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Error bounds for the finite-element approximation of the exterior Stokes equations in two dimensions

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In this paper we design high-order (non)local artificial boundary conditions (ABCs) which are different from those proposed by Han, H. & Bao, W. (1997 *Numer. Math.*, **77**, 347–363) for incompressible materials, and present error bounds for the finite-element approximation of the exterior Stokes equations in two dimensions. The finite-element approximation (especially its corresponding stiff matrix) becomes much simpler (sparser) when it is formulated in a bounded computational domain using the new (non)local approximate ABCs. Our error bounds indicate how the errors of the finite-element approximations depend on the mesh size, terms used in the approximate ABCs and the location of the artificial boundary. Numerical examples of the exterior Stokes equations outside a circle in the plane are presented. Numerical results demonstrate the performance of our error bounds.

Keywords: exterior Stokes equations; finite-element approximation; artificial boundary; (non)local artificial boundary condition (ABC); error bounds.

1. Introduction

Let Γ_i be a smooth bounded simple closed curve in $\mathbb{R}^2$ and Ω be the exterior domain with the boundary Γ_i (see Fig. 1). We consider the following boundary value problem:
(P) Find $(\mathbf{u},\ p)$ such that

$$-\nu\,\Delta\mathbf{u} + \text{ grad } p = \mathbf{f} \qquad \text{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_i, \quad \mathbf{u} \text{ is bounded}, \quad p \to 0, \quad \text{when } r = |\mathbf{x}| \equiv \sqrt{x_1^2 + x_2^2} \to +\infty. \tag{1.2}$$

Here $\mathbf{x} = (x_1, x_2)$ is the Cartesian coordinate system, polar coordinate system is (r, θ), $\mathbf{u} = (u_1, u_2)^T$ is the velocity, p is the pressure, $\nu > 0$ is the viscosity constant and $\mathbf{f} = (f_1,\ f_2)^T$ is a given function with compact support.

The well-posedness of the exterior Stokes problem (P) is treated in Girault & Sequeira (1991). In fact, an equivalent form of (P) is also used to model incompressible materials (Brezzi & Fortin, 1991), i.e.

$$-2\nu\ \text{div}\,\varepsilon(\mathbf{u}) + \text{ grad } p = \mathbf{f}, \qquad \text{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.3}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_i^1, \quad \tilde{\sigma}_{\mathbf{n}}(\mathbf{u}, p) = 2\nu\varepsilon(\mathbf{u})\mathbf{n} - p\mathbf{n} = \mathbf{g}, \quad \text{on } \Gamma_i^2, \tag{1.4}$$

$$\mathbf{u} \text{ is bounded} \quad p \to 0, \quad \text{when } r = |\mathbf{x}| \equiv \sqrt{x_1^2 + x_2^2} \to +\infty; \tag{1.5}$$

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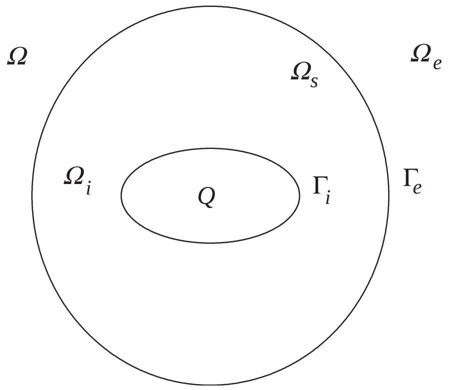

FIG. 1. Set-up of the domain and artificial boundary.

where **n** is the unit outward normal vector, $\mathbf{g} = (g_1, g_2)^T$ is a given function, $\Gamma_i = \Gamma_i^1 \cup \Gamma_i^2$ with $\Gamma_i^1 \cap \Gamma_i^2 = \emptyset$, and $\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))_{i,j=1}^2$ is the strain tensor corresponding to the displacement **u**, which is given by

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad 1 \leqslant i, j \leqslant 2. \tag{1.6}$$

In finding numerical solutions of the problem (P) or (1.3)–(1.5), one of the main numerical difficulties is the unboundedness of the domain Ω which means that the classical finite-element (FEM) or finite-difference method (FDM) can **not** be used directly. In the last two decades, several methods have been proposed to solve boundary value problems in unbounded domains (Givoli, 1992); for example, the boundary element method (BEM) (Kohr, 1997), the infinite-element method (Ying, 1995), coupling of BEM and FEM (Antonio & Meddahi, 2001; Meddahi & Sayas, 2000), etc. One of the most popular methods is to introduce an artificial boundary and set up ABCs on it. Then the original problem is reduced to a boundary value problem in a bounded computational domain. Thus a numerical approximation of the original problem can be obtained by solving the reduced problem. In recent years, many authors have worked on this subject for various problems by different techniques, see Engquist & Majda (1977); Feng (1984); Halpern & Schatzman (1989); Bao (1998, 1997, 2000); Bao & Han (1996); Bao & Xin (2000); Han *et al.* (1994) and references therein.

In the above works, two types of ABC were designed: nonlocal and local ABCs. Each type has its own advantages and disadvantages. For nonlocal ABCs, it is very easy to design high-order approximate boundary conditions, but the stiff matrix of the finite-element approximation for the original problem by using a nonlocal ABC becomes much denser than that of a local ABC. On the other hand, it is usually not easy to implement high-order local boundary conditions because the high-order derivatives (usually higher than second-order) will appear in the conditions (see (3.4)). Furthermore, several authors also gave error estimates for the numerical solution, see Han & Wu (1992) and Givoli *et al.* (1997). But their error estimates only depend on the mesh size and the approximate ABCs. How the error depends on the location of the artificial boundary is unknown. But this is a very interesting problem for engineers. Recently we got new error bounds which depend not only on the mesh size and terms used in the approximate ABCs but also on the location of the artificial boundary for the finite-element approximation of elliptic equations (Han & Bao, 2000; Bao & Han, 2000) and the linear elastic system (Han & Bao, 2001) in an unbounded domain. The key idea was to use an equivalent norm in the error analysis.

In (1.4), if $|\Gamma_i^2| \neq 0$, the variational formulation of the Stokes problem must be based on the partial differential equations (PDEs) (1.3) in order to cope with the normal stress boundary condition (1.4). In this case, to design the ABCs at a given artificial boundary Γ_e should be based on the continuity of the displacement and normal stress related to (1.3), i.e.

$$\mathbf{u}|_{\Gamma_e}, \qquad \tilde{\sigma}_{\mathbf{n}}(\mathbf{u}, p)|_{\Gamma_e} = 2\nu\varepsilon(\mathbf{u})\mathbf{n} - p\mathbf{n}|_{\Gamma_e}. \tag{1.7}$$

This kind of ABC for incompressible materials was studied by Han & Bao (1997) and error bounds for the finite-element approximation were also given there (Han & Bao, 1997). In contrast, if $|\Gamma_i^2| = 0$, i.e. pure Dirichlet boundary condition (1.2) is posed on the inside boundary Γ_i, the variational formulation can be obtained from the PDEs (1.1) instead of (1.3). Thus the variational formulation itself and especially its corresponding stiff matrix will become much simpler and sparser, respectively. In this case, to design ABCs at a given artificial boundary Γ_e must be based on the continuity of the velocity and normal stress related to (1.1), i.e.

$$\mathbf{u}|_{\Gamma_e}, \qquad \sigma_{\mathbf{n}}(\mathbf{u}, p)|_{\Gamma_e} = \nu\frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n}\bigg|_{\Gamma_e}. \tag{1.8}$$

The purpose of this paper is to design high-order (non)local ABCs for the exterior Stokes problem (P) based on (1.8), which are different from those proposed by Han & Bao (1997) for incompressible materials, and provide error estimates for the finite-element approximation of (P). The advantage of the new ABCs is such that the stiff matrix of the finite-element approximation of (P) becomes much sparser and thus could save computational time in solving the corresponding linear system. Our error estimates depend on not only the mesh size and the approximate ABCs but also the location of the artificial boundary. They can be used to determine how to choose the mesh size, terms used in the ABCs and the location of the artificial boundary for practical computations.

The layout of this paper is as follows. In Section 2 we design a family of high-order nonlocal ABCs at a given artificial boundary for the problem (P). In Section 3 we propose a family of high-order local ABCs. In Section 4 we introduce the finite-element formulation of the problem (P) in a bounded computational domain using an approximate nonlocal ABC and prove error bounds for the finite-element approximation. In Section 5 similar results for using high-order local ABCs are presented. In Section 6 we report on some numerical experiments. Finally, in Section 7 we draw some conclusions.

2. High-order nonlocal ABCs

We introduce a circle Γ_e with radius R such that supp $\mathbf{f} \subset B_R(\mathbf{0}) := \{\mathbf{x} \in \mathbb{R}^2 \ : \ |\mathbf{x}| < R\}$, then Ω is divided into two parts: the unbounded part $\Omega_e := \Omega \setminus B_R(\mathbf{0})$ and the bounded part $\Omega_i := \Omega \setminus \bar{\Omega}_e$ (see Fig. 1). The restriction of the solution $(\mathbf{u},\ p)$ of the problem (P) to the unbounded domain Ω_e is then the solution of the following problem:

(P$_e$) Find $(\mathbf{u},\ p)$ such that

$$-\nu\ \Delta\mathbf{u} + \text{ grad } p = \mathbf{0} \qquad \text{div } \mathbf{u} = 0, \qquad \text{in } \Omega_e, \tag{2.1}$$

$$\mathbf{u}|_{\Gamma_e} = \mathbf{u}(R,\theta) \quad \mathbf{u} \text{ is bounded}, \quad p \to 0, \quad \text{when } r = |\mathbf{x}| \equiv \sqrt{x_1^2 + x_2^2} \to +\infty. \tag{2.2}$$

The general solution of (P$_e$) is (see Han & Bao, 1997, pp. 349-351 for details)

$$u_i(r,\theta) = (r^2 - R^2)\frac{\partial W(r,\theta)}{\partial x_i} + G_i(r,\theta), \quad R \leqslant r < +\infty, \quad 0 \leqslant \theta \leqslant 2\pi, \quad i = 1, 2, \tag{2.3}$$

where G_1, G_2 and W are harmonic functions satisfying

$$G_i(r,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{\infty}\left(a_n^i \cos n\theta + b_n^i \sin n\theta\right)\frac{R^n}{r^n}, \qquad R \leqslant r < +\infty, \quad 0 \leqslant \theta \leqslant 2\pi, \qquad i = 1, 2, \tag{2.4}$$

$$W(r,\theta) = -\sum_{n=2}^{\infty}\frac{n-1}{2n}\left[p_n \cos n\theta + q_n \sin n\theta\right]\frac{R^{n-1}}{r^n}, \qquad R \leqslant r < +\infty, \quad 0 \leqslant \theta \leqslant 2\pi; \tag{2.5}$$

with

$$a_n^i = \frac{1}{\pi}\int_0^{2\pi} G_i(R,\theta)\cos n\theta \,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} u_i(R,\theta)\cos n\theta \,\mathrm{d}\theta, \qquad i = 1, 2, \quad n \geqslant 0, \tag{2.6}$$

$$b_n^i = \frac{1}{\pi}\int_0^{2\pi} G_i(R,\theta)\sin n\theta \,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{2\pi} u_i(R,\theta)\sin n\theta \,\mathrm{d}\theta, \qquad i = 1, 2, \quad n \geqslant 1, \tag{2.7}$$

$$p_n = a_{n-1}^1 - b_{n-1}^2, \qquad q_n = b_{n-1}^1 + a_{n-1}^2, \qquad n \geqslant 2\cdot \tag{2.8}$$

