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UNIFORM ERROR ESTIMATES OF FINITE DIFFERENCE METHODS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH WAVE OPERATOR*

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Abstract. We establish uniform error estimates of finite difference methods for the nonlinear Schrödinger equation (NLS) perturbed by the wave operator (NLSW) with a perturbation strength described by a dimensionless parameter ε ($\varepsilon \in (0,1]$). When $\varepsilon \to 0^+$, NLSW collapses to the standard NLS. In the small perturbation parameter regime, i.e., $0 < \varepsilon \ll 1$, the solution of NLSW is perturbed from that of NLS with a function oscillating in time with $O(\varepsilon^2)$-wavelength at $O(\varepsilon^4)$ and $O(\varepsilon^2)$ amplitudes for well-prepared and ill-prepared initial data, respectively. This high oscillation of the solution in time brings significant difficulties in establishing error estimates uniformly in ε of the standard finite difference methods for NLSW, such as the conservative Crank–Nicolson finite difference (CNFD) method, and the semi-implicit finite difference (SIFD) method. We obtain error bounds uniformly in ε, at the order of $O(h^2+\tau)$ and $O(h^2+\tau^{2/3})$ with time step τ and mesh size h for well-prepared and ill-prepared initial data, respectively, for both CNFD and SIFD in the l^2-norm and discrete semi-H^1 norm. Our error bounds are valid for general nonlinearity in NLSW and for one, two, and three dimensions. To derive these uniform error bounds, we combine ε-dependent error estimates of NLSW, ε-dependent error bounds between the numerical approximate solutions of NLSW and the solution of NLS, together with error bounds between the solutions of NLSW and NLS. Other key techniques in the analysis include the energy method, cut-off of the nonlinearity, and a posterior bound of the numerical solutions by using the inverse inequality and discrete semi-H^1 norm estimate. Finally, numerical results are reported to confirm our error estimates of the numerical methods and show that the convergence rates are sharp in the respective parameter regimes.

Key words. nonlinear Schrödinger equation with wave operator, energy method, error estimates, conservative Crank–Nicolson finite difference method, semi-implicit finite difference method

AMS subject classifications. 35Q55, 65M06, 65M12, 65M15, 81-08

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1. Introduction.

In this paper, we establish error estimates for finite difference approximations of the nonlinear Schrödinger equation with wave operator (NLSW) in d ($d = 1, 2, 3$) dimensions as

(1.1)
$$\begin{cases} i\partial_t u^\varepsilon(\mathbf{x},t) - \varepsilon^2\partial_{tt}u^\varepsilon(\mathbf{x},t) + \nabla^2 u^\varepsilon(\mathbf{x},t) + f(|u^\varepsilon|^2)u^\varepsilon(\mathbf{x},t) = 0, & \mathbf{x}\in\mathbb{R}^d,\ t>0, \\ u^\varepsilon(\mathbf{x},0) = u_0(\mathbf{x}), \qquad \partial_t u^\varepsilon(\mathbf{x},0) = u_1^\varepsilon(\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d, \end{cases}$$

where t is time, $\mathbf{x}$ is the spatial variable, $u^\varepsilon := u^\varepsilon(\mathbf{x},t)$ is a complex-valued function, $0 < \varepsilon \le 1$ is a dimensionless parameter, $f : [0,+\infty) \to \mathbb{R}$ is a real-valued function, and $\nabla^2 = \Delta$ is the d-dimensional Laplace operator. The above NLSW arises from different physics applications, such as the nonrelativistic limit of the Klein–Gordon equation [18, 22, 25], the Langmuir wave envelope approximation in plasma [7, 10],

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and modulated planar pulse approximation of the sine-Gordon equation for light bullets [5, 27]. It is easy to see that NLSW has the following two important conserved quantities, i.e., the *mass*

(1.2)
$$N^\varepsilon(t) := \int_{\mathbb{R}^d} |u^\varepsilon(\mathbf{x},t)|^2\, d\mathbf{x} - 2\varepsilon^2 \int_{\mathbb{R}^d} \mathrm{Im}\left(\overline{u^\varepsilon(\mathbf{x},t)}\partial_t u^\varepsilon(\mathbf{x},t)\right)\, d\mathbf{x} \equiv N^\varepsilon(0), \quad t \ge 0,$$

and the *energy*
(1.3)
$$E^\varepsilon(t) := \int_{\mathbb{R}^d} \left[\varepsilon^2|\partial_t u^\varepsilon(\mathbf{x},t)|^2 + |\nabla u^\varepsilon(\mathbf{x},t)|^2 - F(|u^\varepsilon(\mathbf{x},t)|^2)\right]\, d\mathbf{x} \equiv E^\varepsilon(0), \quad t \ge 0,$$

where $\bar{c}$ and $\mathrm{Im}(c)$ denote the conjugate and imaginary part of c, respectively, and F is the primitive function of f defined as

$$F(s) = \int_0^s f(\rho)\, d\rho, \qquad s \ge 0. \tag{1.4}$$

In the nonrelativistic limit of the Klein–Gordon equation and the singular limit of the Langmuir wave envelope approximation, i.e., $\varepsilon \to 0^+$, NLSW (1.1) collapses to the standard nonlinear Schrödinger equation (NLS) [7, 18, 22, 25]

$$\begin{cases} i\partial_t u(\mathbf{x},t) + \nabla^2 u(\mathbf{x},t) + f(|u|^2)u(\mathbf{x},t) = 0, & \mathbf{x} \in \mathbb{R}^d,\ t > 0, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d, \end{cases} \tag{1.5}$$

and the corresponding conservation laws (1.2) and (1.3) hold for NLS with $\varepsilon = 0$. In particular, it is proved in [7] that if the nonlinearity satisfies

$$|\partial^k f(\rho)| \le K\rho^{\sigma-k} \quad \text{for some constant } K > 0 \text{ and } \sigma \ge 1, \quad k = 0,1,2,$$

then for the initial data $(u_0, u_1^\varepsilon) \in H^2 \times H^2$ with $\|u_1^\varepsilon\|_{H^2}$ uniformly bounded, there exists a constant $T > 0$ independent of ε such that the solution u^ε of NLSW (1.1) and the solution u of NLS (1.5) exist on $[0,T]$ [18, 22, 25]. Furthermore, the following convergence rate can be established (following [7]):

$$\|u^\varepsilon - u\|_{L^\infty([0,T];H^2)} \le C\varepsilon^2. \tag{1.6}$$

Formally, as $\varepsilon \to 0^+$, the solution of NLSW (1.1) exhibits oscillation in time t with wavelength $O(\varepsilon^2)$ due to the wave operator and/or the initial data u_1^ε. Actually, suppose the initial data u_1^ε satisfies the condition

$$u_1^\varepsilon(\mathbf{x}) = i\left(\nabla^2 u_0(\mathbf{x}) + f(|u_0(\mathbf{x})|^2)u_0(\mathbf{x})\right) + \varepsilon^\alpha w(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \qquad \alpha \ge 0; \tag{1.7}$$

then we would have the asymptotic expansion for the solution $u^\varepsilon(\mathbf{x},t)$ of NLSW (1.1) as

$$\begin{aligned} u^\varepsilon(\mathbf{x},t) = u(\mathbf{x},t) &+ \varepsilon^2\{\text{terms without oscillation}\} \\ &+ \varepsilon^{2+\min\{\alpha,2\}} v(\mathbf{x}, t/\varepsilon^2) \\ &+ \text{higher order terms with oscillation}, \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \ge 0, \end{aligned} \tag{1.8}$$

where $u := u(\mathbf{x},t)$ satisfies NLS (1.5). The expansion (1.8) can be verified in the spirit

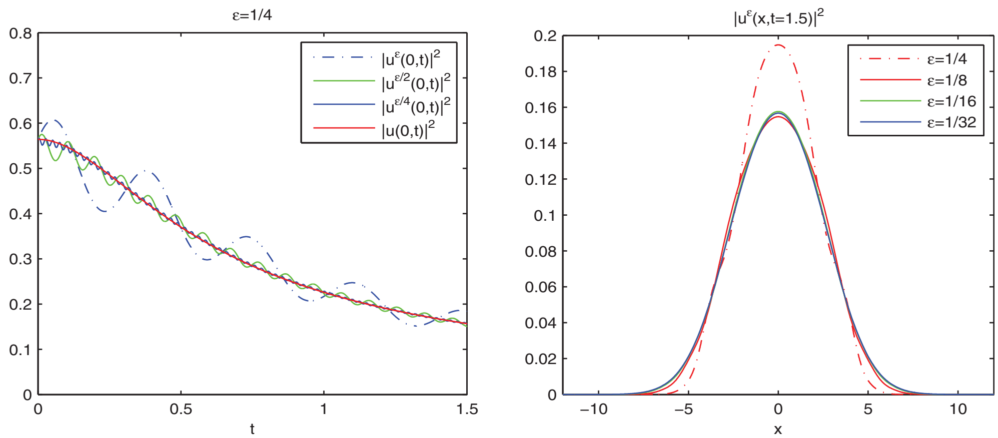

FIG. 1.1. *Temporal profile of* $|u^\varepsilon(0,t)|^2$ *and* $|u(0,t)|^2$ *(left) and spatial profile of* $|u^\varepsilon(\mathbf{x}, t = 1.5)|^2$ *(right) for different* ε *with* $\alpha = 0$ *and* u_0*,* w*,* f *as given in section* 5.

of [7], and we plot the densities $|u^\varepsilon(0,t)|^2$ and $|u^\varepsilon(\mathbf{x}, t = 1.5)|^2$ in the case of $\alpha = 0$ and $d = 1$ (cf. Figure 1.1).

Based on the above observation, we can make assumptions (A) and (B) (cf. section 2.2) on the solution of NLSW. Furthermore, from (1.8), we can classify the initial data into well-prepared ($\alpha \geq 2$) and ill-prepared ($0 \leq \alpha < 2$) cases. In fact, when $\alpha > 2$, the leading order oscillation term comes from the perturbation of the wave operator, and, respectively, when $0 \leq \alpha < 2$, it comes from the initial data.

Different kinds of numerical methods have been proposed for NLS in the literature, such as the time-splitting pseudospectral method [6, 15, 21, 23] and the finite difference methods [1]. However, few numerical methods have been considered for NLSW in the literature, and most of them are the conservative finite difference methods [10, 14, 26]. For the corresponding error analysis on the split error for NLS, see [8, 11, 17, 19] and the references therein. For the error estimates of the implicit Runge–Kutta finite element method for NLS, see [2, 20]. Error bounds of conservative Crank–Nicolson finite difference (CNFD) for NLS in one dimension (1D) have been established in [9, 13]. For NLSW in 1D with $\varepsilon = O(1)$, the error estimates of conservative finite difference schemes have been obtained in [26]. However, the proofs in [26] rely strongly on the conservative properties of the schemes and the discrete version of the Sobolev inequality in 1D,

$$\|g\|_{L^\infty}^2 \leq \|g\|_{L^2}\|g'\|_{L^2}, \qquad \forall g \in H_0^1(U), \quad U \subset \mathbb{R}, \tag{1.9}$$

while the corresponding Sobolev inequality is unavailable in two (2D) and three (3D) dimensions. (See [3] for a discussion on the NLS case.) Thus their proof cannot be extended to either higher dimensions (2D or 3D) or nonconservative schemes. Noticing the above asymptotic expansion for NLSW, there exists high oscillation in time for small ε, which would cause trouble in analyzing the discretizations for NLSW (1.1), especially in the regime $0 < \varepsilon \ll 1$. The main aim of this paper is to develop a unified approach for establishing uniform error estimates in terms of $\varepsilon \in (0, 1]$ for conservative CNFD and semi-implicit finite difference (SIFD) for NLSW (1.1) in d-dimensions ($d = 1, 2, 3$). Our approach includes the energy method, the cut-off technique for dealing with general nonlinearity, and the inverse inequality for obtaining a uniform posterior bound of the numerical solution.

The paper is organized as follows. In section 2, we introduce CNFD and SIFD for the discretization of NLSW and state our main results. In section 3, we prove in detail the uniform error estimates of SIFD by using the energy method, the cut-off technique for nonlinearity, and inverse inequality, and a similar proof with key steps for CNFD is presented in section 4. Numerical results are reported in section 5 to confirm our theoretical analysis. Finally, some conclusions are drawn in section 6. Throughout the paper, we adopt the standard Sobolev spaces and their corresponding norms, let C denote a generic constant independent of ε, mesh size h, and time step τ, and use the notation $A \lesssim B$ to mean that there exists a generic constant C which is independent of ε, τ, and h such that $|A| \le C\,B$.

2. Finite difference schemes and main results. In practical computation, NLSW (1.1) is usually truncated on a bounded interval $\Omega = (a, b)$ in 1D, or a bounded rectangle $\Omega = (a, b) \times (c, d)$ in 2D, or a bounded box $\Omega = (a, b) \times (c, d) \times (e, f)$ in 3D, with zero Dirichlet boundary condition. For simplicity of notation, we only deal with the case in 1D, i.e., $d = 1$ and $\Omega = (a, b)$. Extensions to 2D and 3D are straightforward, and the error estimates in l^2-norm and discrete semi-H^1 norm are the same in 2D and 3D. In 1D, NLSW (1.1) is truncated on an interval $\Omega = (a, b)$ as

(2.1)
$$\begin{cases} i\partial_t u^\varepsilon(x,t) - \varepsilon^2\partial_{tt}u^\varepsilon(x,t) + \partial_{xx}u^\varepsilon(x,t) + f(|u^\varepsilon|^2)u^\varepsilon(x,t) = 0, & x \in \Omega = (a,b) \subset \mathbb{R},\ t > 0,\\ u^\varepsilon(x,0) = u_0(x), \qquad \partial_t u^\varepsilon(x,0) = u_1^\varepsilon(x), & x \in \overline{\Omega} = [a,b],\\ u^\varepsilon(x,t)\big|_{\partial\Omega} = 0, & t > 0. \end{cases}$$

Formally, as $\varepsilon \to 0^+$, (2.1) collapses to the standard NLS [7, 22, 25]

$$\begin{cases} i\partial_t u(x,t) + \partial_{xx}u(x,t) + f(|u|^2)u(x,t) = 0, & x \in \Omega \subset \mathbb{R},\ t > 0,\\ u(x,0) = u_0(x), & x \in \overline{\Omega},\\ u(x,t)\big|_{\partial\Omega} = 0, & t > 0. \end{cases} \tag{2.2}$$

We assume that the initial data u_1^ε satisfies the condition

$$\begin{aligned} &u_1^\varepsilon(x) = u_1(x) + \varepsilon^\alpha w^\varepsilon(x),\\ &u_1(x) := \partial_t u(x,t)|_{t=0} = i\left[\partial_{xx}u_0(x) + f(|u_0(x)|^2)u_0(x)\right], \qquad x \in \Omega, \end{aligned} \tag{2.3}$$

where w^ε is uniformly bounded in H^2 (w.r.t. ε) with $\liminf_{\varepsilon\to 0^+} \|w^\varepsilon\|_{H^2} > 0$ and $\alpha \ge 0$ is a parameter describing the consistency of the initial data w.r.t. NLS (2.2).

2.1. Numerical methods. Choose time step $\tau := \Delta t$ and denote time steps as $t_n := n\,\tau$ for $n = 0, 1, 2, \dots$; choose mesh size $\Delta x := \frac{b-a}{M}$ with M being a positive integer and denote $h := \Delta x$ and grid points as $x_j := a + j\,\Delta x$, $j = 0, 1, \dots, M$. Define the index sets

$$\mathcal{T}_M = \{j \mid j = 1, 2, \dots, M-1\}, \qquad \mathcal{T}_M^0 = \{j \mid j = 0, 1, 2, \dots, M\}.$$

Let $u_j^{\varepsilon,n}$ and u_j^n be the numerical approximations of $u^\varepsilon(x_j, t_n)$ and $u(x_j, t_n)$, respectively, for $j \in \mathcal{T}_M^0$ and $n \ge 0$, and denote $u^{\varepsilon,n}$, $u^n \in \mathbb{C}^{(M+1)}$ to be the numerical solutions at time $t = t_n$. Introduce the following finite difference operators:

$$\delta_x^+ u_j^n = \frac{1}{h}(u_{j+1}^n - u_j^n), \quad \delta_x^- u_j^n = \frac{1}{h}(u_j^n - u_{j-1}^n),$$
$$\delta_x u_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\,h}, \quad \delta_x^2 u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2},$$
$$\delta_t^+ u_j^n = \frac{1}{\tau}(u_j^{n+1} - u_j^n), \quad \delta_t^- u_j^n = \frac{1}{\tau}(u_j^n - u_j^{n-1}),$$
$$\delta_t u_j^n = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau}, \quad \delta_t^2 u_j^n = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2}.$$

The CNFD discretization of NLSW (2.1) reads as

(2.4)
$$(i\delta_t - \varepsilon^2\delta_t^2)u_j^{\varepsilon,n} = -\frac{1}{2}\left[\delta_x^2 u_j^{\varepsilon,n+1} + \delta_x^2 u_j^{\varepsilon,n-1}\right] - G(u_j^{\varepsilon,n+1}, u_j^{\varepsilon,n-1}), \quad j \in \mathcal{T}_M,\ n \geq 1,$$

where $G(z_1, z_2)$ is defined for $z_1, z_2 \in \mathbb{C}$ as

(2.5)
$$G(z_1, z_2) := \int_0^1 f(\theta|z_1|^2 + (1-\theta)|z_2|^2)\,d\theta \cdot \frac{z_1 + z_2}{2} = \frac{F(|z_1|^2) - F(|z_2|^2)}{|z_1|^2 - |z_2|^2} \cdot \frac{z_1 + z_2}{2}.$$

This conservative CNFD type method is widely used for discretizing NLS and NLSW in the mathematics literature since it can keep the mass and the energy conservation in the discretized level which is analogous to conservation in the continuous level. However, for such a scheme, at each time step, a fully nonlinear system has to be solved very accurately, which may be very time-consuming, especially in 2D and 3D. In fact, if the nonlinear system is not solved very accurately, the numerical solution computed doesn't conserve the energy and/or mass *exactly* [3]. This motivates us also to consider the following SIFD discretization for NLSW in which at each time step only a linear system needs to be solved and it usually can be solved by fast direct Poisson solver. The SIFD discretization for NLSW (2.1) is to apply Crank–Nicolson/leap-frog schemes for discretizing linear/nonlinear terms, respectively, as

(2.6) $$i\delta_t u_j^{\varepsilon,n} = \varepsilon^2\delta_t^2 u_j^{\varepsilon,n} - \frac{1}{2}\left[\delta_x^2 u_j^{\varepsilon,n+1} + \delta_x^2 u_j^{\varepsilon,n-1}\right] - f(|u_j^{\varepsilon,n}|^2)u_j^{\varepsilon,n}, \quad j \in \mathcal{T}_M,\ n \geq 1.$$

For both schemes, the boundary and initial conditions are discretized as

(2.7) $$u_0^{\varepsilon,n} = u_M^{\varepsilon,n} = 0, \quad n \geq 0; \qquad u_j^{\varepsilon,0} = u_0(x_j), \quad j \in \mathcal{T}_M^0.$$

Since CNFD (2.4) and SIFD (2.6) are three-level schemes, value at time step $n = 1$ should be assigned.

