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Error bounds for the finite element approximation of an incompressible material in an unbounded domain

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Summary. In this paper we design high-order local artificial boundary conditions and present error bounds for the finite element approximation of an incompressible elastic material in an unbounded domain. The finite element approximation is formulated in a bounded computational domain using a nonlocal approximate artificial boundary condition or a local one. In fact there are a family of nonlocal approximate artificial boundary conditions with increasing accuracy (and computational cost) and a family of local ones for a given artificial boundary. Our error bounds indicate how the errors of the finite element approximations depend on the mesh size, the terms used in the approximate artificial boundary condition and the location of the artificial boundary. Numerical examples of an incompressible elastic material outside a circle in the plane is presented. Numerical results demonstrate the performance of our error bounds.

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1 Introduction

Let Γ_i be a smooth bounded simple closed curve in $\mathbb{R}^2$ and Ω be the unbounded domain with the boundary Γ_i (see Fig. 1). We consider the following boundary value problem:

(1.1) $\quad -2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = f \qquad \text{in } \Omega,$

(1.2) $\quad \operatorname{div} u = 0 \qquad \text{in } \Omega,$

(1.3) $\quad u = 0 \qquad \text{on } \Gamma_i,$

(1.4) $\quad u \text{ is bounded, } p \to 0 \quad \text{when} \quad r = |x| \equiv \sqrt{x_1^2 + x_2^2} \to +\infty;$

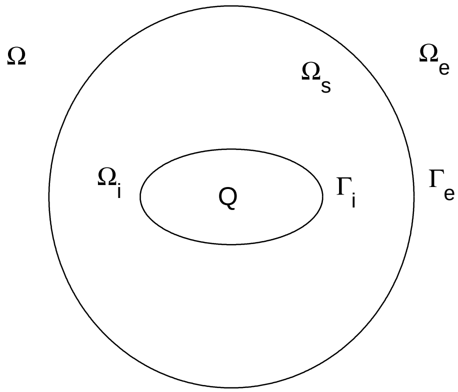

Fig. 1. Set up of the domain and artificial boundary

where $x = (x_1, x_2)$ is the Cartesian coordinate system and the corresponding polar coordinate system is (r, θ), $u = (u_1, u_2)^{\mathrm{T}}$ is the displacement and p is the hydrostatic pressure of an incompressible elastic material, $\mu > 0$ is the Lamé constant, $f = (f_1, f_2)^{\mathrm{T}}$ is the density of applied body force whose support is compact and $\varepsilon(u) = (\varepsilon_{ij}(u))_{2\times 2}$ is the strain tensor corresponding to the displacement u which satisfies:

$$(1.5) \qquad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad 1 \le i, j \le 2.$$

The boundary value problem of Stokes equations (1.1)-(1.4) describes the deformation of an incompressible elastic material in the unbounded domain Ω [28]. The finite element approximations of incompressible materials in bounded domains were studied by many authors using various penalty or mixed methods, for example, Zienkiewicz [29], Fried [11], Kikuchi and Oden [24], Brezzi and Fortin [7] and the references therein. Since the problem (1.1)-(1.4) is defined in the unbounded domain Ω, in finding the numerical solution of this problem, it is often difficult to use the classical finite element method or finite difference method directly. In the last two decades, several methods were proposed to solve boundary value problems in unbounded domains [13]. One of the most popular methods is to introduce an artificial boundary and set up artificial boundary conditions on it. Then the original problem is reduced to a boundary value problem in a bounded computational domain. Thus a numerical approximation of the original problem can be obtained by solving the reduced problem. Recent years, many authors have worked on this subject for various problems by different techniques, see Engquist and Majda [9], Goldstein [15], Feng [10], Han and Wu [23], Hagstrom and Keller [16,17], Halpern and Schatzman [18], Bao et. al. [2–5], Han et al. [22], Han and Bao [19] and the references therein.

In the above works, several authors also gave error bounds for the numerical solution, see [23, 14]. But their error bounds only depend on the mesh size and the approximate artificial boundary condition. How does the error depend on the location of the artificial boundary is unknown? But this is a very interesting problem for engineers because it can be used to determine how large the bounded computational domain should be chosen. In this paper, we will design high-order local artificial boundary conditions and provide new error bounds for the finite element approximation of the exterior problem (1.1)-(1.4). Our error bounds depend on not only the mesh size and the approximate artificial boundary condition but also the location of the artificial boundary. This kind of error bounds can be used to choose the location of the artificial boundary, the mesh size and the terms to be used in the approximate artificial boundary conditions for practical computations of an incompressible elastic material in an unbounded domain.

The layout of this paper is as follows. In the next section we design high-order local artificial boundary conditions at a given artificial boundary for the problem (1.1)–(1.4). In Sect. 3 we introduce the finite element formulation of the problem (1.1)–(1.4) in a bounded computational domain using an approximate nonlocal artificial boundary condition and prove new error bounds for the finite element approximation. In Sect. 4 we propose the finite element formulation of the problem (1.1)–(1.4) in a bounded computational domain using a high-order local artificial boundary condition and establish error bounds for the finite element approximation. In Sect. 5 we report on some numerical experiments. Finally in Sect. 6 we draw a conclusion, summarizing the main results.

2 High-order local artificial boundary conditions

In order to design high-order local artificial boundary conditions, we recall here the derivation of the exact boundary condition at a given artificial boundary for the problem (1.1)–(1.4) as described in [19].

Introducing a circle $\varGamma_e$ with radius R such that supp $f \subset B_R(0) := \{x \in \mathbb{R}^2 \ : \ |x| < R\}$, then $\varOmega$ is divided into two parts: the unbounded part $\varOmega_e := \varOmega \setminus B_R(0)$ and the bounded part $\varOmega_i := \varOmega \setminus \bar{\varOmega}_e$ (see Fig. 1). The restriction of the solution $(u,\ p)$ of the problem (1.1)–(1.4) to the unbounded domain $\varOmega_e$ is then the solution of the following problem:

$$-2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = 0 \qquad \text{in } \varOmega_e, \tag{2.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \varOmega_e, \tag{2.2}$$

$$u|_{\varGamma_e} = u(R, \theta), \tag{2.3}$$

$$u \quad \text{is bounded}, \quad p \to 0 \qquad \text{when} \quad r \to +\infty. \tag{2.4}$$

We know that the general solution of (2.1)–(2.4) is (see [19] for details):

$$u_i(r,\theta) = (r^2 - R^2)\frac{\partial W(r,\theta)}{\partial x_i} + G_i(r,\theta)$$

(2.5) $$R \le r < +\infty \quad 0 \le \theta \le 2\pi \qquad i = 1, 2,$$

where G_1, G_2 and W are harmonic functions and satisfying:

$$G_i(r,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{\infty} \left(a_n^i \cos n\theta + b_n^i \sin n\theta\right) \frac{R^n}{r^n}$$

(2.6) $$R \le r < +\infty \quad 0 \le \theta \le 2\pi \qquad i = 1, 2,$$

$$W(r,\theta) = -\sum_{n=2}^{\infty} \frac{n-1}{2n} \left[p_n \cos n\theta + q_n \sin n\theta\right] \frac{R^{n-1}}{r^n}$$

(2.7) $$R \le r < +\infty \quad 0 \le \theta \le 2\pi;$$

with

$$a_n^i = \frac{1}{\pi}\int_0^{2\pi} G_i(R,\theta)\cos n\theta \, d\theta = \frac{1}{\pi}\int_0^{2\pi} u_i(R,\theta)\cos n\theta \, d\theta$$

(2.8) $$i = 1, 2 \quad n \ge 0,$$

$$b_n^i = \frac{1}{\pi}\int_0^{2\pi} G_i(R,\theta)\sin n\theta \, d\theta = \frac{1}{\pi}\int_0^{2\pi} u_i(R,\theta)\sin n\theta \, d\theta$$

(2.9) $$i = 1, 2 \quad n \ge 1,$$

(2.10) $$p_n = a_{n-1}^1 - b_{n-1}^2 \qquad q_n = b_{n-1}^1 + a_{n-1}^2 \qquad n \ge 2.$$

Combining (2.5), (2.6), (2.7) and (2.1) on noting the boundary condition at infinity for p in (2.4), a computation shows

$$p(r,\theta) = 2\mu \sum_{n=2}^{\infty} (n-1)\left[p_n \cos n\theta + q_n \sin n\theta\right] \frac{R^{n-1}}{r^n}$$

(2.11) $$R \le r < +\infty \quad 0 \le \theta \le 2\pi.$$

Let $\sigma(u,p) = (\sigma_{ij}(u,p))_{2\times 2}$ be the stress tensor which satisfies:

(2.12) $$\sigma_{ij}(u,p) = 2\mu\, \varepsilon_{ij}(u) - p\, \delta_{ij} \qquad 1 \le i, j \le 2;$$

where δ_{ij} is the Kronecker Delta. Furthermore let $\sigma_n(u,p) = (\sigma_{n_1}(u,p), \sigma_{n_2}(u,p))^{\mathrm{T}}$ be the normal stress corresponding to the displacement u and the hydrostatic pressure p at the artificial boundary Γ_e, say

(2.13) $$\sigma_{n_i}(u,p) = \sigma_{i1}(u,p)\cos\theta + \sigma_{i2}(u,p)\sin\theta|_{\Gamma_e} \qquad i = 1, 2.$$

Combining (2.5), (2.6), (2.7) and (2.11) with $r = R$, (1.5), (2.12) and (2.13), a computation shows (see details in [19])

$$\sigma_n(u,p) = \begin{pmatrix} \sigma_{n_1}(u,p) \\ \sigma_{n_2}(u,p) \end{pmatrix} = \begin{pmatrix} 2\mu \left.\dfrac{\partial G_1}{\partial r}\right|_{r=R} \\ \\ 2\mu \left.\dfrac{\partial G_1}{\partial r}\right|_{r=R} \end{pmatrix}. \tag{2.14}$$

Differentiating (2.6) with respect to r on noting (2.5) and (2.14) and set $r = R$, we obtain

$$\begin{aligned} \sigma_{n_i}(u,p) &= \frac{2\mu}{\pi R} \sum_{n=1}^{\infty} \frac{\partial}{\partial \theta} \int_0^{2\pi} \frac{\cos n(\phi-\theta)}{n} \frac{\partial u_i(R,\phi)}{\partial \phi}\, d\phi \\ &\equiv T_i(u). \end{aligned} \tag{2.15}$$