Combining (2.3)–(2.5) and (2.1) on noting the boundary condition at infinity for p in (2.2), a computation shows

$$p(r,\theta) = 2\nu\sum_{n=2}^{\infty}(n-1)\left[p_n \cos n\theta + q_n \sin n\theta\right]\frac{R^{n-1}}{r^n}, \qquad R \leqslant r < +\infty, \quad 0 \leqslant \theta \leqslant 2\pi. \tag{2.9}$$

Applying the standard method for elliptic problems (Givoli, 1992; Han & Bao, 2000), Helmholtz-type equations (Goldstein, 1982), the linear elastic system (Han & Wu, 1992; Han & Bao, 2001) etc., we now design the nonlocal ABCs for (P) at Γ_e with the transmission conditions

$$\mathbf{u}|_{r=R^+} = \mathbf{u}|_{r=R^-}, \qquad \sigma_{\mathbf{n}}(\mathbf{u}, p)|_{r=R^+} = \sigma_{\mathbf{n}}(\mathbf{u}, p)|_{r=R^-}, \qquad \text{on } \Gamma_e, \tag{2.10}$$

where $\sigma_{\mathbf{n}} = (\sigma_{n_1}, \sigma_{n_2})^T$ is the normal stress defined in (1.8) and related to (1.1) with $\mathbf{n} = (\cos\theta, \sin\theta)^T$ the unit outward normal vector at Γ_e. In (2.10) $\pm$ refer to the limits from Ω_e and Ω_i, respectively. The transmission condition (2.10) was also used in Halpern (1996, 2001) for numerical solutions of the exterior Stokes problem defined in the whole plane by using the spectral method. Combining (1.8), (2.10), noting (2.3), (2.4), (2.9), (2.6) and (2.7), one obtains

$$\begin{aligned}
\sigma_{n_1} &= \cos\theta\left(\nu\frac{\partial u_1}{\partial x_1} - p\right) + \nu\sin\theta\frac{\partial u_1}{\partial x_2}\Big|_{\Gamma_e} \\
&= \left[\nu\cos\theta\left(\cos\theta\frac{\partial u_1}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u_1}{\partial\theta}\right) - p\right] + \nu\sin\theta\left(\sin\theta\frac{\partial u_1}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u_1}{\partial\theta}\right)\Big|_{\Gamma_e} \\
&= \nu\frac{\partial u_1}{\partial r} - p\ \cos\theta\Big|_{\Gamma_e} = \nu\left[\frac{\partial G_1}{\partial r} + 2r\frac{\partial W}{\partial x_1} + (r^2 - R^2)\frac{\partial^2 W}{\partial r\,\partial x_1}\right] - p\ \cos\theta\Big|_{\Gamma_e} \\
&= \nu\frac{\partial G_1}{\partial r} + 2r\nu\left(\cos\theta\frac{\partial W}{\partial r} - \frac{\sin\theta}{r}\frac{\partial W}{\partial\theta}\right) - 4\nu r\cos\theta\frac{\partial W}{\partial r}\Big|_{\Gamma_e} \\
&= \nu\frac{\partial G_1}{\partial r} - 2\nu\left(x_1\frac{\partial W}{\partial r} + \frac{x_2}{r}\frac{\partial W}{\partial\theta}\right)\Big|_{\Gamma_e}.
\end{aligned} \tag{2.11}$$

Plugging (2.5) into (2.11), noting (2.8), one has

$$\begin{aligned}
\sigma_{n_1} &= \nu\frac{\partial G_1}{\partial r}\Big|_{\Gamma_e} - 2\nu\left[R\cos\theta\sum_{n=2}^{\infty}\frac{n-1}{2}\left(p_n\cos n\theta + q_n\sin n\theta\right)\frac{1}{R^2}\right. \\
&\quad \left. -\sin\theta\sum_{n=2}^{\infty}\frac{n-1}{2}\left(-p_n\sin n\theta + q_n\cos n\theta\right)\frac{1}{R}\right] \\
&= \nu\frac{\partial G_1}{\partial r}\Big|_{r=R} - \nu\sum_{n=1}^{\infty}\frac{n}{R}(p_{n+1}\cos n\theta + q_{n+1}\sin n\theta) = \nu\frac{\partial G_1}{\partial r} + \nu\frac{\partial G_1}{\partial r} + \frac{\nu}{R}\frac{\partial G_2}{\partial\theta}\Big|_{r=R} \\
&= 2\nu\frac{\partial G_1}{\partial r} + \frac{\nu}{R}\frac{\partial G_2}{\partial\theta}\Big|_{r=R} = \frac{\nu}{R}\frac{\partial u_2}{\partial\theta}(R,\theta) - \frac{2\nu}{\pi R}\sum_{n=1}^{\infty} n\int_0^{2\pi} u_1(R,\phi)\cos n(\phi-\theta)\,\mathrm{d}\phi \\
&= -\frac{\nu}{\pi R}\sum_{n=1}^{\infty} n\int_0^{2\pi}\left[2u_1(R,\phi)\cos n(\phi-\theta) - u_2(R,\phi)\sin n(\phi-\theta)\right]\mathrm{d}\phi \\
&= \frac{\nu}{\pi R}\sum_{n=1}^{\infty}\frac{\partial}{\partial\theta}\int_0^{2\pi}\left[\frac{2\cos n(\phi-\theta)}{n}\frac{\partial u_1(R,\phi)}{\partial\phi} - \frac{\sin n(\phi-\theta)}{n}\frac{\partial u_2(R,\phi)}{\partial\phi}\right]\mathrm{d}\phi \\
&\equiv T_1(\mathbf{u}).
\end{aligned} \tag{2.12}$$

Similarly, we obtain

$$\sigma_{n_2} = \frac{\nu}{\pi R} \sum_{n=1}^{\infty} \frac{\partial}{\partial \theta} \int_0^{2\pi} \left[\frac{2\cos n(\phi-\theta)}{n} \frac{\partial u_2(R,\phi)}{\partial \phi} + \frac{\sin n(\phi-\theta)}{n} \frac{\partial u_1(R,\phi)}{\partial \phi} \right] \mathrm{d}\phi$$
$$\equiv T_2(\mathbf{u}). \tag{2.13}$$

It is easy to see that the boundary conditions (2.12), (2.13) are different from those designed in Han & Bao (1997). In fact, (2.12)–(2.13) are the exact boundary conditions at Γ_e for the problem (P). Thus the restriction of the solution $(\mathbf{u},\ p)$ of the problem (P) to the bounded domain Ω_i is the solution of the following problem:
(P_i) Find $(\mathbf{u},\ p)$ such that

$$-\nu\, \Delta \mathbf{u} + \text{ grad } p = \mathbf{f} \qquad \text{div}\, \mathbf{u} = 0, \qquad \text{in } \Omega_i, \tag{2.14}$$
$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_i, \quad \sigma_{\mathbf{n}}(\mathbf{u}, p) = T(\mathbf{u}) \equiv (T_1(\mathbf{u}), T_2(\mathbf{u}))^T, \quad \text{on} \quad \Gamma_e. \tag{2.15}$$

In the exact boundary conditions (2.12)–(2.13), there are infinite terms in the right-hand sides. When we solve the problem numerically, we need to truncate the right-hand sides of (2.12)–(2.13) into finite terms. Let

$$T_i^N(\mathbf{u}) = \frac{\nu}{\pi R} \sum_{n=1}^{N} \frac{\partial}{\partial \theta} \int_0^{2\pi}$$
$$\left[\frac{2\cos n(\phi-\theta)}{n} \frac{\partial u_i(R,\phi)}{\partial \phi} + (-1)^i \frac{\sin n(\phi-\theta)}{n} \frac{\partial u_{i+(-1)^{i-1}}(R,\phi)}{\partial \phi} \right] \mathrm{d}\phi$$
$$\equiv T_i^N(\mathbf{u}), \qquad i = 1, 2. \tag{2.16}$$

Then we derive a series of approximate ABCs at Γ_e:

$$\sigma_n(\mathbf{u}, p) = T^N(\mathbf{u}) \equiv \left(T_1^N(\mathbf{u}), T_2^N(\mathbf{u}) \right)^T, \qquad \text{on } \Gamma_e, \qquad N = 0, 1, 2, \ldots, \tag{2.17}$$

where $T^0(\mathbf{u}) = (0,0)^T$ is the stress free boundary condition which is often used in engineering literature. Then the original problem (P) can be reduced to the following problem defined in the bounded domain Ω_i approximately for $N = 0, 1, 2, \ldots$:
(P_N) Find $(\mathbf{u}^N,\ p^N)$ such that

$$-\nu\, \Delta \mathbf{u}^N + \text{ grad } p^N = \mathbf{f} \qquad \text{div}\, \mathbf{u}^N = 0, \qquad \text{in } \Omega_i, \tag{2.18}$$
$$\mathbf{u}^N = \mathbf{0} \quad \text{on } \Gamma_i, \quad \sigma_{\mathbf{n}}(\mathbf{u}^N, p^N) = T^N(\mathbf{u}^N), \quad \text{on} \quad \Gamma_e. \tag{2.19}$$

3. High-order local ABCs

In this section we design high-order local ABCs at Γ_e for the problem (P). We consider a solution $(\mathbf{u},\ p)$ of (P), which consists of the first N harmonics at Γ_e. Thus we assume

$$u_i(R,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{N} \left(a_n^i \cos n\theta + b_n^i \sin n\theta \right), \qquad i = 1, 2; \tag{3.1}$$

