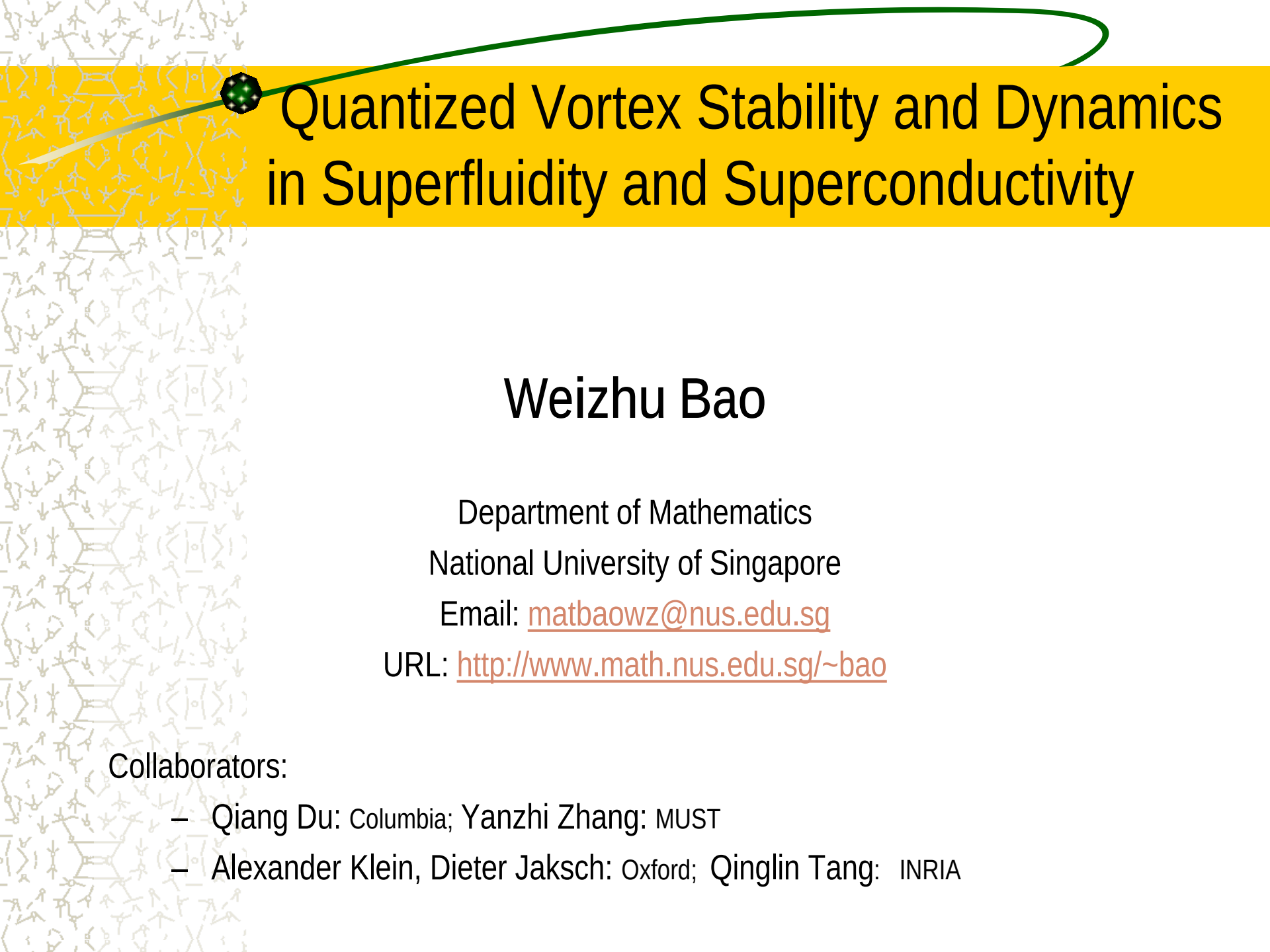




Quantized Vortex Stability and Dynamics in Superfluidity and Superconductivity

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National University of Singapore

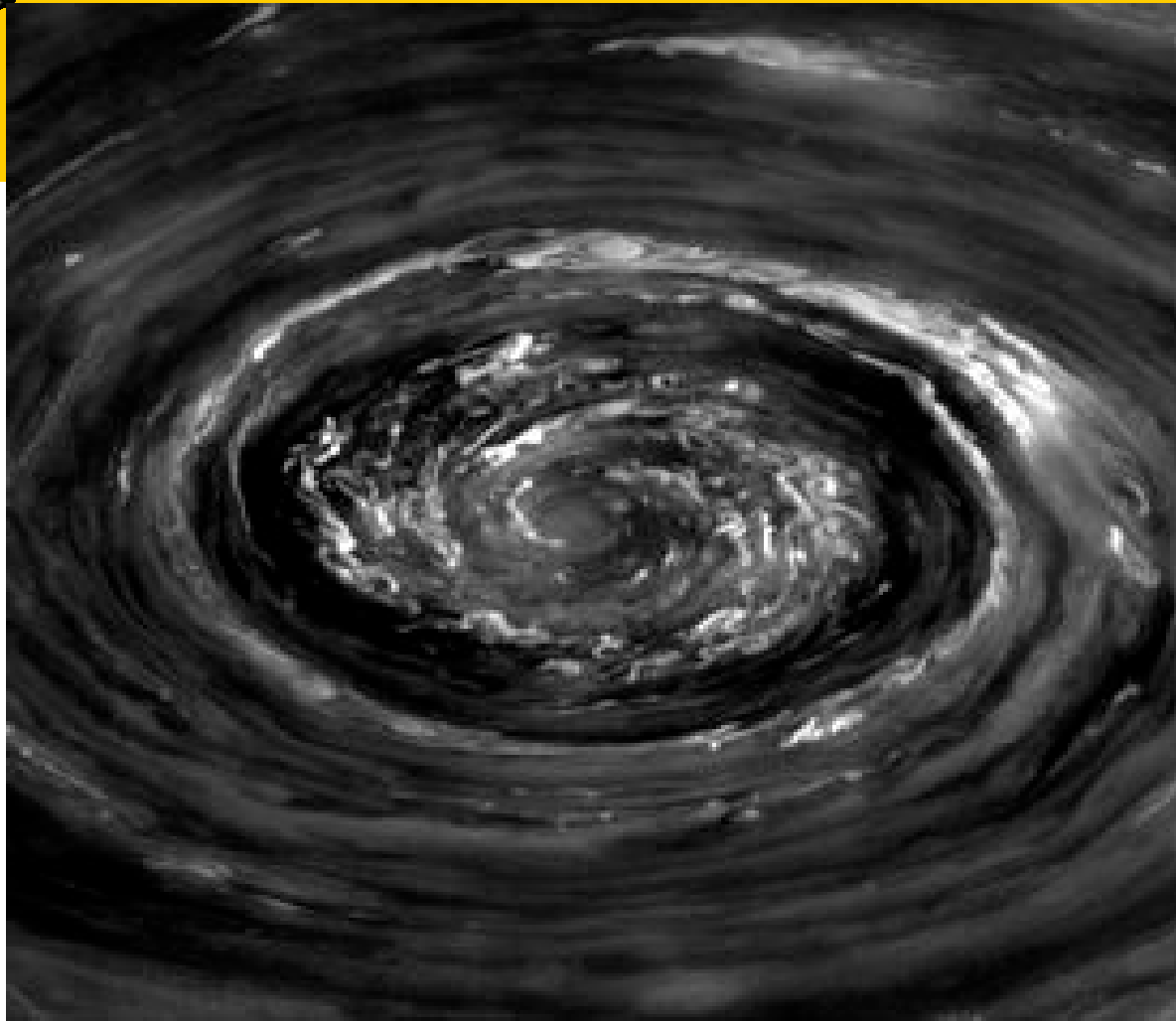
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Collaborators:

- Qiang Du: Columbia; Yanzhi Zhang: MUST
- Alexander Klein, Dieter Jaksch: Oxford; Qinglin Tang: INRIA