This is the desired exact boundary condition at Γ_e for the problem (1.1)–(1.4). Thus the restriction of the solution $(u,\ p)$ of the problem (1.1)–(1.4) to the bounded domain Ω_i is the solution of the following problem:

(P) Find $(u,\ p)$ such that

$$-2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = f \qquad \text{in } \Omega_i, \tag{2.16}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega_i, \tag{2.17}$$

$$u = 0 \qquad \text{on } \Gamma_i, \tag{2.18}$$

$$\sigma_n(u,p) = T(u) \equiv (T_1(u), T_2(u))^{\mathrm{T}} \qquad \text{on} \quad \Gamma_e. \tag{2.19}$$

Let

$$T_i^N(u) = \frac{2\mu}{\pi R} \sum_{n=1}^{N} \frac{\partial}{\partial \theta} \int_0^{2\pi} \frac{\cos n(\phi-\theta)}{n} \frac{\partial u_i(R,\phi)}{\partial \phi}\, d\phi. \tag{2.20}$$

Then we derive a series of approximate artificial boundary conditions at Γ_e:

$$\sigma_n(u,p) = T^N(u) \equiv \left(T_1^N(u), T_2^N(u)\right)^{\mathrm{T}} \quad \text{on } \Gamma_e \quad N = 0,1,2,\cdots; \tag{2.21}$$

where $T^0(u) = (0,0)^{\mathrm{T}}$ is the stress free boundary condition which is often used in engineering literature. Then the original problem (1.1)–(1.4) can be reduced to the following problem defined in the bounded domain Ω_i approximately for $N = 0,1,2,\cdots$

(P_N) Find $(u_N,\ p_N)$ such that

$$-2\mu \operatorname{div} \varepsilon(u_N) + \operatorname{grad} p_N = f \qquad \text{in } \Omega_i, \tag{2.22}$$

$$\operatorname{div} u_N = 0 \qquad \text{in } \Omega_i, \tag{2.23}$$

$$u_N = 0 \qquad \text{on } \Gamma_i, \tag{2.24}$$

$$\sigma_n(u_N, p_N) = T^N(u_N) \qquad \text{on } \Gamma_e. \tag{2.25}$$

Now we design high-order local artificial boundary conditions at Γ_e for the problem (1.1)–(1.4). We consider a solution $(u,\ p)$ of the problem (1.1)–(1.4), which consists of the first N harmonics at Γ_e. Thus we assume

$$u_i(R,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{N} \left(a_n^i \cos n\theta + b_n^i \sin n\theta\right) \qquad i = 1, 2; \tag{2.26}$$

where the a_n^1, b_n^1, a_n^2 and b_n^2 are constants (Fourier coefficients, see (2.8) and (2.9)). Substituting (2.26) into (2.15), we get

$$\sigma_{n_i}(u,p) = -\frac{2\mu}{R} \sum_{n=1}^{N} n \left(a_n^i \cos n\theta + b_n^i \sin n\theta\right) \qquad i = 1, 2. \tag{2.27}$$

In order to design high-order local artificial boundary conditions, noting the form of $\sigma_{n_i}(u,p)$ in (2.27) and the form of $u_i(R,\theta)$ in (2.26), it is desired to find a linear differential operator L_N which does not depend on n, such that

$$\begin{aligned} &L_N[1] = 0 \quad L_N[\cos n\theta] = n \cos n\theta \\ &L_N[\sin n\theta] = n \sin n\theta \qquad n = 1, 2, \cdots, N. \end{aligned} \tag{2.28}$$

With such an operator at hand, noting (2.26), then (2.27) can be written

$$\begin{aligned} \sigma_{n_i}(u,p) &= -\frac{2\mu}{R} \left[0 + \sum_{n=1}^{N} \left(a_n^i n \cos n\theta + b_n^i n \sin n\theta\right)\right] \\ &= -\frac{2\mu}{R} \left[\frac{a_0^i}{2}\ L_N[1] + \sum_{n=1}^{N} \left(a_n^i\ L_N[\cos n\theta] + b_n^i\ L_N[\sin n\theta]\right)\right] \\ &= -\frac{2\mu}{R}\ L_N \left[\frac{a_0^i}{2} + \sum_{n=1}^{N} \left(a_n^i \cos n\theta + b_n^i \sin n\theta\right)\right] \\ &= -\frac{2\mu}{R}\ L_N[u_i(R,\theta)] \qquad i = 1, 2. \end{aligned} \tag{2.29}$$

The equality (2.29) is a local artificial boundary condition at Γ_e which is exact for all solutions consisting of at most the first N harmonics at Γ_e. Noting the fact

$$\begin{aligned} &\frac{d^{2m}}{d\theta^{2m}} \cos n\theta = (-1)^m n^{2m} \cos n\theta, \\ &\frac{d^{2m}}{d\theta^{2m}} \sin n\theta = (-1)^m n^{2m} \sin n\theta \qquad m \geq 0 \ \ n \geq 0, \end{aligned} \tag{2.30}$$

we can assume the operator L_N has the following form

Table 1. The coefficients $\alpha_m^{(N)}$ in the first five local artificial boundary conditions

	$\alpha_1^{(N)}$	$\alpha_2^{(N)}$	$\alpha_3^{(N)}$	$\alpha_4^{(N)}$	$\alpha_5^{(N)}$
$N=1$	1				
$N=2$	$7/6$	$-1/6$			
$N=3$	$74/60$	$-15/60$	$1/60$		
$N=4$	$533/420$	$-43/144$	$11/360$	$-1/1008$	
$N=5$	$3881/3780$	$-214/643$	$71/1728$	$-13/6048$	$1/25920$

$$L_N[u(R,\theta)] = \sum_{m=1}^{N}(-1)^m\alpha_m^{(N)}\frac{\partial^{2m}}{\partial\theta^{2m}}u(R,\theta). \tag{2.31}$$

Inserting (2.31) into (2.29), noting (2.27) and (2.26), we obtain

$$\sum_{m=1}^{N} n^{2m}\alpha_m^{(N)} = n \qquad n = 1,2,\cdots,N. \tag{2.32}$$

It is straightforward to check that the linear system (2.32) has a unique solution for any $N \in \mathbb{N}$. Table 1 shows the coefficients $\alpha_m^{(N)}$ in the first five local artificial boundary conditions. Combining (2.29) and (2.31), we get high-order local artificial boundary conditions at Γ_e for the problem (1.1)–(1.4):

$$\sigma_n(u,p) = \tilde{T}^N(u) \equiv \left(\tilde{T}_1^N(u), \tilde{T}_2^N(u)\right)^{\mathrm{T}} \qquad N = 1,2,\cdots; \tag{2.33}$$

where

$$\tilde{T}_i^N(u) = -\frac{2\mu}{R}\sum_{m=1}^{N}(-1)^m\alpha_m^{(N)}\frac{\partial^{2m}u_i(R,\theta)}{\partial\theta^{2m}} \qquad i = 1,2. \tag{2.34}$$

Then the original problem (1.1)–(1.4) can be reduced to the following problem defined in the bounded domain Ω_i approximately for $N = 1,2,\cdots$

($\tilde{\mathrm{P}}_N$) Find $(\tilde{u}_N,\ \tilde{p}_N)$ such that

$$-2\mu \operatorname{div} \varepsilon(\tilde{u}_N) + \operatorname{grad} \tilde{p}_N = f \qquad \text{in } \Omega_i, \tag{2.35}$$

$$\operatorname{div} \tilde{u}_N = 0 \qquad \text{in } \Omega_i, \tag{2.36}$$

$$\tilde{u}_N = 0 \qquad \text{on } \Gamma_i, \tag{2.37}$$

$$\sigma_n(\tilde{u}_N, \tilde{p}_N) = \tilde{T}^N(\tilde{u}_N) \qquad \text{on } \Gamma_e. \tag{2.38}$$

3 Error bounds for the case of using nonlocal artificial boundary conditions

In the work [19], the authors have already given error bounds for the finite element approximation of the problem (P_N). But from their error bounds, one doesn't know how the errors depend on the location of the artificial boundary. In this section, we will present new error bounds for the finite element approximations of problems (P_N). These error bounds depend on not only the mesh size and the approximate artificial boundary condition but also the location of the artificial boundary. This kind of error bounds is very useful in engineering applications.

Let $H^m(\Omega_i)$ and $H^s(\Gamma_e)$ be the usual Sobolev spaces on the domain Ω_i and the boundary Γ_e with integer m and real number s. Furthermore $\|\cdot\|_{m,\Omega_i}$ and $|\cdot|_{m,\Omega_i}$ denote the usual norm and semi-norm on $H^m(\Omega_i)$, respectively [1]. Suppose

$$\begin{aligned} V &= \left\{v = (v_1, v_2)^{\mathrm{T}} \in [H^1(\Omega_i)]^2 \ : \ v|_{\Gamma_i} = 0\right\}; \\ W &= L^2(\Omega_i). \end{aligned}$$

Then the boundary value problem (P) is equivalent to the following variational problem:

(VP) Find $(u,\ p) \in V \times W$ such that

$$a(u,v) + a_0(u,v) + b(p,v) = f(v) \qquad \forall v \in V, \tag{3.1}$$

$$b(q,u) = 0 \qquad \forall q \in W. \tag{3.2}$$

where

$$\begin{aligned} a(u,v) &= 2\mu \int_{\Omega_i} \sum_{i,j=1}^{2} \varepsilon_{ij}(u)\varepsilon_{ij}(v)\, dx \\ &\equiv 2\mu \int_{\Omega_i} \varepsilon(u) : \varepsilon(v)\, dx \qquad \forall u, v \in V, \end{aligned} \tag{3.3}$$

$$\begin{aligned} a_0(u,v) &= -\int_{\Gamma_e} T(u)\cdot v\, ds \\ &= \frac{2\mu}{\pi} \sum_{n=1}^{\infty} n \int_0^{2\pi}\int_0^{2\pi} \\ &\quad \times \sum_{i=1}^{2} u_i(R,\phi) v_i(R,\theta) \cos n(\phi-\theta)\, d\theta\, d\phi \\ &= \frac{2\mu}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi}\int_0^{2\pi} \\ &\quad \times \sum_{i=1}^{2} \frac{\partial u_i(R,\phi)}{\partial \phi} \frac{\partial v_i(R,\theta)}{\partial \theta} \frac{\cos n(\phi-\theta)}{n} d\theta\, d\phi \\ &\quad \forall u, v \in V, \end{aligned} \tag{3.4}$$

$$b(q,v) = -\int_{\Omega_i} q \operatorname{div} v \, dx \qquad \forall v \in V \quad q \in W, \tag{3.5}$$

$$f(v) = \int_{\Omega_i} f \cdot v \, dx \qquad \forall v \in V. \tag{3.6}$$

Furthermore let

$$a_0^N(u,v) = \frac{2\mu}{\pi} \sum_{n=1}^{N} n \int_0^{2\pi} \int_0^{2\pi} \sum_{i=1}^{2} u_i(R,\phi) v_i(R,\theta) \cos n(\phi - \theta) \, d\theta \, d\phi$$
$$\forall u, v \in V. \tag{3.7}$$