TABLE 1 *The coefficients $\alpha_m^{(N)}$ in the first five local ABCs*

	$\alpha_1^{(N)}$	$\alpha_2^{(N)}$	$\alpha_3^{(N)}$	$\alpha_4^{(N)}$	$\alpha_5^{(N)}$
$N = 1$	1				
$N = 2$	7/6	−1/6			
$N = 3$	74/60	−15/60	1/60		
$N = 4$	533/420	−43/144	11/360	−1/1008	
$N = 5$	3881/3780	−214/643	71/1728	−13/6048	1/25920

where the a_n^1, b_n^1, a_n^2 and b_n^2 are constants (Fourier coefficients, see (2.6) and (2.7)). Substituting (3.1) into (1.8), we obtain

$$\sigma_{n_i}(\mathbf{u}, p) = -\frac{2\nu}{R}\sum_{n=1}^{N} n\left(a_n^i \cos n\theta + b_n^i \sin n\theta\right) + (-1)^{i-1}\frac{\nu}{R}\frac{\partial u_{i+(-1)^{i-1}}}{\partial\theta}(R,\theta), \qquad i = 1, 2. \tag{3.2}$$

Following the standard method used for elliptic problems (Bao & Han, 2000) and the linear elastic system (Han & Bao, 2001), we design high-order local ABCs at Γ_e for the problem (P):

$$\sigma_n(\mathbf{u}, p) = \tilde{T}^N(\mathbf{u}) \equiv \left(\tilde{T}_1^N(\mathbf{u}), \tilde{T}_2^N(\mathbf{u})\right)^T, \qquad N = 1, 2, \ldots; \tag{3.3}$$

where

$$\tilde{T}_i^N(\mathbf{u}) = -\frac{2\nu}{R}\sum_{m=1}^{N}(-1)^m \alpha_m^{(N)}\frac{\partial^{2m} u_i(R,\theta)}{\partial\theta^{2m}} + (-1)^{i-1}\frac{\nu}{R}\frac{\partial u_{i+(-1)^{i-1}}}{\partial\theta}(R,\theta), \qquad i = 1, 2; \tag{3.4}$$

and the coefficients $\alpha_m^{(N)}$ are listed in Table 1 for the first five local ABCs.

Then the original problem (P) can be reduced to the following problem defined in the bounded domain Ω_i approximately for $N = 1, 2, \ldots$:

$(\tilde{\mathrm{P}}_N)$ Find $(\tilde{\mathbf{u}}^N, \tilde{p}^N)$ such that

$$-\nu\,\Delta\tilde{\mathbf{u}}^N + \operatorname{grad}\tilde{p}^N = \mathbf{f} \qquad \operatorname{div}\tilde{\mathbf{u}}^N = 0, \qquad \text{in } \Omega_i, \tag{3.5}$$

$$\tilde{\mathbf{u}}^N = \mathbf{0} \quad \text{on } \Gamma_i, \quad \sigma_{\mathbf{n}}(\tilde{\mathbf{u}}^N, \tilde{p}^N) = T^N(\tilde{\mathbf{u}}^N), \quad \text{on } \Gamma_e. \tag{3.6}$$

4. Error bounds for the case of using nonlocal ABCs

In this section, we present error bounds for the finite-element approximations of problems (P_N). These error bounds depend on not only the mesh size, h, and terms used in the approximate ABCs, N (see (2.16)), but also the location of the artificial boundary, R. This kind of error bound is very useful in engineering applications.

Let $H^m(\Omega_i)$ and $H^s(\Gamma_e)$ be the usual Sobolev spaces on the domain Ω_i and the boundary Γ_e with integer m and real number s. Furthermore, $\|\cdot\|_{m,\Omega_i}$ and $|\cdot|_{m,\Omega_i}$ denote the usual norm and semi-norm on $H^m(\Omega_i)$, respectively (Adams, 1975). Suppose

$$V = \left\{ \mathbf{v} = (v_1, v_2)^T \in [H^1(\Omega_i)]^2 \ : \ \mathbf{v}|_{\Gamma_i} = \mathbf{0} \right\}, \qquad W = L^2(\Omega_i).$$

Then the boundary value problem (P_i) is equivalent to the following variational problem:
(VP) Find $(\mathbf{u},\ p) \in V \times W$ such that

$$a(\mathbf{u},\mathbf{v}) + a_0(\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) = f(\mathbf{v}), \qquad \forall \mathbf{v} \in V, \tag{4.1}$$

$$b(q,\mathbf{u}) = 0, \qquad \forall q \in W; \tag{4.2}$$

where

$$a(\mathbf{u},\mathbf{v}) = \nu \int_{\Omega_i} \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \,\mathrm{d}\mathbf{x} \equiv \nu \int_{\Omega_i} \nabla \mathbf{u} : \nabla \mathbf{v} \,\mathrm{d}\mathbf{x}, \qquad \forall \mathbf{u},\mathbf{v} \in V, \tag{4.3}$$

$$\begin{aligned} a_0(\mathbf{u},\mathbf{v}) &= -\int_{\Gamma_e} T(\mathbf{u}) \cdot \mathbf{v} \,\mathrm{d}s = \frac{\nu}{\pi} \sum_{n=1}^{\infty} n \int_0^{2\pi} \int_0^{2\pi} \sum_{i=1}^{2} \Big[2u_i(R,\phi) v_i(R,\theta) \cos n(\phi-\theta) \\ &\quad + (-1)^{i-1} u_i(R,\phi)\, v_{i+(-1)^{i-1}}(R,\theta) \sin n(\phi-\theta) \Big] \mathrm{d}\phi\,\mathrm{d}\theta \\ &= \frac{\nu}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \sum_{i=1}^{2} \left[2 \frac{\partial u_i(R,\phi)}{\partial \phi} \frac{\partial v_i(R,\theta)}{\partial \theta} \frac{\cos n(\phi-\theta)}{n} \right. \\ &\quad \left. + (-1)^{i-1} \frac{\partial u_i(R,\phi)}{\partial \phi} \frac{\partial v_{i+(-1)^{i-1}}(R,\theta)}{\partial \theta} \frac{\sin n(\phi-\theta)}{n} \right] \mathrm{d}\theta\,\mathrm{d}\phi, \ \ \forall \mathbf{u},\mathbf{v} \in V, \end{aligned} \tag{4.4}$$

$$b(q,\mathbf{v}) = -\int_{\Omega_i} q \ \mathrm{div}\, \mathbf{v} \,\mathrm{d}\mathbf{x}, \qquad \forall \mathbf{v} \in V, \quad q \in W, \tag{4.5}$$

$$f(\mathbf{v}) = \int_{\Omega_i} \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x}, \qquad \forall \mathbf{v} \in V. \tag{4.6}$$

Notice that, in the bilinear functional $a(\mathbf{u},\mathbf{v})$, the integrand is now $\nabla\mathbf{u} : \nabla\mathbf{v}$ instead of $\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})$ which is used in Han & Bao (1997) for incompressible materials. Thus, the stiff matrix of the corresponding finite-element approximation becomes much sparser and could save computational time in solving the corresponding linear system. Furthermore, let

$$\begin{aligned} a_0^N(\mathbf{u},\mathbf{v}) &= \frac{\nu}{\pi} \sum_{n=1}^{N} \int_0^{2\pi} \int_0^{2\pi} \sum_{i=1}^{2} \left[2 \frac{\partial u_i(R,\phi)}{\partial \phi} \frac{\partial v_i(R,\theta)}{\partial \theta} \frac{\cos n(\phi-\theta)}{n} \right. \\ &\quad \left. + (-1)^{i-1} \frac{\partial u_i(R,\phi)}{\partial \phi} \frac{\partial v_{i+(-1)^{i-1}}(R,\theta)}{\partial \theta} \frac{\sin n(\phi-\theta)}{n} \right] \mathrm{d}\theta\,\mathrm{d}\phi, \ \ \forall \mathbf{u},\mathbf{v} \in V. \end{aligned} \tag{4.7}$$

Then the boundary value problem (P_N) is equivalent to the following variational problem:

(VP_N) Find $(\mathbf{u}^N, p^N) \in V \times W$ such that

$$a(\mathbf{u}^N, \mathbf{v}) + a_0^N(\mathbf{u}^N, \mathbf{v}) + b(p^N, \mathbf{v}) = f(\mathbf{v}), \qquad \forall \mathbf{v} \in V, \tag{4.8}$$

$$b(q, \mathbf{u}^N) = 0, \qquad \forall q \in W. \tag{4.9}$$

If we replace V and W by their conforming finite-element subspaces $V^h \subset V$ and $W^h \subset W$ in which h represents the mesh size (Ciarlet, 1978), then the finite-element approximation of the problem (VP_N) is

(VP_N^h) Find $(\mathbf{u}^{h,N}, p^{h,N}) \in V^h \times W^h$ such that

$$a(\mathbf{u}^{h,N}, \mathbf{v}^h) + a_0^N(\mathbf{u}^{h,N}, \mathbf{v}^h) + b(p^{h,N}, \mathbf{v}^h) = f(\mathbf{v}^h), \qquad \forall \mathbf{v}^h \in V^h, \tag{4.10}$$

$$b(q^h, \mathbf{u}^{h,N}) = 0, \qquad \forall q^h \in W^h. \tag{4.11}$$

We note that the symmetric bilinear form $a(\cdot,\cdot)$ is bounded and coercive on $V \times V$ from the Poincaré inequality (Adams, 1975), i.e. there exist two positive constants M_1, M_2 such that

$$|a(\mathbf{u}, \mathbf{v})| \leqslant M_1 \|\mathbf{u}\|_V \cdot \|\mathbf{v}\|_V, \qquad \forall \mathbf{u}, \mathbf{v} \in V, \tag{4.12}$$

$$M_2 \|\mathbf{v}\|_V^2 \leqslant a(\mathbf{v}, \mathbf{v}), \qquad \forall \mathbf{v} \in V. \tag{4.13}$$

Thus we can define an equivalent norm on the space V:

$$\|\mathbf{v}\|_* = [a(\mathbf{v}, \mathbf{v})]^{1/2} = \sqrt{\nu}|\mathbf{v}|_{1,2,\Omega_i}, \qquad \forall \mathbf{v} \in V. \tag{4.14}$$

Therefore, we have that

$$|a(\mathbf{u}, \mathbf{v})| \leqslant \|\mathbf{u}\|_* \cdot \|\mathbf{v}\|_*, \qquad \forall \mathbf{u}, \mathbf{v} \in V, \tag{4.15}$$

$$\|\mathbf{v}\|_*^2 \leqslant a(\mathbf{v}, \mathbf{v}), \qquad \forall \mathbf{v} \in V. \tag{4.16}$$

For the bilinear forms $a_0(\cdot,\cdot)$ and $a_0^N(\cdot,\cdot)$, we have the following lemma.