Vortex: From macroscale to microscale



Cloud vortex — Saturn hexagon at the north pole of the planet Saturn

http://en.wikipedia.org/wiki/Saturn%27s_hexagon

Vortex: From macroscale to microscale



Vortex galaxy – vortex in cosmos

Vortex: From macroscale to microscale



Tornado – Vortex in air

Vortex: From macroscale to microscale



Vortex in water – generated by a boat

Vortex: From macroscale to microscale



Vortex in water – generated by an airplane

Vortex: From macroscale to microscale



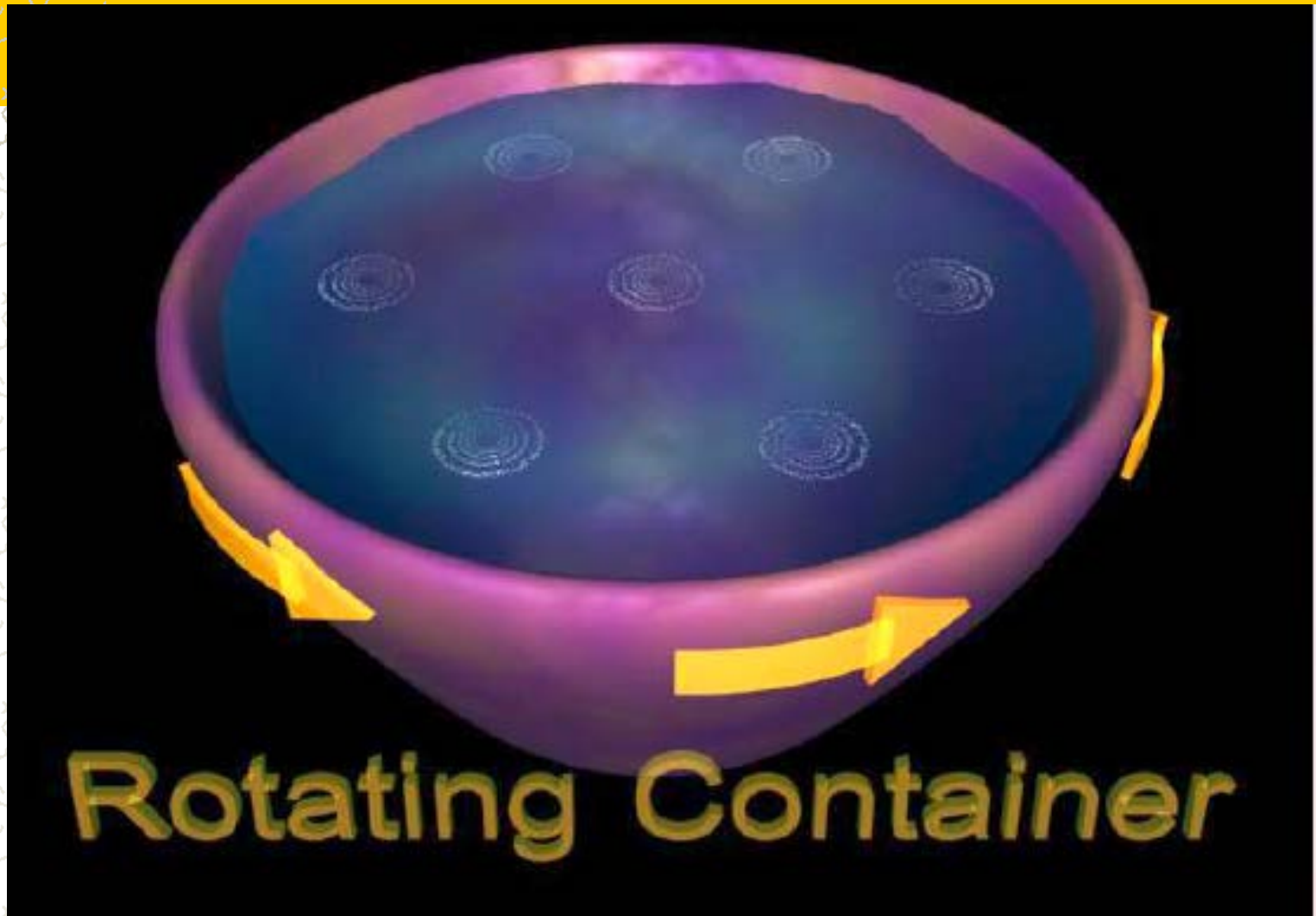
Magnetic vortex –vortex in plasma

Vortex: From macroscale to microscale



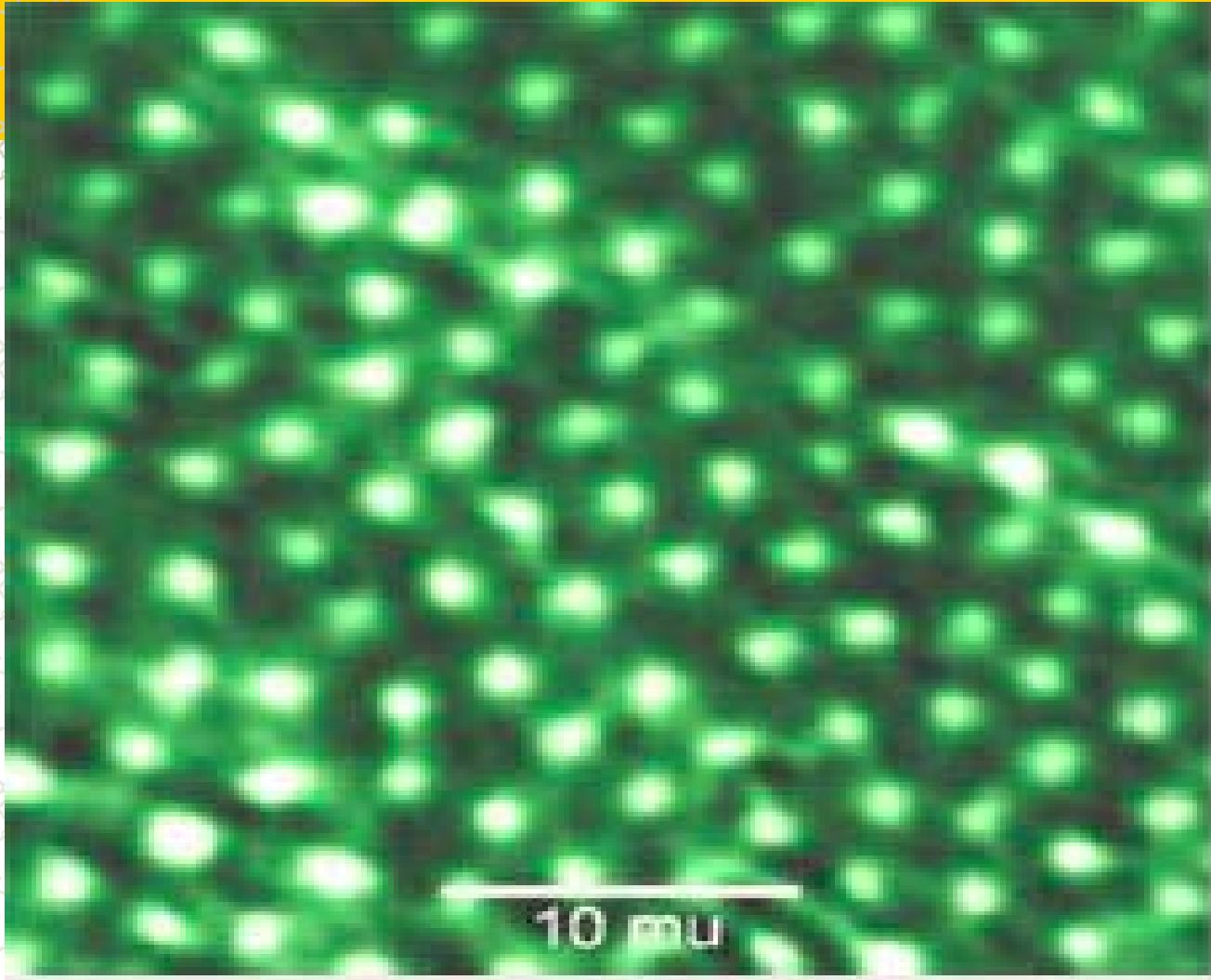
Vortex in plant

Vortex: From macroscale to microscale



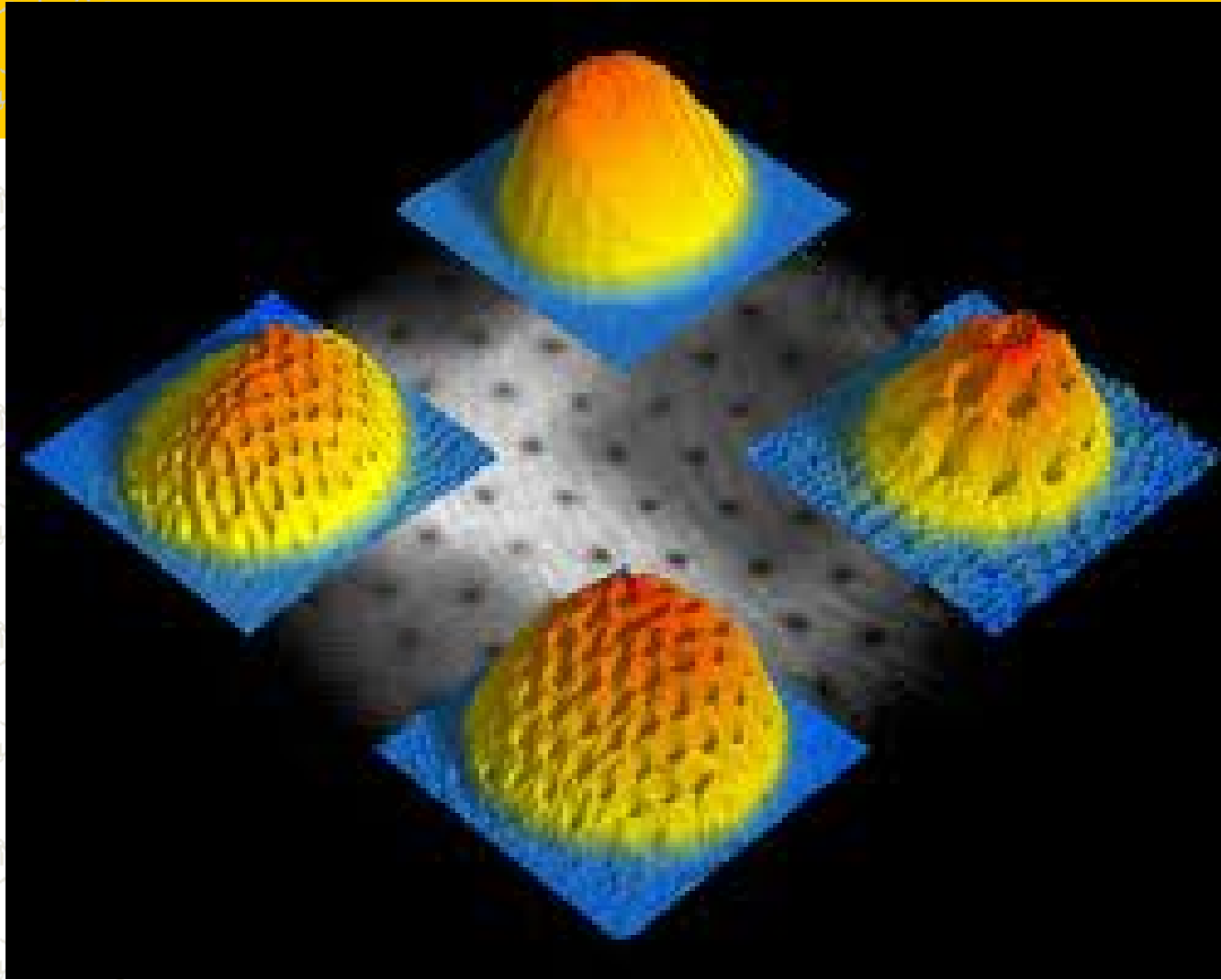
Quantized Vortex in liquid Helium 3

Vortex: From macroscale to microscale

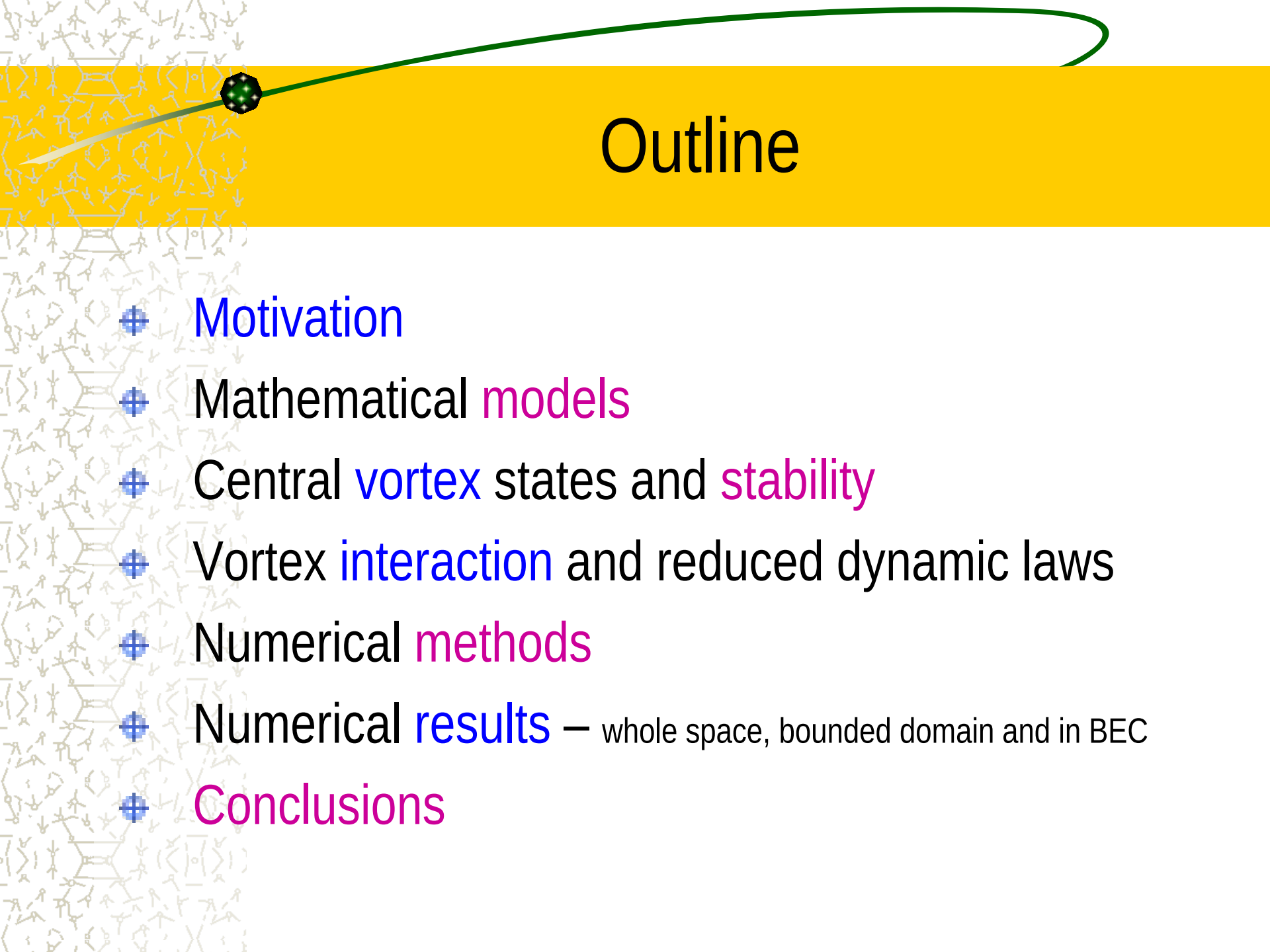


Quantized Vortex in a type-II superconductor

Vortex: From macroscale to microscale



Quantized Vortex in Bose-Einstein condensation (BEC)

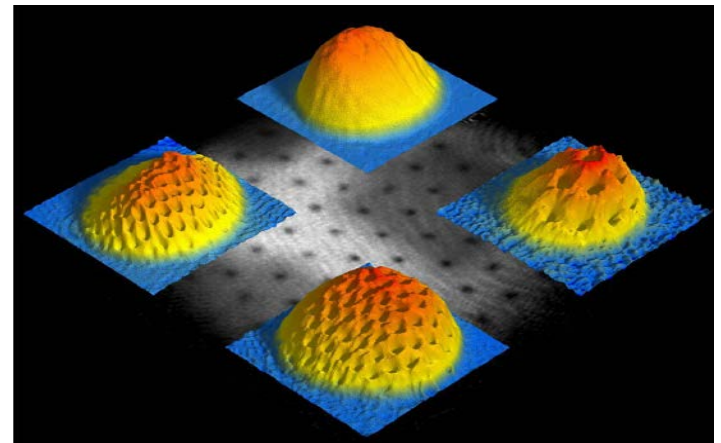
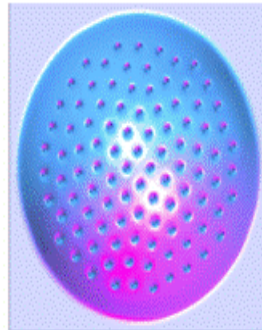
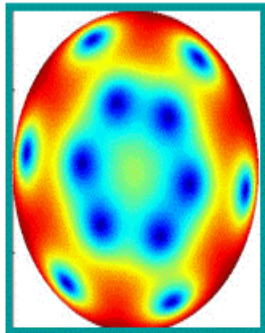
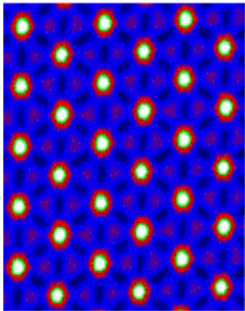


Outline

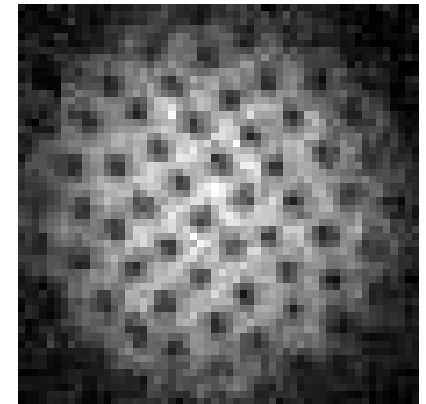
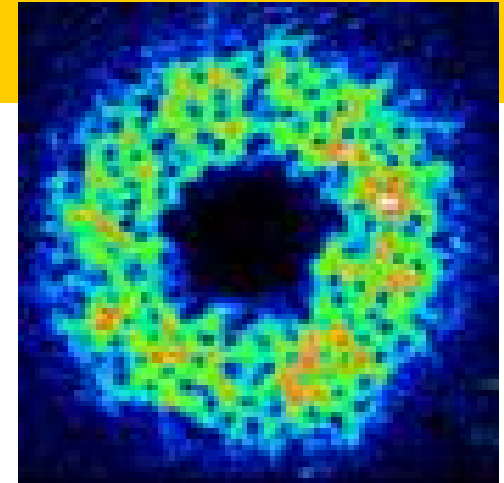
- ⊕ Motivation
- ⊕ Mathematical models
- ⊕ Central vortex states and stability
- ⊕ Vortex interaction and reduced dynamic laws
- ⊕ Numerical methods
- ⊕ Numerical results — whole space, bounded domain and in BEC
- ⊕ Conclusions

Motivation

- **Quantized Vortex:** Particle-like (topological) defect
 - Zero of the complex scalar field
 - localized phase singularities with integer topological charge:
 $\psi(\vec{x}_0) = 0, \quad \psi(\vec{x}) = \sqrt{\rho} e^{i\phi}, \quad \int_{\Gamma} d(\arg \psi) = \int_{\Gamma} d\phi = 2\pi n \neq 0$
 - Key of superfluidity: ability to support dissipationless flow