Then the boundary value problem (P_N) is equivalent to the following variational problem:

(VP_N) Find $(u_N,\ p_N) \in V \times W$ such that

$$a(u_N, v) + a_0^N(u_N, v) + b(p_N, v) = f(v) \qquad \forall v \in V, \tag{3.8}$$
$$b(q, u_N) = 0 \qquad \forall q \in W. \tag{3.9}$$

If we replace V and W by their finite element subspaces V^h and W^h in which h represents the mesh size [8], then the finite element approximation of the problem (VP_N) is:

(VP_N^h) Find $(u_N^h,\ p_N^h) \in V^h \times W^h$ such that

$$a(u_N^h, v^h) + a_0^N(u_N^h, v^h) + b(p_N^h, v^h) = f(v^h) \qquad \forall v^h \in V^h, \tag{3.10}$$
$$b(q^h, u_N^h) = 0 \qquad \forall q^h \in W^h. \tag{3.11}$$

We note that the symmetric bilinear form $a(\cdot,\cdot)$ is bounded and coercive on $V \times V$ from the Körn inequality [26] and Poincaré inequality [1], i.e. there exist two positive constants M_1, M_2 such that

$$|a(u,v)| \leq M_1 \|u\|_V \cdot \|v\|_V \qquad \forall u, v \in V, \tag{3.12}$$
$$M_2 \|v\|_V^2 \leq a(v,v) \qquad \forall v \in V. \tag{3.13}$$

Thus we can define an equivalent norm on the space V:

$$\|v\|_* = [a(v,v)]^{1/2} \qquad \forall v \in V. \tag{3.14}$$

Therefore we have that

$$|a(u,v)| \leq \|u\|_* \cdot \|v\|_* \qquad \forall u, v \in V, \tag{3.15}$$
$$\|v\|_*^2 \leq a(v,v) \qquad \forall v \in V. \tag{3.16}$$

For the bilinear forms $a_0(\cdot,\cdot)$ and $a_0^N(\cdot,\cdot)$, we have that

Lemma 3.1 *The following inequality holds:*

$$0 \le a_0^N(v,v) \le a_0(v,v) \le 3a(v,v) \equiv 3\|v\|_*^2 \qquad \forall v \in V \quad N \ge 0, \tag{3.17}$$

$$|a_0(u,v)| \le 3\|u\|_* \cdot \|v\|_*, \quad |a_0^N(u,v)| \le 3\|u\|_* \cdot \|v\|_* \qquad \forall u,v \in V \quad N \ge 0. \tag{3.18}$$

where $a_0(u,v)$ *and* $a_0^N(u,v)$ *are defined in (3.4) and (3.7), respectively.*

Proof. See Appendix.

For the bilinear form $b(q,v)$, we have that

Lemma 3.2 *There exists a generic constant* $\beta_0 > 0$ *independent of* R*, such that*

$$\sup_{v \in V\setminus\{0\}} \frac{b(q,v)}{\|v\|_*} \ge \beta_0 \, \|q\|_W \qquad \forall q \in W. \tag{3.19}$$

Proof. See Appendix.

In order to derive the well-posedness of the problem (VP_N^h) and an error bound of the finite element approximation, we suppose the following discrete inf-sup condition between V^h and W^h holds:

$$\sup_{v^h \in V^h\setminus\{0\}} \frac{b(q^h,v^h)}{\|v^h\|_*} \ge \beta_0^* \, \|q^h\|_W \qquad \forall q^h \in W^h, \tag{3.20}$$

where β_0^* is a constant independent of h, N and R.

It follows immediately from (3.15), (3.16), (3.18), (3.17), (3.19), (3.20) and Theorem 4.1 in Chapter I of [12] that the variational problems (VP), (VP_N) and (VP_N^h) are well-posed; that is, for $f \in V'$, the dual of V, there exists a unique $(u,\ p) \in V \times W$ solving (VP), a unique $(u_{_N},\ p_{_N}) \in V \times W$ solving (VP_N), a unique $(u_{_N}^h, p_{_N}^h) \in V^h \times W^h$ solving (VP_N^h), and

$$\|u\|_* + \|u_{_N}\|_* + \|u_{_N}^h\|_* + \|p\|_W + \|p_{_N}\|_W + \|p_{_N}^h\|_W \le C\, \|f\|_{V'}. \tag{3.21}$$

Note that the well-posedness of (VP) implies immediately the well-posedness of the original problem (1.1)–(1.4).

Let $R_0 = \max\{|x| \,:\, x \in \ \text{supp}\, f \cup \Gamma_i\}, \Gamma_0 = \{(R_0,\theta) \,:\, 0 \le \theta \le 2\pi\}$ and $\Omega_0 = \{x \in \Omega_i \ :\ |x| < R_0\}$ and $\Gamma_r = \{(r,\theta) \ :\ 0 \le \theta \le 2\pi\}$. We recall an equivalent definition of Sobolev space $H^s(\Gamma_r)$ for any real number s [25]:

$$w \in H^s(\Gamma_r) \Longleftrightarrow w(r,\theta) = \frac{p_0}{2} + \sum_{m=1}^{\infty} (p_m \cos m\theta + q_m \sin m\theta) \quad \text{and}$$

$$\frac{\pi p_0^2}{2} + \sum_{m=1}^{\infty} \pi(1+m^2)^s(p_m^2 + q_m^2) < \infty.$$

Thus we use

$$|w|_{s,\Gamma_r} = \left[\sum_{m=1}^{\infty} \pi m^{2s}(p_m^2 + q_m^2)\right]^{1/2} \tag{3.22}$$

as a semi-norm of the space $H^s(\Gamma_r)$. Then we have the following estimate:

Lemma 3.3 *Suppose* $(u,\ p) \in V \times W$ *be the solution of the exterior problem (1.1)–(1.4) and there exists an integer* $k \geq 1$ *such that* $u|_{\Gamma_0} \in [H^{k+\frac{1}{2}}(\Gamma_0)]^2$. *Then we have that*

$$|a_0(u,v) - a_0^N(u,v)| \leq \frac{C_0}{(N+1)^{k-1}}\left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{k+\frac{1}{2},\Gamma_0} \cdot \|v\|_* \quad \forall v \in V, \tag{3.23}$$

where C_0 *is a generic constant independent of* u, N, h *and* R.

Proof. Assume that

$$u_i(R_0,\theta) = \frac{p_0^i}{2} + \sum_{n=1}^{\infty}(p_n^i \cos n\theta + q_n^i \sin n\theta) \qquad i = 1,2, \tag{3.24}$$

$$v_i(R,\theta) = \frac{c_0^i}{2} + \sum_{n=1}^{\infty}(c_n^i \cos n\theta + d_n^i \sin n\theta) \qquad i = 1,2; \tag{3.25}$$

where

$$p_n^i = \frac{1}{\pi}\int_0^{2\pi} u_i(R_0,\theta)\cos n\theta\ d\theta,$$

$$q_n^i = \frac{1}{\pi}\int_0^{2\pi} u_i(R_0,\theta)\sin n\theta\ d\theta \qquad i = 1,2 \quad n \geq 0; \tag{3.26}$$

$$c_n^i = \frac{1}{\pi}\int_0^{2\pi} v_i(R,\theta)\cos n\theta\ d\theta,$$

$$d_n^i = \frac{1}{\pi}\int_0^{2\pi} v_i(R,\theta)\sin n\theta\ d\theta \qquad i = 1,2 \quad n \geq 0. \tag{3.27}$$

Noting that $(u,\ p)$ satisfies the homogeneous Stokes equations (say (1.1)–(1.2) with $f = 0$) in the domain $\{x\ :\ |x| > R_0\}$, by separation of variables, we get

$$u_1(r,\theta) = \frac{r^2 - R_0^2}{2} \sum_{n=3}^{\infty} (n-2) \Big[(p_{n-2}^1 - q_{n-2}^2) \cos n\theta$$
$$+ (q_{n-2}^1 + p_{n-2}^2) \sin n\theta \Big] \frac{R_0^{n-2}}{r^n}$$
$$+ \frac{p_0^1}{2} + \sum_{n=1}^{\infty} (p_n^1 \cos n\theta + q_n^1 \sin n\theta) \frac{R_0^n}{r^n}$$
$$R_0 \le r \quad 0 \le \theta \le 2\pi, \tag{3.28}$$

$$u_2(r,\theta) = \frac{r^2 - R_0^2}{2} \sum_{n=3}^{\infty} (n-2) \Big[-(q_{n-2}^1 + p_{n-2}^2) \cos n\theta$$
$$+ (p_{n-2}^1 - q_{n-2}^2) \sin n\theta \Big] \frac{R_0^{n-2}}{r^n}$$
$$+ \frac{p_0^2}{2} + \sum_{n=1}^{\infty} (p_n^2 \cos n\theta + q_n^2 \sin n\theta) \frac{R_0^n}{r^n}$$
$$R_0 \le r \quad 0 \le \theta \le 2\pi. \tag{3.29}$$

