta$, located in the higher spectrum and representing the rotational modes, are in fact zero frequency modes in (4.1). Obviously, using (4.2) instead of (4.1) can significantly reduce the number of zero frequency modes.

Recall that in the numerical inf–sup tests for both the $\mathbf{u}/p$ and $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulations, as presented in Ref. 20, we need to solve two similar eigenvalue problems by replacing the mass matrix $\mathbf{M}_{uu}$ with $\mathbf{S}_{uu}$, and modifying the stiffness matrices, i.e. we solve

$$\tilde{\mathbf{K}}_a \mathbf{U} = \lambda_a \mathbf{S}_{uu} \mathbf{U} \,, \tag{4.3}$$

and

$$\tilde{\mathbf{K}}_b \mathbf{U} = \lambda_b \mathbf{S}_{uu} \mathbf{U} \,, \tag{4.4}$$

where $\tilde{\mathbf{K}}_a = \frac{1}{\beta}\mathbf{K}_a$ and $\tilde{\mathbf{K}}_b = -\frac{1}{\beta}\mathbf{K}_{up}\mathbf{K}_{pp}^{-1}\mathbf{K}_{pu} - \frac{1}{\alpha}\mathbf{K}_{u\lambda}\mathbf{K}_{\lambda\lambda}^{-1}\mathbf{K}_{\lambda u}$.

Notice that the matrices $\mathbf{K}_a$, $\tilde{\mathbf{K}}_a$, $\mathbf{K}_b$, $\tilde{\mathbf{K}}_b$, $\mathbf{M}_{uu}$ and $\mathbf{S}_{uu}$ are all symmetric. In addition, $\mathbf{M}_{uu}$ and $\mathbf{S}_{uu}$ are positive definite. To have stable and reliable finite element discretizations for both mixed formulations, we must have

$$\inf_{q^h \in \mathcal{Q}_0^h \setminus \{0\}} \sup_{\mathbf{v}^h \in \mathcal{V}_0^h \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1 \|q^h\|_0} = \sqrt{\hat{\lambda}_a(h)} \,, \tag{4.5}$$

with $\lim_{h\to 0} \sqrt{\hat{\lambda}_a(h)} = \sqrt{a_\lambda} > 0$; and

$$\inf_{(q^h,\mu^h) \in \mathcal{Q}_0^h \times \mathcal{P}^h \setminus \{(0,0)\}} \sup_{\mathbf{v}^h \in \mathcal{V}_0^h \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}^h, q^h) + c(\mathbf{v}^h, \mu^h)}{\|\mathbf{v}^h\|_1 (\|q^h\|_0 + \|\mu^h\|_0)} = \sqrt{\hat{\lambda}_b(h)} \,, \tag{4.6}$$

with $\lim_{h\to 0} \sqrt{\hat{\lambda}_b(h)} = \sqrt{b_\lambda} > 0$, where $\hat{\lambda}_a(h)$ and $\hat{\lambda}_b(h)$ are the smallest nonzero eigenvalues of the problems (4.3) and (4.4), respectively; and a_λ and b_λ are positive constants independent of h (and of course, material properties).

Comparing Eq. (4.1) with Eq. (4.3), it is easy to confirm that

$$\inf_{q^h \in \mathcal{Q}_0^h \setminus \{0\}} \sup_{\mathbf{v}^h \in \mathcal{V}_0^h \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_0 \|q^h\|_0} = \frac{\sqrt{\rho}}{\sqrt{\beta}} \hat{\omega}_a(h) \,,$$

where $\hat{\omega}_a$ is the smallest nonzero eigenvalue of the problem (4.1).

Since $\|\mathbf{v}^h\|_0 \le \|\mathbf{v}^h\|_1$, $\forall \mathbf{v}^h \in \mathcal{V}_0^h$, we obtain

$$\frac{\sqrt{\rho}}{\sqrt{\beta}} \hat{\omega}_a(h) \ge \sqrt{\hat{\lambda}_a(h)} \,.$$

Thus, we have

$$\lim_{h\to 0} \hat{\omega}_a(h) \ge \frac{\sqrt{\beta a_\lambda}}{\sqrt{\rho}} > 0 \,,$$

and in addition, because of the same matrix structure of $\mathbf{K}_a$ and $\tilde{\mathbf{K}}_a$, the number of zero frequencies of the eigenvalue problems (4.1) and (4.3) are the same. Therefore, we can in this case simply calculate the lowest frequency of the problem in (4.1) for increasingly refined meshes, and if this frequency approaches zero, the inf–sup test is not passed. On the other hand, if the smallest frequency does not approach zero, we cannot, strictly, say anything about the discretization scheme.