Choice of the first step value. Under the hypothesis of suitable regularity of $u^\varepsilon(x,t)$, one may use Taylor expansion to have

(2.8)
$$u_j^{\varepsilon,1} \approx u_0^\varepsilon(x_j) + \tau u_t^\varepsilon(x_j, 0) + \frac{\tau^2}{2} u_{tt}^\varepsilon(x_j, 0), \qquad u_t^\varepsilon(x_j, 0) = u_1^\varepsilon(x_j), \qquad j \in \mathcal{T}_M,$$

(2.9)
$$u_{tt}^\varepsilon(x_j, 0) = \frac{1}{\varepsilon^2}[iu_1^\varepsilon(x_j) + \partial_{xx}u_0(x_j) + f(|u_0|^2)u_0(x_j)] = i\varepsilon^{\alpha-2}w^\varepsilon(x_j), \quad j \in \mathcal{T}_M.$$

Due to the oscillation in time especially for the ill-prepared initial data case ($0 \leq \alpha < 2$), approximation (2.8) is not appropriate if $\varepsilon \ll 1$. In such case, τ has to be very small to resolve the error from the Taylor expansion (2.8). Our aim is to obtain a suitable choice of the first step value $u_j^{\varepsilon,1}$ which is uniformly accurate for all $\varepsilon \in (0,1]$. Denote

$$\Theta(v) = \partial_{xx} v + f(|v|^2)v, \qquad v \in H^2(\Omega); \tag{2.10}$$

then by integrating NLSW (2.1) w.r.t. t, we can write the solution $u^\varepsilon(x,t)$ as

$$u^\varepsilon(x,t) = u_0(x) - i\varepsilon^2(e^{it/\varepsilon^2} - 1)u_1^\varepsilon(x) - i\int_0^t (e^{i(t-s)/\varepsilon^2} - 1)\Theta(u^\varepsilon(x,s))\,ds. \tag{2.11}$$

Rewriting the integral term as

$$\begin{aligned}
&\int_0^t (e^{i(t-s)/\varepsilon^2} - 1)\Theta(u^\varepsilon(s))\,ds \\
&\quad = \int_0^t (e^{i(t-s)/\varepsilon^2} - 1)\left[\Theta(u^\varepsilon(s)) - \Theta(u^\varepsilon(0)) + \Theta(u^\varepsilon(0))\right]\,ds \\
&\quad = \left[-i\varepsilon^2(e^{it/\varepsilon^2} - 1) - t\right]\Theta(u^\varepsilon(0)) \\
&\qquad + \int_0^t \left(e^{i(t-s)/\varepsilon^2} - 1\right)\left[\Theta(u^\varepsilon(s)) - \Theta(u^\varepsilon(0))\right]\,ds,
\end{aligned}$$

then applying the trapezoidal rule to the integral in the right-hand side, we could obtain a second order approximation of $u^\varepsilon(x,\tau)$ as

$$u^\varepsilon(x,\tau) \approx u_0(x) - \varepsilon^2(e^{i\tau/\varepsilon^2} - 1)(iu_1^\varepsilon(x) + \Theta(u^\varepsilon(x,0))) + i\tau\Theta(u^\varepsilon(x,0)). \tag{2.12}$$

Hence, we propose the first step as

$$u_j^{\varepsilon,1} = u_0(x_j) - i\varepsilon^{2+\alpha}(e^{i\tau/\varepsilon^2} - 1)w^\varepsilon(x_j) + i\tau\Theta_j, \quad j \in \mathcal{T}_M, \tag{2.13}$$

where Θ_j is given by

$$\Theta_j = \delta_x^2 u_0(x_j) + f(|u_0(x_j)|^2)u_0(x_j), \quad j \in \mathcal{T}_M. \tag{2.14}$$

Now (2.4) or (2.6), together with (2.7) and (2.13), complete the scheme CNFD or SIFD for NLSW (2.1). For both CNFD and SIFD schemes, we can prove the uniform convergence rates at the order of $O(h^2+\tau)$ and $O(h^2+\tau^{2/3})$ for well-prepared and ill-prepared initial data, respectively.

2.2. Main results. Before introducing our main results, we denote

$$X_M = \left\{v = (v_j)_{j\in\mathcal{T}_M^0} \mid v_0 = v_M = 0\right\} \subset \mathbb{C}^{M+1}$$

and define the norms and inner product over X_M as

$$\begin{aligned}
&\|v\|_2^2 = h\sum_{j=0}^{M-1}|v_j|^2,\ \|\delta_x^+ v\|_2^2 = h\sum_{j=0}^{M-1}\left|\delta_x^+ v_j\right|^2, \\
&\|\delta_x^2 v\|_2^2 = h\sum_{j=1}^{M-1}\left|\delta_x^2 v_j\right|^2,\ \|v\|_\infty = \sup_{j\in\mathcal{T}_M^0}|v_j|, \\
&(u,v) = h\sum_{j=0}^{M-1}u_j\bar{v}_j, \quad \langle u,v\rangle = h\sum_{j=1}^{M-1}u_j\bar{v}_j \qquad \forall u,v \in X_M.
\end{aligned} \tag{2.15}$$

For simplicity of notation, we also define

$$\alpha^* = \min\{\alpha, 2\}. \tag{2.16}$$

According to the known results in [7, 18, 22, 25] and the asymptotic expansion in section 1, we can make assumptions on the initial data (2.3) for (2.1),

(A) $\quad 1 \lesssim \|w^\varepsilon(x)\|_{L^\infty(\Omega)} + \|\partial_x w^\varepsilon(x)\|_{L^\infty(\Omega)} + \|\partial_{xx} w^\varepsilon(x)\|_{L^\infty(\Omega)} \lesssim 1,$

and assumptions on $u^\varepsilon(\cdot,t)$ and $u(\cdot,t)$ for $0 < T < T_{\max}$ with $T_{\max}$ being the maximal common existing time and $\Omega_T = \Omega \times [0,T]$,

(B)
$$\begin{aligned} & u, u^\varepsilon \in C^4\left([0,T]; W^{1,\infty}(\Omega)\right) \cap C^2\left([0,T]; W^{3,\infty}(\Omega)\right) \\ & \quad \cap C^0\left([0,T]; W^{5,\infty}(\Omega) \cap H_0^1(\Omega)\right), \\ & \|u^\varepsilon\|_{L^\infty(\Omega_T)} + \|\partial_t u^\varepsilon\|_{L^\infty(\Omega_T)} + \sum_{m=1}^{5} \left\| \frac{\partial^m}{\partial x^m} u^\varepsilon \right\|_{L^\infty(\Omega_T)} \lesssim 1, \\ & \text{and } \left\|\frac{\partial^{m+n}}{\partial t^m \partial x^n} u^\varepsilon\right\|_{L^\infty(\Omega_T)} \lesssim \frac{1}{\varepsilon^{2m-2-\alpha^*}}, \qquad 2 \le m \le 4,\ m+n \le 5. \end{aligned}$$

Under assumptions (A) and (B), the following convergence rate holds:

$$\|u(t) - u^\varepsilon(t)\|_{W^{2,\infty}(\Omega)} \lesssim \varepsilon^2, \quad t \in [0,T]. \tag{2.17}$$

Define the "error" function $e^{\varepsilon,n} \in X_M$ for $n \ge 0$ as

$$e_j^{\varepsilon,n} = u^\varepsilon(x_j, t_n) - u_j^{\varepsilon,n}, \quad j \in \mathcal{T}_M. \tag{2.18}$$

Then we have the following estimates.

Theorem 2.1 (convergence of CNFD). *Assume $f(s) \in C^3([0,+\infty))$; under assumptions* (A) *and* (B), *there exist $h_0 > 0$ and $\tau_0 > 0$ sufficiently small, when $0 < h \le h_0$ and $0 < \tau \le \tau_0$, the CNFD method* (2.6) *with* (2.7) *and* (2.13) *admits a solution such that the following optimal error estimates hold:*

$$\|e^{\varepsilon,n}\|_2 + \|\delta_x^+ e^{\varepsilon,n}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}, \quad 0 \le n \le \frac{T}{\tau}, \tag{2.19}$$

$$\|e^{\varepsilon,n}\|_2 + \|\delta_x^+ e^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^2 + \varepsilon^2, \quad 0 \le n \le \frac{T}{\tau}. \tag{2.20}$$

Thus, by taking the minimum, we have the ε-independent convergence rate as

$$\|e^{\varepsilon,n}\|_2 + \|\delta_x^+ e^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^{4/(6-\alpha^*)}, \quad 0 \le n \le \frac{T}{\tau}. \tag{2.21}$$

Similarly, for the SIFD method, we have the next theorem.

Theorem 2.2 (convergence of SIFD). *Assume $f(s) \in C^2([0,+\infty))$; under assumptions* (A) *and* (B), *there exists $h_0 > 0$ and $\tau_0 > 0$ sufficiently small, when $0 < h \le h_0$ and $0 < \tau \le \tau_0$, the SIFD discretization* (2.6) *with* (2.7) *and* (2.13) *admits a unique solution and the following optimal error estimates hold:*

$$\|e^{\varepsilon,n}\|_2 + \|\delta_x^+ e^{\varepsilon,n}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}, \qquad 0 \le n \le \frac{T}{\tau}, \tag{2.22}$$

$$\|e^{\varepsilon,n}\|_2 + \|\delta_x^+ e^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^2 + \varepsilon^2, \qquad 0 \le n \le \frac{T}{\tau}. \tag{2.23}$$

Thus, by taking the minimum, we have the ε-independent convergence rate as

$$\|e^{\varepsilon,n}\|_2 + \|\delta_x^+ e^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^{4/(6-\alpha^*)}, \quad 0 \le n \le \frac{T}{\tau}. \tag{2.24}$$

3. Convergence of the SIFD scheme. In order to prove Theorem 2.2 for SIFD, we first establish the following lemmas.

LEMMA 3.1 (solvability of SIFD). *For any given* $u^{\varepsilon,0}, u^{\varepsilon,1} \in X_M$, *there exists a unique solution* $u^{\varepsilon,n} \in X_M$ *of* (2.6) *with* (2.7) *for* $n > 1$.

Proof. Standard fixed point arguments apply (see [3]) and we omit the proof for brevity. □

Denote the local truncation error $\eta^{\varepsilon,n} \in X_M$ of SIFD (2.6) with (2.7) and (2.13) for $n \geq 1$ and $j \in \mathcal{T}_M$ as

$$\eta_j^{\varepsilon,n} := (i\delta_t - \varepsilon^2\delta_t^2)u^\varepsilon(x_j,t_n) + \frac{1}{2}(\delta_x^2 u^\varepsilon(x_j,t_{n+1}) + \delta_x^2 u^\varepsilon(x_j,t_{n-1})) + f(|u^\varepsilon(x_j,t_n)|^2)u^\varepsilon(x_j,t_n).$$

LEMMA 3.2 (local truncation error for SIFD). *Under assumption* (B), *assume* $f \in C^1([0,\infty))$*; we have*

$$\|\eta^{\varepsilon,n}\|_2 + \|\delta_x^+\eta^{\varepsilon,n}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}, \quad 1 \leq n \leq \frac{T}{\tau} - 1. \tag{3.1}$$

Proof. Using the Taylor expansion and NLSW (2.1), we obtain for $j \in \mathcal{T}_M$ and $n \geq 1$,

$$\begin{aligned}\eta_j^{\varepsilon,n} = &\frac{i\tau^2}{2}\int_0^1\int_0^\theta\int_{-s}^{s} u_{ttt}^\varepsilon(x_j,\sigma\tau+t_n)\,d\sigma ds d\theta \\ &- \varepsilon^2\tau^2\int_0^1\int_0^\theta\int_0^s\int_{-\sigma}^{\sigma} u_{tttt}^\varepsilon(x_j,s_1\tau+t_n)\,ds_1 d\sigma ds d\theta \\ &+ \frac{h^2}{2}\int_0^1\int_0^\theta\int_0^s\int_{-\sigma}^{\sigma}\sum_{k=\pm1} u_{xxxx}^\varepsilon(x_j+s_1h,t_n+k\tau)ds_1 d\sigma ds d\theta \\ &+ \frac{\tau^2}{2}\int_0^1\int_{-\theta}^{\theta} u_{xxtt}^\varepsilon(x_j,s\tau+t_n)ds d\theta.\end{aligned}$$

Under assumption (B), using the triangle inequality, for $j \in \mathcal{T}_M$ and $n \geq 1$, we get

$$\begin{aligned}|\eta_j^{\varepsilon,n}| &\lesssim h^2\|\partial_{xxxx}u^\varepsilon\|_{L^\infty} + \tau^2\Big(\|\partial_{ttt}u^\varepsilon\|_{L^\infty} + \varepsilon^2\|\partial_{tttt}u^\varepsilon\|_{L^\infty} + \|\partial_{xxtt}u^\varepsilon\|_{L^\infty}\Big) \\ &\lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}},\end{aligned}$$

where the L^∞-norm means $\|u\|_{L^\infty} := \sup_{0\leq t\leq T}\sup_{x\in\Omega}|u(x,t)|$. The first conclusion of the lemma then follows. For $1 \leq j \leq M-2$, applying δ_x^+ to $\eta_j^{\varepsilon,n}$ and using the formula above, noticing $f \in C^1([0,\infty))$, it is easy to check that

$$\begin{aligned}|\delta_x^+\eta_j^{\varepsilon,n}| &\lesssim h^2\|\partial_{xxxxx}u^\varepsilon\|_{L^\infty} + \tau^2\Big(\|\partial_{tttx}u^\varepsilon\|_{L^\infty} + \varepsilon^2\|\partial_{ttttx}u^\varepsilon\|_{L^\infty} + \|\partial_{xxxtt}u^\varepsilon\|_{L^\infty}\Big) \\ &\lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}.\end{aligned}$$

For $j=0$ and $M-1$, we apply the boundary condition to deduce that $\frac{\partial^k}{\partial t^k}u^\varepsilon(x,t)|_{x\in\partial\Omega} = 0$ for $k \geq 0$, and (2.1) shows that $u_{xx}^\varepsilon(x,t)|_{x\in\partial\Omega} = 0$, $u_{xxtt}^\varepsilon(x,t)|_{x\in\partial\Omega} = 0$ and

$u^\varepsilon_{xxxx}(x,t)|_{x\in\partial\Omega} = 0$. Similar to the above, we can get the same estimates for $j = 0, M-1$. Thus, we complete the proof. ∎

Since $u^{\varepsilon,0}$ and $u^{\varepsilon,1}$ are known, we have the error estimates at the first step.