LEMMA 4.1 The following inequality holds:

$$0 \leqslant a_0^N(\mathbf{v}, \mathbf{v}) \leqslant a_0(\mathbf{v}, \mathbf{v}) \leqslant 3a(\mathbf{v}, \mathbf{v}) \equiv 3\|\mathbf{v}\|_*^2, \qquad \forall \mathbf{v} \in V, \quad N \geqslant 0, \tag{4.17}$$

$$|a_0(\mathbf{u}, \mathbf{v})| \leqslant 3\|\mathbf{u}\|_* \cdot \|\mathbf{v}\|_*, |a_0^N(\mathbf{u}, \mathbf{v})| \leqslant 3\|\mathbf{u}\|_* \cdot \|\mathbf{v}\|_*, \forall \mathbf{u}, \mathbf{v} \in V, N \geqslant 0, \tag{4.18}$$

where $a_0(\mathbf{u}, \mathbf{v})$ and $a_0^N(\mathbf{u}, \mathbf{v})$ are defined in (4.4) and (4.7), respectively.

Proof. Follow the proof for elliptic problems in Han & Bao (2000) and notice the inequality ((3.42) on page 1110 of Han & Bao, 2000). □

For the bilinear form $b(q, \mathbf{v})$, we have the following lemma.

LEMMA 4.2 There exists a generic constant $\beta_0 > 0$ independent of R, such that

$$\sup_{\mathbf{v} \in V \setminus \{\mathbf{0}\}} \frac{b(q, \mathbf{v})}{\|\mathbf{v}\|_*} \geqslant \beta_0 \|q\|_W, \qquad \forall q \in W. \tag{4.19}$$

Proof. Follow the proof in Han & Bao (1997). □

In order to derive the well-posedness of the problem (VP_N^h) and an error bound of the finite-element approximation, we suppose that the following discrete inf–sup condition between V^h and W^h holds:

$$\sup_{\mathbf{v}^h\in V^h\setminus\{\mathbf{0}\}} \frac{b(q^h,\mathbf{v}^h)}{\|\mathbf{v}^h\|_*} \geqslant \beta_0^* \, \|q^h\|_W, \qquad \forall q^h \in W^h, \tag{4.20}$$

where β_0^* is a constant independent of h, N and R.

It follows immediately from (4.15)–(4.20) and Theorem 4.1 in Chapter I of Girault & Raviart (1986) that the variational problems (VP), (VP_N) and (VP_N^h) are well-posed; that is, for $f \in V'$, the dual of V, there exists a unique $(\mathbf{u},\ p) \in V \times W$ solving (VP), a unique $(\mathbf{u}^N,\ p^N) \in V \times W$ solving (VP_N), a unique $(\mathbf{u}^{h,N},\ p^{h,N}) \in V^h \times W^h$ solving (VP_N^h), and

$$\|\mathbf{u}\|_* + \|\mathbf{u}^N\|_* + \|\mathbf{u}^{h,N}\|_* + \|p\|_W + \|p^N\|_W + \|p^{h,N}\|_W \leqslant C\ \|\mathbf{f}\|_{V'}. \tag{4.21}$$

Note that the well-posedness of (VP) implies immediately the well-posedness of the original problem (P).

Let $R_0 = \max\{|\mathbf{x}| \ :\ \mathbf{x} \in \ \mathrm{supp}\ \mathbf{f} \cup \Gamma_i\}$, $\Gamma_0 = \{(R_0,\theta) \ :\ 0 \leqslant \theta \leqslant 2\pi\}$ and $\Omega_0 = \{\mathbf{x} \in \Omega_i \ :\ |\mathbf{x}| < R_0\}$ and $\Gamma_r = \{(r,\theta) \ :\ 0 \leqslant \theta \leqslant 2\pi\}$. We recall an equivalent definition of Sobolev space $H^s(\Gamma_r)$ for any real number s (Adams, 1975):

$$w \in H^s(\Gamma_r) \Longleftrightarrow w(r,\theta) = \frac{p_0}{2} + \sum_{m=1}^{\infty} (p_m \cos m\theta + q_m \sin m\theta)$$

and

$$\frac{\pi p_0^2}{2} + \sum_{m=1}^{\infty} \pi (1+m^2)^s (p_m^2 + q_m^2) < \infty.$$

Thus we use

$$|w|_{s,\Gamma_r} = \left[\sum_{m=1}^{\infty} \pi m^{2s} (p_m^2 + q_m^2)\right]^{1/2} \tag{4.22}$$

as a semi-norm of the space $H^s(\Gamma_r)$. Then we have the following estimate.

LEMMA 4.3 Suppose that $(\mathbf{u},\ p) \in V \times W$ is the solution of the exterior Stokes problem (P) and there exists an integer $k \geqslant 1$ such that $\mathbf{u}|_{\Gamma_0} \in [H^{k+\frac{1}{2}}(\Gamma_0)]^2$. Then we have that

$$|a_0(\mathbf{u},\mathbf{v}) - a_0^N(\mathbf{u},\mathbf{v})| \leqslant \frac{C_0}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |\mathbf{u}|_{k+\frac{1}{2},\Gamma_0} \cdot \|\mathbf{v}\|_*, \qquad \forall \mathbf{v} \in V, \tag{4.23}$$

where C_0 is a generic constant independent of $\mathbf{u}$, N, h and R.

Proof. Assume that

$$u_i(R_0,\theta) = \frac{p_0^i}{2} + \sum_{n=1}^{\infty}(p_n^i \cos n\theta + q_n^i \sin n\theta), \qquad i = 1, 2, \tag{4.24}$$

$$v_i(R,\theta) = \frac{c_0^i}{2} + \sum_{n=1}^{\infty}(c_n^i \cos n\theta + d_n^i \sin n\theta), \qquad i = 1, 2; \tag{4.25}$$

where

$$p_n^i = \frac{1}{\pi}\int_0^{2\pi} u_i(R_0,\theta)\cos n\theta \,\mathrm{d}\theta, \quad q_n^i = \frac{1}{\pi}\int_0^{2\pi} u_i(R_0,\theta)\sin n\theta \,\mathrm{d}\theta, \qquad i = 1, 2, \quad n \geqslant 0; \tag{4.26}$$

$$c_n^i = \frac{1}{\pi}\int_0^{2\pi} v_i(R,\theta)\cos n\theta \,\mathrm{d}\theta, \quad d_n^i = \frac{1}{\pi}\int_0^{2\pi} v_i(R,\theta)\sin n\theta \,\mathrm{d}\theta, \qquad i = 1, 2, \quad n \geqslant 0. \tag{4.27}$$

Noting that $(\mathbf{u},\ p)$ satisfies the homogeneous exterior Stokes equations (say (1.1) with $\mathbf{f} = \mathbf{0}$) in the domain $\{\mathbf{x} \,:\, |\mathbf{x}| > R_0\}$, by the method of separation of variables, we obtain

$$\begin{aligned} u_1(r,\theta) = {} & \frac{r^2 - R_0^2}{2}\sum_{n=3}^{\infty}(n-2)\left[(p_{n-2}^1 - q_{n-2}^2)\cos n\theta + (q_{n-2}^1 + p_{n-2}^2)\sin n\theta\right]\frac{R_0^{n-2}}{r^n} \\ & + \frac{p_0^1}{2} + \sum_{n=1}^{\infty}(p_n^1 \cos n\theta + q_n^1 \sin n\theta)\frac{R_0^n}{r^n}, \qquad R_0 \leqslant r, \quad 0 \leqslant \theta \leqslant 2\pi, \end{aligned} \tag{4.28}$$

$$\begin{aligned} u_2(r,\theta) = {} & \frac{r^2 - R_0^2}{2}\sum_{n=3}^{\infty}(n-2)\left[-(q_{n-2}^1 + p_{n-2}^2)\cos n\theta + (p_{n-2}^1 - q_{n-2}^2)\sin n\theta\right]\frac{R_0^{n-2}}{r^n} \\ & + \frac{p_0^2}{2} + \sum_{n=1}^{\infty}(p_n^2 \cos n\theta + q_n^2 \sin n\theta)\frac{R_0^n}{r^n}, \qquad R_0 \leqslant r, \quad 0 \leqslant \theta \leqslant 2\pi. \end{aligned} \tag{4.29}$$

Setting $r = R$ in (4.28) and (4.29), we obtain

$$u_i(R,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{\infty}(a_n^i \cos n\theta + b_n^i \sin n\theta), \qquad i = 1, 2; \tag{4.30}$$

where

$$a_n^1 = \left(\frac{R_0}{R}\right)^n \begin{cases} p_n^1, & n = 0, 1, 2, \\ p_n^1 + \frac{(n-2)(R^2 - R_0^2)}{2R_0^2}(p_{n-2}^1 - q_{n-2}^2), & n \geqslant 3; \end{cases} \tag{4.31}$$

$$b_n^1 = \left(\frac{R_0}{R}\right)^n \begin{cases} q_n^1, & n = 1, 2, \\ q_n^1 + \frac{(n-2)(R^2-R_0^2)}{2R_0^2}(q_{n-2}^1 + p_{n-2}^2), & n \geqslant 3; \end{cases} \tag{4.32}$$

$$a_n^2 = \left(\frac{R_0}{R}\right)^n \begin{cases} p_n^2, & n = 0, 1, 2, \\ p_n^2 - \frac{(n-2)(R^2-R_0^2)}{2R_0^2}(q_{n-2}^1 + p_{n-2}^2), & n \geqslant 3; \end{cases} \tag{4.33}$$

$$b_n^2 = \left(\frac{R_0}{R}\right)^n \begin{cases} q_n^2, & n = 1, 2, \\ q_n^2 + \frac{(n-2)(R^2-R_0^2)}{2R_0^2}(p_{n-2}^1 - q_{n-2}^2), & n \geqslant 3. \end{cases} \tag{4.34}$$

Inserting (4.30), (4.25) into (4.4) and (4.7), using the orthogonality of the cosines and sines, noting (4.31)–(4.34), (4.26), (4.18), (4.17), (4.22), (4.24) and (4.25), we obtain

$$\begin{aligned}
&|a_0(\mathbf{u}, \mathbf{v}) - a_0^N(\mathbf{u}, \mathbf{v})| \\
&= \Bigg| \nu\pi \sum_{n=1}^{\infty} n \left[b_n^1\, c_n^2 - a_n^1\, d_n^2 - b_n^2\, c_n^1 + a_n^2\, d_n^1 + 2 \sum_{i=1}^{2} \left(a_n^i\, c_n^i + b_n^i\, d_n^i \right) \right] \\
&\quad - \nu\pi \sum_{n=1}^{N} n \left[b_n^1\, c_n^2 - a_n^1\, d_n^2 - b_n^2\, c_n^1 + a_n^2\, d_n^1 + 2 \sum_{i=1}^{2} \left(a_n^i\, c_n^i + b_n^i\, d_n^i \right) \right] \Bigg| \\
&= \Bigg| \nu\pi \sum_{n=N+1}^{\infty} n \left[b_n^1\, c_n^2 - a_n^1\, d_n^2 - b_n^2\, c_n^1 + a_n^2\, d_n^1 + 2 \sum_{i=1}^{2} \left(a_n^i\, c_n^i + b_n^i\, d_n^i \right) \right] \Bigg| \\
&\leqslant 3\nu\pi \left[\sum_{n=N+1}^{\infty} n \sum_{i=1}^{2} \left((a_n^i)^2 + (b_n^i)^2 \right) \right]^{1/2} \cdot \left[\sum_{n=N+1}^{\infty} n \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2 \right) \right]^{1/2} \\
&\leqslant C_0 \Bigg[\sum_{n=N+1}^{\infty} n \sum_{i=1}^{2} \left((p_n^i)^2 + (q_n^i)^2 \right) \frac{R_0^{2n}}{R^{2n}} \\
&\quad + \sum_{n=\max\{1, N-1\}}^{\infty} n^3 \sum_{i=1}^{2} \left((p_n^i)^2 + (q_n^i)^2 \right) \frac{R_0^{2n}}{R^{2n}} \Bigg]^{1/2} \cdot \|v\|_* \\
&\leqslant \frac{C_0}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{\max\{1, N-1\}} |\mathbf{u}|_{k+\frac{1}{2}, \Gamma_0} \cdot \|v\|_*.
\end{aligned} \tag{4.35}$$

□

Combining Lemmas 4.1–4.3, we obtain the following error bound.