Considering the **u**-p-$\mathbf{\Lambda}$ formulation, assume that $\alpha = K\beta$, then we have $\mathbf{K}_b = \beta(\mathbf{K}_{up}\tilde{\mathbf{K}}_{pp}^{-1}\mathbf{K}_{pu} + K\mathbf{K}_{u\lambda}\tilde{\mathbf{K}}_{\lambda\lambda}^{-1}\mathbf{K}_{\lambda u})$ and $\tilde{\mathbf{K}}_b = \mathbf{K}_{up}\tilde{\mathbf{K}}_{pp}^{-1}\mathbf{K}_{pu} + \mathbf{K}_{u\lambda}\tilde{\mathbf{K}}_{\lambda\lambda}^{-1}\mathbf{K}_{\lambda u}$.

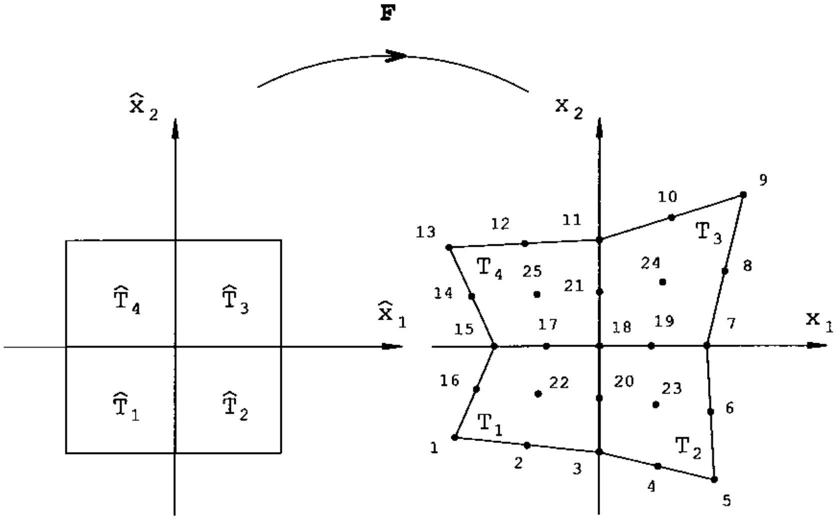

Fig. 1. A typical macroelement.

Therefore, it is obvious that we could only have a similar relationship when $K = 1$, based then on $\mathbf{K}_b = \beta\tilde{\mathbf{K}}_b$, between the numerical inf–sup test value and the vibration frequency for the $\mathbf{u}$-p-$\mathbf{\Lambda}$ formulation. However, in practice, we want that $K \gg 1$.

5. Applications

In this section, we will apply the results discussed in Sec. 3 to two examples.

Example 5.1. The 9-3-1 element.

Let $\mathcal{T}^h$ be a partition of V comprised of convex quadrilateral elements and the finite element spaces be defined as

$$\mathcal{V}_0^h = \{\mathbf{v}^h \in \mathcal{V}_0 | \mathbf{v}^h|_T \in [Q_2(T)]^2\,, \quad T \in \mathcal{T}^h\}\,,$$
$$\mathcal{Q}_0^h = \{q^h \in \mathcal{Q}_0 | q^h|_T \in P_1(T)\,, \quad T \in \mathcal{T}^h\}\,,$$
$$\mathcal{P}^h = \{\mu^h \in \mathcal{P} | \mu^h|_T \in P_0(T)\,, \quad T \in \mathcal{T}^h\}\,.$$

Suppose that the partition $\mathcal{T}^h$ satisfies: For each $T \in \mathcal{T}^h$, there exists at least one node of T in V. Then for this method the macroelement condition in Lemma 3.2 is valid for a macroelement consisting of four elements, as shown in Fig. 1.

To prove this we consider a macroelement $M = T_1 \cup T_2 \cup T_3 \cup T_4$ and the corresponding reference macroelement $\hat{M} = \hat{T}_1 \cup \hat{T}_2 \cup \hat{T}_3 \cup \hat{T}_4$ as shown in Fig. 1. Let $\mathbf{F} = (F_1, F_2)$ be the continuous piecewise bilinear mapping from $\hat{M}$ onto M. Suppose $(q^h, \mu^h) \in \mathcal{N}_M$, i.e.

$$(\text{div } \mathbf{v}^h, q^h)_M + (\text{rot } \mathbf{v}^h, \mu^h)_M = 0\,, \quad \forall \mathbf{v}^h \in \mathcal{V}_{0,M}^h\,.$$