LEMMA 3.3 (error bounds at $n = 1$). *Under assumptions* (A) *and* (B), *we have*

$$\|e^{\varepsilon,1}\|_2 + \|\delta_x^+ e^{\varepsilon,1}\|_2 + \|\delta_x^2 e^{\varepsilon,1}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}, \tag{3.2}$$

$$\|\delta_t^+ e^{\varepsilon,0}\|_2 + \|\delta_t^+\delta_x^+ e^{\varepsilon,0}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}},$$

$$\|e^{\varepsilon,1}\|_2 + \|\delta_x^+ e^{\varepsilon,1}\|_2 + \|\delta_x^2 e^{\varepsilon,1}\|_2 \lesssim h^2 + \tau^2 + \varepsilon^2, \tag{3.3}$$

$$\|\delta_t^+ e^{\varepsilon,0}\|_2 + \|\delta_t^+\delta_x^+ e^{\varepsilon,0}\|_2 \lesssim 1.$$

Proof. By definition, $e^{\varepsilon,0} = \mathbf{0} \in \mathbb{C}^{M+1}$. For $n = 1$, recalling NLSW (2.1) and the choice of $u^{\varepsilon,1}$ (2.13), using the Taylor expansion, we see that for $j \in \mathcal{T}_M$

$$\begin{aligned}
u^\varepsilon(x_j,\tau) &= u_0(x_j) + \tau(i\left(\partial_{xx}u_0(x_j) + f(|u_0(x_j)|^2)u_0(x_j)\right) + \varepsilon^\alpha w^\varepsilon(x_j)) \\
&\quad + \frac{i\tau^2}{2}\varepsilon^{\alpha-2}w^\varepsilon(x_j) + \frac{1}{2}\int_0^\tau u^\varepsilon_{ttt}(x_j,s)\cdot(\tau-s)^2\,ds, \\
u_j^{\varepsilon,1} &= u_0(x_j) + \tau\left[i\left(\partial_{xx}u_0(x_j) + f(|u_0(x_j)|^2)u_0(x_j)\right) + \varepsilon^\alpha w^\varepsilon(x_j)\right] \\
&\quad + \left[-\tau - i\varepsilon^2\left(i\frac{\tau}{\varepsilon^2} - \frac{\tau^2}{2\varepsilon^4} + O(\tau^3\varepsilon^{-6})\right)\right]\varepsilon^\alpha w^\varepsilon(x_j) \\
&\quad + \frac{i\tau h^2}{12}\partial_{xxxx}u_0(x_j + \theta_j^{(1)}h), \\
e_j^{\varepsilon,1} &= u^\varepsilon(x_j,\tau) - u_j^{\varepsilon,1} = -\frac{i\tau h^2}{12}\partial_{xxxx}u_0(x_j+\theta_j^{(1)}h) \\
&\quad + \frac{\tau^3}{6}\partial_{ttt}u^\varepsilon(x_j,\theta_j^{(2)}\tau) + O\left(\frac{\tau^3}{\varepsilon^{4-\alpha}}\right)w^\varepsilon(x_j),
\end{aligned}$$

where $\theta_j^{(1)} \in [-1,1]$, $\theta_j^{(2)} \in [0,1]$ are constants. Noticing that for $\varepsilon \in (0,1]$, $\frac{1}{\varepsilon^{4-\alpha}} \le \frac{1}{\varepsilon^{4-\alpha^*}}$, it is easy to get the conclusion in (3.2) for $\|e^{\varepsilon,1}\|_2 + \|\delta_x^+ e^{\varepsilon,1}\|_2$ (the boundary case is the same as that in Lemma 3.2) and $\|\delta_t^+ e^{\varepsilon,0}\|_2$. For $1 \le j \le M-1$, we can get

$$\begin{aligned}
|\delta_x^2 e_j^{\varepsilon,1}| &\lesssim O\left(\frac{\tau^3}{\varepsilon^{4-\alpha}}\right)\cdot\|\partial_{xx}w^\varepsilon\|_{L^\infty(\Omega)} + \tau h\|\partial_{xxxxx}u_0\|_{L^\infty(\Omega)} \\
&\quad + \int_0^\tau s^2 ds\|\partial_{tttxx}u^\varepsilon\|_{L^\infty(\Omega_T)} \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}, \\
|\delta_t^+\delta_x^+ e_j^{\varepsilon,0}| &\lesssim O\left(\frac{\tau^2}{\varepsilon^{4-\alpha}}\right)\|\partial_x w^\varepsilon\|_{L^\infty(\Omega)} + h^2\|\partial_{xxxxx}u_0\|_{L^\infty(\Omega)} \\
&\quad + \tau^2\|\partial_{tttx}u^\varepsilon\|_{L^\infty(\Omega_T)} \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}},
\end{aligned}$$

which implies the results for $\|\delta_t^+\delta_x^+ e^{\varepsilon,0}\|_2$ (the boundary case is similar to the above) and $\|\delta_x^2 e^{\varepsilon,1}\|_2$ in (3.2).

For the assertion (3.3), we use the relation between $u(x,t)$ and $u^\varepsilon(x,t)$. Taylor expansion would give for $1 \le j \le M-1$

$$u(x_j,\tau) - u_j^{\varepsilon,1} = -\frac{i\tau h^2}{12}\partial_{xxxx}u_0(x_j+\theta_j^{(2)}h) + \int_0^\tau \partial_{tt}u(x_j,s)(\tau-s)\,ds + i\varepsilon^{2+\alpha}(e^{i\tau/\varepsilon^2}-1)w^\varepsilon(x_j)$$

and

$$\left|\delta_x^+\left(u(x_j,\tau)-u_j^{\varepsilon,1}\right)\right| \lesssim \tau h^2\|\partial_{xxxxx}u_0\|_{L^\infty} + \tau^2\|\partial_{ttx}u\|_{L^\infty} + \varepsilon^2\|\partial_x w^\varepsilon\|_{L^\infty},$$
$$\left|\delta_x^2\left(u(x_j,\tau)-u_j^{\varepsilon,1}\right)\right| \lesssim \tau h\|\partial_{xxxxxx}u_0\|_{L^\infty} + \tau^2\|\partial_{ttxx}u\|_{L^\infty} + \varepsilon^2\|\partial_{xx} w^\varepsilon\|_{L^\infty},$$

and it is convenient to use the boundary condition as before to find that

(3.4)
$$\|u(x_j,\tau)-u_j^{\varepsilon,1}\|_2 + \|\delta_x^+\left(u(x_j,\tau)-u_j^{\varepsilon,1}\right)\|_2 + \|\delta_x^2\left(u(x_j,\tau)-u_j^{\varepsilon,1}\right)\|_2 \lesssim h^2+\tau^2+\varepsilon^2.$$

Recalling the convergence $|u^\varepsilon(x_j,\tau)-u(x_j,\tau)| \lesssim \varepsilon^2$ and

$$\left|\delta_x^+\left[u^\varepsilon(x_j,\tau)-u(x_j,\tau)\right]\right| \lesssim \varepsilon^2 + h^2(\|u_{xxx}\|_{L^\infty(\Omega_T)} + \|u^\varepsilon_{xxx}\|_{L^\infty(\Omega_T)}),$$
$$\left|\delta_x^2\left[u^\varepsilon(x_j,\tau)-u(x_j,\tau)\right]\right| \lesssim \varepsilon^2 + h^2(\|u_{xxxx}\|_{L^\infty(\Omega_T)} + \|u^\varepsilon_{xxxx}\|_{L^\infty(\Omega_T)}),$$

the triangular inequality then gives the conclusion for $\|e^{\varepsilon,1}\|_2 + \|\delta_x^+e^{\varepsilon,1}\|_2 + \|\delta_x^2 e^{\varepsilon,1}\|_2$ in (3.3).

Similarly, for $0 \le j \le M-1$

$$\left|\delta_t^+\left(u(x_j,0)-u_j^{\varepsilon,0}\right)\right| \lesssim h^2\|\partial_{xxxx}u_0\|_{L^\infty(\Omega)} + \tau\|\partial_{tt}u\|_{L^\infty(\Omega_T)} + \varepsilon^\alpha\|w^\varepsilon\|_{L^\infty(\Omega)},$$
$$\left|\delta_t^+\delta_x^+\left(u(x_j,0)-u_j^{\varepsilon,0}\right)\right| \lesssim h^2\|\partial_{xxxxx}u_0\|_{L^\infty(\Omega)} + \tau\|\partial_{ttx}u\|_{L^\infty(\Omega_T)} + \varepsilon^\alpha\|\partial_x w^\varepsilon\|_{L^\infty(\Omega)},$$

combined with the triangle inequality and assumption (B), which implies

(3.5) $$\left|\delta_t^+u^\varepsilon(x_j,0)-\delta_t^+u(x_j,0)\right| + \left|\delta_t^+\delta_x^+u^\varepsilon(x_j,0)-\delta_t^+\delta_x^+u(x_j,0)\right| \lesssim 1,$$

we draw conclusion (3.3) for $\|\delta_t^+e^{\varepsilon,0}\|_2 + \|\delta_t^+\delta_x^+e^{\varepsilon,0}\|_2$. ∎

One main difficulty in deriving error bounds for SIFD and/or in high dimensions is the l^∞ bounds for the finite difference solutions. In [2, 3, 4, 24], this difficulty was overcome by truncating the nonlinearity f to a global Lipschitz function with compact support in d-dimensions ($d = 1,2,3$). This is guaranteed if the continuous solution is bounded and the numerical solution is close to the continuous solution. Here, we apply the same idea. Choose $M_0 > 0$ and a smooth function $\rho(s) \in C^\infty(\mathbb{R}^1)$ such that

(3.6) $$M_0 = \max\left\{\|u(x,t)\|_{L^\infty(\Omega_T)},\ \sup_{\varepsilon\in(0,1]}\|u^\varepsilon(x,t)\|_{L^\infty(\Omega_T)}\right\},$$

$$\rho(s) = s\begin{cases}1, & 0 \le |s| \le 1\,,\\ \in[0,1], & 1 \le |s| \le 2\,,\\ 0, & |s| \ge 2\,.\end{cases}$$

By assumption (B), M_0 is well defined and let us denote $B = (M_0+1)^2$. For $s \geq 0$ and $z \in \mathbb{C}$, define

(3.7)
$$f_B(s) = f(s)\rho(s/B),\ F_B(s) = \int_0^s f_B(\sigma)d\sigma,\ \rho_{_B}(s) = \rho(s/B),\ g_{_B}(z) = z\rho_{_B}(|z|^2).$$

Then $f_B(s)$ and $g_{_B}(z)$ are global Lipschitz and

$$|f_B(s_1) - f_B(s_2)| \leq C_B|\sqrt{s_1} - \sqrt{s_2}|, \quad \forall s_1, s_2 \geq 0. \tag{3.8}$$

Choose $v^{\varepsilon,0} = u^{\varepsilon,0}$, $v^{\varepsilon,1} = u^{\varepsilon,1}$, and define $v^{\varepsilon,n+1} \in X_M$ $(n \geq 1)$ for $j \in \mathcal{T}_M$ as

$$\left(i\delta_t - \varepsilon^2\delta_t^2\right) v_j^{\varepsilon,n} + \frac{1}{2}(\delta_x^2 v_j^{\varepsilon,n+1} + \delta_x^2 v_j^{\varepsilon,n-1}) + f_B(|v_j^{\varepsilon,n}|^2)v_j^{\varepsilon,n} = 0. \tag{3.9}$$

In fact, $v^{\varepsilon,n}$ can be viewed as another approximation of $u^\varepsilon(x, t_n)$.

Define the "error" function $\hat{e}^{\varepsilon,n} \in X_M$ as

$$\hat{e}_j^{\varepsilon,n} := u^\varepsilon(x_j, t_n) - v_j^{\varepsilon,n}, \quad j \in \mathcal{T}_M^0,\ n \geq 0, \tag{3.10}$$

and the local truncation error $\hat{\eta}^{\varepsilon,n} \in X_M$ for $n \geq 1$ and $j \in \mathcal{T}_M$ as

(3.11)
$$\hat{\eta}_j^{\varepsilon,n} := (i\delta_t - \varepsilon^2\delta_t^2 + f_B(|u^\varepsilon(x_j, t_n)|^2))u^\varepsilon(x_j, t_n) + \frac{1}{2}(\delta_x^2 u^\varepsilon(x_j, t_{n+1}) + \delta_x^2 u^\varepsilon(x_j, t_{n-1})).$$

Similar as Lemma 3.2, we have the bounds for $\hat{\eta}^{\varepsilon,n}$ $(n \geq 1)$ as

$$\|\hat{\eta}^{\varepsilon,n}\|_2 + \|\delta_x^+ \hat{\eta}^{\varepsilon,n}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}. \tag{3.12}$$

Subtracting (3.9) from (3.11), we obtain the error equation for $\hat{e}^{\varepsilon,n} \in X_M$ as

$$\left(i\delta_t - \varepsilon^2\delta_t^2\right) \hat{e}_j^{\varepsilon,n} + \frac{1}{2}(\delta_x^2 \hat{e}_j^{\varepsilon,n+1} + \delta_x^2 \hat{e}_j^{\varepsilon,n-1}) - \hat{\eta}_j^{\varepsilon,n} + \xi_j^{\varepsilon,n} = 0, \quad n \geq 1, \tag{3.13}$$

where $\xi^{\varepsilon,n} \in X_M$ $(n \geq 1)$ is defined for $j \in \mathcal{T}_M$ as

$$\xi_j^{\varepsilon,n} = f_B(|v_j^{\varepsilon,n}|^2)\hat{e}_j^{\varepsilon,n} + u^\varepsilon(x_j, t_n)\left(f_B(|u^\varepsilon(x_j, t_n)|^2) - f_B(|v_j^{\varepsilon,n}|^2)\right). \tag{3.14}$$

For $\xi^{\varepsilon,n}$, we have the following properties.

LEMMA 3.4. *Under the assumptions in Theorem* 2.2, *for* $\xi^{\varepsilon,n} \in X_M$ $(n \geq 1)$ *in* (3.14), *we have*

(3.15)
$$|\xi_j^{\varepsilon,n}| \lesssim |\hat{e}_j^{\varepsilon,n}|, \qquad |\delta_x^+ \xi_j^{\varepsilon,n}| \lesssim |\hat{e}_j^{\varepsilon,n}| + |\hat{e}_{j+1}^{\varepsilon,n}| + |\delta_x^+ \hat{e}_j^{\varepsilon,n}|, \quad 0 \leq j \leq M-1,\ 1 \leq n \leq \frac{T}{\tau}.$$

Proof. Using the properties of $f_B(s)$, it is easy to obtain

$$|\xi_j^{\varepsilon,n}| \lesssim |\hat{e}_j^{\varepsilon,n}|, \quad j \in \mathcal{T}_M^0,\ n \geq 1. \tag{3.16}$$

For $0 \leq j \leq M-1$, $n \geq 1$, and $\theta \in [0,1]$, denote

$$u_{j,\theta}^\varepsilon = \theta u^\varepsilon(x_{j+1}, t_n) + (1-\theta)u^\varepsilon(x_j, t_n), \quad v_{j,\theta}^\varepsilon = \theta v_{j+1}^{\varepsilon,n} + (1-\theta)v_j^{\varepsilon,n}; \tag{3.17}$$

then we have

$$\delta_x^+ \xi_j^{\varepsilon,n} = \delta_x^+ \left(f(|u^\varepsilon(x_j,t_n)|^2)u^\varepsilon(x_j,t_n)\right) - \delta_x^+ \left(f(|v_j^{\varepsilon,n}|^2)v_j^{\varepsilon,n}\right) = I_1 - I_2 \qquad \text{with}$$

$$I_1 = \int_0^1 \Big[\left(f_B(|u_{j,\theta}^\varepsilon|^2) + f_B'(|u_{j,\theta}^\varepsilon|^2)|u_{j,\theta}^\varepsilon|^2\right) \delta_x^+ u^\varepsilon(x_j,t_n) + f_B'(|u_{j,\theta}^\varepsilon|^2)(u_{j,\theta}^\varepsilon)^2 \delta_x^+ \overline{u^\varepsilon(x_j,t_n)}\Big]\, d\theta,$$

$$I_2 = \int_0^1 \Big[\left(f_B(|v_{j,\theta}^\varepsilon|^2) + f_B'(|v_{j,\theta}^\varepsilon|^2)|v_{j,\theta}^\varepsilon|^2\right) \delta_x^+ v_j^{\varepsilon,n} + f_B'(|v_{j,\theta}^\varepsilon|^2)(v_{j,\theta}^\varepsilon)^2 \overline{\delta_x^+ v_j^{\varepsilon,n}}\Big]\, d\theta.$$

Using the definition of f_B, it is easy to see $f_B \in C_0^2(\mathbb{R})$ and the following holds:

$$\begin{aligned}
&\left| \left[\left(f_B(|u_{j,\theta}^\varepsilon|^2) + f_B'(|u_{j,\theta}^\varepsilon|^2)|u_{j,\theta}^\varepsilon|^2\right) - \left(f_B(|v_{j,\theta}^\varepsilon|^2) + f_B'(|v_{j,\theta}^\varepsilon|^2)|v_{j,\theta}^\varepsilon|^2\right)\right] \delta_x^+ u^\varepsilon(x_j,t_n)\right| \\
&\quad \lesssim \big|\, |u_{j,\theta}^\varepsilon| - |v_{j,\theta}^\varepsilon|\, \big| \lesssim |\hat{e}_j^{\varepsilon,n}| + |\hat{e}_{j+1}^{\varepsilon,n}|, \\
&\quad \left| \left[f_B'(|u_{j,\theta}^\varepsilon|^2)(u_{j,\theta}^\varepsilon)^2 - f_B'(|v_{j,\theta}^\varepsilon|^2)(v_{j,\theta}^\varepsilon)^2\right] \overline{\delta_x^+ u^\varepsilon(x_j,t_n)}\right| \\
&\quad \lesssim \big||u_{j,\theta}^\varepsilon| - |v_{j,\theta}^\varepsilon|\big| \lesssim |\hat{e}_j^{\varepsilon,n}| + |\hat{e}_{j+1}^{\varepsilon,n}|, \\
&\quad \left| \left[f_B(|v_{j,\theta}^\varepsilon|^2) + f_B'(|v_{j,\theta}^\varepsilon|^2)|v_{j,\theta}^\varepsilon|^2\right] \left(\delta_x^+ u^\varepsilon(x_j,t_n) - \delta_x^+ v_j^{\varepsilon,n}\right)\right| \lesssim |\delta_x^+ \hat{e}_j^{\varepsilon,n}|, \\
&\quad \left| f_B'(|v_{j,\theta}^\varepsilon|^2)(v_{j,\theta}^\varepsilon)^2 \left(\overline{\delta_x^+ u^\varepsilon(x_j,t_n)} - \overline{\delta_x^+ v_j^{\varepsilon,n}}\right)\right| \lesssim |\delta_x^+ \hat{e}_j^{\varepsilon,n}|.
\end{aligned}$$

Hence, we get the desired conclusion. The proof is complete. ∎

Proof of Theorem 2.2. The proof is divided into three main steps.