Motivation



- Existing in
 - Superconductors:
 - Ginzburg-Landau equations (GLE)
 - Liquid helium:
 - Two-fluid model
 - Gross-Pitaevskii equation (GPE)
 - Bose-Einstein condensation (BEC):
 - Nonlinear Schroedinger equation (NLSE) or GPE
 - Nonlinear optics & propagation of laser beams
 - Nonlinear Schroedinger equation (NLSE)
 - Nonlinear wave equation (NLWE)

Mathematical models

- Ginzburg-Landau equation (GLE): Superconductivity, nonlinear heat flow, etc.

$$\psi_t = \Delta \psi + \frac{1}{\varepsilon^2} \left(V(\vec{x}) - |\psi|^2 \right) \psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0,$$

- Gross-Pitaevskii equation (GPE): nonlinear optics, BEC, superfluidity

$$-i \psi_t = \Delta \psi + \frac{1}{\varepsilon^2} \left(V(\vec{x}) - |\psi|^2 \right) \psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0,$$

- Nonlinear wave equation (NLWE): wave motion

$$\psi_{tt} = \Delta \psi + \frac{1}{\varepsilon^2} \left(V(\vec{x}) - |\psi|^2 \right) \psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0,$$

– Here

ψ : complex-valued wave function or order parameter,

$\varepsilon > 0$: dimensionless constant,

$V(\vec{x})$: real-valued external potential

Mathematical models

- 'Free Energy' or Lyapunov functional:

$$E(\psi) = \int_{\mathbb{R}^2} \left[|\nabla \psi|^2 + \frac{1}{2\varepsilon^2} \left(V(\vec{x}) - |\psi|^2 \right)^2 \right] d\vec{x},$$

- Dispersive system (GPE): $-i \frac{\partial \psi}{\partial t} = -\frac{\delta E(\psi)}{\delta \psi^*}$

- Energy conservation: $E(\psi) \equiv E(\psi_0), \quad t \geq 0$

- Density conservation: $\int_{\mathbb{R}^2} \left(|\psi(\vec{x}, t)|^2 - |\psi_0(\vec{x})|^2 \right) d\vec{x} \equiv 0, \quad t \geq 0$

- Admits particle like solutions: solitons, kinks & vortices

- Dissipative system (GLE): $\frac{\partial \psi}{\partial t} = -\frac{\delta E(\psi)}{\delta \psi^*}$

- Energy diminishing, etc.

Two kinds of vortices

✶ 'Bright-tail' vortex: GLE, GPE & NLWE ('order parameter')

$$V(\vec{x}) \rightarrow 1, \quad \text{when } |\vec{x}| \rightarrow \infty, \Rightarrow |\psi(\vec{x}, t)| \rightarrow 1, \quad |\vec{x}| \rightarrow \infty, \quad (\text{e.g. } \psi \rightarrow e^{im\theta}),$$

– Time independent case (*Neu, 90'*): $\varepsilon = 1$, $V(\vec{x}) \equiv 1$

– Vortex solutions: $\psi(\vec{x}) = \phi_n(\vec{x}) = f_n(r) e^{in\theta}$

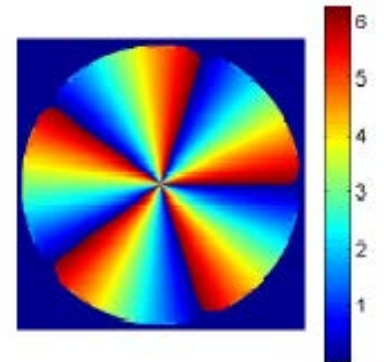
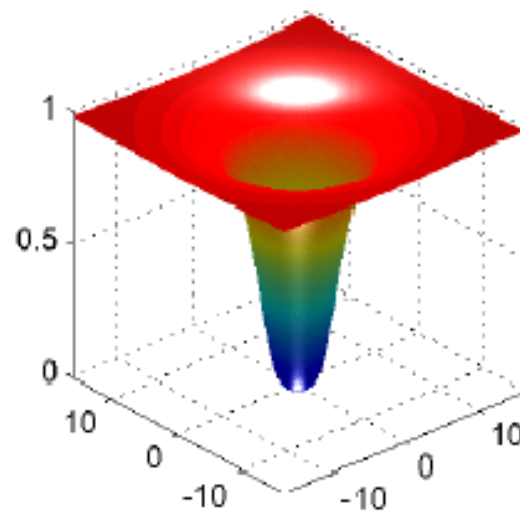
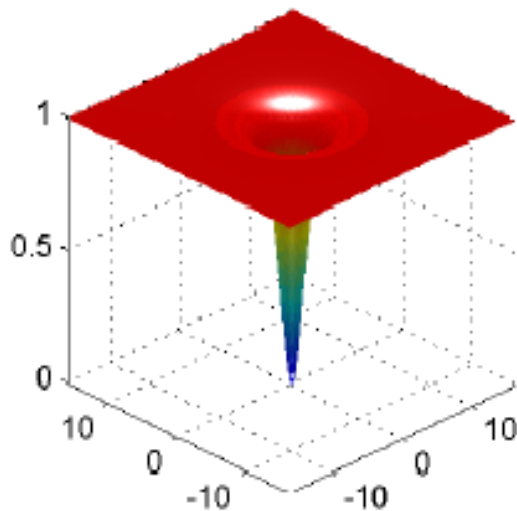
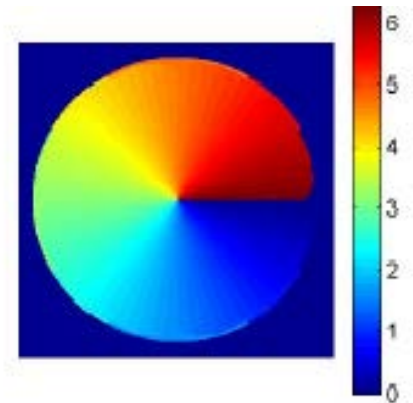
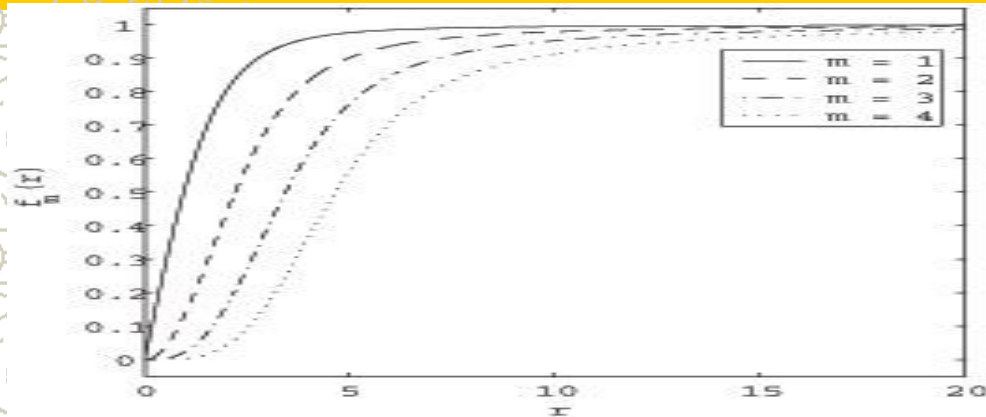
$$f_n''(r) + \frac{1}{r} f_n'(r) - \frac{n^2}{r^2} f_n(r) + (1 - f_n^2(r)) f_n(r) = 0, \quad 0 < r < \infty,$$

$$f_n(0) = 0, \quad f_n(\infty) = 1.$$

– Asymptotic results

$$f_n(r) \approx \begin{cases} ar^{|n|} + O(r^{|n|+2}), & r \rightarrow 0, \\ 1 - n^2 / 2r^2 + O(1/r^4), & r \rightarrow \infty. \end{cases}$$

'Bright-tail' vortex



Two kinds of vortices

✱ 'Dark-tail' vortex: BEC (wave function)

$$i \psi_t = -\Delta \psi + \frac{|\vec{x}|^2}{2} \psi + \beta |\psi|^2 \psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0, \quad \|\psi\|^2 = \int |\psi|^2 d\vec{x} = 1$$

– Center vortex states: $\psi(\vec{x}, t) = e^{-i\mu_n t} \phi_n(\vec{x}) = e^{-i\mu_n t} f_n(r) e^{in\theta}$

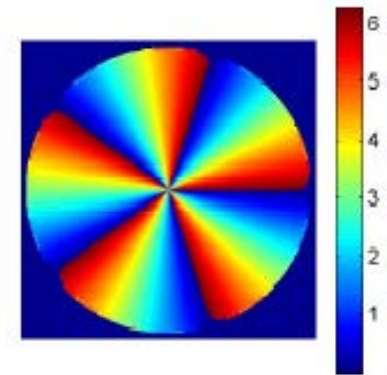
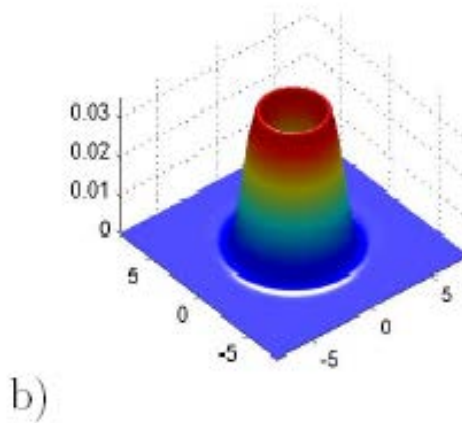
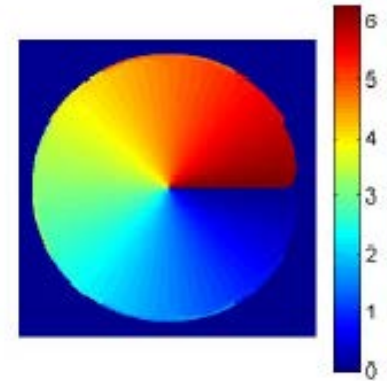
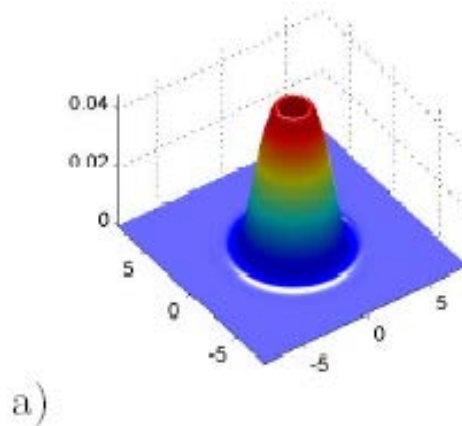
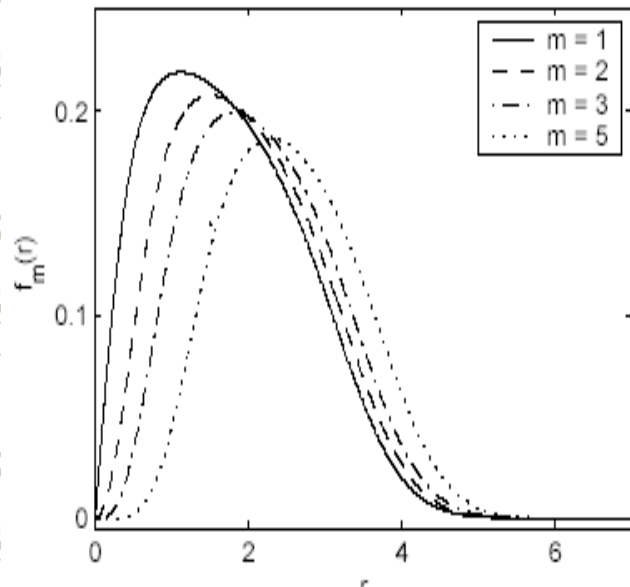
$$\mu_n f_n(r) = -\frac{d}{dr} \left(r \frac{df_n(r)}{dr} \right) + \left(\frac{n^2}{r^2} + \frac{r^2}{2} \right) f_n(r) + \beta f_n^3(r), \quad 0 < r < \infty,$$

$$f_n(0) = 0, \quad f_n(\infty) = 0, \quad 2\pi \int_0^\infty f_n^2(r) r dr = 1$$

– Asymptotic results

$$f_n(r) \approx \begin{cases} a r^{|n|} + O(r^{|n|+2}), & r \rightarrow 0, \\ b r^{|n|} e^{-r^2}, & r \rightarrow \infty. \end{cases}$$

'Dark-tail' vortex



Stability of vortices

- Data chosen

$$\varepsilon = 1, \quad V(\vec{x}, t) \equiv 1 \quad (+W(\vec{x}, t)), \quad \psi_0(\vec{x}) = \phi_n^h(\vec{x}) \quad (+ \text{noise})$$

- Results (Bao, Du & Zhang, Eur. J. Appl. Math., 07')