THEOREM 4.1 Let $(\mathbf{u},\ p)$ be the solution of the problem (P) and $(\mathbf{u}^{h,N},\ p^{h,N})$ be the solution of the problem (VP_N^h). Suppose the discrete inf–sup condition (4.20) holds, $\mathbf{f} \in$

$[L^2(\Omega_i)]^2$ and $\mathbf{u}|_{\Gamma_0} \in [H^{k+\frac{1}{2}}(\Gamma_0)]^2$ ($k \geqslant 1$). Then we have the following error bound:

$$\|\mathbf{u} - \mathbf{u}^{h,N}\|_* + \|p - p^{h,N}\|_W \leqslant C_0 \Bigg[\inf_{\mathbf{v}^h \in V^h} \|\mathbf{u} - \mathbf{v}^h\|_* + \inf_{q^h \in W^h} \|p - q^h\|_W + \frac{1}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{\max\{1, N-1\}} |\mathbf{u}|_{k+\frac{1}{2}, \Gamma_0} \Bigg]. \quad (4.36)$$

Proof. The proof of this theorem is by using a standard technique of the mixed finite-element method (Girault & Raviart, 1986) and noting the estimate (4.23). The details are omitted here. □

Suppose that $\mathbf{u} \in [H^{k+1}(\Omega_i)]^2$, $p \in [H^k(\Omega_i)]^2$, $\mathbf{u}|_{\Gamma_0} \in [H^{k+\frac{1}{2}}(\Gamma_0)]^2$ and the interpolation errors of V^h to V and W^h to W (Ciarlet, 1978; Girault & Raviart, 1986) satisfy

$$\inf_{\mathbf{v}^h \in V^h} \|\mathbf{u} - \mathbf{v}^h\|_V + \inf_{q^h \in W^h} \|p - p^h\|_W \leqslant C_0 h^k \left(|\mathbf{u}|_{k+1, \Omega_i} + |p|_{k, \Omega_i}\right). \quad (4.37)$$

Then combining (4.37) and (4.36), noting (4.3), (4.14), and the Poincaré inequality, we obtain

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^{h,N}\|_{1,\Omega_0} + \|p - p^{h,N}\|_{0,\Omega_0} \\
&\leqslant C_0 \left(|\mathbf{u} - \mathbf{u}^{h,N}|_{1,\Omega_0} + \|p - p^{h,N}\|_{0,\Omega_0}\right) \leqslant C_0 \left(|\mathbf{u} - \mathbf{u}^{h,N}|_{1,\Omega_i} + \|p - p^{h,N}\|_{0,\Omega_i}\right) \\
&\leqslant C_0 \left(\|\mathbf{u} - \mathbf{u}^{h,N}\|_* + \|p - p^{h,N}\|_W\right) \\
&\leqslant C_0 \left[h^k \left(|\mathbf{u}|_{k+1,\Omega_i} + |p|_{k,\Omega_i}\right) + \frac{1}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |\mathbf{u}|_{k+\frac{1}{2},\Gamma_0} \right]. \quad (4.38)
\end{aligned}$$

For example, for the Taylor–Hood (P2/P1) element which satisfies (4.20) (Girault & Raviart, 1986), the error bound (4.38) holds for $k = 2$.

5. Error bounds for the case of using local ABCs

In this section, we present the finite-element formulation of the problem ($\tilde{\mathrm{P}}_N$) and provide an error bound for the finite-element approximation. To cope with the high-order local ABCs (3.3), we define

$$\tilde{V} = \{\mathbf{v} \in [H^1(\Omega_i)]^2 \ : \ \mathbf{v}|_{\Gamma_e} \in [H^N(\Gamma_e)]^2, \quad \mathbf{v}|_{\Gamma_i} = \mathbf{0}\}.$$

Let

$$\tilde{a}_0^N(\mathbf{u}, \mathbf{v}) = -\int_{\Gamma_e} \mathbf{v} \cdot \tilde{T}^N(\mathbf{u})\, \mathrm{d}s = \nu \int_0^{2\pi} \Bigg[v_2(R,\theta) \frac{\partial u_1}{\partial \theta}(R,\theta) - v_1(R,\theta) \frac{\partial u_2}{\partial \theta}(R,\theta) + 2 \sum_{m=1}^{N} \alpha_m^{(N)} \sum_{i=1}^{2} \frac{\partial^m u_i(R,\theta)}{\partial \theta^m} \frac{\partial^m v_i(R,\theta)}{\partial \theta^m} \Bigg] \mathrm{d}\theta, \quad \forall \mathbf{u}, \mathbf{v} \in \tilde{V}\cdot \quad (5.1)$$

Then the weak form of the problem $(\tilde{\mathrm{P}}_N)$ is:

$(\tilde{\mathrm{V}}\mathrm{P}_N)$ Find $(\tilde{\mathbf{u}}^N,\ \tilde{p}^N) \in \tilde{V} \times W$ such that

$$a(\tilde{\mathbf{u}}^N, \mathbf{v}) + \tilde{a}_0^N(\tilde{\mathbf{u}}^N, \mathbf{v}) + b(\tilde{p}^N, \mathbf{v}) = f(\mathbf{v}), \quad \forall \mathbf{v} \in \tilde{V}, \tag{5.2}$$

$$b(q, \tilde{\mathbf{u}}^N) = 0, \quad \forall q \in W. \tag{5.3}$$

If we replace $\tilde{V}$ and W by their conforming finite-dimensional subspaces, $\tilde{V}^h \subset \tilde{V}$ and $W^h \subset W$ in which h is the mesh size (Ciarlet, 1978), then the finite-element approximation of the problem $(\tilde{\mathrm{V}}\mathrm{P}_N)$ is:

$(\tilde{\mathrm{V}}\mathrm{P}_N^h)$ Find $(\tilde{\mathbf{u}}^{h,N},\ \tilde{p}^{h,N}) \in \tilde{V}^h \times W^h$ such that

$$a(\tilde{\mathbf{u}}^{h,N}, \mathbf{v}^h) + \tilde{a}_0^N(\tilde{\mathbf{u}}^{h,N}, \mathbf{v}^h) + b(\tilde{p}^{h,N}, \mathbf{v}^h) = f(\mathbf{v}^h), \quad \forall \mathbf{v}^h \in \tilde{V}^h, \tag{5.4}$$

$$b(q^h, \tilde{\mathbf{u}}^{h,N}) = 0, \quad \forall q^h \in W^h. \tag{5.5}$$

From (4.15), (4.16) and (4.19), the well-posedness of the problem $(\tilde{\mathrm{V}}\mathrm{P}_N)$ depends on the property of the bilinear form $\tilde{a}_0^N(\mathbf{u}, \mathbf{v})$. For any $\mathbf{u}, \mathbf{v} \in \tilde{V}$, we can also formally expand $\mathbf{u}|_{\Gamma_e} = \mathbf{u}(R, \theta)$ and $\mathbf{v}|_{\Gamma_e} = \mathbf{v}(R, \theta)$ in Fourier series (see (4.30) and (4.25)). Substituting (4.30) and (4.25) into (5.1) and using the orthogonality of the cosines and sines, we obtain

$$\tilde{a}_0^N(\mathbf{u}, \mathbf{v}) = \nu\pi \sum_{n=1}^{\infty} \left[n \left(b_n^1 c_n^2 - a_n^1 d_n^2 - b_n^2 c_n^1 + a_n^2 d_n^1 \right) \right.$$
$$\left. + 2\gamma_n^{(N)} \sum_{i=1}^{2} \left(a_n^i c_n^i + b_n^i d_n^i \right) \right], \ \forall \mathbf{u}, \mathbf{v} \in \tilde{V}, \tag{5.6}$$

where

$$\gamma_n^{(N)} = \sum_{m=1}^{N} n^{2m} \alpha_m^{(N)}, \qquad \forall n \in \mathbb{N}. \tag{5.7}$$

Thus the property of $\tilde{a}_0^N(\mathbf{u}, \mathbf{v})$ depends on the property of $\gamma_n^{(N)}$. In fact, we have the following lemma.