Setting $r = R$ in (3.28) and (3.29), we obtain

$$u_i(R,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{\infty} (a_n^i \cos n\theta + b_n^i \sin n\theta) \qquad i = 1,2; \tag{3.30}$$

where

$$a_n^1 = \left(\frac{R_0}{R}\right)^n \begin{cases} p_n^1 & n = 0,1,2, \\ p_n^1 + \frac{(n-2)(R^2-R_0^2)}{2R_0^2}(p_{n-2}^1 - q_{n-2}^2) & n \ge 3; \end{cases} \tag{3.31}$$

$$b_n^1 = \left(\frac{R_0}{R}\right)^n \begin{cases} q_n^1 & n = 1,2, \\ q_n^1 + \frac{(n-2)(R^2-R_0^2)}{2R_0^2}(q_{n-2}^1 + p_{n-2}^2) & n \ge 3; \end{cases} \tag{3.32}$$

$$a_n^2 = \left(\frac{R_0}{R}\right)^n \begin{cases} p_n^2 & n = 0,1,2, \\ p_n^2 - \frac{(n-2)(R^2-R_0^2)}{2R_0^2}(q_{n-2}^1 + p_{n-2}^2) & n \ge 3; \end{cases} \tag{3.33}$$

$$b_n^2 = \left(\frac{R_0}{R}\right)^n \begin{cases} q_n^2 & n = 1,2, \\ q_n^2 + \frac{(n-2)(R^2-R_0^2)}{2R_0^2}(p_{n-2}^1 - q_{n-2}^2) & n \ge 3. \end{cases} \tag{3.34}$$

Inserting (3.30), (3.25) into (3.4) and (3.7), using the orthogonality of the cosines and sines, noting (3.31)–(3.34), (3.26), (3.18) and (3.17), we obtain

$$|a_0(u,v) - a_0^N(u,v)|$$

$$
\begin{aligned}
&= \left| 2\mu\pi \sum_{n=1}^{\infty}\sum_{i=1}^{2} n(a_n^i c_n^i + b_n^i d_n^i) - 2\mu\pi \sum_{n=1}^{N}\sum_{i=1}^{2} n(a_n^i c_n^i + b_n^i d_n^i) \right| \\
&= 2\mu\pi \left| \sum_{n=N+1}^{\infty} \sum_{i=1}^{2} n(a_n^i c_n^i + b_n^i d_n^i) \right| \\
&\le 2\mu\pi \left[\sum_{n=N+1}^{\infty} n \sum_{i=1}^{2} \left((a_n^i)^2 + (b_n^i)^2\right) \right]^{1/2} \\
&\quad \cdot \left[\sum_{n=N+1}^{\infty} n \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2\right) \right]^{1/2} \\
&\le C_0 \left[\sum_{n=N+1}^{\infty} n \sum_{i=1}^{2} \left((p_n^i)^2 + (q_n^i)^2\right) \frac{R_0^{2n}}{R^{2n}} \right. \\
&\quad \left. + \sum_{n=\max\{1,N-1\}}^{\infty} n^3 \sum_{i=1}^{2} \left((p_n^i)^2 + (q_n^i)^2\right) \frac{R_0^{2n}}{R^{2n}} \right]^{1/2} \cdot \|v\|_* \\
&\le \frac{C_0}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{k+\frac{1}{2},\Gamma_0} \cdot \|v\|_*.
\end{aligned}
\tag{3.35}
$$

Combining Lemmas 3.1–3.3, we get the following error bound:

Theorem 3.1 *Let* $(u,\ p)$ *be the solution of the problem (1.1)–(1.4) and* $(u_N^h,\ p_N^h)$ *be the solution of the problem (*VP_N^h*). Suppose the discrete inf-sup condition (3.20) holds,* $f \in [L^2(\Omega_i)]^2$ *and* $u|_{\Gamma_0} \in [H^{k+\frac{1}{2}}(\Gamma_0)]^2$ *(*$k \ge 1$*). Then we have the following error bound:*

$$
\begin{aligned}
\|u - u_N^h\|_* + \|p - p_N^h\|_W \le C_0 \Bigg[& \inf_{v^h \in V^h} \|u - v^h\|_* + \inf_{q^h \in W^h} \|p - q^h\|_W \\
& + \frac{1}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{k+\frac{1}{2},\Gamma_0} \Bigg].
\end{aligned}
\tag{3.36}
$$

Proof. The proof of this theorem is by using a standard technique of mixed finite element method [12] and noting the estimate (3.23). The detail is omitted here.

Suppose $u \in [H^{k+1}(\Omega_i)]^2$, $p \in [H^k(\Omega_i)]^2$, $u|_{\Gamma_0} \in [H^{k+\frac{1}{2}}(\Gamma_0)]^2$ and the interpolation errors of V^h to V and W^h to W [8, 12]

$$
\inf_{v^h \in V^h} \|u - v^h\|_V + \inf_{q^h \in W^h} \|p - p^h\|_W
$$

$$\leq C_0 h^k \left(|u|_{k+1,\Omega_i} + |p|_{k,\Omega_i}\right). \tag{3.37}$$

Then combining (3.37) and (3.36), noting the Körn inequality and Poincaré inequality, (1.5), (3.3) and (3.14), we get

$$\begin{aligned}
&\|u - u_N^h\|_{1,\Omega_0} + \|p - p_N^h\|_{0,\Omega_0} \\
&\leq C_0 \left(|u - u_N^h|_{1,\Omega_0} + \|p - p_N^h\|_{0,\Omega_0}\right) \\
&\leq C_0 \left(\|\varepsilon(u - u_N^h)\|_{0,\Omega_0} + \|p - p_N^h\|_W\right) \\
&\leq C_0 \left(\|u - u_N^h\|_* + \|p - p_N^h\|_W\right) \\
&\leq C_0 \Bigg[h^k \left(|u|_{k+1,\Omega_i} + |p|_{k,\Omega_i}\right) \\
&\qquad + \frac{1}{(N+1)^{k-1}} \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{k+\frac{1}{2},\Gamma_0}\Bigg].
\end{aligned} \tag{3.38}$$

For example, for the Taylor-Hood (P2/P1) element which satisfies (3.20) [6,12], the error bound (3.38) holds for $k = 2$.

4 Error bounds for the case of using high-order local artificial boundary conditions

In this section, we will present the finite element formulation of the problem ($\tilde{\mathrm{P}}_N$) and provide an error bound for the finite element approximation. To cope with the high-order local artificial boundary condition (2.33), we define

$$\tilde{V} = \{v \in [H^1(\Omega_i)]^2 \ : \ v|_{\Gamma_e} \in [H^N(\Gamma_e)]^2 \quad v|_{\Gamma_i} = 0\}.$$

Let

$$\begin{aligned}
\tilde{a}_0^N(u,v) &= -\int_{\Gamma_e} v \cdot \tilde{T}^N(u)\, ds \\
&= 2\mu \int_0^{2\pi} \sum_{m=1}^{N} \alpha_m^{(N)} \sum_{i=1}^{2} \frac{\partial^m u_i(R,\theta)}{\partial \theta^m} \frac{\partial^m v_i(R,\theta)}{\partial \theta^m} d\theta \\
&\qquad \forall u, v \in \tilde{V}.
\end{aligned} \tag{4.1}$$

Then the weak form of the problem ($\tilde{\mathrm{P}}_N$) is:

($\tilde{\mathrm{V}}\mathrm{P}_N$) Find $(\tilde{u}_N,\ \tilde{p}_N) \in \tilde{V} \times W$ such that

$$a(\tilde{u}_N, v) + \tilde{a}_0^N(\tilde{u}_N, v) + b(\tilde{p}_N, v) = f(v) \qquad \forall v \in \tilde{V}, \tag{4.2}$$

$$b(q, \tilde{u}_N) = 0 \qquad \forall q \in W. \tag{4.3}$$

If we replace $\tilde{V}$ and W by their finite dimensional subspaces, $\tilde{V}^h \subset \tilde{V}$ and $W^h \subset W$ in which h is the mesh size [8], then the finite element approximation of the problem ($\tilde{\mathrm{V}}\mathrm{P}_N$) is:

($\tilde{\mathrm{V}}\mathrm{P}_N^h$) Find $(\tilde{u}_{_N}^h,\ \tilde{p}_{_N}^h) \in \tilde{V}^h \times W^h$ such that

$$a(\tilde{u}_{_N}^h, v^h) + \tilde{a}_0^N(\tilde{u}_{_N}^h, v^h) + b(\tilde{p}_{_N}^h, v^h) = f(v^h) \qquad \forall v^h \in \tilde{V}^h, \tag{4.4}$$

$$b(q^h, \tilde{u}_{_N}^h) = 0 \qquad \forall q^h \in W^h. \tag{4.5}$$

From (3.15), (3.16) and (3.19), the well-posedness of the problem ($\tilde{\mathrm{V}}\mathrm{P}_N$) depends on the property of the bilinear form $\tilde{a}_0^N(u,v)$. For any $u, v \in \tilde{V}$, we can also formally expand $u|_{\Gamma_e} = u(R,\theta)$ and $v|_{\Gamma_e} = v(R,\theta)$ in Fourier series (see (3.30) and (3.25)). Substituting (3.30) and (3.25) into (4.1) and using the orthogonality of the cosines and sines, we obtain

$$\tilde{a}_0^N(u,v) = 2\mu\pi \sum_{n=1}^{\infty} \gamma_n^{(N)} \sum_{i=1}^{2} \left(a_n^i c_n^i + b_n^i d_n^i\right) \qquad \forall u, v \in \tilde{V}, \tag{4.6}$$

where

$$\gamma_n^{(N)} = \sum_{m=1}^{N} n^{2m} \alpha_m^{(N)} \qquad \forall n \in \mathbb{N}. \tag{4.7}$$

Thus the property of $\tilde{a}_0^N(u,v)$ depends on the property of $\gamma_n^{(N)}$.