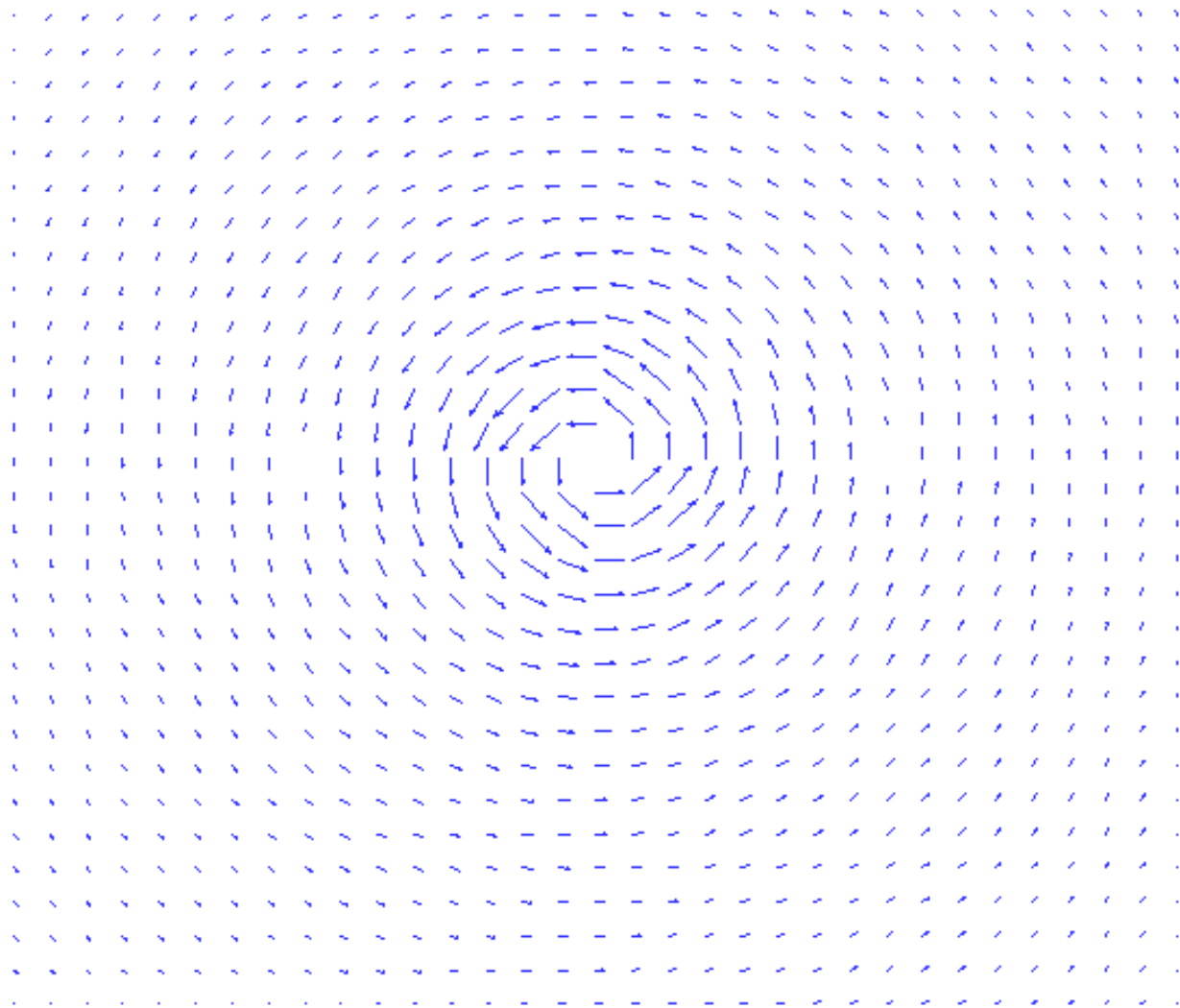
- For GLE or NLWE with initial data perturbed

- n=1: velocity density
- N=3: velocity density

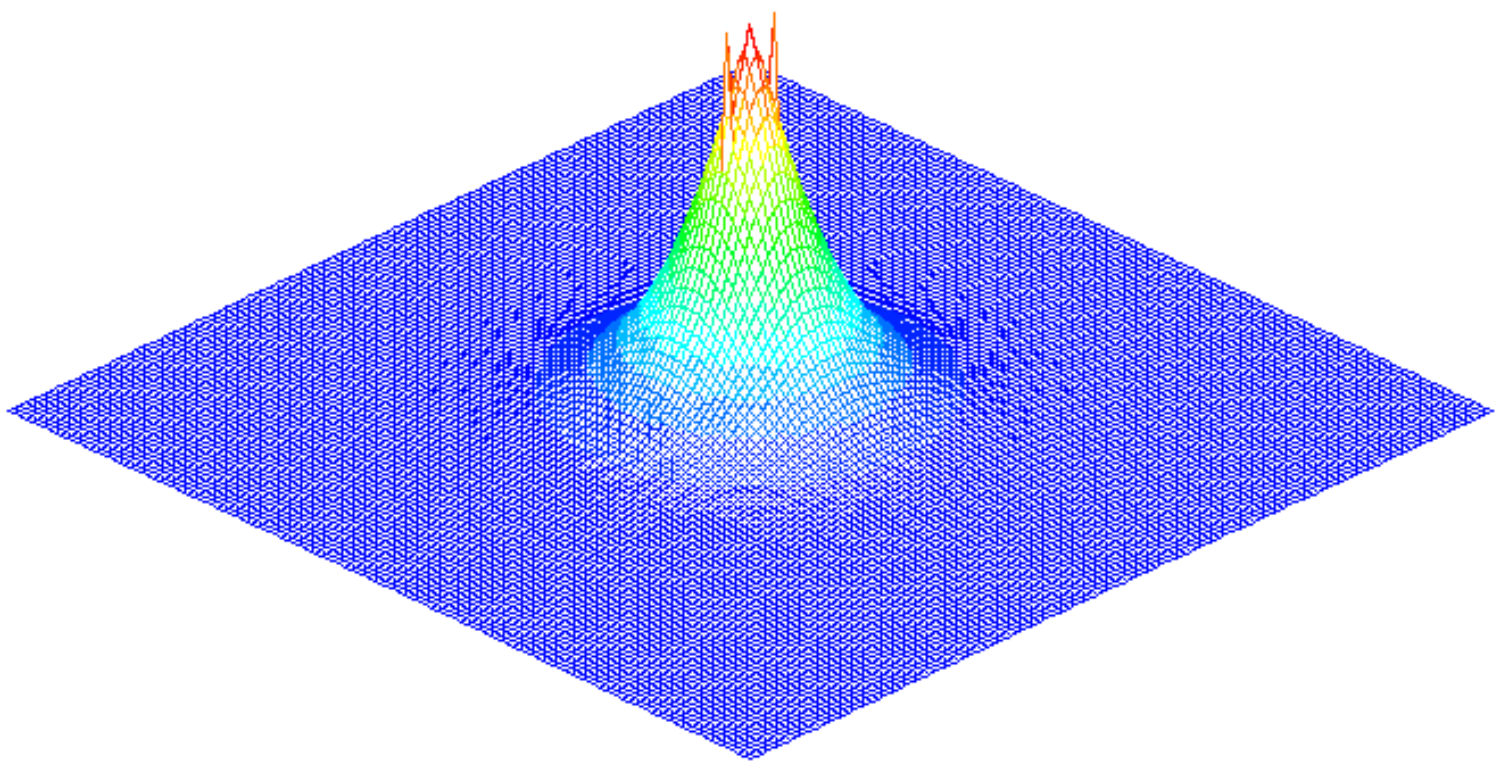
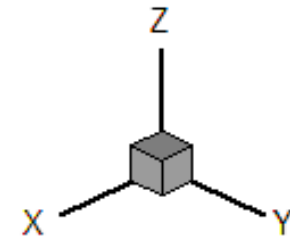
- For NLSE with external potential perturbed