LEMMA 5.1 For any odd $1 \leqslant N \leqslant 19$, there exist two generic positive constants $C_N^{(1)}$ and $C_N^{(2)}$ depending only on N such that

$$\left|\tilde{a}_0^N(\mathbf{u}, \mathbf{v})\right| \leqslant C_N^{(2)} |\mathbf{u}|_{N,\Gamma_R} \cdot |\mathbf{v}|_{N,\Gamma_R}, \qquad \forall \mathbf{u}, \mathbf{v} \in \tilde{V}, \tag{5.8}$$

$$C_N^{(1)} |\mathbf{v}|^2_{N,\Gamma_R} \leqslant \tilde{a}_0^N(\mathbf{v}, \mathbf{v}), \qquad \forall \mathbf{v} \in \tilde{V}. \tag{5.9}$$

Proof. For any odd $1 \leqslant N \leqslant 19$, noting (5.7) and Lemma 3.1 in Bao & Han (2000), we know that there exist positive constants $C_N^{(1)}$ and $C_N^{(2)}$ such that

$$\gamma_n^{(N)} \geqslant n, \qquad C_N^{(1)} n^{2N} \leqslant \gamma_n^{(N)} \leqslant C_N^{(2)} n^{2N}, \qquad n = 1, 2, 3, \ldots. \tag{5.10}$$

If $2 \leqslant N \leqslant 20$ is even, then

$$\gamma_n^{(N)} < 0, \ \text{when } n \text{ is sufficient large}, \ \lim_{n \to +\infty} \frac{\gamma_n^{(N)}}{n^{2N}} = \alpha_N^{(N)} < 0. \tag{5.11}$$

From (5.10) and (5.6), noting (4.22) with $r = R$, (4.30) and (4.25), we have that

$$\begin{aligned}
|\tilde{a}_0^N(\mathbf{u},\mathbf{v})| &\leqslant C_N^{(2)} \sum_{n=1}^{\infty} \left[n \left| b_n^1 c_n^2 - a_n^1 d_n^2 - b_n^2 c_n^1 + a_n^2 d_n^1 \right| + n^{2N} \left| \sum_{i=1}^{2} \left(a_n^i c_n^i + b_n^i d_n^i \right) \right| \right] \\
&\leqslant C_N^{(2)} \left[\sum_{n=1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((a_n^i)^2 + (b_n^i)^2 \right) \right]^{1/2} \cdot \left[\sum_{n=1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((d_n^i)^2 + (c_n^i)^2 \right) \right]^{1/2} \\
&\leqslant C_N^{(2)} |\mathbf{u}|_{N,\Gamma_R} \cdot |\mathbf{v}|_{N,\Gamma_R}, \qquad \forall \mathbf{u},\mathbf{v} \in \tilde{V}.
\end{aligned} \tag{5.12}$$

Furthermore, from (5.10) and (5.6) with $\mathbf{u} = \mathbf{v}$, noting (4.22) with $r = R$ and (4.25), we obtain

$$\begin{aligned}
\tilde{a}_0^N(\mathbf{v},\mathbf{v}) &\geqslant \nu\pi \sum_{n=1}^{\infty} \gamma_n^{(N)} \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2 \right) \\
&\quad + \nu\pi \sum_{n=1}^{\infty} n \left[2 d_n^1 c_n^2 - 2 c_n^1 d_n^2 + \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2 \right) \right] \\
&\geqslant \nu\pi \sum_{n=1}^{\infty} \gamma_n^{(N)} \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2 \right) + \nu\pi \sum_{n=1}^{\infty} n \left[\left(d_n^1 - c_n^2 \right)^2 + \left(c_n^1 - d_n^2 \right)^2 \right] \\
&\geqslant C_N^{(1)} \sum_{n=1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2 \right) = C_N^{(1)} |\mathbf{v}|_{N,\Gamma_R}, \qquad \forall \mathbf{v} \in \tilde{V}.
\end{aligned} \tag{5.13}$$

Thus the desired inequalities (5.8) and (5.9) are proved. □

REMARK 5.1 For any even $2 \leqslant N \leqslant 20$, noting (5.11) and (5.6), the inequality (5.9) does not hold. In fact, it is easy to construct a function $\mathbf{v} \in \tilde{V}$ such that $\tilde{a}_0^N(\mathbf{v},\mathbf{v}) < 0$ for an even $2 \leqslant N \leqslant 20$. Thus the bilinear form $a(\mathbf{u},\mathbf{v}) + \tilde{a}_0^N(\mathbf{u},\mathbf{v})$ is not coercive on $\tilde{V}$. Therefore one cannot prove the well-posedness of the problem $(\tilde{V}P_N)$. This phenomenon was also observed in numerical simulations of elliptic equations (Bao & Han, 2000; Givoli *et al.*, 1997), and the linear elastic system (Han & Bao, 2001) in an unbounded domain: the errors of the finite-element solution when choosing $N = 2, 4, \ldots$ in high-order local ABCs are much larger than those when choosing $N = 0, 1, 3, \ldots$.

From the discussion above, noting the Poincaré inequality, we assign the following norm on $\tilde{\mathrm{V}}$:

$$\|\mathbf{v}\|_\triangle := \left[a(\mathbf{v},\mathbf{v}) + |\mathbf{v}|_{N,\Gamma_R}^2 \right]^{1/2} \equiv \left[\|\mathbf{v}\|_*^2 + |\mathbf{v}|_{N,\Gamma_R}^2 \right]^{1/2}, \qquad \forall \mathbf{v} \in \tilde{V}. \tag{5.14}$$

For the well-posedness of the problem $(\tilde{\mathrm{V}}\mathrm{P}_N^h)$, we assume that the following discrete inf–sup condition between $\tilde{V}^h$ and W^h holds:

$$\sup_{\mathbf{v}^h \in \tilde{V}^h \setminus \{0\}} \frac{b(q^h, \mathbf{v}^h)}{\|\mathbf{v}^h\|_\triangle} \geqslant \beta_0^* \, \|q^h\|_W, \qquad \forall q^h \in W^h, \tag{5.15}$$

where β_0^* is a constant independent of h, N and R.

It follows immediately from (4.15), (4.16), (4.19), (5.15), (5.8) and (5.9) that the variational problems ($\tilde{\mathrm{V}}\mathrm{P}_N$) and ($\tilde{\mathrm{V}}\mathrm{P}_N^h$) are well-posed in the case of odd $1 \leqslant N \leqslant 19$ or $N = 0$; that is, for $\mathbf{f} \in \tilde{V}'$, the dual of $\tilde{V}$, there exists a unique $(\tilde{\mathbf{u}}^N, \ \tilde{p}^N) \in \tilde{V} \times W$ solving ($\tilde{\mathrm{V}}\mathrm{P}_N$), a unique $(\tilde{\mathbf{u}}^{h,N}, \tilde{p}^{h,N}) \in \tilde{V}^h \times W^h$ solving ($\tilde{\mathrm{V}}\mathrm{P}_N^h$), and

$$\|\tilde{\mathbf{u}}^N\|_\triangle + \|\tilde{\mathbf{u}}^{h,N}\|_\triangle + \|\tilde{p}^N\|_W + \|\tilde{p}^{h,N}\|_W \leqslant M_N \ \|\mathbf{f}\|_{\tilde{V}'}, \qquad \forall \text{ odd } 1 \leqslant N \leqslant 19, \tag{5.16}$$

where M_N is a constant.

When $N = 0$, then $\tilde{a}_0^N(\mathbf{u}, \mathbf{v}) = a_0^N(\mathbf{u}, \mathbf{v}) \equiv 0$. We have dealt with this case in the previous section. From now on, we always assume that $1 \leqslant N \leqslant 19$ is an odd integer. For the bilinear form $\tilde{a}_0^N(\mathbf{u}, \mathbf{v})$, we have the following estimate.

LEMMA 5.2 Suppose that $(\mathbf{u}, \ p) \in V \times W$ is the solution of the exterior problem (P) and $\mathbf{u}|_{\Gamma_0} \in [H^{N+1}(\Gamma_0)]^2$. Then we have the following estimate for any odd $1 \leqslant N \leqslant 19$:

$$|a_0(\mathbf{u}, \mathbf{v}) - \tilde{a}_0^N(\mathbf{u}, \mathbf{v})| \leqslant C_{(N)} \left(\frac{R_0}{R}\right)^{\max\{1, N-1\}} |\mathbf{u}|_{N+1, \Gamma_0} \cdot |\mathbf{v}|_{N, \Gamma_R}, \qquad \forall \mathbf{v} \in \tilde{V}, \tag{5.17}$$

where $C_{(N)}$ is a generic constant independent of $\mathbf{u}$, R and h.

Proof. Inserting (4.30), (4.25) into (4.4), using the orthogonality of the cosines and sines, noting (5.6), (5.10), (4.31)–(4.34), (5.7) and (4.22), we obtain for any odd $1 \leqslant N \leqslant 19$

$$\begin{aligned}
&\left|a_0(\mathbf{u}, \mathbf{v}) - \tilde{a}_0^N(\mathbf{u}, \mathbf{v})\right| \\
&= 2\nu\pi \left| \sum_{n=N+1}^{\infty} \left(\gamma_n^{(N)} - n\right) \sum_{i=1}^{2} \left(a_n^i c_n^i + b_n^i d_n^i\right) \right| \leqslant C_{(N)} \sum_{n=N+1}^{\infty} n^{2N} \sum_{i=1}^{2} \left|a_n^i c_n^i + b_n^i d_n^i\right| \\
&\leqslant C_{(N)} \left[\sum_{n=N+1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((a_n^i)^2 + (b_n^i)^2\right) \right]^{1/2} \cdot \left[\sum_{n=N+1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2\right) \right]^{1/2} \\
&\leqslant C_{(N)} \left[\sum_{n=\max\{1, N-1\}}^{\infty} n^{2N+2} \sum_{i=1}^{2} \left((p_n^i)^2 + (q_n^i)^2\right) \frac{R_0^{2n}}{R^{2n}} \right]^{1/2} \cdot |v|_{N, \Gamma_R} \\
&\leqslant C_{(N)} \left(\frac{R_0}{R}\right)^{\max\{1, N-1\}} |\mathbf{u}|_{N+1, \Gamma_0} \cdot |\mathbf{v}|_{N, \Gamma_R}, \qquad \forall \mathbf{v} \in \tilde{V}.
\end{aligned} \tag{5.18}$$

Combining Lemmas 5.1 and 5.2, we obtain the following error bound. □

THEOREM 5.1 Let $(\mathbf{u}, \ p)$ be the solution of the problem (P) and $(\tilde{\mathbf{u}}^{h,N}, \ \tilde{p}^{h,N})$ be the solution of the problem ($\tilde{\mathrm{V}}\mathrm{P}_N^h$). Suppose the discrete inf–sup condition (5.15) holds, $\mathbf{f} \in [L^2(\Omega_i)]^2$ and $\mathbf{u}|_{\Gamma_0} \in [H^{N+1}(\Gamma_0)]^2$. Then we have the following error bound for any odd

$1 \leqslant N \leqslant 19$:

$$\|\mathbf{u} - \tilde{\mathbf{u}}^{h,N}\|_{\triangle} + \|p - \tilde{p}^{h,N}\|_W \leqslant C_{(N)} \left[\inf_{\mathbf{v}^h \in \tilde{V}^h} \|\mathbf{u} - \mathbf{v}^h\|_{\triangle} + \inf_{q^h \in W^h} \|p - q^h\|_W + \left(\frac{R_0}{R}\right)^{\max\{1, N-1\}} |\mathbf{u}|_{N+1,\Gamma_0} \right]. \tag{5.19}$$

Proof. The proof of this theorem is also by using a standard technique of the mixed finite-element method (Girault & Raviart, 1986) and noting $\|\mathbf{v}\|_* \leqslant \|\mathbf{v}\|_{\triangle}$ for all $\mathbf{v} \in \tilde{V}$ and the estimate (5.17). The details are also omitted here. □