Step 1: *Establish* (2.22)*-type error bound for* $\hat{e}^{\varepsilon,n}$. From the error equation (3.13), multiplying both sides of (3.13) by $\overline{\hat{e}_j^{\varepsilon,n+1} + \hat{e}_j^{\varepsilon,n-1}}$ and summing for $j \in \mathcal{T}_M$, using the summation by parts formula, and taking imaginary parts, we have

$$\begin{aligned}
&\|\hat{e}^{\varepsilon,n+1}\|_2^2 + 4\varepsilon^2 \,\mathrm{Im}\left(\hat{e}^{\varepsilon,n}, \delta_t^+ \hat{e}^{\varepsilon,n}\right) - \left\{\|\hat{e}^{\varepsilon,n-1}\|_2^2 + 4\varepsilon^2 \,\mathrm{Im}\left(\hat{e}^{\varepsilon,n-1}, \delta_t^+ \hat{e}^{\varepsilon,n-1}\right)\right\} \\
&\qquad = -2\tau \mathrm{Im}\left(\xi^{\varepsilon,n} - \hat{\eta}^{\varepsilon,n}, \hat{e}^{\varepsilon,n+1} + \hat{e}^{\varepsilon,n-1}\right), \quad n \geq 1.
\end{aligned} \tag{3.18}$$

Adding (3.18) for $1, 2, \ldots, n$ ($n \leq \frac{T}{\tau} - 1$), in view of Lemma 3.4 and the local truncation error (3.11), we have

$$\|\hat{e}^{\varepsilon,n+1}\|_2^2 + \|\hat{e}^{\varepsilon,n}\|_2^2 + 4\varepsilon^2 \mathrm{Im}\left(\hat{e}^n, \delta_t^+ \hat{e}^{\varepsilon,n}\right) \lesssim n\tau \left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2 + \tau \sum_{m=1}^{n+1} \|\hat{e}^{\varepsilon,m}\|_2^2. \tag{3.19}$$

Multiplying both sides of (3.13) by $\overline{\hat{e}_j^{\varepsilon,n+1} - \hat{e}_j^{\varepsilon,n-1}}$ and summing for $j \in \mathcal{T}_M$, using the summation by parts formula, and taking real parts, we have

$$\begin{aligned}
&-\left(\varepsilon^2 \|\delta_t^+ \hat{e}^{\varepsilon,n}\|_2^2 + \frac{1}{2}\|\delta_x^+ \hat{e}^{\varepsilon,n+1}\|_2^2\right) + \left(\varepsilon^2 \|\delta_t^+ \hat{e}^{\varepsilon,n-1}\|_2^2 + \frac{1}{2}\|\delta_x^+ \hat{e}^{\varepsilon,n-1}\|_2^2\right) \\
&\quad = -\,\mathrm{Re}\left(\xi^{\varepsilon,n} - \hat{\eta}^{\varepsilon,n}, \hat{e}^{\varepsilon,n+1} - \hat{e}^{\varepsilon,n-1}\right),
\end{aligned} \tag{3.20}$$

where $\mathrm{Re}(c)$ denotes the real part of c. Noticing that

$$\begin{aligned}\left|\mathrm{Re}\left(\xi^{\varepsilon,n}-\hat{\eta}^{\varepsilon,n},\hat{e}^{\varepsilon,n+1}-\hat{e}^{\varepsilon,n-1}\right)\right| &= \tau\left|\mathrm{Re}\left(\xi^{\varepsilon,n}-\hat{\eta}^{\varepsilon,n},\delta_t^+\hat{e}^{\varepsilon,n}+\delta_t^+\hat{e}^{\varepsilon,n-1}\right)\right|\\ &\le \frac{C\tau}{\varepsilon^2}\left[\left(h^2+\frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2+\ \|\hat{e}^{\varepsilon,n}\|_2^2\right]\\ &\quad+\tau\varepsilon^2\left(\|\delta_t^+\hat{e}^{\varepsilon,n-1}\|_2^2+\|\delta_t^+\hat{e}^{\varepsilon,n}\|_2^2\right),\end{aligned}$$

combined with (3.20), taking summation for $1,2,\ldots,n$, and using Lemma 3.3, we find that

(3.21)

$$\begin{aligned}&\varepsilon^2\|\delta_t^+\hat{e}^{\varepsilon,n}\|_2^2+\frac{1}{2}\|\delta_x^+\hat{e}^{\varepsilon,n+1}\|_2^2+\frac{1}{2}\|\delta_x^+\hat{e}^{\varepsilon,n}\|_2^2\\ &\lesssim \frac{n\tau}{\varepsilon^2}\left(h^2+\frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2+\tau\sum_{m=1}^{n}\varepsilon^2\|\delta_t^+\hat{e}^{\varepsilon,m}\|_2^2+\tau\sum_{m=1}^{n+1}\frac{1}{\varepsilon^2}\|\hat{e}^{\varepsilon,m}\|_2^2,\quad 1\le n\le\frac{T}{\tau}-1.\end{aligned}$$

For $1\le n\le \frac{T}{\tau}-1$, define

(3.22)

$$\mathcal{S}^n=8\left(\varepsilon^2\|\delta_t^+\hat{e}^{\varepsilon,n}\|_2^2+\frac{1}{2}\|\delta_x^+\hat{e}^{\varepsilon,n+1}\|_2^2+\frac{1}{2}\|\delta_x^+\hat{e}^{\varepsilon,n}\|_2^2\right)+\frac{1}{2\varepsilon^2}\left(\|\hat{e}^{\varepsilon,n+1}\|_2^2+\|\hat{e}^{\varepsilon,n}\|_2^2\right).$$

In view of the Cauchy inequality which implies

$$8\varepsilon^2\|\delta_t^+\hat{e}^{\varepsilon,n}\|_2^2+\frac{1}{2\varepsilon^2}\|\hat{e}^{\varepsilon,n}\|_2^2\ge 4|\left(\delta_t^+\hat{e}^{\varepsilon,n},\hat{e}^{\varepsilon,n}\right)|,$$

together with $\frac{1}{\varepsilon^2}\times(3.19)+16\times(3.21)$, we obtain

$$\mathcal{S}^n\lesssim\frac{n\tau}{\varepsilon^2}\left(h^2+\frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2+\tau\sum_{m=1}^{n}\mathcal{S}^m,\quad 1\le n\le\frac{T}{\tau}-1. \tag{3.23}$$

Hence, the discrete Gronwall inequality [9, 13] implies that for τ small enough,

$$\mathcal{S}^n\lesssim\frac{1}{\varepsilon^2}\left(h^2+\frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2,\quad 1\le n\le\frac{T}{\tau}-1. \tag{3.24}$$

In particular, we have established the l^2 error bounds

$$\|\hat{e}^{\varepsilon,n}\|_2\lesssim h^2+\frac{\tau^2}{\varepsilon^{4-\alpha^*}},\quad n\le\frac{T}{\tau}. \tag{3.25}$$

However, the discrete semi-H^1 convergence is not optimal. In order to derive the optimal convergence rate in discrete semi-H^1 norm, multiplying both sides of (3.13) by $\overline{\delta_x^2(\hat{e}_j^{\varepsilon,n+1}+\hat{e}_j^{\varepsilon,n-1})}$, then summing together for $j=1,2,\ldots,M-1$, after taking the imaginary parts of both sides and applying the summation by parts formula, and using the l^2 error estimates (3.25), we have for $1 \le n \le \frac{T}{\tau} - 1$

$$\begin{aligned}
&\|\delta_x^+ \hat{e}^{\varepsilon,n+1}\|_2^2 + 4\varepsilon^2 \operatorname{Im}\left(\delta_x^+ \hat{e}^{\varepsilon,n}, \delta_t^+ \delta_x^+ \hat{e}^{\varepsilon,n}\right) \\
&\qquad - \left\{\|\delta_x^+ \hat{e}^{\varepsilon,n-1}\|_2^2 + 4\varepsilon^2 \operatorname{Im}\left(\delta_x^+ \hat{e}^{\varepsilon,n-1}, \delta_t^+ \delta_x^+ \hat{e}^{\varepsilon,n-1}\right)\right\} \\
&\quad = -2\tau \operatorname{Im}\left\langle \xi^{\varepsilon,n} - \hat{\eta}^{\varepsilon,n}, \delta_x^2(\hat{e}^{\varepsilon,n+1} + \hat{e}^{\varepsilon,n-1})\right\rangle \\
&\quad = 2\tau \operatorname{Im}\left(\delta_x^+ \xi^{\varepsilon,n} - \delta_x^+ \hat{\eta}^{\varepsilon,n}, \delta_x^+ \hat{e}^{\varepsilon,n+1} + \delta_x^+ \hat{e}^{\varepsilon,n-1}\right) \\
&\quad \le C\tau \left[\|\delta_x^+ \hat{e}^{\varepsilon,n+1}\|_2^2 + \|\delta_x^+ \hat{e}^{\varepsilon,n}\|_2^2 + \|\delta_x^+ \hat{e}^{\varepsilon,n-1}\|_2^2 + \|\hat{e}^{\varepsilon,n}\|_2^2 + \left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2\right] \\
&\quad \le C\tau \left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2 + C\tau(\|\delta_x^+ \hat{e}^{\varepsilon,n+1}\|_2^2 + \|\delta_x^+ \hat{e}^{\varepsilon,n}\|_2^2 + \|\delta_x^+ \hat{e}^{\varepsilon,n-1}\|_2^2).
\end{aligned}$$

Summing the above inequalities for $1, 2, \ldots, n$ and making use of Lemma 3.3, we then have

$$\begin{aligned}
&\|\delta_x^+ \hat{e}^{\varepsilon,n+1}\|_2^2 + \|\delta_x^+ \hat{e}^{\varepsilon,n}\|_2^2 + 4\varepsilon^2 \operatorname{Im}\left(\delta_x^+ \hat{e}^{\varepsilon,n}, \delta_t^+ \delta_x^+ \hat{e}^{\varepsilon,n}\right) \\
&\quad \le n\tau \left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2 + \tau \sum_{m=1}^{n+1} \|\delta_x^+ \hat{e}^{\varepsilon,m}\|_2^2 + \|\delta_x^+ \hat{e}^{\varepsilon,1}\|_2^2 \\
&\qquad + \|\delta_x^+ \hat{e}^{\varepsilon,0}\|_2^2 + 4\varepsilon^2 \operatorname{Im}\left(\delta_x^+ \hat{e}^{\varepsilon,0}, \delta_t^+ \delta_x^+ \hat{e}^{\varepsilon,0}\right) \\
&\quad \lesssim \left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2 + \tau \sum_{m=1}^{n+1} \|\delta_x^+ \hat{e}^{\varepsilon,m}\|_2^2, \qquad 1 \le n \le \frac{T}{\tau} - 1. \qquad (3.26)
\end{aligned}$$

Multiplying both sides of (3.13) by $\overline{\delta_x^2(\hat{e}_j^{\varepsilon,n+1} - \hat{e}_j^{\varepsilon,n-1})}$, summing up together for $j = 1, 2, \ldots, M-1$, then taking the real parts of both sides and applying the summation by parts formula, and using the l^2 error estimates (3.25) and the local truncation error (3.11), we have for $n \ge 1$

$$\begin{aligned}
&\varepsilon^2 \|\delta_t^+ \delta_x^+ \hat{e}^{\varepsilon,n}\|_2^2 + \frac{1}{2}\|\delta_x^2 \hat{e}^{\varepsilon,n+1}\|_2^2 - \varepsilon^2 \|\delta_t^+ \delta_x^+ \hat{e}^{\varepsilon,n-1}\|_2^2 - \frac{1}{2}\|\delta_x^2 \hat{e}^{\varepsilon,n-1}\|_2^2 \\
&\quad = \operatorname{Re}\left\langle \xi^{\varepsilon,n} - \hat{\eta}^{\varepsilon,n}, \delta_x^2(\hat{e}_j^{\varepsilon,n+1} - \hat{e}_j^{\varepsilon,n-1})\right\rangle \\
&\quad = -\operatorname{Re}\left(\delta_x^+ \xi^{\varepsilon,n} - \delta_x^+ \hat{\eta}^{\varepsilon,n}, \delta_x^+(\hat{e}_j^{\varepsilon,n+1} - \hat{e}_j^{\varepsilon,n-1})\right) \\
&\quad = -\tau \operatorname{Re}\left(\delta_x^+ \xi^{\varepsilon,n} - \delta_x^+ \hat{\eta}^{\varepsilon,n}, \delta_t^+ \delta_x^+ \hat{e}_j^{\varepsilon,n} + \delta_t^+ \delta_x^+ \hat{e}_j^{\varepsilon,n-1}\right) \\
&\quad \le \tau\ \left(\varepsilon^2 \|\delta_t^+ \delta_x^+ \hat{e}^{\varepsilon,n-1}\|_2^2 + \varepsilon^2 \|\delta_t^+ \delta_x^+ \hat{e}^{\varepsilon,n}\|_2^2\right) + \frac{C\tau}{\varepsilon^2}\|\delta_x^+ \hat{e}^{\varepsilon,n}\|_2^2 + \frac{C\tau}{\varepsilon^2}\left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2.
\end{aligned}$$

Summing the above inequalities together for $1, 2, \ldots, n$ and using Lemma 3.3, we find that for $1 \le n \le \frac{T}{\tau} - 1$

$$
\begin{aligned}
&\varepsilon^2\|\delta_t^+\delta_x^+\hat{e}^{\varepsilon,n}\|_2^2 + \|\delta_x^2\hat{e}^{\varepsilon,n+1}\|_2^2 \cdot 1/2 + \|\delta_x^2\hat{e}^{\varepsilon,n}\|_2^2 \cdot 1/2 \\
&\quad \lesssim \tau \sum_{m=1}^{n} \varepsilon^2\|\delta_t^+\hat{e}^{\varepsilon,m}\|_2^2 + \frac{\tau}{\varepsilon^2}\sum_{m=1}^{n+1}\|\delta_x^+\hat{e}^{\varepsilon,m}\|_2^2 + \|\delta_x^2\hat{e}^{\varepsilon,1}\|_2^2 \\
&\qquad + \varepsilon^2\|\delta_t^+\delta_x^+\hat{e}^{\varepsilon,0}\|_2^2 + \frac{1}{\varepsilon^2}\left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2 \\
&\quad \lesssim \tau \sum_{m=1}^{n} \varepsilon^2\|\delta_t^+\delta_x^+\hat{e}^{\varepsilon,m}\|_2^2 + \frac{\tau}{\varepsilon^2}\sum_{m=1}^{n+1}\|\delta_x^+\hat{e}^{\varepsilon,m}\|_2^2 + \frac{1}{\varepsilon^2}\left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2. \qquad (3.27)
\end{aligned}
$$

In view of (3.26) and (3.27), define $\mathcal{T}^n$ for $n \ge 1$ as

$$
\begin{aligned}
\mathcal{T}^n = {}& 8\left(\varepsilon^2\|\delta_t^+\delta_x^+\hat{e}^{\varepsilon,n}\|_2^2 + \frac{1}{2}\|\delta_x^2\hat{e}^{\varepsilon,n+1}\|_2^2 + \frac{1}{2}\|\delta_x^2\hat{e}^{\varepsilon,n}\|_2^2\right) \\
&+ \frac{1}{2\varepsilon^2}\left(\|\delta_x^+\hat{e}^{\varepsilon,n+1}\|_2^2 + \|\delta_x^+\hat{e}^{\varepsilon,n}\|_2^2\right).
\end{aligned} \qquad (3.28)
$$

Again, the Cauchy inequality with $\frac{1}{\varepsilon^2} \times (3.26) + 16 \times (3.27)$ will give that

$$
\mathcal{T}^n \lesssim \frac{1}{\varepsilon^2}\left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2 + \tau\sum_{m=1}^{n}\mathcal{T}^m, \quad 1 \le n \le \frac{T}{\tau} - 1. \qquad (3.29)
$$

Then the discrete Gronwall inequality [9, 13] will imply that for τ small enough,

$$
\mathcal{T}^n \lesssim \frac{1}{\varepsilon^2}\left(h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}\right)^2, \qquad 1 \le n \le \frac{T}{\tau} - 1. \qquad (3.30)
$$

Hence, the discrete semi-H^1 bounds for the error $\hat{e}^{\varepsilon,n}$ holds as

$$
\|\delta_x^+\hat{e}^{\varepsilon,n}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}, \qquad n \le \frac{T}{\tau}. \qquad (3.31)
$$

Step 2: *Prove* (2.23)*-type error bound for* $\hat{e}^{\varepsilon,n}$. For the approximation $v^{\varepsilon,n} \in X_M$ defined in (3.9), introduce the "biased error" function $\tilde{e}^{\varepsilon,n} \in X_M$, i.e., the difference between $v^{\varepsilon,n}$ and the solution $u(x,t_n)$ of NLS (2.2), for $j \in \mathcal{T}_M$ as

$$
\tilde{e}_j^{\varepsilon,n} = u(x_j,t_n) - v_j^{\varepsilon,n}, \quad n \ge 0. \qquad (3.32)
$$

Define the "local truncation error" $\tilde{\eta}^{\varepsilon,n} \in X_M$ for $n \ge 1$ and $j \in \mathcal{T}_M$ as

(3.33)

$$
\tilde{\eta}_j^{\varepsilon,n} := (i\delta_t - \varepsilon^2\delta_t^2 + f_B(|u(x_j,t_n)|^2))u(x_j,t_n) + \frac{1}{2}(\delta_x^2 u(x_j,t_{n+1}) + \delta_x^2 u(x_j,t_{n-1})).
$$

Similar to Lemma 3.2, we can prove that under the assumptions in Theorem 2.2,

$$
\|\tilde{\eta}^{\varepsilon,n}\|_2 + \|\delta_x^+\tilde{\eta}^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^2 + \varepsilon^2, \quad 1 \le n \le \frac{T}{\tau} - 1. \qquad (3.34)
$$

Subtracting (3.9) from (3.33), we obtain the error equation for $\tilde{e}^{\varepsilon,n} \in X_M$ as

$$\left(i\delta_t - \varepsilon^2\delta_t^2\right)\tilde{e}_j^{\varepsilon,n} + \frac{1}{2}(\delta_x^2\tilde{e}_j^{\varepsilon,n+1} + \delta_x^2\tilde{e}_j^{\varepsilon,n-1}) - \tilde{\eta}_j^{\varepsilon,n} + \tilde{\xi}_j^{\varepsilon,n} = 0, \tag{3.35}$$

where $\tilde{\xi}^{\varepsilon,n} \in X_M$ $(n \geq 1)$ is defined for $j \in \mathcal{T}_M$ as

$$\tilde{\xi}_j^{\varepsilon,n} = f_B(|v_j^{\varepsilon,n}|^2)\tilde{e}_j^{\varepsilon,n} + u(x_j,t_n)\left(f_B(|u(x_j,t_n)|^2) - f_B(|v_j^{\varepsilon,n}|^2)\right). \tag{3.36}$$