- n=1: velocity density
- N=3: velocity density

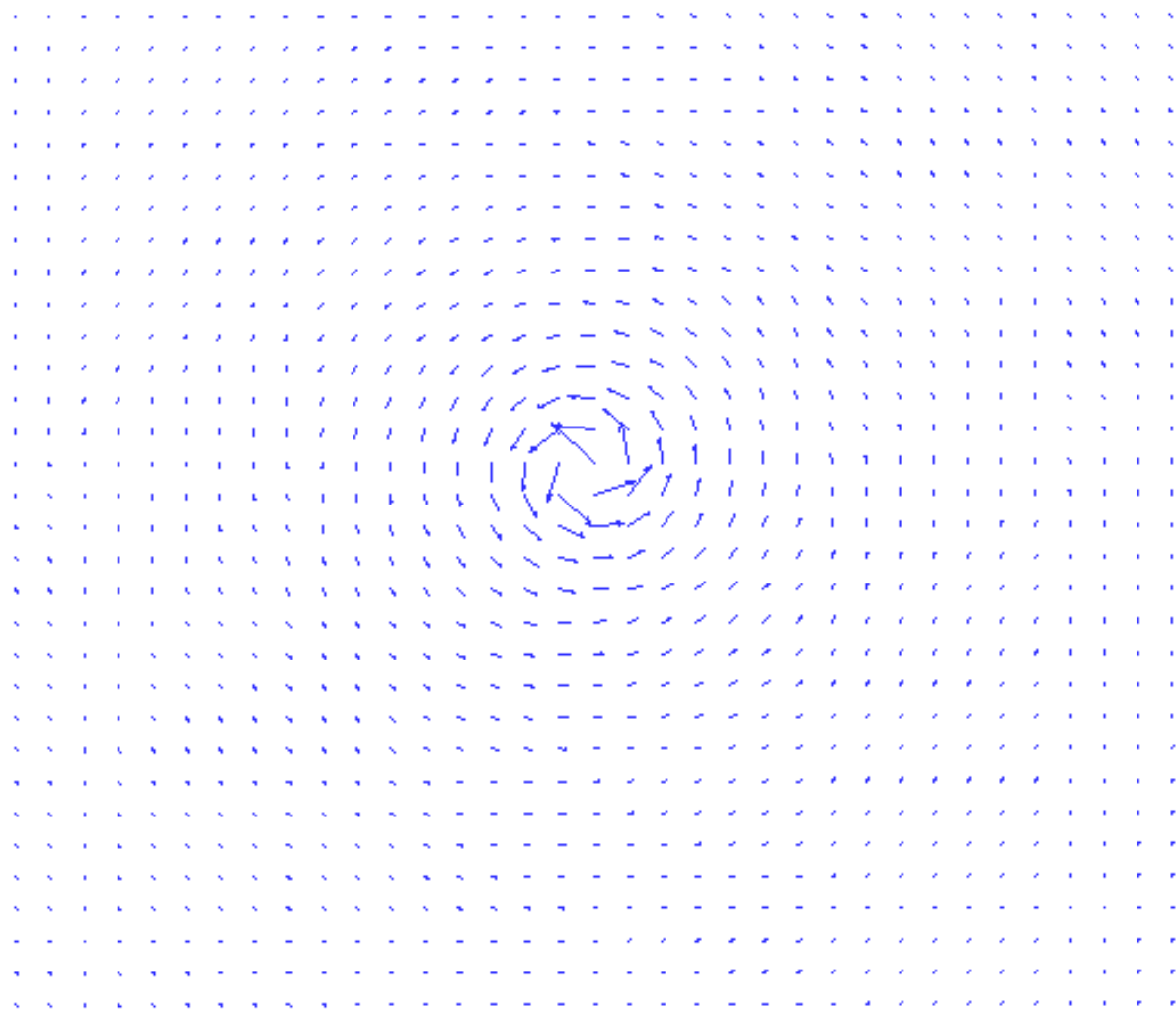
- Results



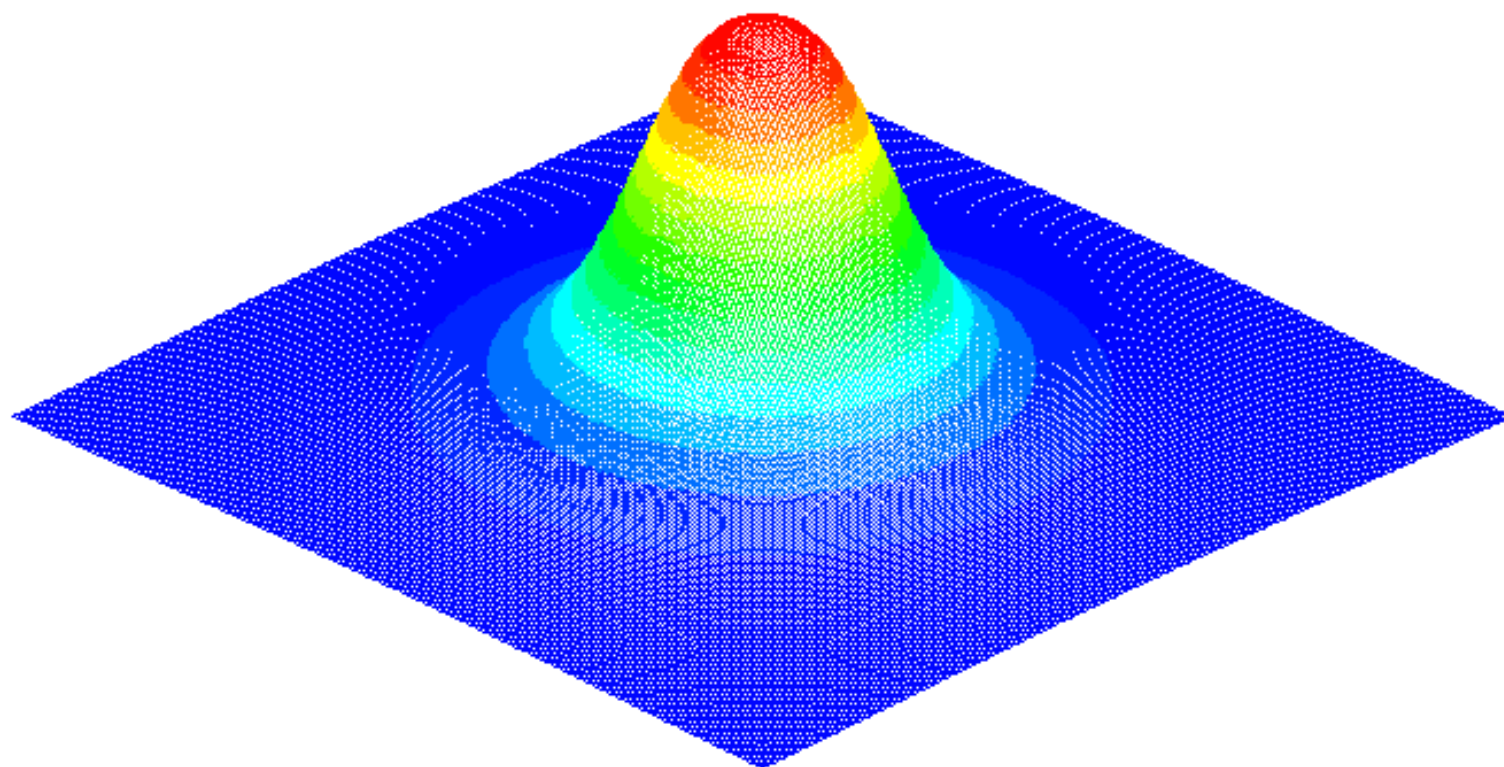
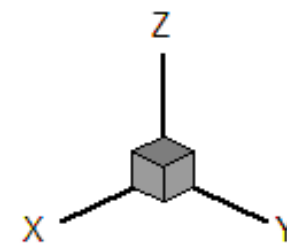
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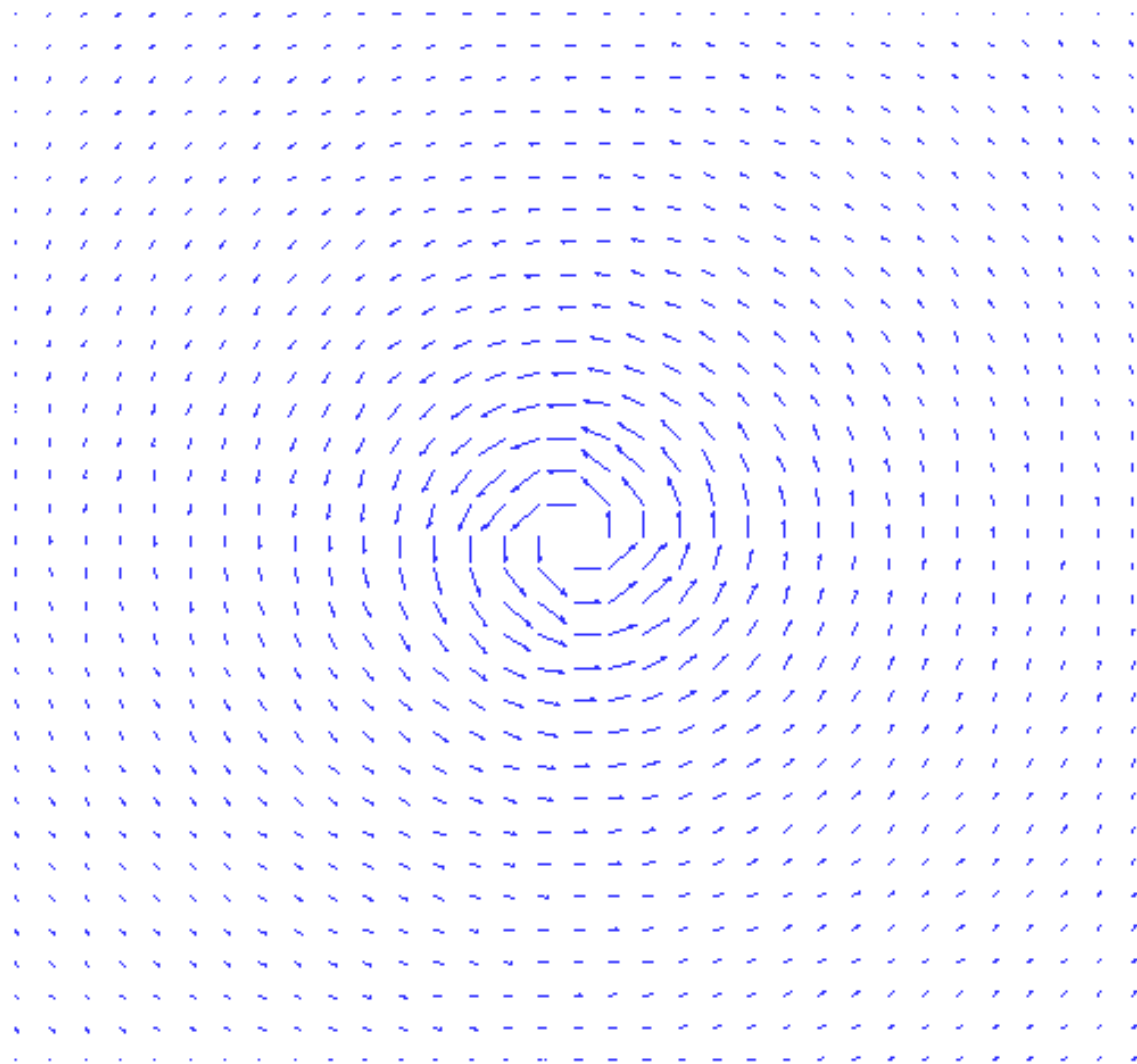
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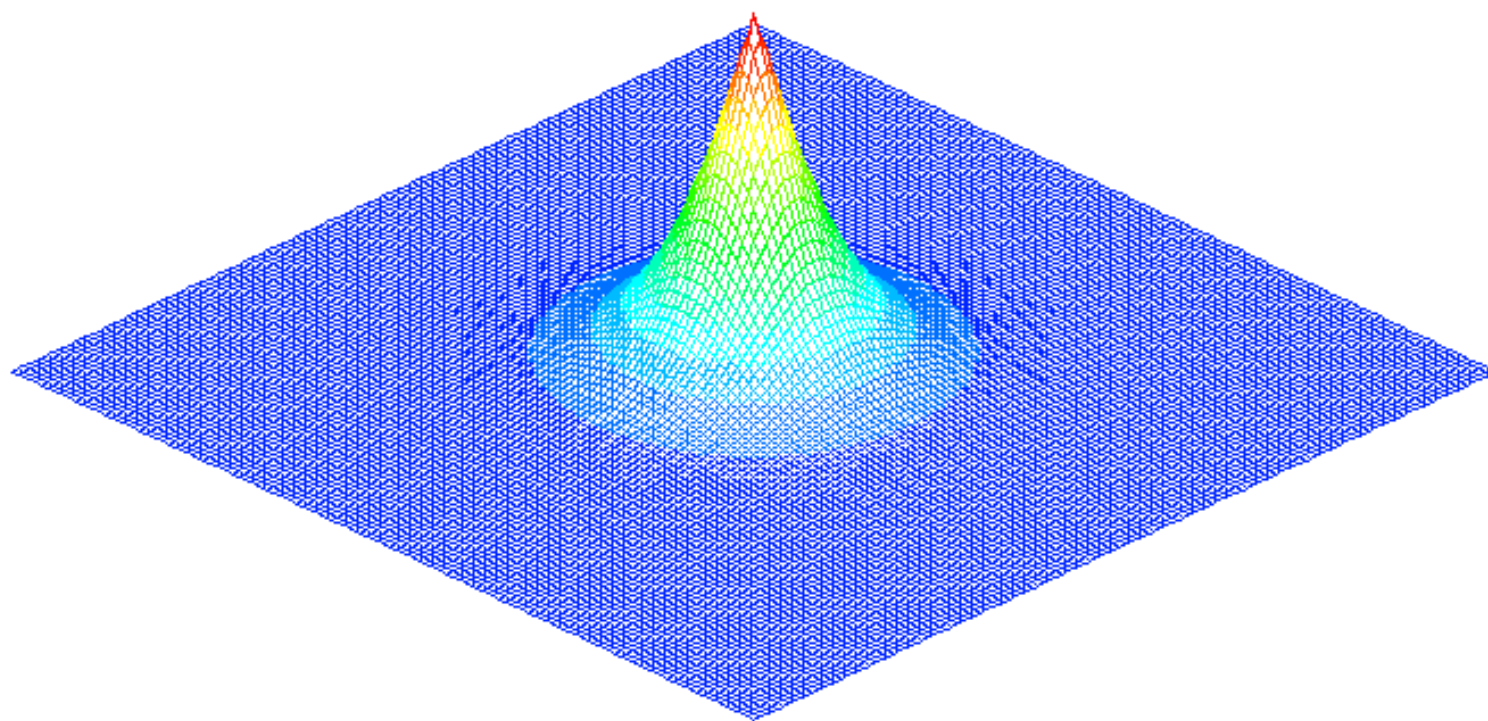
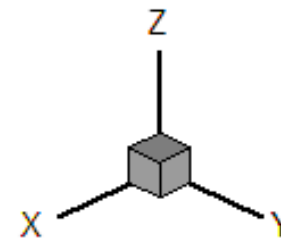
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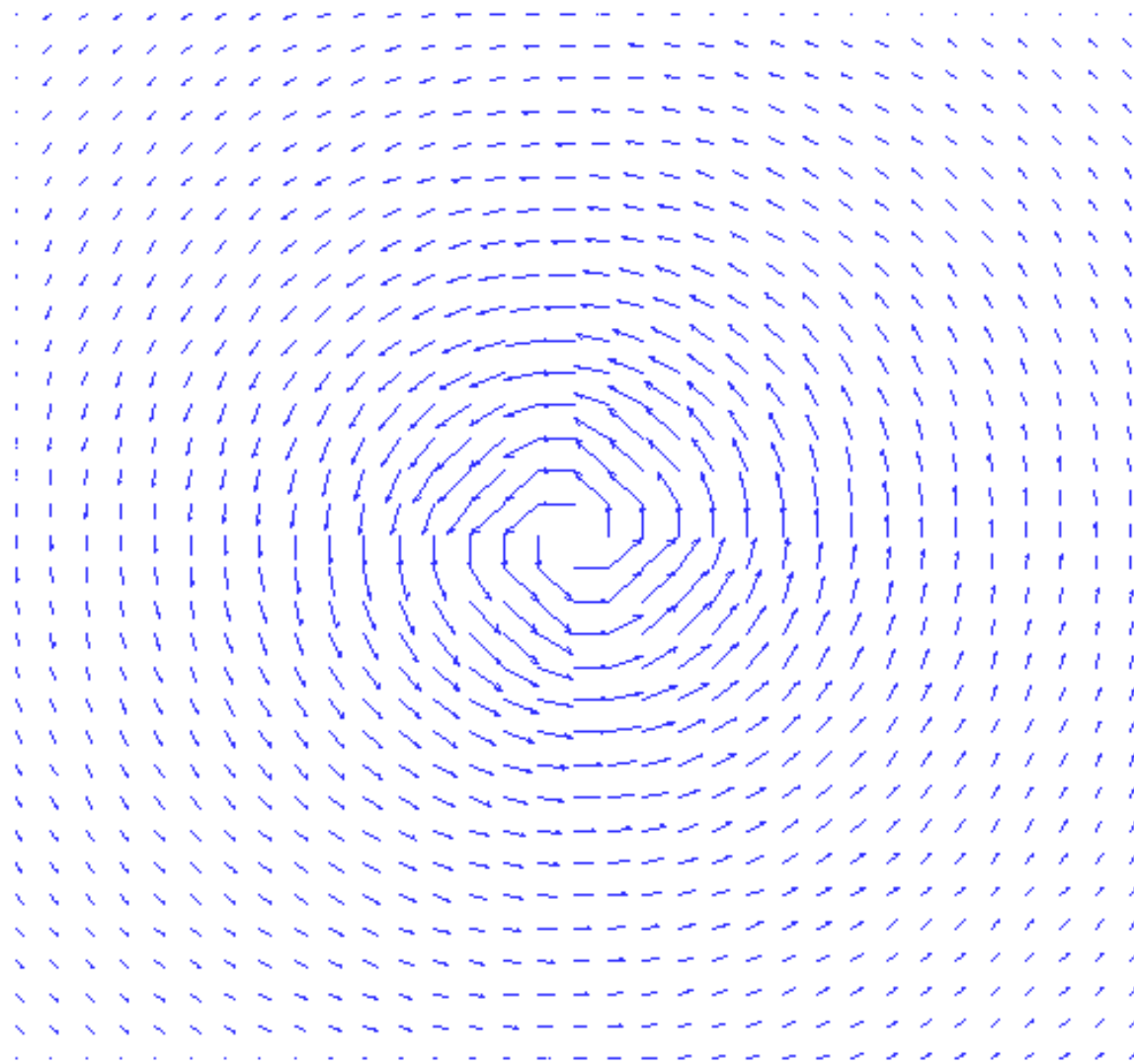
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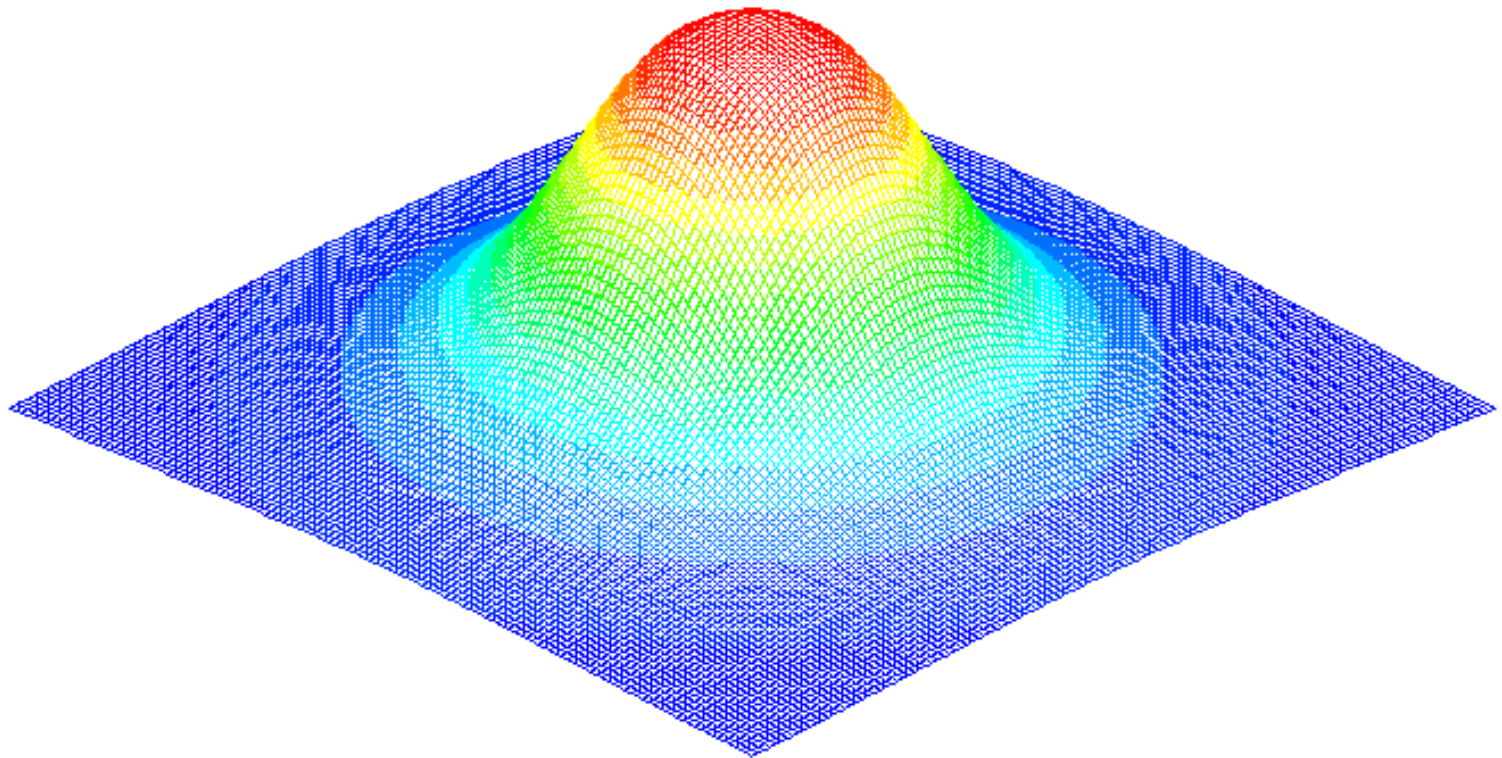
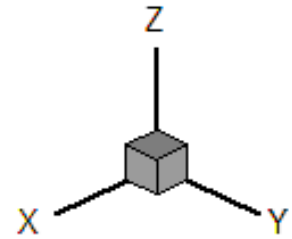
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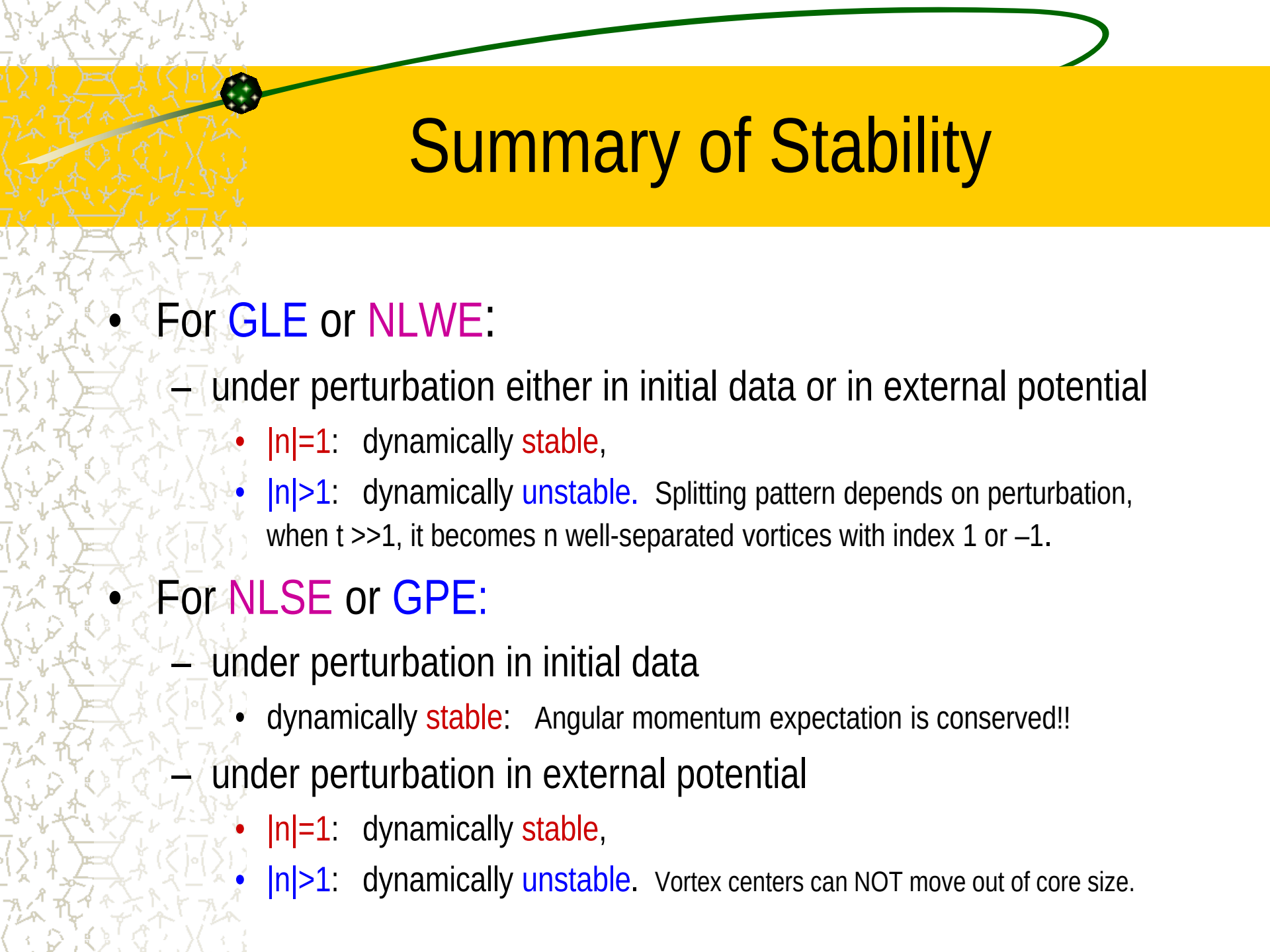
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Summary of Stability