Suppose $\mathbf{u} \in [H^{k+1}(\Omega_i)]^2$, $p \in [H^k(\Omega_i)]^2$, $\mathbf{u}|_{\Gamma_R} \in [H^{k+N}(\Gamma_R)]^2$ and the interpolation errors of $\tilde{V}^h$ to $\tilde{V}$ and W^h to W (Ciarlet, 1978; Girault & Raviart, 1986; Givoli *et al.*, 1997) satisfy:

$$\inf_{\mathbf{v}^h \in \tilde{V}^h} \|\mathbf{u} - \mathbf{v}^h\|_{\triangle} + \inf_{q^h \in W^h} \|p - q^h\|_W \leqslant C_0 h^k \left[|\mathbf{u}|_{k+1,\Omega_i} + |p|_{k,\Omega_i} + |\mathbf{u}|_{k+N,\Gamma_R} \right]. \tag{5.20}$$

Then combining (5.20) and (5.19), noting the Poincaré inequality, (4.22) and (4.14), we obtain for any odd $1 \leqslant N \leqslant 19$

$$\begin{aligned} &\|\mathbf{u} - \tilde{\mathbf{u}}^{h,N}\|_{1,\Omega_0} + \|p - \tilde{p}^{h,N}\|_{0,\Omega_0} \\ &\leqslant C_0 \left(|\mathbf{u} - \tilde{\mathbf{u}}^{h,N}|_{1,\Omega_0} + \|p - \tilde{p}^{h,N}\|_{0,\Omega_0} \right) \leqslant C_0 \left(|\mathbf{u} - \tilde{\mathbf{u}}^{h,N}|_{1,\Omega_i} + \|p - \tilde{p}^{h,N}\|_{0,\Omega_i} \right) \\ &\leqslant C_0 \left(\|\mathbf{u} - \tilde{\mathbf{u}}^h_N\|_{\triangle} + \|p - \tilde{p}^{h,N}\|_W \right) \\ &\leqslant C_{(N)} \left[h^k \left(|\mathbf{u}|_{k+1,\Omega_i} + |p|_{k,\Omega_i} + |\mathbf{u}|_{k+N,\Gamma_R} \right) + \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |\mathbf{u}|_{N+1,\Gamma_0} \right]. \end{aligned} \tag{5.21}$$

6. Numerical results

In this section we present numerical results which demonstrate the performance of the error bounds (4.38) and (5.21). We consider the numerical implementation for the finite-element approximation by using a (non)local ABC at Γ_e. When dealing with high-order local ABCs, we only consider the case of $N = 1$ in this section. In this case, the usual finite-element subspace V^h of V proposed in Girault & Raviart (1986) can be used as $\tilde{V}^h$ directly. When dealing with the case of $N > 1$ in local ABCs, special subspace $\tilde{V}^h$ which has a higher regularity at Γ_e should be used. A family of this kind of spaces was introduced in Givoli *et al.* (1997). In our computations, the Taylor–Hood element (i.e. P2/P1) which satisfies the discrete inf–sup condition (4.20) (Girault & Raviart, 1986) was used to construct the finite-element subspaces V^h and W^h. That is to say, $k = 2$ in the interpolation errors (4.37) and (5.20) (Ciarlet, 1978). The integrations on the circle in (5.1) and (4.7) are evaluated on each element numerically by a Gaussian quadrature.

EXAMPLE An exterior Stokes problem. We consider the exterior Stokes equations in the planar domain outside a circular obstacle of radius $a = 0{\cdot}5$ (see Fig. 1). The problem is governed by the following boundary value problem:

$$-\nu\,\triangle\mathbf{u} + \text{ grad } p = \mathbf{f} \quad \text{in } \Omega = \{\mathbf{x} \;:\; 0{\cdot}5 < |\mathbf{x}|\}, \quad \text{div } \mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \tag{6.1}$$

$$\mathbf{u}(0{\cdot}5, \theta) = \mathbf{g}(\theta) \equiv (g_1(\theta), g_2(\theta))^T, \quad \text{on } \Gamma_i = \partial\Omega, \tag{6.2}$$

$$\mathbf{u} \quad \text{is bounded}, \quad p \to 0, \quad \text{when} \quad r = \sqrt{x_1^2 + x_2^2} \to +\infty; \tag{6.3}$$

where

$$f_1(\mathbf{x}) = \begin{cases} x_2[(5-48\nu)x_1^2 + (1-48\nu)x_2^2 + 24\nu - 1](|\mathbf{x}|^2 - 1), & 0{\cdot}5 \leqslant |\mathbf{x}| < 1{\cdot}0, \\ 0, & 1{\cdot}0 \leqslant |\mathbf{x}|; \end{cases}$$

$$f_2(\mathbf{x}) = \begin{cases} x_1[(1+48\nu)x_1^2 + (5+48\nu)x_2^2 - 1 - 24\nu](|\mathbf{x}|^2 - 1), & 0{\cdot}5 \leqslant |\mathbf{x}| < 1{\cdot}0, \\ 0, & 1{\cdot}0 \leqslant |\mathbf{x}|; \end{cases}$$

$$g_1(\theta) = \frac{1}{4\nu}\left[\frac{2\cos^2\theta\sin\theta}{1{\cdot}5625 - \sin^2\theta} - \frac{1}{2}\ln\frac{1{\cdot}25 - \sin\theta}{1{\cdot}25 + \sin\theta}\right] + \frac{27}{128}\sin\theta, \qquad 0 \leqslant \theta \leqslant 2\pi;$$

$$g_2(\theta) = \frac{\cos\theta(2\sin^2\theta - 1{\cdot}25)}{4\nu(1{\cdot}5625 - \sin^2\theta)} - \frac{27}{128}\cos\theta, \qquad 0 \leqslant \theta \leqslant 2\pi.$$

This problem has an exact solution:

$$u_1(\mathbf{x}) = \begin{cases} \frac{1}{4\nu}\left[\frac{x_1^2}{x_1^2 + (x_2 - 0{\cdot}25)^2} - \frac{x_1^2}{x_1^2 + (x_2 + 0{\cdot}25)^2} - \frac{1}{2}\ln\frac{x_1^2 + (x_2 - 0{\cdot}25)^2}{x_1^2 + (x_2 + 0{\cdot}25)^2}\right] + x_2(|\mathbf{x}|^2 - 1)^3, & 0{\cdot}5 \leqslant |\mathbf{x}| < 1{\cdot}0, \\ \frac{1}{4\nu}\left[\frac{x_1^2}{x_1^2+(x_2-0{\cdot}25)^2} - \frac{x_1^2}{x_1^2+(x_2-0{\cdot}25)^2} - \frac{1}{2}\ln\frac{x_1^2+(x_2-0{\cdot}25)^2}{x_1^2+(x_2+0{\cdot}25)^2}\right], & 1{\cdot}0 \leqslant |\mathbf{x}|; \end{cases}$$

$$u_2(\mathbf{x}) = \begin{cases} \frac{1}{4\nu}\left[\frac{x_1(x_2-0{\cdot}25)}{x_1^2+(x_2-0{\cdot}25)^2} - \frac{x_1(x_2+0{\cdot}25)}{x_1^2+(x_2+0{\cdot}25)^2}\right] - x_1(|\mathbf{x}|^2 - 1)^3, & 0{\cdot}5 \leqslant |\mathbf{x}| < 1{\cdot}0, \\ \frac{1}{4\nu}\left[\frac{x_1(x_2-0{\cdot}25)}{x_1^2+(x_2-0{\cdot}25)^2} - \frac{x_1(x_2+0{\cdot}25)}{x_1^2+(x_2+0{\cdot}25)^2}\right], & 1{\cdot}0 \leqslant |\mathbf{x}|; \end{cases}$$

$$p(\mathbf{x}) = \begin{cases} \frac{1}{2}\left[\frac{x_1}{x_1^2+(x_2-0{\cdot}25)^2} - \frac{x_1}{x_1^2+(x_2+0{\cdot}25)^2}\right] + x_1x_2(|\mathbf{x}|^2 - 1)^2, & 0{\cdot}5 \leqslant |\mathbf{x}| < 1{\cdot}0, \\ \frac{1}{2}\left[\frac{x_1}{x_1^2+(x_2-0{\cdot}25)^2} - \frac{x_1}{x_1^2+(x_2+0{\cdot}25)^2}\right], & 1{\cdot}0 \leqslant |\mathbf{x}|\cdot \end{cases}$$

In this example, we take $\nu = 1$ and the unbounded domain $\Omega = \{\mathbf{x} \in \mathbb{R}^2 \;:\; 0{\cdot}5 < |\mathbf{x}|\}$ which is the exterior domain outside a circle $\Gamma_i = \{x \in \mathbb{R}^2 \;:\; |\mathbf{x}| = 0{\cdot}5\}$.

TABLE 2 *The effect of the mesh size h using nonlocal ABCs*

Mesh	$h = 0{\cdot}7368$	$h = 0{\cdot}3684$	$h = 0{\cdot}1842$	$h = 0{\cdot}0921$
$\max \lvert \mathbf{u} - \mathbf{u}^{h,N} \rvert$	5·6674E−2	6·7481E−3	6·9003E−4	1·5893E−4
$\max \lvert p - p^{h,N} \rvert$	1·1739	0·32159	0·10791	0·03316
$\lVert \mathbf{u} - \mathbf{u}^{h,N} \rVert_{0,\Omega_0}$	4·7154E−2	4·6029E−3	6·7270E−4	1·3840E−4
$\lVert \mathbf{u} - \mathbf{u}^{h,N} \rVert_{1,\Omega_0}$	0·66928	0·19194	0·06400	0·02548
$\lVert p - p^{h,N} \rVert_{0,\Omega_0}$	3·834E−1	6·921E−2	1·716E−2	4·290E−3

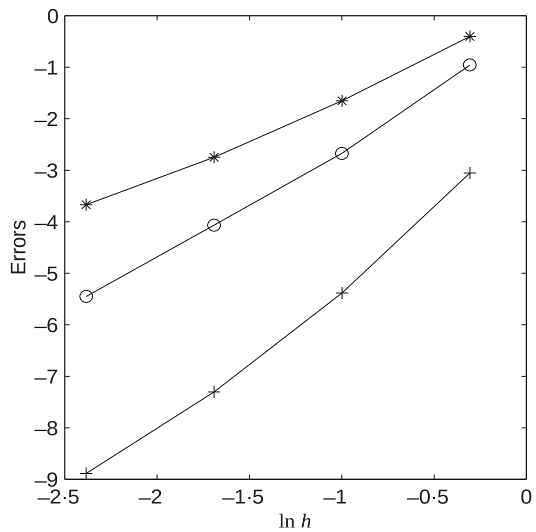

FIG. 2. The effect of the mesh size h by using nonlocal ABCs. + +: $\ln \lVert \mathbf{u} - \mathbf{u}^{h,N} \rVert_{0,\Omega_0}$, * *: $\ln \lVert \mathbf{u} - \mathbf{u}^{h,N} \rVert_{1,\Omega_0}$, o o: $\ln \lVert p - p^{h,N} \rVert_{0,\Omega_0}$.