Table 1 shows $\alpha_m^{(N)}$ is positive for odd m, and is negative for even $m > 0$ for $1 \le N \le 5$. This property can be demonstrated numerically for $1 \le N \le 20$. Thus we have that

$$\alpha_1^{(N)} > 0 \qquad \alpha_N^{(N)} = \begin{cases} > 0 & N \text{ is odd} \\ < 0 & N \text{ is even} \end{cases} \qquad 1 \le N \le 20. \tag{4.8}$$

From the engineering application point of view, the parameter N in (2.33) is always less than 10. Therefore in this paper we assume $N \le 20$ in (2.33). Then for the $\gamma_n^{(N)}$ we have that

Lemma 4.1 *If* $1 \le N \le 20$ *is odd, then*

$$\gamma_n^{(N)} \ge n \qquad \forall n \ge 1 \qquad \lim_{n\to+\infty} \frac{\gamma_n^{(N)}}{n^{2N}} = \alpha_N^{(N)} > 0. \tag{4.9}$$

If $1 \le N \le 20$ *is even, then*

$$\gamma_n^{(N)} < 0 \quad \text{when } n \text{ is sufficient large} \qquad \lim_{n\to+\infty} \frac{\gamma_n^{(N)}}{n^{2N}} = \alpha_N^{(N)} < 0. \tag{4.10}$$

Proof. We set a polynomial function whose degree is $2N$, say

$$\eta_N(t) = \sum_{m=1}^{N} \alpha_m^{(N)} t^{2m} - t. \tag{4.11}$$

Since $\eta_N''(t)$ is an even polynomial function whose degree is $2N-2$ and $\eta_N''(0) = 2\alpha_1^{(N)} > 0$ for $1 \le N \le 20$ by noting (4.8), we know that $\eta_N''(t) = 0$ has at most $N-1$ non-negative roots. Thus $\eta_N(t) = 0$ has at most $N+1$ non-negative roots. From (4.11) and (2.32), we know that $t = 0, 1, 2, \cdots, N$ are roots of $\eta_N(t) = 0$. Thus for $1 \le N \le 20$, we have that

$$\eta_N(t) \neq 0 \quad \forall t > N \qquad \lim_{t \to +\infty} \frac{\eta_N(t)}{t^{2N}} = \alpha_N^{(N)}. \tag{4.12}$$

Then the desired inequalities (4.9) and (4.10) follows immediately from (4.12) and (4.8).

From the above discussion we have that

Lemma 4.2 *For odd $1 \le N \le 20$, there exist two generic positive constants $C_N^{(1)}$ and $C_N^{(2)}$ depending only on N such that*

$$|\tilde{a}_0^N(u,v)| \le C_N^{(2)} |u|_{N,\Gamma_R} \cdot |v|_{N,\Gamma_R} \qquad \forall u, v \in \tilde{V}, \tag{4.13}$$

$$C_N^{(1)} |v|_{N,\Gamma_R}^2 \le \tilde{a}_0^N(v,v) \qquad \forall v \in \tilde{V}. \tag{4.14}$$

Proof. For any odd $1 \le N \le 20$, noting (4.7) and (4.9), we know that there exist positive constants $C_N^{(1)}$ and $C_N^{(2)}$ such that

$$C_N^{(1)} n^{2N} \le \gamma_n^{(N)} \le C_N^{(2)} n^{2N} \qquad n = 1, 2, 3, \cdots \tag{4.15}$$

From (4.15) and (4.6), noting (3.22) with $r = R$, (3.30) and (3.25), we have that

$$\begin{aligned}
|\tilde{a}_0^N(u,v)| &\le C_N^{(2)} \sum_{n=1}^{\infty} n^{2N} \left| a_n^1 c_n^1 + b_n^1 d_n^1 + a_n^2 c_n^2 + b_n^2 d_n^2 \right| \\
&\le C_N^{(2)} \left[\sum_{n=1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((a_n^i)^2 + (b_n^i)^2 \right) \right]^{1/2} \\
&\quad \cdot \left[\sum_{n=1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((d_n^i)^2 + (c_n^i)^2 \right) \right]^{1/2} \\
&\le C_N^{(2)} |u|_{N,\Gamma_R} \cdot |v|_{N,\Gamma_R} \qquad \forall u, v \in \tilde{V}.
\end{aligned} \tag{4.16}$$

Furthermore from (4.15) and (4.6) with $u = v$, noting (3.22) with $r = R$ and (3.25), we obtain

$$\tilde{a}_0^N(v,v) \geq 2\mu\pi \sum_{n=1}^{\infty} \gamma_n^{(N)} \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2\right)$$
$$\geq C_N^{(1)} \sum_{n=1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2\right)$$
$$= C_N^{(1)} |v|_{N,\Gamma_R} \qquad \forall v \in \tilde{V}. \tag{4.17}$$

Thus the desired inequalities (4.13) and (4.14) are proved.

Remark 4.1 For even $2 \leq N \leq 20$, noting (4.10) and (4.6), the inequality (4.14) doesn't hold. In fact, it is easy to construct a function $v \in \tilde{V}$ such that $\tilde{a}_0^N(v,v) < 0$ for even $2 \leq N \leq 20$. Thus the bilinear form $a(u,v) + \tilde{a}_0^N(u,v)$ is not coercive on $\tilde{V}$. Therefore one can't prove the well-posedness of the problem $(\tilde{V}P_N)$. This phenomena was also observed in numerical simulation of Poisson equation in unbounded domain: the errors of the finite element solution when choosing $N = 2, 4, \cdots$ in high-order local artificial boundary conditions are much more larger than those when choosing $N = 0, 1, 3, \cdots$, see [21] and [14] for details.

From the discussion above, noting the Körn inequality and Poincaré inequality, we assign the following norm on Ṽ:

$$\|v\|_{\triangle} := \left[a(v,v) + |v|_{N,\Gamma_R}^2\right]^{1/2}$$
$$\equiv \left[\|v\|_*^2 + |v|_{N,\Gamma_R}^2\right]^{1/2} \qquad \forall v \in \tilde{V}. \tag{4.18}$$

For the well-posedness of the problem $(\tilde{\mathrm{V}}\mathrm{P}_N^h)$, We assume the following discrete inf-sup condition between $\tilde{V}^h$ and W^h holds:

$$\sup_{v^h \in \tilde{V}^h \setminus \{0\}} \frac{b(q^h, v^h)}{\|v^h\|_{\triangle}} \geq \beta_0^* \|q^h\|_W \qquad \forall q^h \in W^h, \tag{4.19}$$

where β_0^* is a constant independent of h, N and R.

Remark 4.2 The discrete inf-sup condition (4.19) is an assumption for the following error estimates. When $N = 1$ in (4.1), the usual finite element subspace V^h of V proposed in [12] can be used as $\tilde{V}^h$ directly, e.g. the Taylor-Hood element (i.e. P2/P1). From our simple numerical study, the P2/P1 element seems to satisfy (4.19) with $N = 1$ in (4.18). When deal with the case of $N > 1$ in (4.1), special subspace $\tilde{V}^h$ which has higher regularity at Γ_e should be used. A family of this kind of spaces was introduced in [14].

It follows immediately from (3.15), (3.16), (3.19), (4.19), (4.13) and (4.14) that the variational problems $(\tilde{\mathrm{V}}\mathrm{P}_N)$ and $(\tilde{\mathrm{V}}\mathrm{P}_N^h)$ are well-posed in the case of odd $1 \le N \le 20$ or $N = 0$; that is, for $f \in \tilde{V}'$, the dual of $\tilde{V}$, there exists a unique $(\tilde{u}_N,\ \tilde{p}_N) \in \tilde{V} \times W$ solving ($\tilde{\mathrm{V}}\mathrm{P}_N$), a unique $(\tilde{u}_N^h, \tilde{p}_N^h) \in \tilde{V}^h \times W^h$ solving ($\tilde{\mathrm{V}}\mathrm{P}_N^h$), and

$$\|\tilde{u}_N\|_\triangle + \|\tilde{u}_N^h\|_\triangle + \|\tilde{p}_N\|_W + \|\tilde{p}_N^h\|_W \le M_N \ \|f\|_{\tilde{V}'} \qquad \forall \text{ odd } 1 \le N \le 20, \tag{4.20}$$

where M_N is a constant.

When $N = 0$, then $\tilde{a}_0^N(u,v) = a_0^N(u,v) \equiv 0$. We have dealt with this case in last section. From now on, we always assume $1 \le N \le 20$ be an odd integer. For the bilinear form $\tilde{a}_0^N(u,v)$, we have the following estimate:

Lemma 4.3 *Suppose* $(u,\ p) \in V \times W$ *be the solution of the exterior problem (1.1)–(1.4) and* $u|_{\Gamma_0} \in [H^{N+1}(\Gamma_0)]^2$*. Then we have the following estimate for odd* $1 \le N \le 20$*:*

$$|a_0(u,v) - \tilde{a}_0^N(u,v)| \le C_{(N)} \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{N+1,\Gamma_0} \cdot |v|_{N,\Gamma_R} \qquad \forall v \in \tilde{V}, \tag{4.21}$$

where $C_{(N)}$ *is a generic constant independent of* u*,* R *and* h*.*

Proof. Inserting (3.30), (3.25) into (3.4), using the orthogonality of the cosines and sines, noting (4.6), (4.15), (3.31)–(3.34), (4.7), (2.32) and (3.22), we obtain for odd $1 \le N \le 20$

$$\begin{aligned}
&|a_0(u,v) - \tilde{a}_0^N(u,v)| \\
&\quad = 2\mu\pi \left| \sum_{n=N+1}^{\infty} \left(\gamma_n^{(N)} - n\right) \sum_{i=1}^{2} \left(a_n^i c_n^i + b_n^i d_n^i\right) \right| \\
&\quad \le C_{(N)} \sum_{n=N+1}^{\infty} n^{2N} \sum_{i=1}^{2} \left|a_n^i c_n^i + b_n^i d_n^i\right| \\
&\quad \le C_{(N)} \left[\sum_{n=N+1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((a_n^i)^2 + (b_n^i)^2\right) \right]^{1/2} \\
&\qquad \cdot \left[\sum_{n=N+1}^{\infty} n^{2N} \sum_{i=1}^{2} \left((c_n^i)^2 + (d_n^i)^2\right) \right]^{1/2} \\
&\quad \le C_{(N)} \left[\sum_{n=\max\{1,N-1\}}^{\infty} n^{2N+2} \sum_{i=1}^{2} \left((p_n^i)^2 + (q_n^i)^2\right) \frac{R_0^{2n}}{R^{2n}} \right]^{1/2} \cdot |v|_{N,\Gamma_R}
\end{aligned}$$

$$\leq C_{(N)} \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{N+1,\Gamma_0} \cdot |v|_{N,\Gamma_R} \qquad \forall v \in \tilde{V}. \tag{4.22}$$

Combining Lemmas 4.2-4.3, we get the following error bound:

Theorem 4.1 *Let* $(u,\ p)$ *be the solution of the problem (1.1)–(1.4) and* $(\tilde{u}^h_N,\ \tilde{p}^h_N)$ *be the solution of the problem (*$\tilde{V}P^h_N$*). Suppose the discrete inf-sup condition (4.19) holds,* $f \in [L^2(\Omega_i)]^2$ *and* $u|_{\Gamma_0} \in [H^{N+1}(\Gamma_0)]^2$*. Then we have the following error bound for odd* $1 \leq N \leq 20$*:*

$$\begin{aligned} &\|u - \tilde{u}^h_N\|_\triangle + \|p - \tilde{p}^h_N\|_W \\ &\leq C_{(N)} \left[\inf_{v^h \in \tilde{V}^h} \|u - v^h\|_\triangle + \inf_{q^h \in W^h} \|p - q^h\|_W \right. \\ &\qquad \left. + \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{N+1,\Gamma_0} \right]. \end{aligned} \tag{4.23}$$

Proof. The proof of this theorem is also using a standard technique of mixed finite element method [12] and noting $\|v\|_* \leq \|v\|_\triangle$ for all $v \in \tilde{V}$ and the estimate (4.21). The detail is also omitted here.