Then we have the following properties on $\tilde{\xi}^{\varepsilon,n}$, similar to Lemma 3.4:

$$|\tilde{\xi}_j^{\varepsilon,n}| \lesssim |\tilde{e}_j^{\varepsilon,n}|, \qquad |\delta_x^+\tilde{\xi}_j^{\varepsilon,n}| \lesssim |\tilde{e}_j^{\varepsilon,n}| + |\tilde{e}_{j+1}^{\varepsilon,n}| + |\delta_x^+\tilde{e}_j^{\varepsilon,n}|, \quad 0 \leq j \leq M-1,\ n \geq 1. \tag{3.37}$$

As shown in Lemma 3.3, we have $\tilde{e}^{\varepsilon,0} = \mathbf{0}$ and

$$\|\tilde{e}^{\varepsilon,1}\|_2 + \|\delta_x^+\tilde{e}^{\varepsilon,1}\|_2 + \|\delta_x^2\tilde{e}^{\varepsilon,1}\|_2 \lesssim h^2 + \tau^2 + \varepsilon^2, \quad \|\delta_t^+\tilde{e}^{\varepsilon,0}\|_2 + \|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,0}\|_2 \lesssim 1. \tag{3.38}$$

From error equation (3.35), multiplying both sides of (3.35) by $\overline{\tilde{e}_j^{\varepsilon,n+1} + \tilde{e}_j^{\varepsilon,n-1}}$ and summing for $j \in \mathcal{T}_M$, using summation by parts formula, and taking imaginary parts, we have

$$\begin{aligned}&\|\tilde{e}^{\varepsilon,n+1}\|_2^2 + 4\varepsilon^2\,\mathrm{Im}\left(\tilde{e}^{\varepsilon,n}, \delta_t^+\tilde{e}^{\varepsilon,n}\right) - \left\{\|\tilde{e}^{\varepsilon,n-1}\|_2^2 + 4\varepsilon^2\,\mathrm{Im}\left(\tilde{e}^{\varepsilon,n-1}, \delta_t^+\tilde{e}^{\varepsilon,n-1}\right)\right\}\\ &\quad = -2\tau\,\mathrm{Im}\left(\tilde{\xi}^{\varepsilon,n} - \tilde{\eta}^{\varepsilon,n}, \tilde{e}^{\varepsilon,n+1} + \tilde{e}^{\varepsilon,n-1}\right), \quad n \geq 1.\end{aligned} \tag{3.39}$$

Adding (3.39) for $1, 2, \ldots, n$ $(n \leq \frac{T}{\tau} - 1)$, similar to the proof of (3.19) for $\hat{e}^{\varepsilon,n}$, we have

$$\|\tilde{e}^{\varepsilon,n+1}\|_2^2 + \|\tilde{e}^{\varepsilon,n}\|_2^2 + 4\varepsilon^2\mathrm{Im}\left(\tilde{e}^n, \delta_t^+\tilde{e}^{\varepsilon,n}\right) \lesssim n\tau\left(h^2 + \tau^2 + \varepsilon^2\right)^2 + \tau\sum_{m=1}^{n+1}\|\tilde{e}^{\varepsilon,m}\|_2^2. \tag{3.40}$$

Multiplying both sides of (3.35) by $\overline{\tilde{e}_j^{\varepsilon,n+1} - \tilde{e}_j^{\varepsilon,n-1}}$ and summing for $j \in \mathcal{T}_M$, using the summation by parts formula, and taking real parts, we have

$$\begin{aligned}&-\left(\varepsilon^2\|\delta_t^+\tilde{e}^{\varepsilon,n}\|_2^2 + \frac{1}{2}\|\delta_x^+\tilde{e}^{\varepsilon,n+1}\|_2^2\right) + \left(\varepsilon^2\|\delta_t^+\tilde{e}^{\varepsilon,n-1}\|_2^2 + \frac{1}{2}\|\delta_x^+\tilde{e}^{\varepsilon,n-1}\|_2^2\right)\\ &\quad = -\mathrm{Re}\left(\tilde{\xi}^{\varepsilon,n} - \tilde{\eta}^{\varepsilon,n}, \tilde{e}^{\varepsilon,n+1} - \tilde{e}^{\varepsilon,n-1}\right), \qquad n \geq 1.\end{aligned} \tag{3.41}$$

Noticing that

$$\begin{aligned}\left|\mathrm{Re}\left(\tilde{\xi}^{\varepsilon,n} - \tilde{\eta}^{\varepsilon,n}, \tilde{e}^{\varepsilon,n+1} - \tilde{e}^{\varepsilon,n-1}\right)\right| &= \tau\left|\mathrm{Re}\left(\tilde{\xi}^{\varepsilon,n} - \tilde{\eta}^{\varepsilon,n}, \delta_t^+\tilde{e}^{\varepsilon,n} + \delta_t^+\tilde{e}^{\varepsilon,n-1}\right)\right|\\ &\leq \frac{C\tau}{\varepsilon^2}\left((h^2 + \tau^2 + \varepsilon^2)^2 + \|\tilde{e}^{\varepsilon,n}\|_2^2\right)\\ &\quad + \frac{1}{2}\tau\varepsilon^2\left(\|\delta_t^+\tilde{e}^{\varepsilon,n-1}\|_2^2 + \|\delta_t^+\tilde{e}^{\varepsilon,n}\|_2^2\right),\end{aligned}$$

summing (3.41) for $1, 2, \ldots, n$ and making use of (3.38), we have

$$\begin{aligned}&\varepsilon^2\|\delta_t^+\tilde{e}^{\varepsilon,n}\|_2^2 + \frac{1}{2}\|\delta_x^+\tilde{e}^{\varepsilon,n+1}\|_2^2 + \frac{1}{2}\|\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2 \leq \tau\sum_{m=1}^{n}\varepsilon^2\|\delta_t^+\tilde{e}^{\varepsilon,m}\|_2^2 + \tau\sum_{m=1}^{n+1}\frac{1}{\varepsilon^2}\|\tilde{e}^{\varepsilon,m}\|_2^2\\ &\qquad + n\tau\frac{C}{\varepsilon^2}(h^2 + \tau^2 + \varepsilon^2)^2 + C\varepsilon^2 + C(h^2 + \tau^2 + \varepsilon^2)^2, \quad 1 \leq n \leq \frac{T}{\tau} - 1.\end{aligned} \tag{3.42}$$

Let

$$\mathcal{E}^n = 8\left(\varepsilon^2\|\delta_t^+\tilde{e}^{\varepsilon,n}\|_2^2 + \frac{1}{2}\|\delta_x^+\tilde{e}^{\varepsilon,n+1}\|_2^2 + \frac{1}{2}\|\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2\right) + \frac{1}{2\varepsilon^2}\left(\|\tilde{e}^{\varepsilon,n+1}\|_2^2 + \|\tilde{e}^{\varepsilon,n}\|_2^2\right),\ n \geq 1; \tag{3.43}$$

then similar to the case of $\hat{e}^{\varepsilon,n}$, using the Cauchy inequality together with (3.42) and (3.40), we have

$$\mathcal{E}^n \lesssim \frac{1}{\varepsilon^2}(h^2+\tau^2+\varepsilon^2)^2 + \tau\sum_{m=1}^{n}\mathcal{E}^m, \tag{3.44}$$

and the discrete Gronwall inequality [9, 13] will imply for small τ

$$\mathcal{E}^n \lesssim \frac{1}{\varepsilon^2}(h^2+\tau^2+\varepsilon^2)^2, \quad 1 \leq n \leq \frac{T}{\tau} - 1. \tag{3.45}$$

Hence the l^2 estimate holds

$$\|\tilde{e}^{\varepsilon,n}\|_2 \lesssim h^2+\tau^2+\varepsilon^2, \quad n \leq \frac{T}{\tau}. \tag{3.46}$$

To prove the corresponding discrete semi-H^1 error estimates, multiplying both sides of (3.35) by $\delta_x^2(\overline{\tilde{e}_j^{\varepsilon,n+1}} + \overline{\tilde{e}_j^{\varepsilon,n-1}})$, summing together for $j = 1, 2, \dots, M-1$, summation by parts, taking imaginary parts of both sides, and making use of the l^2 estimates and (3.34), we then have for $1 \leq n \leq \frac{T}{\tau} - 1$

$$\begin{aligned}
&\|\delta_x^+\tilde{e}^{\varepsilon,n+1}\|_2^2 + 4\varepsilon^2\mathrm{Im}\left(\delta_x^+\tilde{e}^{\varepsilon,n}, \delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n}\right) - \|\delta_x^+\tilde{e}^{\varepsilon,n-1}\|_2^2 \\
&\qquad - 4\varepsilon^2\mathrm{Im}\left(\delta_x^+\tilde{e}^{\varepsilon,n-1}, \delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n-1}\right) \\
&\quad = -2\tau\,\mathrm{Im}\left\langle\tilde{\xi}^{\varepsilon,n} - \tilde{\eta}^{\varepsilon,n}, \delta_x^2(\tilde{e}^{\varepsilon,n+1} + \tilde{e}^{\varepsilon,n-1})\right\rangle \\
&\quad = 2\tau\,\mathrm{Im}\left(\delta_x^+\tilde{\xi}^{\varepsilon,n} - \delta_x^+\tilde{\eta}^{\varepsilon,n}, \delta_x^+(\tilde{e}^{\varepsilon,n+1} + \tilde{e}^{\varepsilon,n-1})\right) \\
&\quad \leq C\tau(\|\delta_x^+\tilde{e}^{\varepsilon,n+1}\|_2^2 + \|\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2 + \|\delta_x^+\tilde{e}^{\varepsilon,n-1}\|_2^2 + \|\tilde{e}^{\varepsilon,n}\|_2^2) + C\tau(h^2+\tau^2+\varepsilon^2)^2 \\
&\quad \leq C\tau\left(\|\delta_x^+\tilde{e}^{\varepsilon,n+1}\|_2^2 + \|\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2 + \|\delta_x^+\tilde{e}^{\varepsilon,n-1}\|_2^2\right) + C\tau(h^2+\tau^2+\varepsilon^2)^2.
\end{aligned}$$

Adding the above inequalities together for time steps $1, 2, \dots, n$, using Lemma 3.3, we have

$$\|\delta_x^+\tilde{e}^{\varepsilon,n+1}\|_2^2 + \|\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2 + 4\varepsilon^2\,\mathrm{Im}\left(\delta_x^+\tilde{e}^{\varepsilon,n}, \delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n}\right) \lesssim (h^2+\tau^2+\varepsilon^2)^2 + \tau\sum_{m=1}^{n+1}\|\delta_x^+\tilde{e}^{\varepsilon,m}\|_2^2. \tag{3.47}$$

Multiplying both sides of (3.35) by $\overline{\delta_x^2(\tilde{e}_j^{\varepsilon,n+1} - \tilde{e}_j^{\varepsilon,n-1})}$, summing together for $j = 1, 2, \dots, M-1$, summation by parts, taking real parts of both sides, and making use of the l^2 estimates and (3.34), we get for $1 \le n \le \frac{T}{\tau} - 1$

$$
\begin{aligned}
&\varepsilon^2\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2 + \frac{1}{2}\|\delta_x^2\tilde{e}^{\varepsilon,n+1}\|_2^2 - \varepsilon^2\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n-1}\|_2^2 - \frac{1}{2}\|\delta_x^2\tilde{e}^{\varepsilon,n-1}\|_2^2 \\
&\quad = \mathrm{Re}\left\langle \tilde{\xi}^{\varepsilon,n} - \tilde{\eta}^{\varepsilon,n}, \delta_x^2(\tilde{e}^{\varepsilon,n+1} - \tilde{e}^{\varepsilon,n-1})\right\rangle \\
&\quad = -\mathrm{Re}\left(\delta_x^+\tilde{\xi}^{\varepsilon,n} - \delta_x^+\tilde{\eta}^{\varepsilon,n}, \delta_x^+(\tilde{e}^{\varepsilon,n+1} - \tilde{e}^{\varepsilon,n-1})\right) \\
&\quad = -\tau\mathrm{Re}\left(\delta_x^+\tilde{\xi}^{\varepsilon,n} - \delta_x^+\tilde{\eta}^{\varepsilon,n}, \delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n} + \delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n-1}\right) \\
&\quad \le C\tau\left(\varepsilon^2\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2 + \varepsilon^2\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n-1}\|_2^2\right) + \frac{C\tau}{\varepsilon^2}\left[\|\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2 + (h^2+\tau^2+\varepsilon^2)^2\right].
\end{aligned}
$$

Summing the above inequalities for time steps $1, 2, \dots, n$ and using Lemma 3.3 on the errors of $\|\delta_x^2\tilde{e}^{\varepsilon,1}\|_2$ and $\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,0}\|_2$, we have for $1 \le n \le \frac{T}{\tau} - 1$

$$
\begin{aligned}
\varepsilon^2\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2 + \frac{1}{2}\|\delta_x^2\tilde{e}^{\varepsilon,n+1}\|_2^2 + \frac{1}{2}\|\delta_x^2\tilde{e}^{\varepsilon,n}\|_2^2 &\le \varepsilon^2\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,0}\|_2^2 + \frac{1}{2}\|\delta_x^2\tilde{e}^{\varepsilon,1}\|_2^2 \\
&\quad + C\varepsilon^2\tau\sum_{m=1}^{n}\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,m}\|_2^2 + \frac{\tau}{\varepsilon^2}\sum_{m=1}^{n+1}\|\delta_x^+\tilde{e}^{\varepsilon,m}\|_2^2 + \frac{n\tau}{\varepsilon^2}C(h^2+\tau^2+\varepsilon^2)^2 \\
&\lesssim \varepsilon^2\tau\sum_{m=1}^{n}\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,m}\|_2^2 + \frac{\tau}{\varepsilon^2}\sum_{m=1}^{n+1}\|\delta_x^+\tilde{e}^{\varepsilon,m}\|_2^2 + \frac{1}{\varepsilon^2}(h^2+\tau^2+\varepsilon^2)^2.
\end{aligned} \tag{3.48}
$$

As before, define $\tilde{\mathcal{E}}^n$ for $n \ge 1$ as

$$
\begin{aligned}
\tilde{\mathcal{E}}^n = 8&\left(\varepsilon^2\|\delta_t^+\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2 + \frac{1}{2}\|\delta_x^2\tilde{e}^{\varepsilon,n+1}\|_2^2 + \frac{1}{2}\|\delta_x^2\tilde{e}^{\varepsilon,n}\|_2^2\right) \\
&+ \frac{1}{2\varepsilon^2}\left(\|\delta_x^+\tilde{e}^{\varepsilon,n+1}\|_2^2 + \|\delta_x^+\tilde{e}^{\varepsilon,n}\|_2^2\right);
\end{aligned} \tag{3.49}
$$

combining $\frac{1}{\varepsilon^2} \times (3.47) + 16 \times (3.48)$ and applying the Cauchy inequality, we get

$$
\tilde{\mathcal{E}}^n \lesssim \frac{1}{\varepsilon^2}(h^2+\tau^2+\varepsilon^2)^2 + \tau\sum_{m=1}^{n}\tilde{\mathcal{E}}^m, \quad 1 \le n \le \frac{T}{\tau} - 1. \tag{3.50}
$$

The discrete Gronwall inequality [9, 13] implies that for small enough τ

$$
\tilde{\mathcal{E}}^n \lesssim \frac{1}{\varepsilon^2}(h^2+\tau^2+\varepsilon^2)^2, \quad 1 \le n \le \frac{T}{\tau} - 1. \tag{3.51}
$$

Hence

$$
\|\delta_x^+\tilde{e}^{\varepsilon,n}\|_2 \lesssim h^2+\tau^2+\varepsilon^2, \quad 1 \le n \le \frac{T}{\tau}. \tag{3.52}
$$

Noticing

$$\hat{e}_j^{\varepsilon,n} = \tilde{e}_j^{\varepsilon,n} + (u^\varepsilon(x_j,t_n) - u(x_j,t_n)), \quad j \in \mathcal{T}_M,\ n \geq 0, \tag{3.53}$$

and assumption (B) which implies

$$\begin{aligned}\|u^\varepsilon(x_j,t_n) - u(x_j,t_n)\|_2 + \|\delta_x^+ u^\varepsilon(x_j,t_n) - \delta_x^+ u(x_j,t_n)\|_2 \\ \lesssim h^2 + \tau^2 + \varepsilon^2, \quad n \geq 0,\end{aligned} \tag{3.54}$$

and combining (3.46) and (3.52), we then conclude that

$$\|\hat{e}^{\varepsilon,n}\|_2 + \|\delta_x^+ \hat{e}^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^2 + \varepsilon^2, \quad 0 \leq n \leq \frac{T}{\tau}. \tag{3.55}$$

Step 3: *Obtain ε-uniform estimate* (2.24). From (3.25), (3.31), and (3.55), taking the minimum of ε^2 and $\frac{\varepsilon^2}{\varepsilon^{4-\alpha^*}}$ [12, 16], we get

$$\|\hat{e}^{\varepsilon,n}\|_2 + \|\delta_x^+ \hat{e}^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^{\frac{4}{6-\alpha^*}}, \quad 0 \leq n \leq \frac{T}{\tau}. \tag{3.56}$$

Noticing that $4/(6-\alpha^*) \geq \frac{2}{3}$, using the discrete Sobolev inequality [24]

$$\|\hat{e}^{\varepsilon,n}\|_\infty \leq C\|\delta_x^+ \hat{e}^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^{\frac{4}{6-\alpha^*}}. \tag{3.57}$$

When τ and h become sufficiently small, we have $\|\hat{e}^{\varepsilon,n}\|_\infty \leq 1$, and

$$\|v^{\varepsilon,n}\|_\infty \leq \|u^\varepsilon\|_{L^\infty(\Omega_T)} + \|\hat{e}^{\varepsilon,n}\|_\infty \leq \|u^\varepsilon\|_{L^\infty(\Omega_T)} + 1 \leq \sqrt{B}, \quad n \leq \frac{T}{\tau}. \tag{3.58}$$

Thus, using the properties of $f_B(s)$, scheme (3.9) collapses to SIFD (2.6), and $v^{\varepsilon,n}$ is the solution of SIFD (2.6). In other words, we have proved the results in Theorem 2.2 for SIFD (2.6). ◻

Remark 3.1. Here we emphasis that our approach can be extended to the higher dimensions, e.g., 2D and 3D directly. The key point is the discrete Sobolev inequality in 2D and 3D as

$$\|u_h\|_\infty \leq C|\ln h|\,\|u_h\|_{H_s^1}, \qquad \|v_h\|_\infty \leq Ch^{-1/2}\|v_h\|_{H_s^1}, \tag{3.59}$$

where u_h and v_h are 2D and 3D mesh functions with zero at the boundary, respectively, and the discrete semi-H^1 norm $\|\cdot\|_{H_s^1}$ and l^∞ norm $\|\cdot\|_\infty$ can be defined similarly as the discrete semi-H^1 norm and the l^∞ norm in (2.15). The same proof in 2D and 3D will lead to (3.56), and the above Sobolev inequalities will imply (3.58) by noticing that $4/(6-\alpha^*) \geq \frac{2}{3} > \frac{1}{2}$ and with the additional technical assumption $\tau \lesssim h$ in 2D and 3D.