Using integration by parts, we have

$$\begin{aligned}(\text{div } \mathbf{v}^h, q^h)_M + (\text{rot } \mathbf{v}^h, \mu^h)_M &= -(\mathbf{v}^h, \text{grad } q^h)_M + (\mathbf{v}^h, \text{rot } \mu^h)_M \\ &\quad + \sum_{E \in \ell^h_{M,V}} \int_E [\mathbf{v}^h \cdot \mathbf{n}_T([q^h]) + \mathbf{v}^h \cdot \boldsymbol{\tau}_T([\mu^h])] dS \\ &\quad + \sum_{E \in \ell^h_{M,S}} \int_E \mathbf{v}^h \cdot \boldsymbol{\tau}_T \mu^h dS = 0 \,. \end{aligned} \tag{5.1}$$

Since grad $q^h|_{Ti}$ for $i = 1, \ldots, 4$ are constants, $\mathbf{v}^h$ grad $q^h|_{Ti} \in Q_2(Ti)$ for $i = 1, \ldots, 4$ and the composite integration (nine-point integration) gives the exact values for the integrals $(\mathbf{v}^h, \text{grad } q^h)_{Ti}$. Following the procedure of Ref. 18 and choosing $\mathbf{v}^h \in \mathcal{V}^h_{0,M}$ such that the only nonvanishing degrees of freedom are the values of both components at the center nodes $\mathbf{x}^{22}$, $\mathbf{x}^{23}$, $\mathbf{x}^{24}$ and $\mathbf{x}^{25}$ of T_1, T_2, T_3 and T_4, respectively, we obtain grad $q^h|_{Ti} = 0$ for $i = 1, \ldots, 4$, which implies that both q^h and μ^h are piecewise constant. Then we choose $\mathbf{v}^h \in \mathcal{V}^h_{0,M}$ such that the only nonvanishing degrees of freedom are $\mathbf{v}^h \cdot \mathbf{n}_1$, $\mathbf{v}^h \cdot \mathbf{n}_2$, $\mathbf{v}^h \cdot \mathbf{n}_3$ and $\mathbf{v}^h \cdot \mathbf{n}_4$, where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ are the normals of the segments $\overline{\mathbf{x}^3\mathbf{x}^{18}}$, $\overline{\mathbf{x}^7\mathbf{x}^{18}}$, $\overline{\mathbf{x}^{11}\mathbf{x}^{18}}$ and $\overline{\mathbf{x}^{15}\mathbf{x}^{18}}$, evaluated at the points $\mathbf{x}^{20}$, $\mathbf{x}^{19}$, $\mathbf{x}^{21}$ and $\mathbf{x}^{17}$, respectively, it is not difficult to confirm that q^h is constant on M. Therefore,

$$\begin{aligned}&(\text{div } \mathbf{v}^h, q^h)_M + (\text{rot } \mathbf{v}^h, \mu^h)_M \\ &\quad = \sum_{E \in \ell^h_{M,V}} \int_E \mathbf{v}^h \cdot \boldsymbol{\tau}_T([\mu^h]) ds + \sum_{E \in \ell^h_{M,S}} \int_E \mathbf{v}^h \cdot \boldsymbol{\tau}_T \mu^h dS = 0 \,. \end{aligned} \tag{5.2}$$

Now we choose $\mathbf{v}^h \in \mathcal{V}^h_{0,M}$ such that the only nonvanishing degrees of freedom are $\mathbf{v}^h \cdot \boldsymbol{\tau}_1$, $\mathbf{v}^h \cdot \boldsymbol{\tau}_2$, $\mathbf{v}^h \cdot \boldsymbol{\tau}_3$ and $\mathbf{v}^h \cdot \boldsymbol{\tau}_4$, where $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$ and $\boldsymbol{\tau}_4$ are tangents of the segments $\overline{\mathbf{x}^3\mathbf{x}^{18}}$, $\overline{\mathbf{x}^7\mathbf{x}^{18}}$, $\overline{\mathbf{x}^{11}\mathbf{x}^{18}}$ and $\overline{\mathbf{x}^{15}\mathbf{x}^{18}}$, evaluated at the points $\mathbf{x}^{20}$, $\mathbf{x}^{19}$, $\mathbf{x}^{21}$ and $\mathbf{x}^{17}$, respectively. We confirm that μ^h is constant on M. If $\partial M \cap S = \emptyset$, we obtain the desired macroelement condition, including the fact that $\mathcal{N}_M$ is two-dimensional. If $\partial M \cap S \neq \emptyset$, without loss of generality, suppose $E_0 = \overline{\mathbf{x}^1\mathbf{x}^3} \in \ell^h_{M,S}$. We need to choose $\mathbf{v}^h \in \mathcal{V}^h_{0,M}$ such that the only nonvanishing degree of freedom is $\mathbf{v}^h \cdot \boldsymbol{\tau}_{E_0}$ evaluated at the point $\mathbf{x}^2$. Then we obtain $\mu^h = 0$ on M, i.e. $\mathcal{N}_M$ is one-dimensional. Thus the desired macroelement condition is proved for a macroelement of four elements, and based on Lemma 3.2 and Theorem 3.1, the inf–sup condition of (2.20) is satisfied.