Suppose $u \in [H^{k+1}(\Omega_i)]^2$, $p \in [H^k(\Omega_i)]^2$, $u|_{\Gamma_R} \in [H^{k+N}(\Gamma_R)]^2$ and the interpolation errors of $\tilde{V}^h$ to $\tilde{V}$ and W^h to W [8,12,14]

$$\begin{aligned} &\inf_{v^h \in \tilde{V}^h} \|u - v^h\|_\triangle + \inf_{q^h \in W^h} \|p - q^h\|_W \\ &\qquad \leq C_0 h^k \left[|u|_{k+1,\Omega_i} + |p|_{k,\Omega_i} + |u|_{k+N,\Gamma_R}\right]. \end{aligned} \tag{4.24}$$

Then combining (4.24) and (4.23), noting the Körn inequality and Poincaré inequality, (1.5), (3.22) and (3.14), we get for odd $1 \leq N \leq 20$

$$\begin{aligned} &\|u - \tilde{u}^h_N\|_{1,\Omega_0} + \|p - \tilde{p}^h_N\|_{0,\Omega_0} \\ &\leq C_0 \left(|u - \tilde{u}^h_N|_{1,\Omega_0} + \|p - \tilde{p}^h_N\|_{0,\Omega_0}\right) \\ &\leq C_0 \left(\|\varepsilon(u - \tilde{u}^h_N)\|_{0,\Omega_0} + \|p - \tilde{p}^h_N\|_{0,\Omega_0}\right) \\ &\leq C_0 \left(\|u - \tilde{u}^h_N\|_\triangle + \|p - \tilde{p}^h_N\|_W\right) \\ &\leq C_{(N)} \left[h^k \left(|u|_{k+1,\Omega_i} + |p|_{k,\Omega_i} + |u|_{k+N,\Gamma_R}\right) \right. \\ &\qquad \left. + \left(\frac{R_0}{R}\right)^{\max\{1,N-1\}} |u|_{N+1,\Gamma_0} \right]. \end{aligned} \tag{4.25}$$

5 Numerical results

In this section we present numerical results which demonstrate the performance of error bounds (3.38) and (4.25). We consider the numerical implementation for the finite element approximation using a nonlocal artificial boundary condition or a high-order local artificial boundary condition at Γ_e. When deal with high-order local artificial boundary conditions, we only consider the case of $N = 1$ in this section. In this case, the usual finite element subspace V^h of V proposed in [12] can be used as $\tilde{V}^h$ directly. When deal with the case of $N > 1$ in local artificial boundary conditions, special subspace $\tilde{V}^h$ which has higher regularity at Γ_e should be used. A family of this kind of spaces was introduced in [14]. In our computation, the Taylor-Hood element (i.e. P2/P1) which satisfies the discrete inf-sup condition (3.20) [6,12]) was used to construct the finite element subspaces V^h and W^h. That is to say, $k = 2$ in the interpolation errors (3.37) and (4.24) [8]. The integrations on the circle in (4.1) and (3.7) are evaluated on each element numerically by Gaussian quadrature.

Example. An exterior problem of an incompressible elastic material

We consider the Stokes equations in the planar domain outside a circular obstacle of radius $a = 0.5$ (see Fig. 1). The problem is governed by the following boundary value problem:

$$-2\mu \operatorname{div} \varepsilon(u) + \operatorname{grad} p = f \equiv (f_1, f_2)^{\mathrm{T}} \quad \text{in } \Omega = \{x \,:\, 0.5 < |x|\}, \tag{5.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{5.2}$$

$$u(0.5,\theta) = g(\theta) \equiv (g_1(\theta), g_2(\theta))^{\mathrm{T}} \qquad \text{on } \Gamma_i = \partial\Omega, \tag{5.3}$$

$$u \quad \text{is bounded}, \quad p \to 0 \qquad \text{when} \quad r = \sqrt{x_1^2 + x_2^2} \to +\infty; \tag{5.4}$$

where

$$f_1(x) = \begin{cases} x_2[(5-48\mu)x_1^2 + (1-48\mu)x_2^2 + 24\mu - 1](|x|^2 - 1) & 0.5 \le |x| < 1.0, \\ 0 & 1.0 \le |x|; \end{cases}$$

$$f_2(x) = \begin{cases} x_1[(1+48\mu)x_1^2 + (5+48\mu)x_2^2 - 1 - 24\mu](|x|^2 - 1) & 0.5 \le |x| < 1.0, \\ 0 & 1.0 \le |x|; \end{cases}$$

$$g_1(\theta) = \frac{1}{4\mu}\left[\frac{2\cos^2\theta\sin\theta}{1.5625 - \sin^2\theta} - \frac{1}{2}\ln\frac{1.25 - \sin\theta}{1.25 + \sin\theta}\right] + \frac{27}{128}\sin\theta \quad 0 \le \theta \le 2\pi;$$

$$g_2(\theta) = \frac{\cos\theta(2\sin^2\theta - 1.25)}{4\mu(1.5625 - \sin^2\theta)} - \frac{27}{128}\cos\theta \qquad 0 \le \theta \le 2\pi.$$

This problem has an exact solution:

$$u_1(x) = \begin{cases} \frac{1}{4\mu}\left[\frac{x_1^2}{|x-x^+|^2} - \frac{x_1^2}{|x-x^-|^2} - \ln\frac{|x-x^+|}{|x-x^-|}\right] + x_2(|x|^2-1)^3 \\ \qquad\qquad 0.5 \le |x| < 1.0, \\ \frac{1}{4\mu}\left[\frac{x_1^2}{|x-x^+|^2} - \frac{x_1^2}{|x-x^-|^2} - \ln\frac{|x-x^+|}{|x-x^-|}\right] \qquad 1.0 \le |x|; \end{cases}$$

$$u_2(x) = \begin{cases} \frac{1}{4\mu}\left[\frac{x_1(x_2-x_2^+)}{|x-x^+|^2} - \frac{x_1(x_2-x_2^-)}{|x-x^-|^2}\right] - x_1(|x|^2-1)^3 \\ \qquad\qquad 0.5 \le |x| < 1.0, \\ \frac{1}{4\mu}\left[\frac{x_1(x_2-x_2^+)}{|x-x^+|^2} - \frac{x_1(x_2-x_2^-)}{|x-x^-|^2}\right] \qquad 1.0 \le |x|; \end{cases}$$

$$p(x) = \begin{cases} \frac{1}{2}\left[\frac{x_1}{|x-x^+|^2} - \frac{x_1}{|x-x^-|^2}\right] + x_1x_2(|x|^2-1)^2 \\ \qquad\qquad 0.5 \le |x| < 1.0, \\ \frac{1}{2}\left[\frac{x_1}{|x-x^+|^2} - \frac{x_1}{|x-x^-|^2}\right] \qquad 1.0 \le |x|. \end{cases}$$

In this example, we take $\mu = 1$, $x^+ = (0, 0.25)$, $x^- = (0, -0.25)$ and the unbounded domain $\Omega = \{x \in \mathbb{R}^2 \,:\, 0.5 < |x|\}$ which is the exterior domain outside a circle $\Gamma_i = \{x \in \mathbb{R}^2 \,:\, |x| = 0.5\}$.

First we test the effect of the mesh size h in the error bound (3.38), we introduce a circular artificial boundary $\Gamma_e = \Gamma_0$ of radius $R = R_0 = 1.0$. On Γ_0 we apply the nonlocal artificial boundary condition (2.21) with $N = 0, 1, 2, \cdots$ In the annular computational domain Ω_0, we use four meshes respectively. The first mesh consists of 1 radial layers of elements, with 16 triangular elements in each layer. We denote it as 1×16. The other three meshes are 2×32, 4×64 and 8×128. Figure 2 shows the errors $\|u - u_N^h\|_{0,\Omega_0}$, $\|u - u_N^h\|_{1,\Omega_0}$ $\|p - p_N^h\|_{0,\Omega_0}$ for large N (say $N = 51$). The results show that the convergent rates of $\|u - u_N^h\|_{1,\Omega_0}$ and $\|p - p_N^h\|_{0,\Omega_0}$ with respect to h are approximately 2 when using nonlocal artificial boundary conditions. Second we test the effect of N in the error bound (3.38). Let $(u_\infty^h,\ p_\infty^h)$ denotes the finite element approximation of the problem on the domain Ω_0 with the mesh size h when N is very large (say $N = 51$). In this case

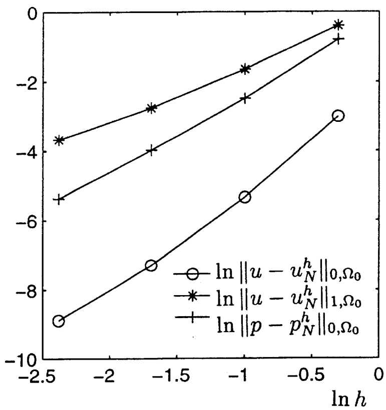

Fig. 2. The effect of the mesh size h using nonlocal artificial boundary conditions