4. Convergence of the CNFD scheme. In order to prove Theorem 2.1 for CNFD, again we first establish the following lemmas.

LEMMA 4.1 (conservation properties of CNFD). *For the CNFD scheme* (2.4) *with* (2.7) *and* (2.13), *for any mesh size $h > 0$, time step $\tau > 0$, and initial data (u_0, u_1^ε), it satisfies the mass and energy conservation laws in the discretized level, i.e., for $n \geq 0$,*

$$N_h^\varepsilon(u^{\varepsilon,n}) := \frac{1}{2}\left(\|u^{\varepsilon,n}\|_2^2 + \|u^{\varepsilon,n+1}\|_2^2\right) - 2\varepsilon^2 \operatorname{Im}(\delta_t^+ u^{\varepsilon,n}, u^{\varepsilon,n}) \equiv N_h^\varepsilon(u^{\varepsilon,0}), \tag{4.1}$$

$$\begin{aligned} E_h^\varepsilon(u^{\varepsilon,n}) &:= \varepsilon^2\|\delta_t^+ u^{\varepsilon,n}\|_2^2 + \frac{1}{2}\left(\|\delta_x^+ u^{\varepsilon,n}\|_2^2 + \|\delta_x^+ u^{\varepsilon,n+1}\|_2^2\right) \\ &\quad - \frac{h}{2}\sum_{j=0}^{M-1}\Big(F(|u_j^{\varepsilon,n}|^2) + F(|u_j^{\varepsilon,n+1}|^2)\Big) \\ &\equiv E_h^\varepsilon(u^{\varepsilon,0}). \end{aligned} \tag{4.2}$$

Proof. Follow the analogous arguments of the CNFD method for NLS [9, 13] and NLSW [14, 26]. We omit the details here for brevity. ∎

LEMMA 4.2 (solvability of the difference equations). *For any given $u^{\varepsilon,n-1}$ and $u^{\varepsilon,n}$, there exists a solution $u^{\varepsilon,n+1}$ of the CNFD discretization* (2.4) *with* (2.7). *In addition, if the nonlinear term $f(|z|^2)z$ ($z \in \mathbb{C}$) is global Lipschitz, i.e., there exists a constant $C > 0$ such that*

$$\left|f(|z_1|^2)z_1 - f(|z_2|^2)z_2\right| \leq C|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C}, \tag{4.3}$$

then there exists $\tau_0 > 0$ such that the solution is unique when $\tau < \tau_0$.

Proof. The proof is quite standard for NLSW [14, 26] and we omit it here for brevity. ∎

Denote the local truncation error $\zeta^{\varepsilon,n} \in X_M$ for CNFD (2.4) with (2.7) and (2.13) for $n \geq 1$ and $j \in \mathcal{T}_M$ as

$$\begin{aligned} \zeta_j^{\varepsilon,n} &:= (i\delta_t - \varepsilon^2\delta_t^2)u^\varepsilon(x_j, t_n) + \frac{1}{2}\left(\delta_x^2 u^\varepsilon(x_j, t_{n+1}) + \delta_x^2 u^\varepsilon(x_j, t_{n-1})\right) \\ &\quad + G(u^\varepsilon(x_j, t_{n+1}), u^\varepsilon(x_j, t_{n-1})). \end{aligned}$$

Similar to Lemma 3.2, we have the following results.

LEMMA 4.3 (local truncation error for CNFD). *Under assumption* (B), *assume $f \in C^3([0,\infty))$, and we have*

$$\|\zeta^{\varepsilon,n}\|_2 + \|\delta_x^+\zeta^{\varepsilon,n}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}, \quad 1 \leq n \leq \frac{T}{\tau} - 1. \tag{4.4}$$

Proof. For $n \geq 1$ and $j \in \mathcal{T}_M$, expanding Taylor series for nonlinear part G at $|u^\varepsilon(x_j,t_n)|^2$ and noticing (2.5) and

$$\begin{aligned}
\Gamma_j^n &:= \frac{1}{\tau}\left(\left|u^\varepsilon(x_j,t_{n+1})\right|^2 - \left|u^\varepsilon(x_j,t_n)\right|^2\right) = \int_0^1 \partial_t(|u^\varepsilon|^2)(x_j,t_n+s\tau)\,ds,\\
\widetilde{\Gamma}_j^n &:= \frac{2}{\tau^2}\left(\frac{1}{2}\left(|u^\varepsilon(x_j,t_{n+1})|^2 + |u^\varepsilon(x_j,t_{n-1})|^2\right) - |u^\varepsilon(x_j,t_n)|^2\right)\\
&= \int_0^1\int_{-\theta}^{\theta} \partial_{tt}(|u^\varepsilon|^2)(x_j,t_n+s\tau)\,dsd\theta,
\end{aligned}$$

then applying the Taylor expansion and NLSW (2.1), we obtain

$$\begin{aligned}
\zeta_j^{\varepsilon,n} = {}& \frac{i\tau^2}{2}\int_0^1\int_0^\theta\int_{-s}^{s} u_{ttt}^\varepsilon(x_j,\sigma\tau+t_n)\,d\sigma dsd\theta\\
&- \varepsilon^2\tau^2\int_0^1\int_0^\theta\int_0^s\int_{-\sigma}^{\sigma} u_{tttt}^\varepsilon(x_j,s_1\tau+t_n)\,ds_1d\sigma dsd\theta\\
&+ \frac{h^2}{2}\int_0^1\int_0^\theta\int_0^s\int_{-\sigma}^{\sigma}\sum_{k=\pm1} u_{xxxx}^\varepsilon(x_j+s_1h,t_n+k\tau)ds_1d\sigma dsd\theta\\
&+ \frac{\tau^2}{2}\int_0^1\int_{-\theta}^{\theta} u_{xxtt}^\varepsilon(x_j,s\tau+t_n)dsd\theta\\
&+ \left(\tau^2\int_0^1\int_0^1(1-\sigma)(\theta\Gamma_j^n-(1-\theta)\Gamma_j^{n-1})^2 f''(\xi_j(\theta,\sigma))\,d\sigma d\theta\right.\\
&\qquad \left. + \frac{\tau^2}{2}f'(|u^\varepsilon(x_j,t_n)|^2)\widetilde{\Gamma}_j^n\right)\\
&\cdot \frac{1}{2}\left(u^\varepsilon(x_j,t_{n+1}) + u^\varepsilon(x_j,t_{n-1})\right)\\
&+ \frac{\tau^2}{2}f(|u^\varepsilon(x_j,t_n)|^2)\int_0^1\int_{-\theta}^{\theta} u_{tt}^\varepsilon(x_j,t_n+s\tau)\,dsd\theta,
\end{aligned}$$

where $\xi_j(\theta,\sigma) = \sigma(\theta|u^\varepsilon(x_j,t_{n+1})|^2 + (1-\theta)|u^\varepsilon(x_j,t_{n-1})|^2) + (1-\sigma)|u^\varepsilon(x_j,t_n)|^2$. Under assumption (B), using the triangle inequality, noticing that $f \in C^2([0,\infty))$, for $j \in \mathcal{T}_M$ and $n \geq 1$, we get

$$\begin{aligned}
|\zeta_j^{\varepsilon,n}| \lesssim{}& h^2\|\partial_{xxxx}u^\varepsilon\|_{L^\infty}\\
&+ \tau^2\Big(\|\partial_{ttt}u^\varepsilon\|_{L^\infty} + \varepsilon^2\|\partial_{tttt}u^\varepsilon\|_{L^\infty} + \|\partial_{tt}u^\varepsilon\|_{L^\infty}\|f(|u^\varepsilon|^2)\|_{L^\infty} + \|\partial_{xxtt}u^\varepsilon\|_{L^\infty}\\
&\quad + \left(\|\partial_t|u^\varepsilon|^2\|_{L^\infty}^2\|f''(|u^\varepsilon|^2)\|_{L^\infty} + \|f'(|u^\varepsilon|^2)\|_{L^\infty}\|\partial_{tt}|u^\varepsilon|^2\|_{L^\infty}\right)\|u^\varepsilon\|_{L^\infty}\Big)\\
\lesssim{}& h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}.
\end{aligned}$$

The first part of the lemma is proved. For $1 \leq j \leq M-1$, in view of the above representation of $\zeta_j^{\varepsilon,n}$ and a similar calculation as above, noticing $f \in C^3([0,\infty))$

when dealing with the nonlinear term G, for $1 \le j \le M-1$, it is easy to check that

$$\begin{aligned}
|\delta_x^+ \zeta_j^{\varepsilon,n}| \lesssim\, & h^2 \|\partial_{xxxxx} u^\varepsilon\|_{L^\infty} \\
& + \tau^2 \Bigg(\|\partial_{tttx} u^\varepsilon\|_{L^\infty} + \varepsilon^2 \|\partial_{ttttx} u^\varepsilon\|_{L^\infty} + \Big[\|\partial_{tt} u^\varepsilon\|_{L^\infty} \|f'(|u^\varepsilon|^2)\|_{L^\infty} \\
& \quad + (\|\partial_t |u^\varepsilon|^2\|_{L^\infty}^2 \|f'''(|u^\varepsilon|^2)\|_{L^\infty} + \|f''(|u^\varepsilon|^2)\|_{L^\infty} \|\partial_{tt}|u^\varepsilon|^2\|_{L^\infty}) \cdot \|u^\varepsilon\|_{L^\infty} \Big] \\
& \quad \cdot \|\partial_x |u^\varepsilon|^2\|_{L^\infty} + (\|\partial_x(\partial_t |u^\varepsilon|^2)^2\|_{L^\infty} \|f''(|u^\varepsilon|^2)\|_{L^\infty} \\
& \quad + \|f'(|u^\varepsilon|^2)\|_{L^\infty} \|\partial_{ttx} |u^\varepsilon|^2\|_{L^\infty}) \cdot \|u^\varepsilon\|_{L^\infty} \\
& \quad + \big(\|\partial_t |u^\varepsilon|^2\|_{L^\infty}^2 \|f''(|u^\varepsilon|^2)\|_{L^\infty} + \|f'(|u^\varepsilon|^2)\|_{L^\infty} \|\partial_{tt}|u^\varepsilon|^2\|_{L^\infty} \big) \\
& \quad \|\partial_x u^\varepsilon\|_{L^\infty} + \|\partial_{ttx} u^\varepsilon\|_{L^\infty} \|f(|u^\varepsilon|^2)\|_{L^\infty} + \|\partial_{xxxtt} u^\varepsilon\|_{L^\infty} \Bigg) \\
\lesssim\, & h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}.
\end{aligned}$$

For $j=0$ and $M-1$, we apply the boundary condition to deduce that $\frac{\partial^k}{\partial t^k} u^\varepsilon(x,t)|_{x\in\partial\Omega} = 0$ for $k \ge 0$, and (2.1) shows that $u_{xx}(x,t)|_{x\in\partial\Omega} = 0$ and $u_{xxxx}(x,t)|_{x\in\partial\Omega} = 0$. As before, we can get the above estimates for $j = 0, M-1$. Thus, we complete the proof. □

The error bounds for $e^{\varepsilon,n}$ at $n = 0,1$ are the same as Lemma 3.3 since the boundary and initial conditions for CNFD (2.4) and SIFD (2.6) are the same.

The proof for the CNFD scheme (2.4) is quite similar to that of the SIFD scheme, and we outline the schedule below, i.e., we prove the key lemmas.

Let $\hat{u}^{\varepsilon,0} = u^{\varepsilon,0}$, $\hat{u}^{\varepsilon,1} = u^{\varepsilon,1}$, and $\hat{u}^{\varepsilon,n+1} \in X_M$ $(n \ge 1)$ be given by

$$(4.5) \quad (i\delta_t - \varepsilon^2\delta_t^2)\hat{u}_j^{\varepsilon,n} + \frac{1}{2}\delta_x^2(\hat{u}_j^{\varepsilon,n+1} + \hat{u}_j^{\varepsilon,n-1}) + G_B(\hat{u}_j^{\varepsilon,n+1}, \hat{u}_j^{\varepsilon,n-1}) = 0, \quad j \in \mathcal{T}_M,$$

where $G_B(z_1,z_2)$ for $z_1, z_2 \in \mathbb{C}$ is given by

$$\begin{aligned}
G_B(z_1,z_2) &= \int_0^1 f_B(\theta|z_1|^2 + (1-\theta)|z_2|^2)\, d\theta \cdot g_{_B}\left(\frac{z_1+z_2}{2}\right) \\
&= \frac{F_B(|z_1|^2) - F_B(|z_2|^2)}{|z_1|^2 - |z_2|^2} \cdot g_{_B}\left(\frac{z_1+z_2}{2}\right)
\end{aligned}$$

with $g_{_B}(z)$, $f_B(\cdot)$ and $F_B(\cdot)$ being defined in (3.7). Actually $\hat{u}_j^{\varepsilon,n}$ can be viewed as another approximation of $u^\varepsilon(x_j,t_n)$. From Lemma 4.2, (4.5) is uniquely solvable for small τ. Define the error $\chi^{\varepsilon,n} \in X_M$ for $n \ge 1$ as

$$(4.6) \qquad \chi_j^{\varepsilon,n} = u^\varepsilon(x_j,t_n) - \hat{u}_j^{\varepsilon,n}, \quad j \in \mathcal{T}_M,$$

and the local truncation error $\hat{\zeta}^{\varepsilon,n} \in X_M$ for $j \in \mathcal{T}_M$ and $n \ge 1$ as

$$\begin{aligned}
(4.7) \qquad \hat{\zeta}_j^{\varepsilon,n} :=\, & (i\delta_t - \varepsilon^2\delta_t^2)u^\varepsilon(x_j,t_n) + \frac{1}{2}\delta_x^2\left(u^\varepsilon(x_j,t_{n+1}) + u^\varepsilon(x_j,t_{n-1})\right) \\
& + G_B\left(u^\varepsilon(x_j,t_{n+1}), u^\varepsilon(x_j,t_{n-1})\right).
\end{aligned}$$

Similar to Lemma 3.2, we can prove that under the assumptions in Theorem 2.1,

$$(4.8) \qquad \|\hat{\zeta}^{\varepsilon,n}\|_2 + \|\delta_x^+\hat{\zeta}^{\varepsilon,n}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}, \quad 1 \le n \le \frac{T}{\tau} - 1,$$

and the estimate for $\|\hat{e}^{\varepsilon,1}\|_2 + \|\delta_x^+\hat{e}^{\varepsilon,1}\|_2$ is proved in Lemma 3.3.

Subtracting (4.5) from (4.7), we obtain

$$i\delta_t\chi_j^{\varepsilon,n} - \varepsilon^2\delta_t^2\chi_j^{\varepsilon,n} + \frac{1}{2}\delta_x^2\left(\chi_j^{\varepsilon,n+1} + \chi_j^{\varepsilon,n-1}\right) + \vartheta_j^{\varepsilon,n} - \hat{\zeta}_j^{\varepsilon,n} = 0, \quad j \in \mathcal{T}_M, \tag{4.9}$$

where $\vartheta^{\varepsilon,n} \in X_M$ is defined for $j \in \mathcal{T}_M$ and $n \geq 1$ as

$$\vartheta_j^{\varepsilon,n} = G_B\left(u^\varepsilon(x_j,t_{n+1}), u^\varepsilon(x_j,t_{n-1})\right) - G_B(\hat{u}_j^{\varepsilon,n+1}, \hat{u}_j^{\varepsilon,n-1}). \tag{4.10}$$

Then we have the following properties on $\vartheta^{\varepsilon,n}$.