- For **GLE** or **NLWE**:
 - under perturbation either in initial data or in external potential
 - $|n|=1$: dynamically **stable**,
 - $|n|>1$: dynamically **unstable**. Splitting pattern depends on perturbation, when $t \gg 1$, it becomes n well-separated vortices with index 1 or -1 .
- For **NLSE** or **GPE**:
 - under perturbation in initial data
 - dynamically **stable**: Angular momentum expectation is conserved!!
 - under perturbation in external potential
 - $|n|=1$: dynamically **stable**,
 - $|n|>1$: dynamically **unstable**. Vortex centers can NOT move out of core size.

Vortex interaction

- For 'bright-tail' vortex:

$$\psi_0(\vec{x}) = \prod_{j=1}^N \phi_{m_j}(\vec{x} - \vec{x}_j^0) = \prod_{j=1}^N \phi_{m_j}(x - x_j^0, y - y_j^0)$$

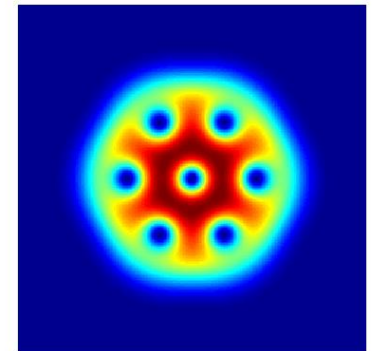
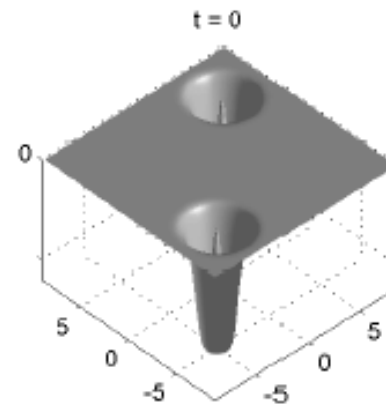
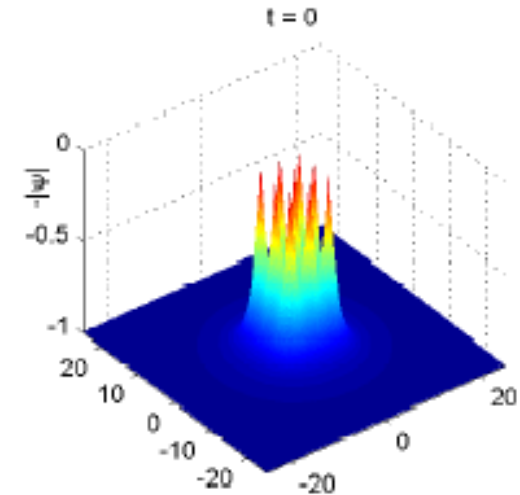
- For 'dark-tail' vortex

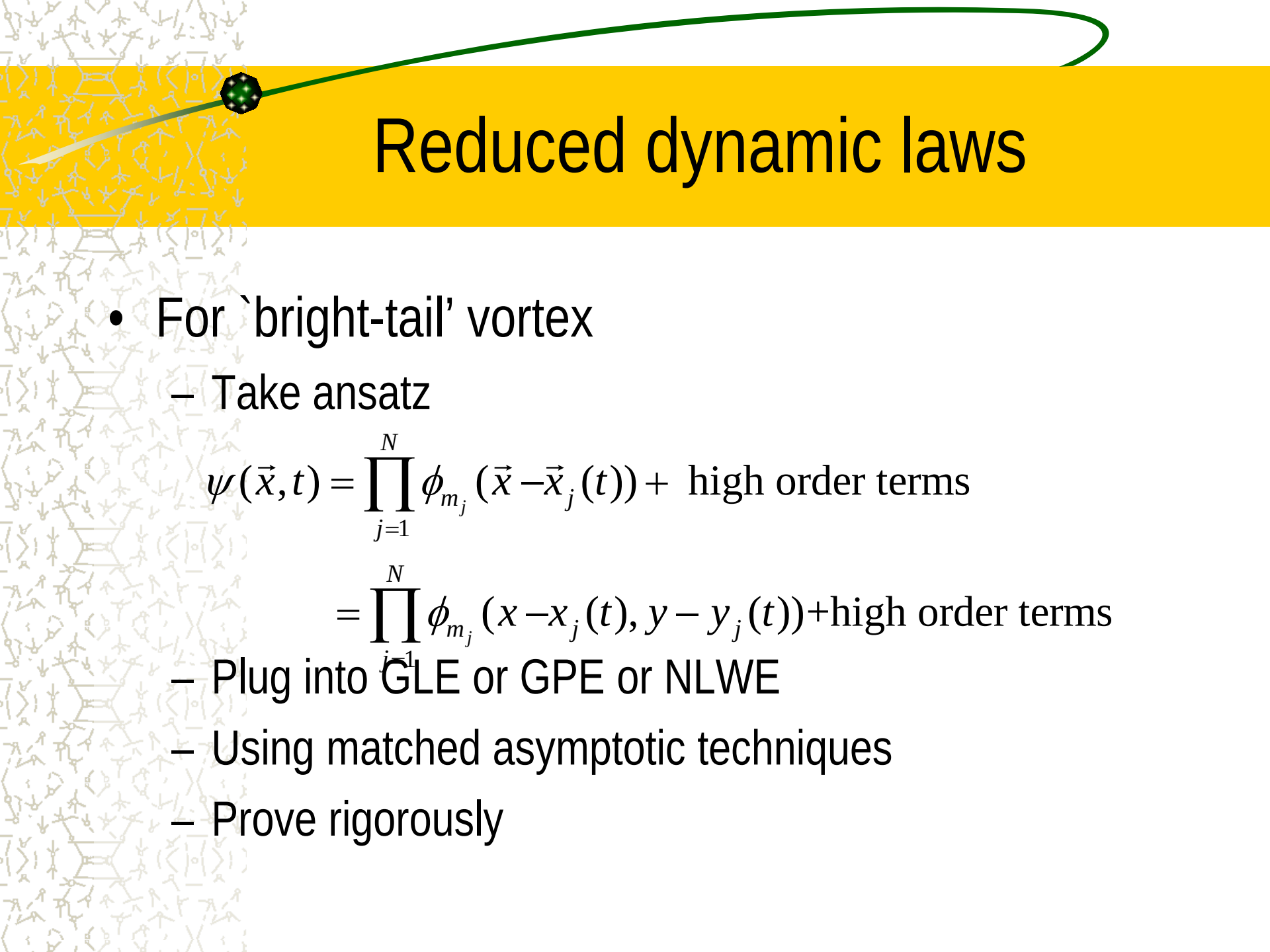
– Well-separate

$$\psi_0(x, y) = \sum_{j=1}^N \phi_{n_j}(x - x_j, y - y_j) \quad / \|\bullet\|$$

– Overlapped

$$\psi(\mathbf{r}, t=0) = \alpha \psi_{\text{gs}}(\mathbf{r}) \prod_{j=1}^n p_{q_j}(\mathbf{r} - \mathbf{r}_j),$$





Reduced dynamic laws

- For 'bright-tail' vortex

- Take ansatz

$$\psi(\vec{x}, t) = \prod_{j=1}^N \phi_{m_j}(\vec{x} - \vec{x}_j(t)) + \text{high order terms}$$

$$= \prod_{j=1}^N \phi_{m_j}(x - x_j(t), y - y_j(t)) + \text{high order terms}$$

- Plug into GLE or GPE or NLWE

- Using matched asymptotic techniques

- Prove rigorously

Reduced dynamic laws

- For 'bright-tail' vortex

- For GLE

$$\kappa \vec{v}_j(t) := \kappa \frac{d \vec{x}_j(t)}{dt} = 2m_j \sum_{l=1, l \neq j}^N m_l \frac{\vec{x}_j(t) - \vec{x}_l(t)}{|\vec{x}_j(t) - \vec{x}_l(t)|^2}, \quad t \geq 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \quad 1 \leq j \leq N.$$