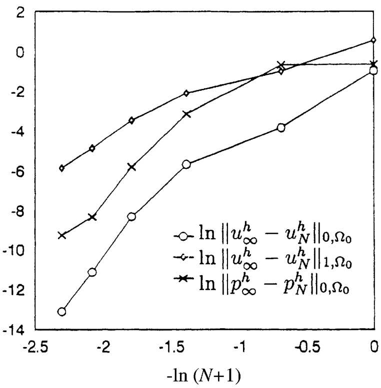

Fig. 3. The effect of N using nonlocal artificial boundary conditions

$R_0 = R$, so the effect of R in the error bound (3.38) disappears. Figure 3 shows the errors $E_N := \|u^h_\infty - u^h_N\|_{k,\Omega_0}$ $(k = 0, 1)$ and $\|p^h_\infty - p^h_N\|_{0,\Omega_0}$ on the mesh 8×128 for $N = 0, 1, 3, 5, 7, 9$. Third we test the effect of the location of the artificial boundary Γ_e. Let $\Omega_R = \{x \ : \ 0.5 < |x| < R\}$ denotes the bounded computational domain with the artificial boundary Γ_R. We choose $R = 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0$ respectively. The corresponding meshes we used were 4×64, 8×64, 12×64, 16×64 and 20×64. That is to say, each computational domain has a mesh with the fixed mesh size $h = 0.1842$. Let $(u^R_N,\ p^R_N)$ denote the finite element approximation of the problem on the domain Ω_R with the corresponding mesh by using the nonlocal artificial boundary condition (2.21) on the artificial boundary Γ_R, $(u^R_\infty,\ p^R_\infty)$ correspond to the solution when N is very large (say $N = 51$) and $(\tilde{u}^R_N,\ \tilde{p}^R_N)$ correspond to the solution using the high-order local artificial boundary condition (2.33) at Γ_R with $N = 1$. Figures 4–5 show the errors $E_R := \|u^R_\infty - u^R_N\|_{1,\Omega_0}$ and $E_R := \|p^R_\infty - p^R_N\|_{0,\Omega_0}$ for $R =$

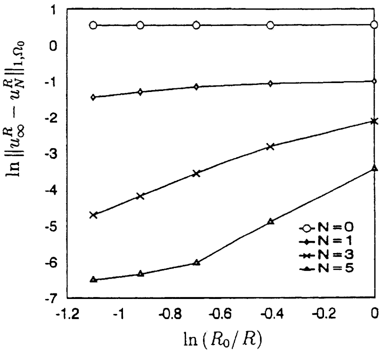

Fig. 4. The effect of R with respect to u using nonlocal artificial boundary conditions

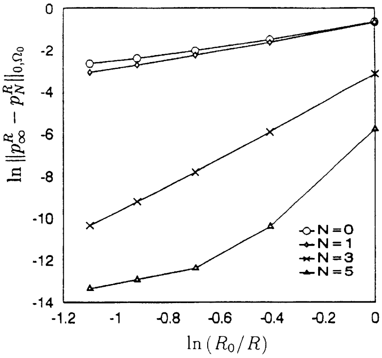

Fig. 5. The effect of R with respect to p using nonlocal artificial boundary conditions

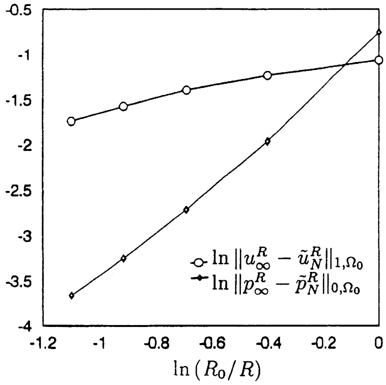

Fig. 6. The effect of R using a local artificial boundary condition ($N = 1$)

$1.0, 1.5, 2.0, 2.5, 3.0$. Figure 6 shows the errors $E_R := \|u_\infty^R - \tilde{u}_N^R\|_{1,\Omega_0}$ and $\|p_\infty^R - \tilde{p}_N^R\|_{0,\Omega_0}$ for $R = 1.0, 1.5, 2.0, 2.5, 3.0$.

Figures 2–6 demonstrate the performance of the error bounds (3.38) and (4.25). In practice, if one wants to use local artificial boundary condition, we recommend to use the one corresponds to $N = 1$. This condition is very simple and easy to be dealt with by using the standard finite elements. From our numerical results, the accuracy of this boundary condition is good.

6 Conclusions

A family of high-order local artificial boundary conditions for numerical simulation of an incompressible elastic material in an unbounded domain is designed. The original problem is then reduced to a problem defined in a bounded computational domain by imposing a nonlocal artificial boundary condition or a local one at a circular artificial boundary. The finite element formulation is presented. New error bounds for the case of using nonlocal artificial boundary conditions are obtained. This kind of error bounds depend on not only the mesh size, the terms used in the artificial boundary condition, but also the location of the artificial boundary. They can be used to choose the mesh size, terms used in the artificial boundary condition and the location of the artificial boundary for practical computations. Error bounds for the case of using local artificial boundary conditions are also obtained. Numerical results demonstrate the performance of our error bounds.

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Appendix: Proof of Lemma 3.1 and Lemma 3.2

Proof of Lemma 3.1. For any given $u, v \in V$, we expand $u|_{\Gamma_e} = u(R,\theta)$ and $v|_{\Gamma_e} = v(R,\theta)$ in Fourier series, i.e.

$$u_i(R,\theta) = \frac{a_0^i}{2} + \sum_{n=1}^{\infty} (a_n^i \cos n\theta + b_n^i \sin n\theta) \qquad i = 1, 2, \tag{A.1}$$

$$v_i(R,\theta) = \frac{c_0^i}{2} + \sum_{n=1}^{\infty} (c_n^i \cos n\theta + d_n^i \sin n\theta) \qquad i = 1, 2; \tag{A.2}$$

where a_n^i, b_n^i are defined in (2.8) and (2.9) and c_n^i, d_n^i are defined in (3.27). Inserting (3.30) and (3.25) into (3.4) and (3.7) and using the orthogonality of cosines and sines, we get

$$a_0(u,v) = 2\mu\pi \sum_{n=1}^{\infty} \sum_{i=1}^{2} n \left(a_n^i c_n^i + b_n^i d_n^i \right) \qquad \forall u, v \in V, \tag{A.3}$$

$$a_0^N(u,v) = 2\mu\pi \sum_{n=1}^{N} \sum_{i=1}^{2} n \left(a_n^i c_n^i + b_n^i d_n^i \right) \qquad \forall u, v \in V. \tag{A.4}$$

Denote Q the domain enclosed by Γ_i, and Ω_s the disk bounded by Γ_e (i.e. $\Omega_s = Q \cup \Omega_i \cup \Gamma_i$) (see Fig. 1). Then for any $v \in V$, we define the function $v^{(0)}$ which satisfies:

$$-2\mu \operatorname{div} \varepsilon(v^{(0)}) = 0 \qquad \text{in } \Omega_i, \tag{A.5}$$

$$v^{(0)} = v \qquad \text{on} \quad \Gamma_e \cup \Gamma_i. \tag{A.6}$$

$$v^{(0)} \equiv 0 \qquad \text{in} \quad Q. \tag{A.7}$$

Then we know $v^{(0)} \in \left[H^1(\Omega_s)\right]^2$. Define the function $v^{(1)} \in \left[H^1(\Omega_s)\right]^2$ which satisfies:

$$-2\mu \operatorname{div} \varepsilon(v^{(1)}) = 0 \qquad \text{in } \Omega_s, \tag{A.8}$$

$$v^{(1)} = v \qquad \text{on} \quad \Gamma_e. \tag{A.9}$$

Then $v^{(0)}|_{\Omega_i}$ minimizes the functional $\int_{\Omega_i} 2\mu|\varepsilon(w)|^2 \, dx$ among all functions $w \in [H^1(\Omega_i)]^2$ which are equal to v on $\Gamma_e \cup \Gamma_i$. Similarly, $v^{(1)}$ minimizes the functional $\int_{\Omega_s} 2\mu|\varepsilon(w)|^2 \, dx$ among all functions $w \in \left[H^1(\Omega_s)\right]^2$ which are equal to v on Γ_e. Therefore we have that

$$\begin{aligned} a(v,v) &= \int_{\Omega_i} 2\mu|\varepsilon(v)|^2 \, dx \geq a(v^{(0)}, v^{(0)}) \\ &= \int_{\Omega_s} 2\mu|\varepsilon(v^{(0)})|^2 \, dx \geq \int_{\Omega_s} 2\mu|\varepsilon(v^{(1)})|^2 \, dx. \end{aligned} \tag{A.10}$$

Recalling that $v^{(1)}$ is the solution of the problem (A.8)–(A.9), by separation of variables, noting $v^{(1)}|_{\Gamma_e} = v^{(1)}(R,\theta) = v(R,\theta)$, we obtain

$$\begin{aligned} v_1^{(1)}(r,\theta) = \frac{R^2 - r^2}{6} &\sum_{n=0}^{\infty} (n+2)\Big[(c_{n+2}^1 - d_{n+2}^2)\cos n\theta \\ &+ (d_{n+2}^1 + c_{n+2}^2)\sin n\theta\Big] \frac{r^n}{R^{n+2}} \\ &+ \frac{c_0^1}{2} + \sum_{n=1}^{\infty} \frac{r^n}{R^n}(c_n^1 \cos n\theta + d_n^1 \sin n\theta) \\ &0 \leq r \leq R \quad 0 \leq \theta \leq 2\pi, \end{aligned} \tag{A.11}$$

$$\begin{aligned} v_2^{(1)}(r,\theta) = \frac{R^2 - r^2}{6} &\sum_{n=0}^{\infty} (n+2)\Big[-(d_{n+2}^1 + c_{n+2}^2)\cos n\theta \\ &+ (c_{n+2}^1 - d_{n+2}^2)\sin n\theta\Big] \frac{r^n}{R^{n+2}} \\ &+ \frac{c_0^2}{2} + \sum_{n=1}^{\infty} \frac{r^n}{R^n}(c_n^2 \cos n\theta + d_n^2 \sin n\theta) \\ &0 \leq r \leq R \quad 0 \leq \theta \leq 2\pi. \end{aligned} \tag{A.12}$$

where c_n^i and d_n^i are defined in (3.27). Combining (A.10), (A.11) and (A.12) with $r = R$, (3.27) with $v|_{\Gamma_e} = v^{(1)}|_{\Gamma_e}$, (1.5) with $u = v^{(1)}$, (A.3) and (A.4) with $u = v$, integration by parts, (A.8), we obtain