LEMMA 4.4. *Under the assumptions in Theorem* 2.1, *for* $\vartheta^{\varepsilon,n} \in X_M$ $(n \geq 1)$ *in* (4.10), *we have*

$$\begin{aligned} &|\vartheta_j^{\varepsilon,n}| \lesssim |\chi_j^{\varepsilon,n+1}| + |\chi_j^{\varepsilon,n-1}|, \quad |\delta_x^+\vartheta_j^{\varepsilon,n}| \\ &\quad \lesssim \sum_{m=n-1,n+1}\left(|\chi_j^{\varepsilon,m}| + |\delta_x^+\chi_j^{\varepsilon,m}| + |\chi_{j+1}^{\varepsilon,m}|\right), \ 0 \leq j \leq M-1, \, n \geq 1. \end{aligned}$$

Proof. For $j \in \mathcal{T}_M^0$, $n \geq 1$ and $\theta \in [0,1]$, denote

$$\begin{aligned} &\rho_j^{\varepsilon,n}(\theta) = \theta|u^\varepsilon(x_j,t_{n+1})|^2 + (1-\theta)|u^\varepsilon(x_j,t_{n-1})|^2, \\ &\hat{\rho}_j^{\varepsilon,n}(\theta) = \theta|\hat{u}_j^{\varepsilon,n+1}|^2 + (1-\theta)|\hat{u}_j^{\varepsilon,n-1}|^2, \\ &\quad \mu_j^{\varepsilon,n} = \frac{1}{2}\left[u^\varepsilon(x_j,t_{n+1}) + u^\varepsilon(x_j,t_{n-1})\right], \quad \hat{\mu}_j^{\varepsilon,n} = \frac{1}{2}\left[\hat{u}_j^{\varepsilon,n+1} + \hat{u}_j^{\varepsilon,n-1}\right], \\ &\quad \pi_j^{\varepsilon,n} = |u^\varepsilon(x_j,t_n)| + |\hat{u}_j^{\varepsilon,n}|, \end{aligned}$$

using the definition of G_B, F_B, and $g_{_B}$, it is easy to get

$$\begin{aligned} \vartheta_j^{\varepsilon,n} = \mu_j^{\varepsilon,n}\int_0^1 &\left[f_B(\rho_j^{\varepsilon,n}(\theta)) - f_B(\hat{\rho}_j^{\varepsilon,n}(\theta))\right] d\theta \\ &+ \left[g_{_B}\left(\mu_j^{\varepsilon,n}\right) - g_{_B}\left(\hat{\mu}_j^{\varepsilon,n}\right)\right]\int_0^1 f_B(\hat{\rho}_j^{\varepsilon,n}(\theta))\, d\theta. \end{aligned}$$

Noticing the Lipschitz property of $f_B(s^2)$ and

$$\begin{aligned} \left|\sqrt{\rho_j^{\varepsilon,n}(\theta)} - \sqrt{\hat{\rho}_j^{\varepsilon,n}(\theta)}\right| &\leq \frac{\theta\pi_j^{\varepsilon,n+1}\left|\chi_j^{\varepsilon,n+1}\right| + (1-\theta)\pi_j^{\varepsilon,n-1}\left|\chi_j^{\varepsilon,n-1}\right|}{\sqrt{\rho_j^{\varepsilon,n}(\theta)} + \sqrt{\hat{\rho}_j^{\varepsilon,n}(\theta)}} \\ &\leq \sqrt{\theta}|\chi_j^{\varepsilon,n+1}| + \sqrt{1-\theta}|\chi_j^{\varepsilon,n-1}|, \end{aligned}$$

combined with the Lipschitz property of $g_{_B}(z)$, we can obtain

$$|\vartheta_j^{\varepsilon,n}| \lesssim |\chi_j^{\varepsilon,n+1}| + |\chi_j^{\varepsilon,n-1}|, \quad j \in \mathcal{T}_M^0. \tag{4.11}$$

Applying δ_x^+ to $\vartheta_j^{\varepsilon,n}$, we can obtain

$$\begin{aligned} \delta_x^+\vartheta_j^{\varepsilon,n} = g_{_B}(\hat{\mu}_j^{\varepsilon,n})&\int_0^1 \delta_x^+\left[f_B(\rho_j^{\varepsilon,n}(\theta)) - f_B(\hat{\rho}_j^{\varepsilon,n}(\theta))\right] d\theta \\ &+ \left[g_{_B}(\mu_j^{\varepsilon,n}) - g_{_B}(\hat{\mu}_j^{\varepsilon,n})\right]\int_0^1 \delta_x^+ f_B(\rho_j^{\varepsilon,n}(\theta))\, d\theta \\ &+ \delta_x^+ g_{_B}(\hat{\mu}_j^{\varepsilon,n})\int_0^1 \left[f_B(\rho_{j+1}^{\varepsilon,n}(\theta)) - f_B(\hat{\rho}_{j+1}^{\varepsilon,n}(\theta))\right] d\theta \\ &+ \delta_x^+\left[g_{_B}(\mu_j^{\varepsilon,n}) - g_{_B}(\hat{\mu}_j^{\varepsilon,n})\right]\int_0^1 f_B(\rho_{j+1}^{\varepsilon,n}(\theta))\, d\theta. \end{aligned}$$

First, for $\theta, s \in [0,1]$, and $n \geq 1$, we denote $\kappa_j^{\varepsilon,n}(\theta,s), \hat{\kappa}_j^{\varepsilon,n}(\theta,s)$ for $0 \leq j \leq M-1$ as

$$\kappa_j^{\varepsilon,n}(\theta,s) = s\rho_{j+1}^{\varepsilon,n}(\theta) + (1-s)\rho_j^{\varepsilon,n}(\theta), \quad \hat{\kappa}_j^{\varepsilon,n}(\theta,s) = s\hat{\rho}_{j+1}^{\varepsilon,n}(\theta) + (1-s)\hat{\rho}_j^{\varepsilon,n}(\theta). \tag{4.12}$$

Noticing that for $1 \leq j \leq M-1$

$$\begin{aligned}
&\delta_x^+ \left[f_B\left(\rho_j^{\varepsilon,n}(\theta)\right) - f_B\left(\hat{\rho}_j^{\varepsilon,n}(\theta)\right)\right] \\
&\quad = \left[\delta_x^+ \rho_j^{\varepsilon,n}(\theta) \int_0^1 f_B'\left(\kappa_j^{\varepsilon,n}(\theta,s)\right) ds - \delta_x^+ \hat{\rho}_j^{\varepsilon,n}(\theta) \int_0^1 f_B'(\hat{\kappa}_j^{\varepsilon,n}(\theta,s))\, ds\right] \\
&\quad = \int_0^1 \left[f_B'\left(\kappa_j^{\varepsilon,n}(\theta,s)\right) - f_B'\left(\hat{\kappa}_j^{\varepsilon,n}(\theta,s)\right)\right] \delta_x \rho_j^{\varepsilon,n}(\theta)\, ds \\
&\qquad + \int_0^1 f_B'\left(\hat{\kappa}_j^{\varepsilon,n}(\theta,s)\right)\left[\delta_x^+ \rho_j^{\varepsilon,n}(\theta) - \delta_x^+ \hat{\rho}_j^{\varepsilon,n}(\theta)\right] ds,
\end{aligned}$$

a careful calculation shows that

$$\begin{aligned}
&\delta_x^+ \left[\rho_j^{\varepsilon,n}(\theta) - \hat{\rho}_j^{\varepsilon,n}(\theta)\right] \\
&\quad = \theta\Bigg[2\,\mathrm{Re}\left(u^\varepsilon(x_j,t_{n+1})\delta_x^+ \overline{\chi_j^{\varepsilon,n+1}} + \overline{\chi_{j+1}^{\varepsilon,n+1}}\delta_x^+ u^\varepsilon(x_j,t_{n+1})\right) \\
&\qquad\quad - \chi_j^{\varepsilon,n+1}\delta_x^+ \overline{\chi_j^{\varepsilon,n+1}} - \overline{\chi_{j+1}^{\varepsilon,n+1}}\delta_x^+ \chi_j^{\varepsilon,n+1}\Bigg] \\
&\qquad + (1-\theta)\Bigg[2\,\mathrm{Re}\left(u^\varepsilon(x_j,t_{n-1})\delta_x^+ \overline{\chi_j^{\varepsilon,n-1}} + \overline{\chi_{j+1}^{\varepsilon,n-1}}\delta_x^+ u^\varepsilon(x_j,t_{n-1})\right) \\
&\qquad\qquad - \chi_j^{\varepsilon,n-1}\delta_x^+ \overline{\chi_j^{\varepsilon,n-1}} - \overline{\chi_{j+1}^{\varepsilon,n-1}}\delta_x^+ \chi_j^{\varepsilon,n-1}\Bigg],
\end{aligned}$$

and $\sqrt{1-\theta}|\chi_{j+1}^{\varepsilon,n-1}| \leq \sqrt{\hat{\rho}_{j+1}^{\varepsilon,n}(\theta)} + |u^\varepsilon(x_{j+1},t_{n-1})|$, $\sqrt{1-\theta}|\chi_j^{\varepsilon,n-1}| \leq \sqrt{\hat{\rho}_j^{\varepsilon,n}(\theta)} + |u^\varepsilon(x_j,t_{n-1})|$, as well as $\sqrt{\theta}|\chi_{j+1}^{\varepsilon,n+1}| \leq \sqrt{\hat{\rho}_{j+1}^{\varepsilon,n}(\theta)} + |u^\varepsilon(x_{j+1},t_{n+1})|$, $\sqrt{\theta}|\chi_j^{\varepsilon,n+1}| \leq \sqrt{\hat{\rho}_j^{\varepsilon,n}(\theta)} + |u^\varepsilon(x_j,t_{n+1})|$. Moreover, from the Lipschitz property of f_B (3.8), we have

$$\left|\int_0^1 f_B'(\hat{\kappa}_j^{\varepsilon,n}(\theta,s))\, ds\right| = \left|\frac{f_B\left(\hat{\rho}_{j+1}^{\varepsilon,n}(\theta)\right) - f_B\left(\hat{\rho}_j^{\varepsilon,n}(\theta)\right)}{\hat{\rho}_{j+1}^{\varepsilon,n}(\theta) - \hat{\rho}_j^{\varepsilon,n}(\theta)}\right| \leq \frac{C}{\sqrt{\hat{\rho}_{j+1}^{\varepsilon,n}(\theta)} + \sqrt{\hat{\rho}_j^{\varepsilon,n}(\theta)}}. \tag{4.13}$$

Recalling the boundedness of $\delta_x^+ \rho_j^{\varepsilon,n}(\theta)$, $g_{B}(\cdot)$ and $f_B'(\cdot)$ as well as the Lipschitz property of $f_B'(s^2)$, i.e., $|f_B'(s_1) - f_B'(s_2)| \leq C|\sqrt{s_1} - \sqrt{s_2}|$, and combining the proof for (4.11), we arrive at

$$\begin{aligned}
&\left|\int_0^1 \delta_x^+ \left[f_B\left(\rho_j^{\varepsilon,n}(\theta)\right) - f_B\left(\hat{\rho}_j^{\varepsilon,n}(\theta)\right)\right] d\theta \cdot g_{B}(\hat{\mu}_j^{\varepsilon,n})\right| \\
&\qquad \lesssim \sum_{m=n+1,n-1} \left(|\chi_j^{\varepsilon,m}| + |\chi_{j+1}^{\varepsilon,m}| + |\delta_x^+ \chi_j^{\varepsilon,m}|\right).
\end{aligned} \tag{4.14}$$

Second, from the property $g_{B}(\cdot) \in C_0^\infty$, we know

$$\begin{aligned}
|\delta_x^+ g_{B}(\hat{\mu}_j^{\varepsilon,n})| &\leq C\left|\delta_x^+ \hat{\mu}_j^{\varepsilon,n}\right| \\
&\leq C\left|\delta_x^+ \chi_j^{\varepsilon,n+1} + \delta_x^+ \chi_j^{\varepsilon,n-1} - \delta_x^+ u^\varepsilon(x_j,t_{n+1}) - \delta_x^+ u^\varepsilon(x_j,t_{n-1})\right|.
\end{aligned}$$

In view of the boundedness of $f_B(s)$ as well as the proof for (4.11), we get

$$\left|\int_0^1 \left[f_B\left(\rho_{j+1}^{\varepsilon,n}(\theta)\right) - f_B\left(\hat{\rho}_{j+1}^{\varepsilon,n}(\theta)\right)\right] d\theta \cdot \delta_x^+ g_{_B}(\hat{\mu}_j^{\varepsilon,n})\right| \lesssim \sum_{m=n-1,n+1} \left(|\chi_{j+1}^{\varepsilon,m}| + |\chi_j^{\varepsilon,m}| + |\delta_x^+ \chi_j^{\varepsilon,m}|\right). \tag{4.15}$$

Third, noticing $\delta_x^+ f_B\left(\rho_j^{\varepsilon,n}(\theta)\right)$ is bounded and $g_{_B}(z)$ is Lipschitz, we have

$$\left|\int_0^1 \delta_x^+ f_B\left(\rho_j^{\varepsilon,n}(\theta)\right) d\theta \cdot \left[g_{_B}(\mu_j^{\varepsilon,n}) - g_{_B}(\hat{\mu}_j^{\varepsilon,n})\right]\right| \lesssim |\chi_j^{\varepsilon,n+1}| + |\chi_j^{\varepsilon,n-1}|. \tag{4.16}$$

Finally, denoting $\sigma_j^n(\theta)$, $\hat{\sigma}_j^n(\theta)$ for $\theta \in [0,1]$ and $0 \le j \le M-1$ as

$$\sigma_j^n(\theta) = \theta\mu_{j+1}^{\varepsilon,n} + (1-\theta)\mu_j^{\varepsilon,n}, \quad \hat{\sigma}_j^n(\theta) = \theta\hat{\mu}_{j+1}^{\varepsilon,n} + (1-\theta)\hat{\mu}_j^{\varepsilon,n},$$

and recalling the definition of $\rho_{_B}(s)$ and $g_{_B}(z)$, we find that

$$\begin{aligned}
\delta_x^+\left(g_{_B}(\mu_j^{\varepsilon,n}) - g_{_B}(\hat{\mu}_j^{\varepsilon,n})\right) &= \delta_x^+\left[\rho_{_B}(|\mu_j^{\varepsilon,n}|^2)\mu_j^{\varepsilon,n} - \rho_{_B}(|\hat{\mu}_j^{\varepsilon,n}|^2)\hat{\mu}_j^{\varepsilon,n}\right] = I_1 + I_2 \\
\text{with } I_1 &= \int_0^1 \left[\delta_x^+\mu_j^{\varepsilon,n}\partial_z g_{_B}(\sigma_j^n(\theta)) - \delta_x^+\hat{\mu}_j^{\varepsilon,n}\partial_z g_{_B}\left(\hat{\sigma}_j^n(\theta)\right)\right] d\theta, \\
\partial_z g_{_B}(z) &= \rho_{_B}(|z|^2) + |z|^2\rho_{_B}'(|z|^2), \\
I_2 &= \int_0^1 \left[\delta_x^+\overline{\mu_j^{\varepsilon,n}}\partial_{\bar{z}} g_{_B}(\sigma_j^n(\theta)) - \delta_x^+\overline{\hat{\mu}_j^{\varepsilon,n}}\partial_{\bar{z}} g_{_B}\left(\hat{\sigma}_j^n(\theta)\right)\right] d\theta, \\
\partial_{\bar{z}} g_{_B}(z) &= z^2\rho_{_B}'(|z|^2).
\end{aligned}$$

Noticing $\delta_x^+\mu_j^{\varepsilon,n}$ is bounded and the C_0^∞ property of $\rho_{_B}(s)$, we know $\partial_z g_{_B}(z)$ is Lipschitz and

$$\begin{aligned}
|I_1| &\le \left|\int_0^1 \left(\partial_z g_{_B}(\sigma_j^n(\theta)) - \partial_z g_{_B}(\hat{\sigma}_j^n(\theta))\right)\delta_x^+\mu_j^{\varepsilon,n}\, d\theta\right| \\
&\quad + \left|\int_0^1 (\delta_x^+\mu_j^{\varepsilon,n} - \delta_x^+\hat{\mu}_j^{\varepsilon,n})\,\partial_z g_{_B}(\hat{\sigma}_j^n(\theta))\, d\theta\right| \\
&\lesssim \max_{\theta\in[0,1]}\left\{\big||\sigma_j^n(\theta)| - |\hat{\sigma}_j^n(\theta)|\big|\right\} + \left|\delta_x^+(\chi_j^{\varepsilon,n+1} + \chi_j^{\varepsilon,n-1})\right| \\
&\lesssim \sum_{m=n+1,n-1}\left(|\chi_j^{\varepsilon,m}| + |\chi_{j+1}^{\varepsilon,m}| + |\delta_x^+\chi_j^{\varepsilon,m}|\right).
\end{aligned}$$

In the same spirit, we can get the same estimates for $|I_2|$ and obtain

$$\left|\int_0^1 f_B\left(\rho_{j+1}^{\varepsilon,n}(\theta)\right) d\theta \cdot \delta_x^+\left[g_{_B}(\mu_j^{\varepsilon,n}) - g_{_B}(\hat{\mu}_j^{\varepsilon,n})\right]\right| \lesssim \sum_{m=n-1,n+1}\left(|\chi_j^{\varepsilon,m}| + |\delta_x^+\chi_j^{\varepsilon,m}| + |\chi_{j+1}^{\varepsilon,m}|\right). \tag{4.17}$$

Combining (4.14), (4.15), (4.16), and (4.17), we finally prove the lemma. ∎

Having Lemma 4.4, the local truncation error (4.8), and the initial error Lemma 3.3, following the analogous proof for SIFD, we could obtain

$$\|\chi^{\varepsilon,n}\|_2 + \|\delta_x^+\chi^{\varepsilon,n}\|_2 \lesssim h^2 + \frac{\tau^2}{\varepsilon^{4-\alpha^*}}, \quad n \le \frac{T}{\tau}. \tag{4.18}$$

To complete the proof, we have to prove the (2.20) type estimate for $\chi^{\varepsilon,n}$. It is a straightforward extension of the proof for SIFD and the proof for Lemma 4.4. More precisely, define

$$\tilde{\chi}_j^{\varepsilon,n} = u(x_j,t_n) - \hat{u}_j^{\varepsilon,n} = \chi_j^{\varepsilon,n} + u(x_j,t_n) - u^\varepsilon(x_j,t_n), \quad j \in \mathcal{T}_M^0, \quad n \ge 0, \tag{4.19}$$

and the local truncation error $\tilde{\zeta}^{\varepsilon,n} \in X_M$ for $n \ge 1$ and $j \in \mathcal{T}_M$ as

$$\begin{aligned} \tilde{\zeta}_j^{\varepsilon,n} := {} & (i\delta_t - \varepsilon^2\delta_t^2)u(x_j,t_n) + \frac{1}{2}\delta_x^2\left(u(x_j,t_{n+1}) + u(x_j,t_{n-1})\right) \\ & + G_B(u(x_j,t_{n+1}), u(x_j,t_{n-1})). \end{aligned} \tag{4.20}$$

Then we have

$$\|\tilde{\zeta}^{\varepsilon,n}\|_2 + \|\delta_x^+\tilde{\zeta}^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^2 + \varepsilon^2, \quad n \ge 1. \tag{4.21}$$

Subtracting (4.5) from (4.20), we obtain for $n \ge 1$

$$i\delta_t\tilde{\chi}_j^{\varepsilon,n} - \varepsilon^2\delta_t^2\tilde{\chi}_j^{\varepsilon,n} + \frac{1}{2}\delta_x^2\left(\tilde{\chi}_j^{\varepsilon,n+1} + \tilde{\chi}_j^{\varepsilon,n-1}\right) + \tilde{\vartheta}_j^{\varepsilon,n} - \tilde{\zeta}_j^{\varepsilon,n} = 0, \quad j \in \mathcal{T}_M, \tag{4.22}$$

where $\tilde{\vartheta}^{\varepsilon,n} \in X_M$ is given for $j \in \mathcal{T}_M$ and $n \ge 1$ as

$$\tilde{\vartheta}_j^{\varepsilon,n} = G_B\left(u(x_j,t_{n+1}), u(x_j,t_{n-1})\right) - G_B\left(\hat{u}_j^{\varepsilon,n+1}, \hat{u}_j^{\varepsilon,n-1}\right). \tag{4.23}$$

Then the following lemma holds and we omit the proof here.