- For GPE

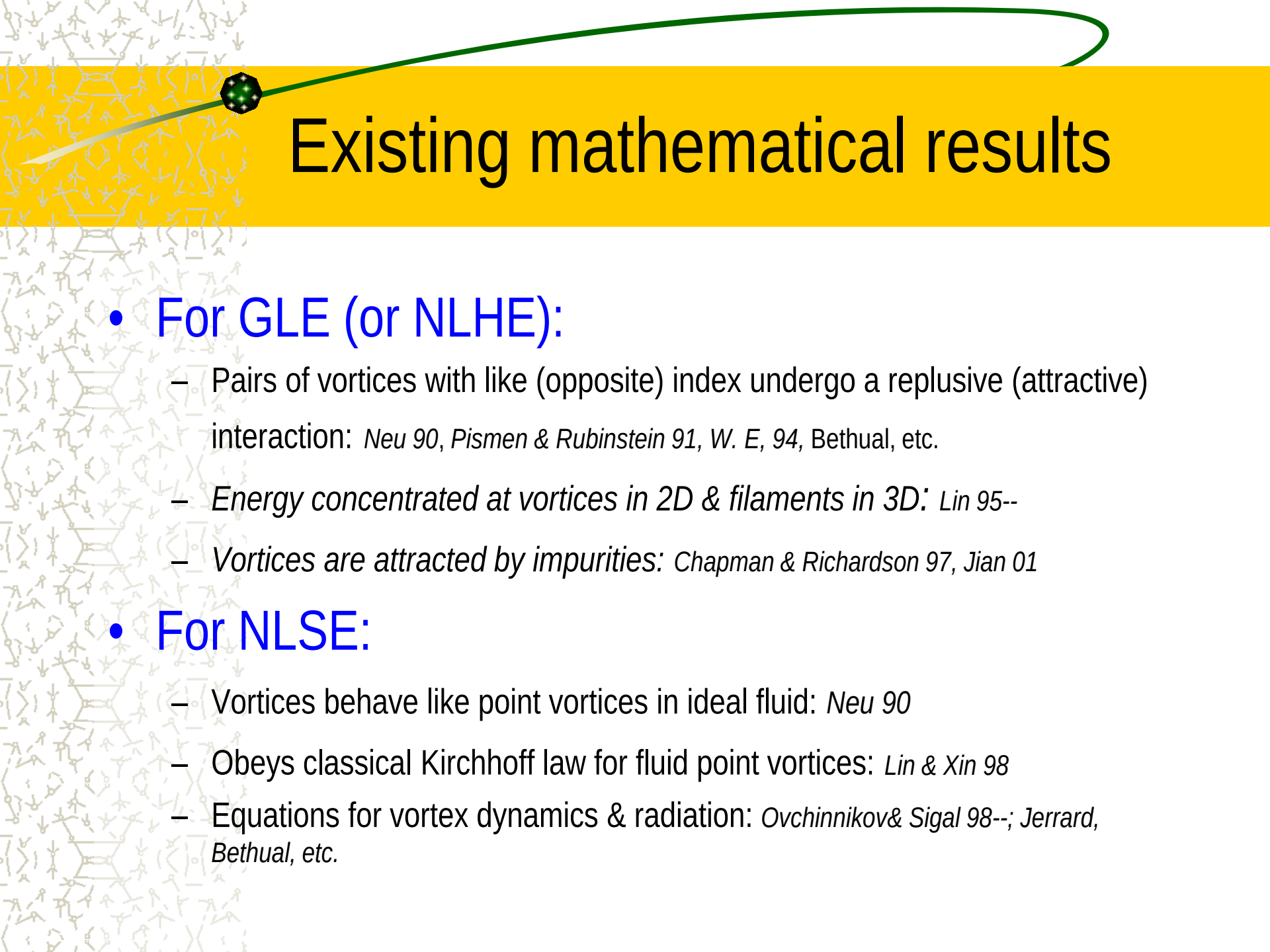
$$\vec{v}_j(t) := \frac{d \vec{x}_j(t)}{dt} = 2 \sum_{l=1, l \neq j}^N m_l \frac{J(\vec{x}_j(t) - \vec{x}_l(t))}{|\vec{x}_j(t) - \vec{x}_l(t)|^2} \quad t \geq 0,$$

$$\vec{x}_j(0) = \vec{x}_j^0 \quad 1 \leq j \leq N; \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- For NLWE

$$\kappa \frac{d^2 \vec{x}_j(t)}{dt^2} = 2m_j \sum_{l=1, l \neq j}^N m_l \frac{\vec{x}_j(t) - \vec{x}_l(t)}{|\vec{x}_j(t) - \vec{x}_l(t)|^2}, \quad t \geq 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \quad \vec{x}_j'(0) = \vec{v}_j^0, \quad 1 \leq j \leq N.$$



Existing mathematical results

- For GLE (or NLHE):

- Pairs of vortices with like (opposite) index undergo a repulsive (attractive) interaction: *Neu 90, Pismen & Rubinstein 91, W. E, 94, Bethual, etc.*
- *Energy concentrated at vortices in 2D & filaments in 3D: Lin 95--*
- *Vortices are attracted by impurities: Chapman & Richardson 97, Jian 01*

- For NLSE:

- Vortices behave like point vortices in ideal fluid: *Neu 90*
- Obeys classical Kirchhoff law for fluid point vortices: *Lin & Xin 98*
- Equations for vortex dynamics & radiation: *Ovchinnikov & Sigal 98--; Jerrard, Bethual, etc.*

Conservation laws

- **Mass center**

$$\bar{\mathbf{x}}(t) := \frac{1}{N} \sum_{j=1}^N \vec{\mathbf{x}}_j(t)$$

Lemma The mass center of the N vortices for GLE is conserved

$$\bar{\mathbf{x}}(t) := \frac{1}{N} \sum_{j=1}^N \vec{\mathbf{x}}_j(t) \equiv \bar{\mathbf{x}}(0) := \frac{1}{N} \sum_{j=1}^N \vec{\mathbf{x}}_j(0) = \frac{1}{N} \sum_{j=1}^N \vec{\mathbf{x}}_j^0$$

Lemma The mass center of the N vortices for NLWE is conserved if initial velocity is zero and it moves linearly if initial velocity is nonzero

$$\bar{\mathbf{x}}(t) := \frac{1}{N} \sum_{j=1}^N \mathbf{x}_j(t) = \bar{\mathbf{x}}(0) + t \bar{\mathbf{v}}(0) = \frac{1}{N} \sum_{j=1}^N \mathbf{x}_j^0 + t \left(\frac{1}{N} \sum_{j=1}^N \mathbf{x}_j^1 \right), \quad t \geq 0,$$

Conservation laws

- Signed mass center

$$\tilde{x}(t) := \frac{1}{N} \sum_{j=1}^N m_j \vec{x}_j(t)$$

Lemma The signed mass center of the N vortices for NLSE is conserved

$$\tilde{x}(t) := \frac{1}{N} \sum_{j=1}^N m_j \vec{x}_j(t) \equiv \tilde{x}(0) := \frac{1}{N} \sum_{j=1}^N m_j \vec{x}_j(0) = \frac{1}{N} \sum_{j=1}^N m_j \vec{x}_j^0$$

Analytical Solutions

- $N (\geq 2)$ like vortices on a **circle** (Bao, Du & Zhang, SIAP, 07')

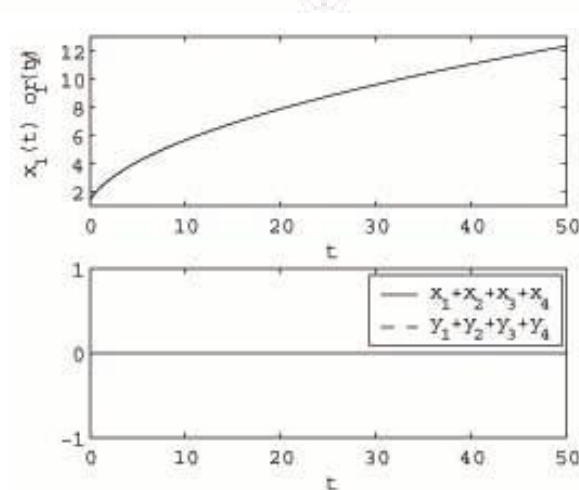
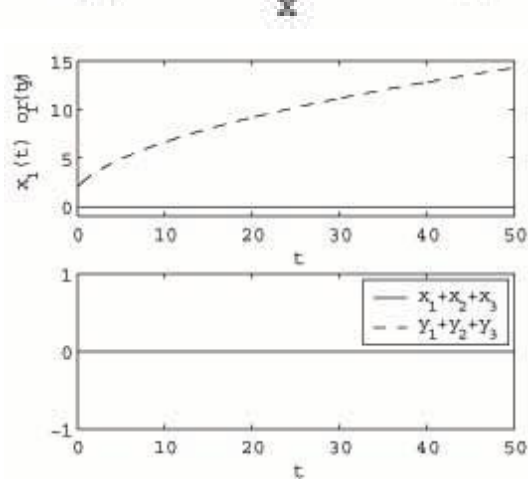
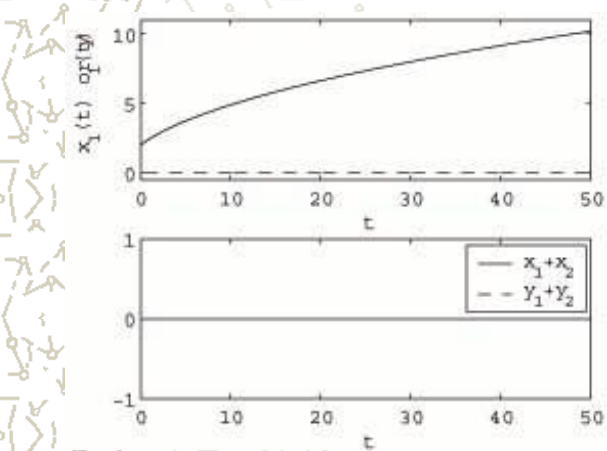
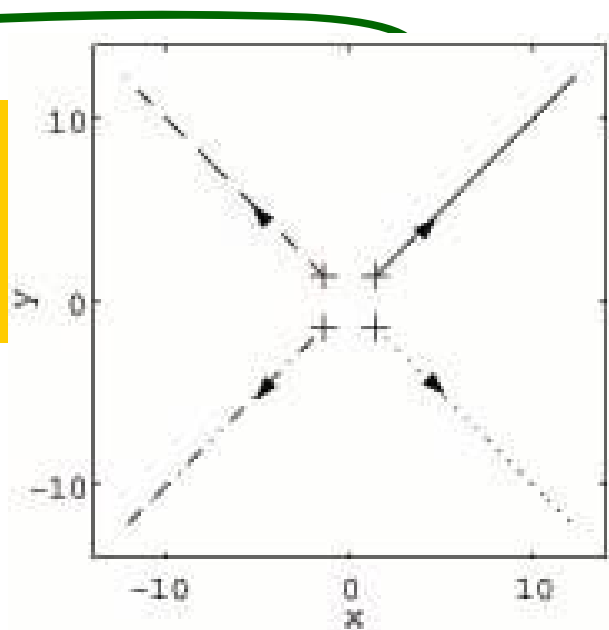
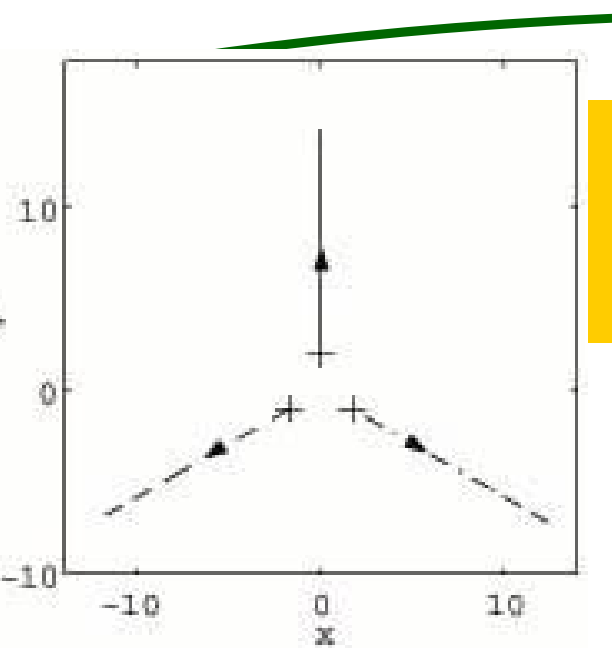
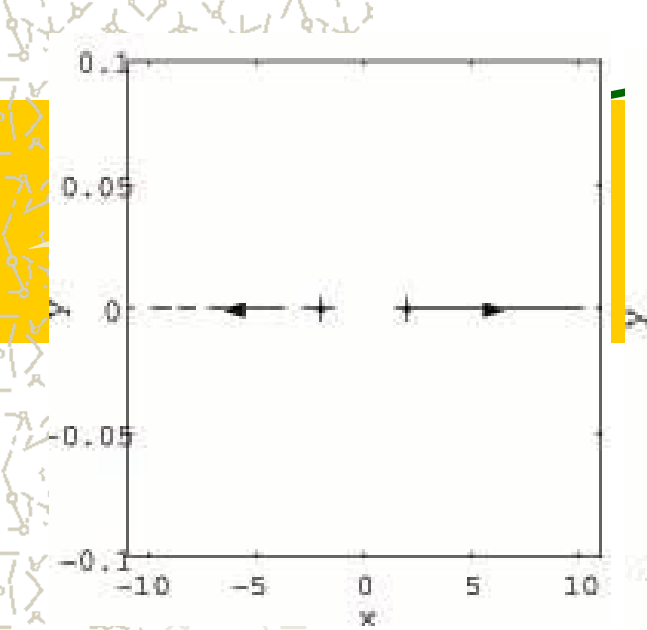
$$\vec{x}_j^0 = a \left(\cos\left(\frac{2j\pi}{N}\right), \sin\left(\frac{2j\pi}{N}\right) \right), \quad m_j = m_0 = \pm 1, \quad 1 \leq j \leq N$$