$$\begin{aligned}
&a(v,v)\\
&\geq \int_{\Omega_s} 2\mu|\varepsilon(v^{(1)})|^2\, dx = 2\mu\int_{\Gamma_e}(\varepsilon(v^{(1)})n)\cdot v^{(1)}\, ds\\
&= \frac{4\mu}{3\pi}\sum_{n=1}^{\infty} n\int_0^{2\pi}\int_0^{2\pi}\left[v_1^{(1)}(R,\phi)v_1^{(1)}(R,\theta)+v_2^{(1)}(R,\phi)v_2^{(1)}(R,\theta)\right]\\
&\quad\times\cos n(\theta-\phi)d\theta\, d\phi\\
&\quad+\frac{2\mu}{3\pi}\sum_{n=1}^{\infty} n\int_0^{2\pi}\int_0^{2\pi}\left[v_1^{(1)}(R,\phi)v_2^{(1)}(R,\theta)-v_2^{(1)}(R,\phi)v_1^{(1)}(R,\theta)\right]\\
&\quad\times\sin n(\theta-\phi)d\theta\, d\phi\\
&= \frac{2\mu\pi}{3}\sum_{n=1}^{\infty} n\left[\sum_{i=1}^{2}[(c_n^i)^2+(d_n^i)^2)+(c_n^1+d_n^2)^2+(d_n^1-c_n^2)^2\right]\\
&\geq \frac{2\mu\pi}{3}\sum_{n=1}^{\infty} n\sum_{i=1}^{2}\left[(c_n^i)^2+(d_n^i)^2\right] = \frac{a_0(v,v)}{3}\\
&\geq \frac{2\mu\pi}{3}\sum_{n=1}^{N} n\sum_{i=1}^{2}\left[(c_n^i)^2+(d_n^i)^2\right] = \frac{a_0^N(v,v)}{3}\geq 0;
\end{aligned}\tag{A.13}$$

where $n = (\cos\theta, \sin\theta)^{\mathrm{T}}$ is the unit normal outward vector at Γ_e. The desired inequality (3.17) is proved. The inequality (3.18) follows from (3.17), (A.3), (A.4) and the Schwarz inequality immediately.

Proof of Lemma 3.2. (i) Let $R_1 = \max\{|x| \;:\; x \in \Gamma_i\}$. Without loss of generality, we can assume $R_1 < R_0 \leq R$. Then we construct $v^{(0)} = (0, v_2^{(0)}) \in V$ and $q^{(0)} \in W$ which satisfy:

$$v_2^{(0)}(x) = \begin{cases} x_2 - R_1 & x \in \Omega_i \text{ and } x_2 > R_1,\\ 0 & \text{otherwise;}\end{cases}$$
$$q^{(0)}(x) = \begin{cases} 1 & x \in \Omega_i \text{ and } x_2 > R_1,\\ 0 & \text{otherwise.}\end{cases}\tag{A.14}$$

Then we have that

$$\operatorname{div} v^{(0)} = q^{(0)} \qquad \int_{\Omega_i} q^{(0)}\, dx \neq 0.\tag{A.15}$$

(ii) For any $q \in L^2(\Omega_i)$, we have that

$$q = q^* + \alpha q^{(0)} \qquad \text{with } \alpha = \frac{\int_{\Omega_i} q\, dx}{\int_{\Omega_i} q^{(0)}\, dx}. \tag{A.16}$$

Hence $q^* \in L_0^2(\Omega_i) \equiv \{w : w \in L^2(\Omega_i) \text{ and } \int_{\Omega_i} w\, dx = 0\}$. Noting (A.14) and Hölder inequality, we have that

$$\begin{aligned} \|\alpha q^{(0)}\|_W &= |\alpha|\, \|q^{(0)}\|_W \\ &= \left| \int_{\Omega_i} q\, dx \right| / \|q^{(0)}\|_W \le \|q\|_W \cdot \|1\|_W / \|q^{(0)}\|_W \\ &\le \frac{\sqrt{\pi} R\, \|q\|_W}{\sqrt{R^2 \cos^{-1}\frac{R_1}{R} - R_1\sqrt{R^2 - R_1^2}}} \\ &\le \frac{\sqrt{\pi}}{\sqrt{\cos^{-1}\frac{R_1}{R_0} - \frac{R_1}{R_0}\sqrt{1 - R_1^2/R_0^2}}}\, \|q\|_W \\ &\equiv \alpha_0 \|q\|_W, \end{aligned} \tag{A.17}$$

where the constant $\alpha_0 = \sqrt{\pi}/\sqrt{\cos^{-1}\frac{R_1}{R_0} - \frac{R_1}{R_0}\sqrt{1 - R_1^2/R_0^2}}$ is independent of R. Noting (A.14), (A.16), (A.17), (3.14), (3.3) and (1.5), we have

$$\begin{aligned} \|q^*\|_W = \|q - \alpha q^{(0)}\|_W &\le \|q\|_W + \|\alpha q^{(0)}\|_W \\ &\le (1 + \alpha_0)\, \|q\|_W, \end{aligned} \tag{A.18}$$

$$\begin{aligned} \|\alpha v^{(0)}\|_* = |\alpha|\, \|v^{(0)}\|_* &\le |\alpha|\, \sqrt{2\mu}\, |v^{(0)}|_{1,\Omega_i} \\ &= \sqrt{2\mu}\, |\alpha|\, \|q^{(0)}\|_W \le \sqrt{2\mu}\, \alpha_0\, \|q\|_W. \end{aligned} \tag{A.19}$$

For $q^* \in L_0^2(\Omega_i)$, as $\partial\Omega_i = \Gamma_i \cup \Gamma_e$ is smooth, there exists a unique $w \in H^2(\Omega_i)$ satisfying [6]

$$-\triangle w = q^* \qquad \text{in } \Omega_i, \tag{A.20}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_i \qquad \text{and} \quad \int_{\Omega_i} w\, dx = 0; \tag{A.21}$$

where n is the unit outer normal vector to Γ_i in Ω_i. By elliptic regularity we have

$$|w|_{2,\Omega_i} = \|\triangle w\|_{0,\Omega_i} = \|q^*\|_W. \tag{A.22}$$

Let $v^{(1)} = -\text{grad}\ w$, then $v^{(1)} \in H^1(\Omega_i)$. Noting (A.20), (A.21) and (A.22), we have

$$\text{div}\ v^{(1)} = q^* \quad \text{in } \Omega_i \qquad \text{and} \qquad v^{(1)} \cdot n|_{\Gamma_i} = -\left.\frac{\partial w}{\partial n}\right|_{\Gamma_i} = 0 \tag{A.23}$$

and

$$|v^{(1)}|_{1,\Omega_i} \le |w|_{2,\Omega_i} = \|q^*\|_W. \tag{A.24}$$

By the trace theorem [1], noting (A.24), there exists $\psi \in H^2(\Omega_0)$ such that

$$\psi|_{\partial\Omega_0} = 0 \qquad \left.\frac{\partial\psi}{\partial n}\right|_{\partial B_{R_0}(0)} = 0 \qquad \left.\frac{\partial\psi}{\partial n}\right|_{\Gamma_i} = v^{(1)} \cdot \tau|_{\Gamma_i} \tag{A.25}$$

and

$$\begin{aligned} |\psi|_{2,\Omega_0} &\le \beta_0 \|v^{(1)}\|_{1,\Omega_0} \\ &\le \beta_0 |v^{(1)}|_{1,\Omega_0} \le \beta_0 |v^{(1)}|_{1,\Omega_i} \le \beta_0 \|q^*\|_W; \end{aligned} \tag{A.26}$$

where τ is the positively oriented unit tangent vector. Extending ψ with 0 outside of Ω_0 to a function on Ω_i, noting (A.25), we have $\psi \in H^2(\Omega_i)$ and $|\psi|_{2,\Omega_i} = |\psi|_{2,\Omega_0}$. Let $v^{(2)} = \text{curl}\ \psi = \left(\frac{\partial\psi}{\partial x_2},\ -\frac{\partial\psi}{\partial x_1}\right)^{\mathrm{T}}$. Noting (A.25) and (A.23), we have

$$v^{(2)} \cdot n|_{\Gamma_i} = \text{curl}\ \psi \cdot n|_{\Gamma_i} = \left.\frac{\partial\psi}{\partial\tau}\right|_{\Gamma_i} = 0 = -v^{(1)} \cdot n|_{\Gamma_i}, \tag{A.27}$$

$$v^{(2)} \cdot \tau|_{\Gamma_i} = \text{curl}\ \psi \cdot \tau|_{\Gamma_i} = -\left.\frac{\partial\psi}{\partial n}\right|_{\Gamma_i} = -v^{(1)} \cdot \tau|_{\Gamma_i}. \tag{A.28}$$

Thus $v^{(2)}|_{\Gamma_i} = -v^{(1)}|_{\Gamma_i}$. Let $v^* = v^{(1)} + v^{(2)}$. Noting (A.23), (A.24) and (A.26), we have

$$\text{div}\ v^* = \text{div}\ v^{(1)} + \text{div}\ v^{(2)} = \text{div}\ v^{(1)} = q^* \quad \text{in}\ \Omega_i, \qquad v^*|_{\Gamma_i} = 0, \tag{A.29}$$

$$|v^*|_{1,\Omega_i} \le |v^{(1)}|_{1,\Omega_i} + |v^{(2)}|_{1,\Omega_i} \le |w|_{2,\Omega_i} + |\psi|_{2,\Omega_i} \le \beta_0 \|q^*\|_W. \tag{A.30}$$

Let $v^q = -v^* - \alpha v^{(0)}$. Noting (A.29), (A.15), (A.16), (3.3), (1.5), (3.14), (A.19) and (A.30), we have

$$-\text{div}\ v^q = \text{div}\ v^* + \alpha\ \text{div}\ v^{(0)} = q^* + \alpha q^{(0)} = q \tag{A.31}$$

and

$$\|v^q\|_* \le \|v^*\|_* + \|\alpha v^{(0)}\|_* \le \sqrt{2\mu}|v^*|_{1,\Omega_i} + \sqrt{2\mu}\alpha_0\|q\|_W \le \beta_0\ \|q\|_W. \tag{A.32}$$

Then the inf-sup condition (3.19) follows from (A.32), (A.31) and (3.5) immediately.

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