Example 5.2. The 9-4c-1 element.

Again, here for the sake of simplicity, we consider a special case. For the general case, one has to check the macroelement condition numerically.

We assume V to be a rectangle and $\mathcal{T}^h$ be a partition of V containing squares with the same size.

$$\mathcal{V}_0^h = \{\mathbf{v}^h \in \mathcal{V}_0 | \mathbf{v}|_T \in [Q_2(T)]^2 , \quad T \in \mathcal{T}^h\} ,$$
$$\mathcal{Q}_0^h = \{q^h \in \mathcal{Q}_0 \cap C(V) | q^h|_T \in Q_1(T) , \quad T \in \mathcal{T}^h\} ,$$
$$\mathcal{P}^h = \{\mu^h \in \mathcal{P} | \mu^h|_T \in P_0(T) , \quad T \in \mathcal{T}^h\} .$$

Suppose the partition $\mathcal{T}^h$ satisifes: For each $T \in \mathcal{T}^h$, there exists at least one node of $\mathcal{T}$ in V. Then, we proceed to prove that the macroelement condition in Lemma 3.2 is valid for a macroelement consisting of four elements.

Following the procedure above, we choose $\mathbf{v}^h \in \mathcal{V}_{0,M}^h$ such that the only nonvanishing degrees of freedom are the values of both components at the center points of T_1, T_2, T_3 and T_4, respectively, as shown in Fig. 1. Since here the elements T_1, T_2, T_3 and T_4 are squares, from grad $q^h = 0$, we get

$$q^h(\mathbf{x}^1) = q^h(\mathbf{x}^{18}) = q^h(\mathbf{x}^5) = q^h(\mathbf{x}^9) = q^h(\mathbf{x}^{13}) = a$$

and

$$q^h(\mathbf{x}^3) = q^h(\mathbf{x}^7) = q^h(\mathbf{x}^{11}) = q^h(\mathbf{x}^{15}) = b ,$$

where a and b are constants.

Then we choose $\mathbf{v}^h \in \mathcal{V}_{0,M}^h$ such that the only nonvanishing degrees of freedom are $\mathbf{v}^h \cdot \boldsymbol{\tau}_1$, $\mathbf{v}^h \cdot \boldsymbol{\tau}_2$, $\mathbf{v}^h \cdot \boldsymbol{\tau}_3$ and $\mathbf{v}^h \cdot \boldsymbol{\tau}_4$, evaluated at the points $\mathbf{x}^{20}$, $\mathbf{x}^{19}$, $\mathbf{x}^{21}$ and $\mathbf{x}^{17}$, respectively. Without much difficulty, using $\mathbf{v}^h \cdot \text{grad } q^h = \mathbf{v}^h \cdot \mathbf{n}\frac{\partial q^h}{\partial n} + \mathbf{v}^h \cdot \boldsymbol{\tau}\frac{\partial q^h}{\partial \tau}$, we obtain $a = b$ and $\mu^h|_{T1} = \mu^h|_{T2} = \mu^h|_{T3} = \mu^h|_{T4} =$ constant. Therefore, q^h and μ^h are always constant functions on M. If $\partial M \cap \partial V \neq \emptyset$, we suppose that one edge, say E_0, of M is such that $E_0 = \overline{\mathbf{x}^1\mathbf{x}^3} \in \ell_{M,S}^h$. Then we choose $\mathbf{v}^h \in \mathcal{V}_{0,M}^h$ such that the only nonvanishing degree of freedom is $\mathbf{v}^h \cdot \boldsymbol{\tau}_{E_0}$ evaluated at the point $\mathbf{x}^2$, and obtain $\mu^h = 0$ on M, i.e. $\mathcal{N}_M$ is one-dimensional. Hence, the desired macroelement condition is proved for a macroelement of four elements and the inf–sup condition (2.20) is satisfied based on Lemma 3.2 and Theorem 3.1.

6. Conclusions

In this paper, we have considered some mixed finite element formulations for an acoustic fluid and have proven that certain finite element discretizations satisfy the inf–sup condition of solvability, stability, and optimal error bounds. The ellipticity condition can be satisfied by using a small shear (material) constant or by using regularization parameters. The procedure employed herein to prove the inf–sup condition can also directly be applied to other discretizations of the mixed finite element formulations considered in the paper. We have also related the numerical inf–sup values calculated in the inf–sup test to the free vibration frequencies of the problem considered. The free vibration frequencies give only insight in some special cases whether an element discretization might be reliable.

Finally, readers are referred to Refs. 4 and 5 for more discussion on the convergence of eigenvalues of mixed finite element formulations.

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