LEMMA 4.5. *Under the assumptions in Theorem* 2.1, *for* $\tilde{\vartheta}^{\varepsilon,n} \in X_M$ $(n \ge 1)$ *in* (4.23), *we have*

$$\begin{aligned} |\tilde{\vartheta}_j^{\varepsilon,n}| & \lesssim |\tilde{\chi}_j^{\varepsilon,n+1}| + |\tilde{\chi}_j^{\varepsilon,n-1}|, \ |\delta_x^+\tilde{\vartheta}_j^{\varepsilon,n}| \\ & \lesssim \sum_{m=n-1,n+1} \left(|\tilde{\chi}_j^{\varepsilon,m}| + |\delta_x^+\tilde{\chi}_j^{\varepsilon,m}| + |\tilde{\chi}_{j+1}^{\varepsilon,m}|\right), \ 0 \le j \le M-1, \ n \ge 1. \end{aligned}$$

Following the analogous proof for the SIFD, in view of Lemma 4.5, local error (4.21), and initial error Lemma 3.3, and recalling assumption (B), we can derive that

$$\|\chi^{\varepsilon,n}\|_2 + \|\delta_x^+\chi^{\varepsilon,n}\|_2 \lesssim h^2 + \tau^2 + \varepsilon^2, \quad n \le \frac{T}{\tau}. \tag{4.24}$$

Proof of Theorem 2.1. Combining (4.18) and (4.24), analogous proof for SIFD applies and the conclusion follows. ∎

5. Numerical results. In this section, we report numerical results for both SIFD (2.6) and CNFD (2.4) schemes applied to NLSW (2.1) with $f(|u^\varepsilon|^2) = -|u^\varepsilon|^2$. The corresponding limiting NLS is the defocusing cubic NLS.

Table 5.1

Spatial error analysis for SIFD scheme (2.6) *with different* ε *and* h *with norm* $\|e\|_{H^1} = \|e\|_2 + \|\delta_x^+ e\|_2$. *The convergence rate is calculated as* $\log_2(\|e(2h)\|_{H^1}/\|e(h)\|_{H^1})$.

Case I, $\alpha = 2$							
$\alpha = 2$	$h = 1/2$	$h = 1/2^2$	$h = 1/2^3$	$h = 1/2^4$	$h = 1/2^5$	$h = 1/2^6$	$h = 1/2^7$
$\varepsilon = 1/2^2$	1.51E-1	4.05E-2	1.03E-2	2.57E-3	6.45E-4	1.60E-4	3.90E-5
		1.90	1.98	2.00	1.99	2.01	2.04
$\varepsilon = 1/2^3$	1.94E-1	5.35E-2	1.36E-2	3.41E-3	8.51E-4	2.10E-4	4.92E-5
		1.89	1.98	2.00	2.00	2.02	2.09
$\varepsilon = 1/2^4$	2.15E-1	6.05E-2	1.55E-2	3.88E-3	9.67E-4	2.39E-4	5.68E-5
		1.83	1.96	2.00	2.00	2.02	2.07
$\varepsilon = 1/2^5$	2.22E-1	6.29E-2	1.61E-2	4.04E-3	1.01E-3	2.49E-4	5.93E-5
		1.82	1.97	1.99	2.00	2.02	2.07
$\varepsilon = 1/2^6$	2.23E-1	6.36E-2	1.63E-2	4.08E-3	1.02E-3	2.52E-4	6.00E-5
		1.81	1.96	2.00	2.00	2.02	2.07
$\varepsilon = 1/2^7$	2.24E-1	6.37E-2	1.63E-2	4.10E-3	1.02E-3	2.52E-4	6.01E-5
		1.81	1.97	1.99	2.01	2.02	2.07
$\varepsilon = 1/2^{10}$	2.24E-1	6.38E-2	1.63E-2	4.10E-3	1.02E-3	2.53E-4	6.02E-5
		1.81	1.97	1.99	2.01	2.01	2.07
$\varepsilon = 1/2^{20}$	2.24E-1	6.38E-2	1.63E-2	4.10E-3	1.02E-3	2.53E-4	6.02E-5
		1.81	1.97	1.99	2.01	2.01	2.07

Case II, $\alpha = 0$							
$\alpha = 0$	$h = 1/2$	$h = 1/2^2$	$h = 1/2^3$	$h = 1/2^4$	$h = 1/2^5$	$h = 1/2^6$	$h = 1/2^7$
$\varepsilon = 1/2^2$	1.52E-1	4.09E-2	1.04E-2	2.60E-3	6.53E-4	1.62E-4	3.94E-5
		1.89	1.98	2.00	1.99	2.01	2.04
$\varepsilon = 1/2^3$	1.95E-1	5.36E-2	1.36E-2	3.41E-3	8.52E-4	2.10E-4	4.93E-5
		1.86	1.98	2.00	2.00	2.02	2.09
$\varepsilon = 1/2^4$	2.15E-1	6.05E-2	1.55E-2	3.88E-3	9.67E-4	2.39E-4	5.68E-5
		1.83	1.96	2.00	2.00	2.02	2.07
$\varepsilon = 1/2^5$	2.22E-1	6.29E-2	1.61E-2	4.04E-3	1.01E-3	2.49E-4	5.93E-5
		1.82	1.97	1.99	2.00	2.02	2.07
$\varepsilon = 1/2^6$	2.23E-1	6.36E-2	1.63E-2	4.08E-3	1.02E-3	2.52E-4	6.00E-5
		1.81	1.96	2.00	2.00	2.02	2.07
$\varepsilon = 1/2^7$	2.24E-1	6.37E-2	1.63E-2	4.10E-3	1.02E-3	2.52E-4	6.01E-5
		1.81	1.97	1.99	2.01	2.02	2.07
$\varepsilon = 1/2^{10}$	2.24E-1	6.38E-2	1.63E-2	4.10E-3	1.02E-3	2.53E-4	6.02E-5
		1.81	1.97	1.99	2.01	2.01	2.07
$\varepsilon = 1/2^{20}$	2.24E-1	6.38E-2	1.63E-2	4.10E-3	1.02E-3	2.53E-4	6.02E-5
		1.81	1.97	1.99	2.01	2.01	2.07

For the numerical tests, we choose $u_0(x) = \pi^{-1/4}e^{-x^2/2}$ and $w^\varepsilon(x) = e^{-x^2/2}$ in (2.1)–(2.3). The computational domain is chosen as $[a, b] = [-16, 16]$. The "exact" solution is computed with a very fine mesh $h = 1/512$ and time step $\tau = 10^{-6}$. We study the following two cases of initial data:

Case I, $\alpha = 2$, i.e., the well-prepared case,

Case II, $\alpha = 0$, i.e., the ill-prepared case.

We measure the error e_h at time $t = 1$ with the discrete H^1 norm $\|e_h\|_{H^1} = \|e_h\|_2 + \|\delta_x^+ e_h\|_2$.

Table 5.1 depicts spatial errors of SIFD for Cases I and II, for different h and ε, with fixed $\tau = 10^{-6}$, where the time step τ is so small that the temporal error can be neglected. From the table, we can conclude that SIFD is uniformly second order accurate in h for all ε. Tables 5.2 and 5.3 list temporal errors of SIFD for Cases I and II, for different ε and τ, with fixed $h = 1/512$. With this very fine mesh $h = 1/512$, the spatial error can be ignored. Table 5.2 shows that when τ is small

TABLE 5.2
Temporal error analysis for SIFD scheme (2.6) *with different* ε *and* τ *with norm* $\|e\|_{H^1}$.

Case I, $\alpha = 2$

$\alpha = 2$	$\tau = 0.1$	$\tau = \frac{0.1}{2}$	$\tau = \frac{0.1}{2^2}$	$\tau = \frac{0.1}{2^3}$	$\tau = \frac{0.1}{2^4}$	$\tau = \frac{0.1}{2^5}$	$\tau = \frac{0.1}{2^6}$	$\tau = \frac{0.1}{2^7}$	$\tau = \frac{0.1}{2^8}$
$\varepsilon = \frac{1}{2^2}$	1.10E-1	4.75E-2	1.49E-2	3.86E-3	9.70E-4	2.43E-4	6.10E-5	1.56E-5	4.47E-6
		1.21	1.67	1.95	1.99	2.00	1.99	1.97	1.80
$\varepsilon = \frac{1}{2^3}$	1.60E-1	5.06E-2	1.46E-2	5.45E-3	3.07E-3	8.27E-4	2.08E-4	5.21E-5	1.32E-5
		1.66	1.79	1.42	0.83	1.89	1.99	2.00	1.98
$\varepsilon = \frac{1}{2^4}$	1.98E-1	6.02E-2	1.85E-2	4.78E-3	1.25E-3	4.14E-4	3.74E-4	1.81E-4	4.70E-5
		1.72	1.70	1.95	1.94	1.59	0.15	1.05	1.95
$\varepsilon = \frac{1}{2^5}$	1.90E-1	7.30E-2	1.92E-2	5.00E-3	1.39E-3	3.49E-4	8.75E-5	2.74E-5	1.65E-5
		1.38	1.93	1.94	1.85	1.99	2.00	1.68	0.73
$\varepsilon = \frac{1}{2^6}$	1.89E-1	6.87E-2	2.18E-2	5.28E-3	1.32E-3	3.34E-4	9.09E-5	2.17E-5	5.72E-6
		1.46	1.66	2.06	2.00	1.98	1.88	2.07	1.92
$\varepsilon = \frac{1}{2^7}$	1.89E-1	6.79E-2	2.06E-2	5.81E-3	1.36E-3	3.38E-4	8.26E-5	2.20E-5	5.54E-6
		1.48	1.72	1.83	2.09	2.01	2.03	1.91	1.98
$\varepsilon = \frac{1}{2^{10}}$	1.89E-1	6.76E-2	2.01E-2	5.37E-3	1.37E-3	3.50E-4	9.27E-5	2.14E-5	5.31E-6
		1.48	1.75	1.90	1.97	1.97	1.92	2.11	2.01
$\varepsilon = \frac{1}{2^{20}}$	1.89E-1	6.76E-2	2.01E-2	5.37E-3	1.36E-3	3.42E-4	8.56E-5	2.14E-5	5.35E-6
		1.48	1.75	1.90	1.98	1.99	2.00	2.00	2.00

Case II, $\alpha = 0$

$\alpha = 0$	$\tau = 0.1$	$\tau = \frac{0.1}{2}$	$\tau = \frac{0.1}{2^2}$	$\tau = \frac{0.1}{2^3}$	$\tau = \frac{0.1}{2^4}$	$\tau = \frac{0.1}{2^5}$	$\tau = \frac{0.1}{2^6}$	$\tau = \frac{0.1}{2^7}$	$\tau = \frac{0.1}{2^8}$
$\varepsilon = \frac{1}{2^2}$	2.91E-1	1.39E-1	4.05E-2	1.04E-2	2.63E-3	6.59E-4	1.66E-4	4.54E-5	1.11E-5
		1.07	1.78	1.96	1.98	2.00	1.99	1.87	2.03
$\varepsilon = \frac{1}{2^3}$	1.76E-1	9.04E-2	6.52E-2	7.35E-2	3.30E-2	8.71E-3	2.19E-3	5.50E-4	1.38E-4
		0.96	0.47	-0.17	1.16	1.92	1.99	1.99	1.99
$\varepsilon = \frac{1}{2^4}$	1.96E-1	6.02E-2	2.10E-2	1.01E-2	1.98E-2	3.81E-3	1.92E-2	8.16E-3	2.11E-3
		1.70	1.52	1.06	-0.97	2.38	-2.33	1.23	1.95
$\varepsilon = \frac{1}{2^5}$	1.90E-1	7.26E-2	1.94E-2	6.11E-3	3.36E-3	3.61E-3	4.69E-3	1.01E-3	2.05E-3
		1.39	1.90	1.67	0.86	-0.10	-0.38	2.22	-1.02
$\varepsilon = \frac{1}{2^6}$	1.89E-1	6.87E-2	2.17E-2	5.32E-3	1.55E-3	8.15E-4	7.31E-4	1.39E-3	6.84E-4
		1.46	1.66	2.03	1.78	0.93	0.16	-0.93	1.02
$\varepsilon = \frac{1}{2^7}$	1.89E-1	6.78E-2	2.05E-2	5.81E-3	1.39E-3	4.37E-4	2.50E-4	2.03E-4	2.07E-4
		1.48	1.73	1.82	2.06	1.67	0.81	0.30	-0.03
$\varepsilon = \frac{1}{2^8}$	1.89E-1	6.77E-2	2.02E-2	5.48E-3	1.47E-3	3.46E-4	1.08E-4	6.21E-5	4.63E-5
		1.48	1.74	1.88	1.90	2.09	1.68	0.80	0.42
$\varepsilon = \frac{1}{2^9}$	1.89E-1	6.76E-2	2.02E-2	5.39E-3	1.39E-3	3.70E-4	8.70E-5	2.35E-5	1.60E-5
		1.48	1.74	1.91	1.96	1.91	2.09	1.89	0.55
$\varepsilon = \frac{1}{2^{10}}$	1.89E-1	6.76E-2	2.01E-2	5.37E-3	1.37E-3	3.50E-4	9.28E-5	2.22E-5	7.78E-6
		1.48	1.75	1.90	1.97	1.97	1.92	2.06	1.51
$\varepsilon = \frac{1}{2^{20}}$	1.89E-1	6.76E-2	2.01E-2	5.37E-3	1.36E-3	3.42E-4	8.56E-5	2.14E-5	5.35E-6
		1.48	1.75	1.90	1.98	1.99	2.00	2.00	2.00

(upper triangle part), the temporal error is of second order for each ε; when ε is small (lower triangle part), the temporal error is also of second order; near the diagonal part (for $\alpha = 2$, slightly upper), the degeneracy of the second order accuracy is observed. This confirms our error estimates (2.22) and (2.23) for SIFD. Table 5.3 presents the errors of SIFD at the degeneracy regime for $\alpha = 2$ in the regime $\tau \sim \varepsilon^2$, and resp., for $\alpha = 0$ in the regime $\tau \sim \varepsilon^3$, predicted by our error estimates. The results clearly demonstrate that SIFD converges at $O(h^2 + \tau)$ and $O(h^2 + \tau^{2/3})$ for $\alpha = 2$ and $\alpha = 0$, respectively. Similar tests were also carried out for CNFD and we obtain a similar conclusion; thus they are omitted here for brevity.

TABLE 5.3

Degeneracy of convergence rates for SIFD with $h = 1/512$*,* Case I *and* Case II*. The convergence rate is calculated as* $\log_2(\|e(2^2\tau, 2\varepsilon)\|_{H^1}/\|e(\tau,\varepsilon)\|_{H^1})/2$ *for* $\alpha = 2$ *(Case* I*) and* $\log_2(\|e(2^3\tau, 2\varepsilon)\|_{H^1}/\|e(\tau,\varepsilon)\|_{H^1})/3$ *for* $\alpha = 0$ *(Case* II*).*

$\alpha = 2$	$\varepsilon = 1$ $\tau = 0.2$	$\varepsilon = 1/2$ $\tau = 0.2/2^2$	$\varepsilon = 1/2^2$ $\tau = 0.2/2^4$	$\varepsilon = 1/2^3$ $\tau = 0.2/2^6$	$\varepsilon = 1/2^4$ $\tau = 0.2/2^8$
$\|e\|_{H^1}$	1.07E-1	1.77E-2 1.30	3.86E-3 1.10	8.27E-4 1.11	1.81E-4 1.10
$\alpha = 0$	$\varepsilon = 1/2^2$ $\tau = 0.1$	$\varepsilon = 1/2^3$ $\tau = 0.1/2^3$	$\varepsilon = 1/2^4$ $\tau = 0.1/2^6$	$\varepsilon = 1/2^5$ $\tau = 0.1/2^9$	$\varepsilon = 1/2^6$ $\tau = 0.1/2^{12}$
$\|e\|_{H^1}$	2.91E-1	7.35E-2 1.99/3	1.92E-2 1.94/3	4.83E-3 1.99/3	1.21E-3 2.00/3

6. Conclusion. We have analyzed the conservative CNFD method and the SIFD method for discretizing the NLSW with perturbation strength of the wave operator described by a dimensionless parameter ε ($0 < \varepsilon \leq 1$) in one, two, and three dimensions. The main difficulty in the analysis was that for $0 < \varepsilon \ll 1$, the solution of NLSW oscillated in time with $O(\varepsilon^2)$ wavelength at amplitude of order $O(\varepsilon^4)$ and $O(\varepsilon^2)$ for well-prepared and ill-prepared initial data, respectively. For both CNFD and SIFD, we established the uniform convergence rates in ε, at the order $O(h^2 + \tau)$ and $O(h^2 + \tau^{2/3})$ for well-prepared and ill-prepared initial data, respectively, in l^2-norm and discrete semi-H^1 norm with time step τ and mesh size h. Numerical results confirmed our theoretical analysis.

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