First we test the effect of the mesh size h in the error bound (4.38), we introduce a circular artificial boundary $\Gamma_e = \Gamma_0$ of radius $R = R_0 = 1{\cdot}0$. On Γ_0 we apply the nonlocal ABC (2.17) with a very large N (say $N = 51$). In this case, the numerical error comes mainly from the finite-element discretization because the error from the approximate ABC is negligible. In the annular computational domain Ω_0, we use four meshes respectively. The first mesh consists of one radial layer of elements, with 16 triangular elements in the layer. We denote it as 1×16. The other three meshes are 2×32, 4×64 and 8×128. Table 2 shows the maximum errors of $\mathbf{u} - \mathbf{u}^{h,N}$, $p - p^{h,N}$ over the mesh points, and $\lVert \mathbf{u} - \mathbf{u}^{h,N} \rVert_{0,\Omega_0}$, $\lVert \mathbf{u} - \mathbf{u}^{h,N} \rVert_{1,\Omega_0}$, $\lVert p - p^{h,N} \rVert_{0,\Omega_0}$ for large N (say $N = 51$).

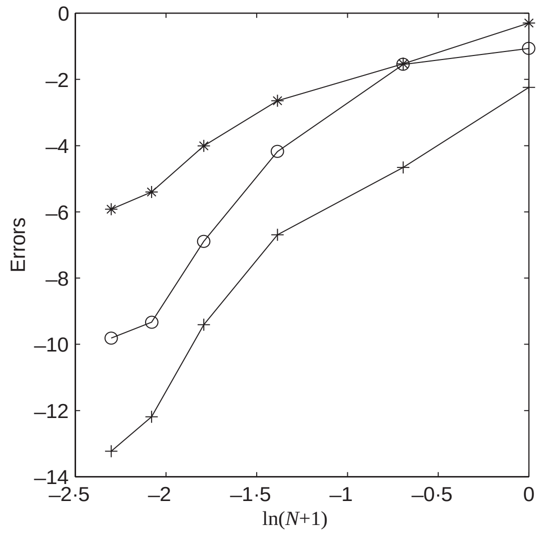

FIG. 3. The effect of N by using nonlocal ABCs. + +: $\ln\|\mathbf{u}^{h,\infty}-\mathbf{u}^{h,N}\|_{0,\Omega_0}$, * *: $\ln\|\mathbf{u}^{h,\infty}-\mathbf{u}^{h,N}\|_{1,\Omega_0}$, o o: $\ln\|p^{h,\infty}-p^{h,N}\|_{0,\Omega_0}$.

Figure 2 shows the errors $\|\mathbf{u}-\mathbf{u}^{h,N}\|_{0,\Omega_0}$, $\|\mathbf{u}-\mathbf{u}^{h,N}\|_{1,\Omega_0}$ and $\|p-p^{h,N}\|_{0,\Omega_0}$ for large N (say $N=51$).

Second we test the effect of N in the error bound (4.38). In order to do so, we choose $\Gamma_e=\Gamma_0$ with $R=R_0$ so that the effect of R in the error bound (4.38) disappears. Let $(\mathbf{u}^{h,\infty},\ p^{h,\infty})$ denote the finite-element approximation of the problem on the domain Ω_0 with the mesh size h when N is very large (say $N=51$). Figure 3 shows the errors $E_N := \|\mathbf{u}^{h,\infty}-u^{h,N}\|_{k,\Omega_0}$ $(k=0,1)$ and $\|p^{h,\infty}-p^{h,N}\|_{0,\Omega_0}$ on the mesh 8×128 for $N=0,1,3,5,7,9$.

Third we test the effect of the location of the artificial boundary Γ_e. Let $\Omega_R=\{x\ :\ 0{\cdot}5<|\mathbf{x}|<R\}$ denote the bounded computational domain with the artificial boundary Γ_R. We choose $R=1{\cdot}0,\ 1{\cdot}5,\ 2{\cdot}0,\ 2{\cdot}5,\ 3{\cdot}0$ respectively. The corresponding meshes we used were 4×64, 8×64, 12×64, 16×64 and 20×64. That is to say, each computational domain has a mesh with the fixed mesh size $h=0{\cdot}1842$. Let $(u^{R,N},\ p^{R,N})$ denote the finite-element approximation of the problem on the domain Ω_R with the corresponding mesh by using the nonlocal ABCs (2.17) on the artificial boundary Γ_R, $(\mathbf{u}^{R,\infty},\ p^{R,\infty})$ correspond to the solution when N is very large (say $N=51$) and $(\tilde{\mathbf{u}}^{R,N},\ \tilde{p}^{R,N})$ correspond to the solution by using the high-order local ABC (3.3) at Γ_R with $N=1$. Figures 4 and 5 show the errors $E_R := \|\mathbf{u}^{R,\infty}-\mathbf{u}^{R,N}\|_{1,\Omega_0}$ and $E_R := \|p^{R,\infty}-p^{R,N}\|_{0,\Omega_0}$ for $R=1{\cdot}0, 1{\cdot}5, 2{\cdot}0, 2{\cdot}5, 3{\cdot}0$. Figure 6 shows the errors $E_R := \|\mathbf{u}^{R,\infty}-\tilde{\mathbf{u}}^{R,N}\|_{1,\Omega_0}$ and $\|p^{R,\infty}-\tilde{p}^{R,N}\|_{0,\Omega_0}$ for $R=1{\cdot}0, 1{\cdot}5, 2{\cdot}0, 2{\cdot}5, 3{\cdot}0$.

From Table 2 and Fig. 2, one can see that the convergent rates of $\|\mathbf{u}-\mathbf{u}^{h,N}\|_{1,\Omega_0}$ and $\|p-p^{h,N}\|_{0,\Omega_0}$ with respect to h are approximately 2 when using nonlocal ABCs with a very large N, which are consistent with the error bound (4.38) with $k=2$. Figure 3 confirms the effect of N in the error bound (4.38). Furthermore, Figs 4 and 5 confirm the

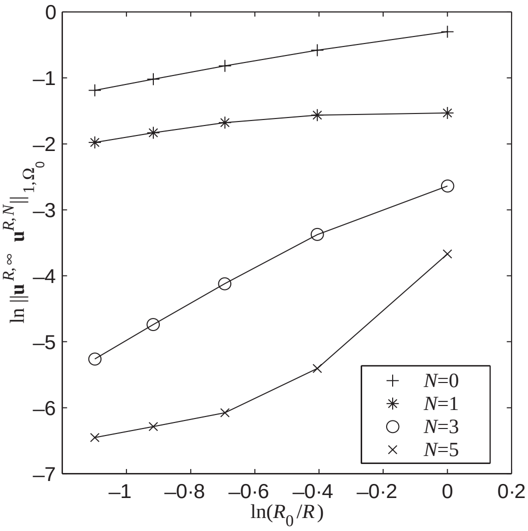

FIG. 4. The effect of R with respect to $\mathbf{u}$ by using nonlocal ABCs.

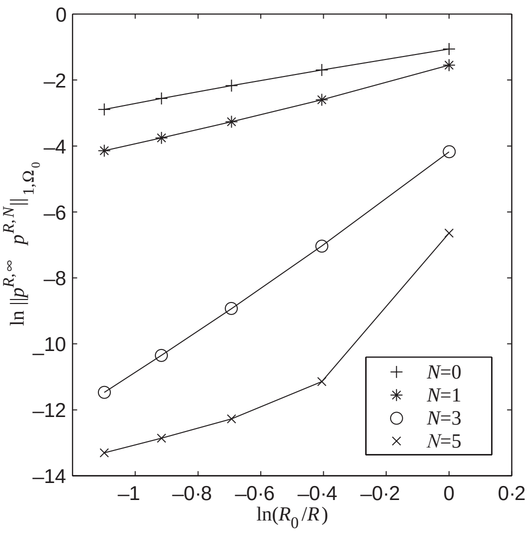

FIG. 5. The effect of R with respect to p by using nonlocal ABCs.

effect of R in the error bound (4.38) and Fig. 6 confirms the effect of R in the error bound (5.21). The minor discrepancies when $N = 5$ in Figs 4 and 5 are due to round-off errors because in this case the errors themselves are very small.

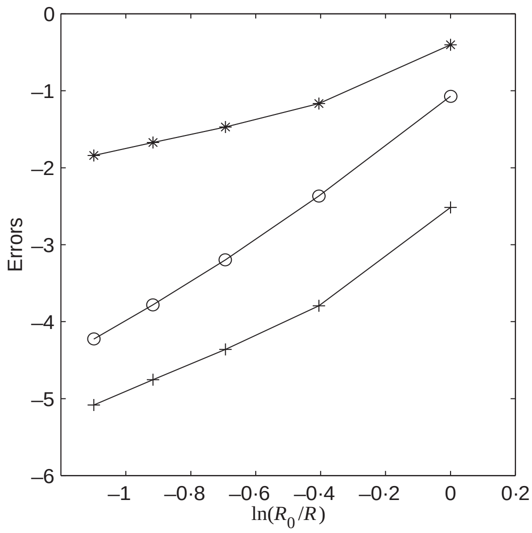

FIG. 6. The effect of R by using a local ABC ($N = 1$). + +: $\ln \|\mathbf{u}^{R,\infty} - \mathbf{u}^{R,N}\|_{0,\Omega_0}$, * *: $\ln \|\mathbf{u}^{R,\infty} - \mathbf{u}^{R,N}\|_{1,\Omega_0}$, o o: $\ln \|p^{R,\infty} - p^{R,N}\|_{0,\Omega_0}$.

7. Conclusions

A family of high-order (non)local ABCs for numerical simulations of the exterior Stokes equations in an unbounded domain is designed. The original problem is then reduced to a problem defined in a bounded computational domain by imposing a (non)local ABC at a circular artificial boundary. The finite-element formulation is presented. Error bounds for the case of using (non)local ABCs are obtained. This kind of error bounds depends on not only the mesh size, terms used in the ABCs, but also the location of the artificial boundary. They can be used to choose the mesh size, terms used in the ABCs and the location of the artificial boundary for practical computations. Numerical results demonstrate the performance of our error bounds.

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