- Analytical solutions for **GLE** [figure](#)

$$\vec{x}_j(t) = \sqrt{a^2 + \frac{2(N-1)}{\kappa}t} \left(\cos\left(\frac{2j\pi}{N}\right), \sin\left(\frac{2j\pi}{N}\right) \right)$$

- Analytical solutions for **NLSE** [figure](#) [next](#)

$$\vec{x}_j(t) = a \left(\cos\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right), \sin\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right) \right)$$

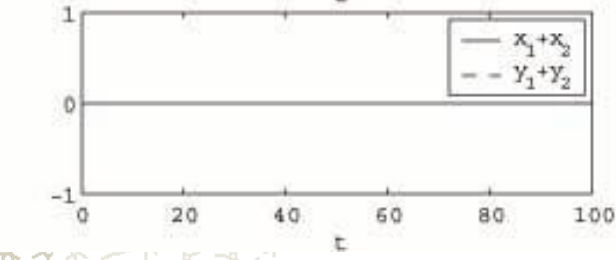
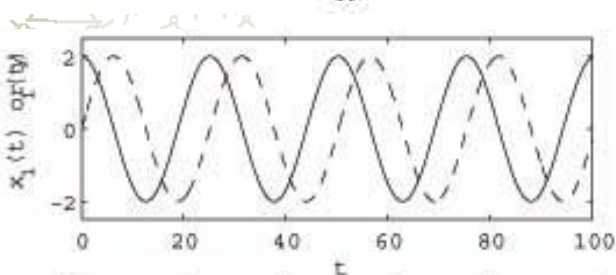
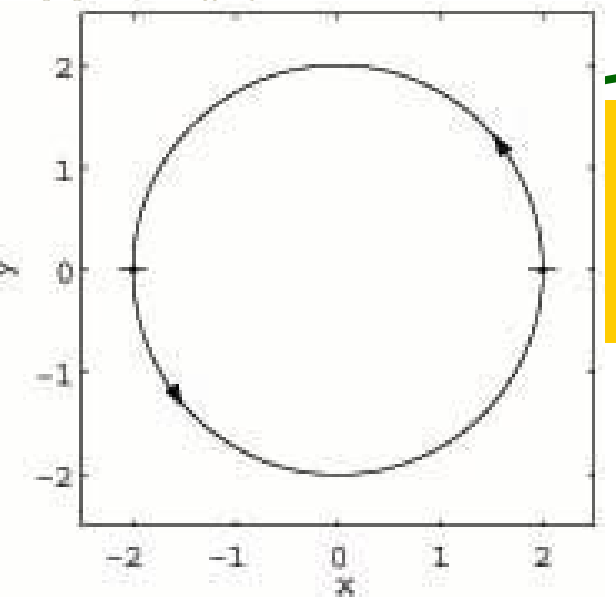


N=2

N=3

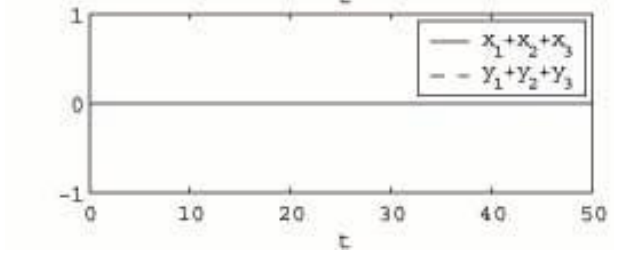
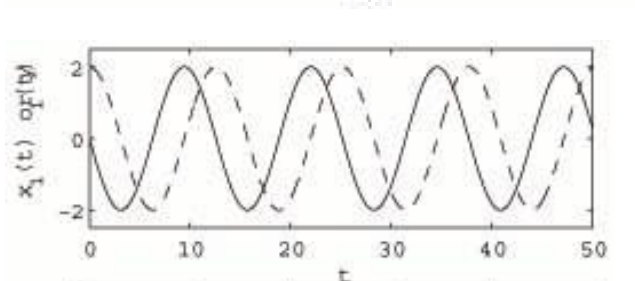
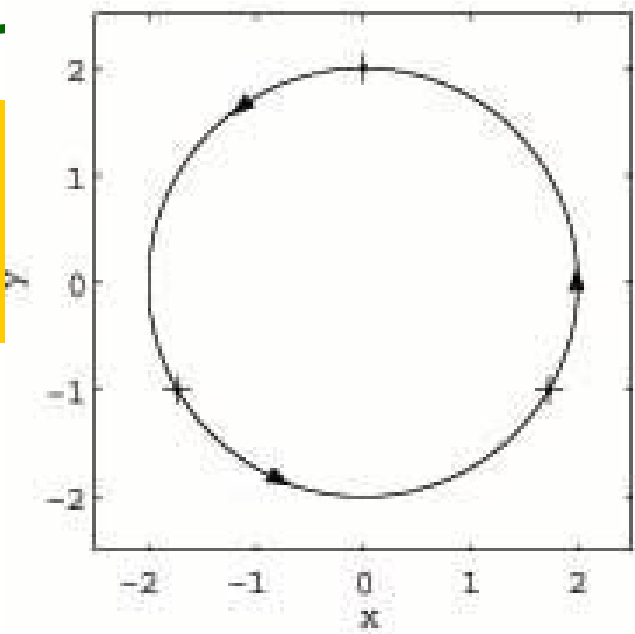
N=4

[back](#)



N=2

[back](#)



N=3

Analytical Solutions

- $N (\geq 3)$ like vortices on a circle and its center (Bao, Du & Zhang, SIAP, 07')

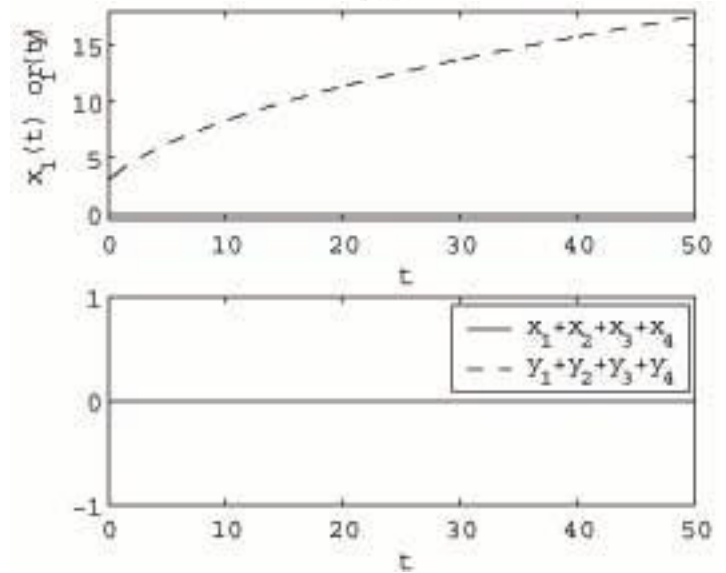
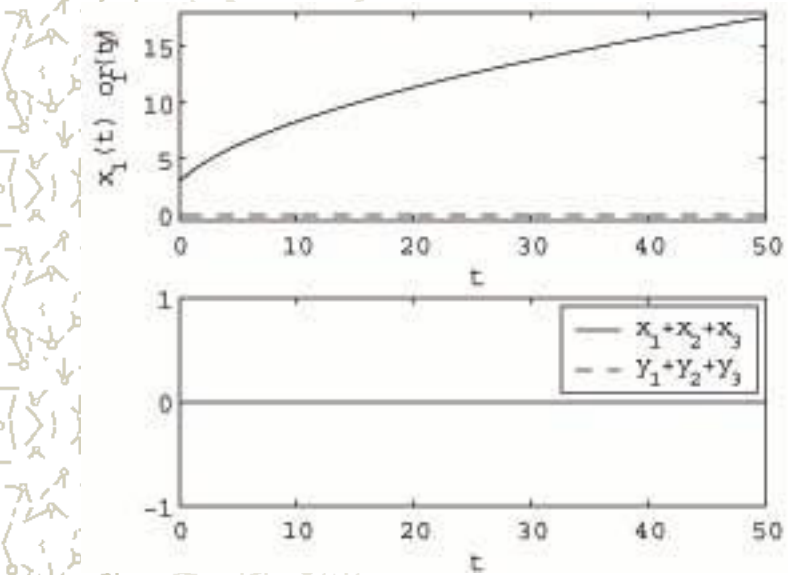
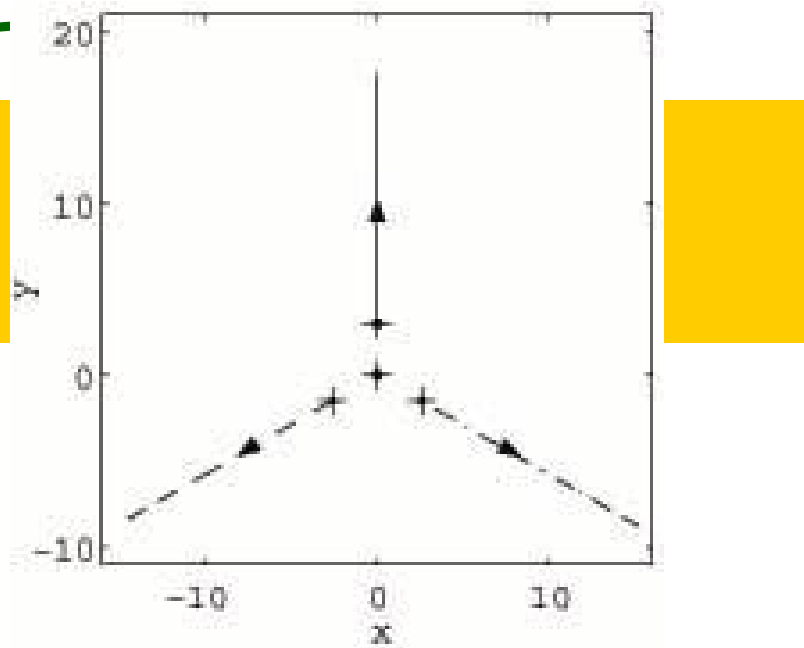
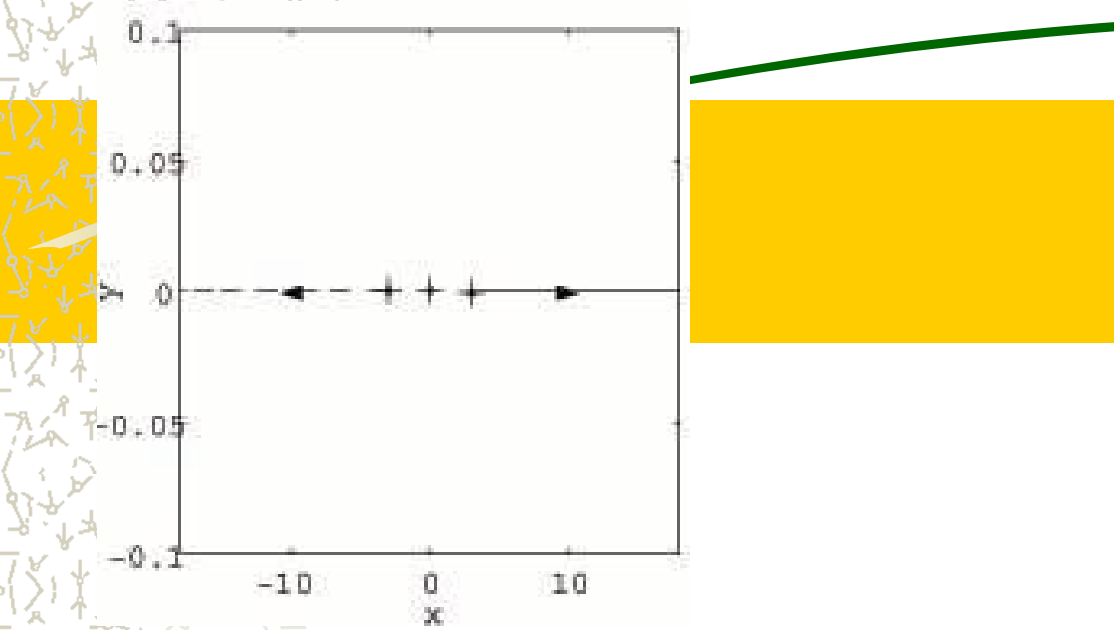
$$\vec{x}_N^0 = \vec{0}; \quad \vec{x}_j^0 = a \left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right) \right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for GLE [figure](#)

$$\vec{x}_N(t) = \vec{0}; \quad \vec{x}_j(t) = \sqrt{a^2 + \frac{2N}{\kappa}t} \left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right) \right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for NLSE [figure](#) [next](#)

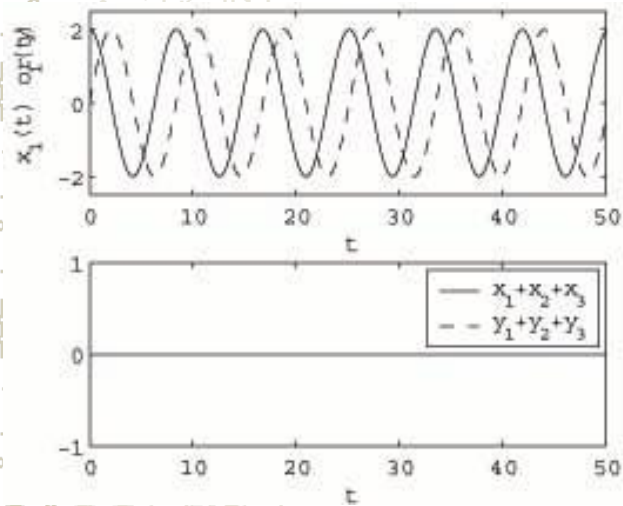
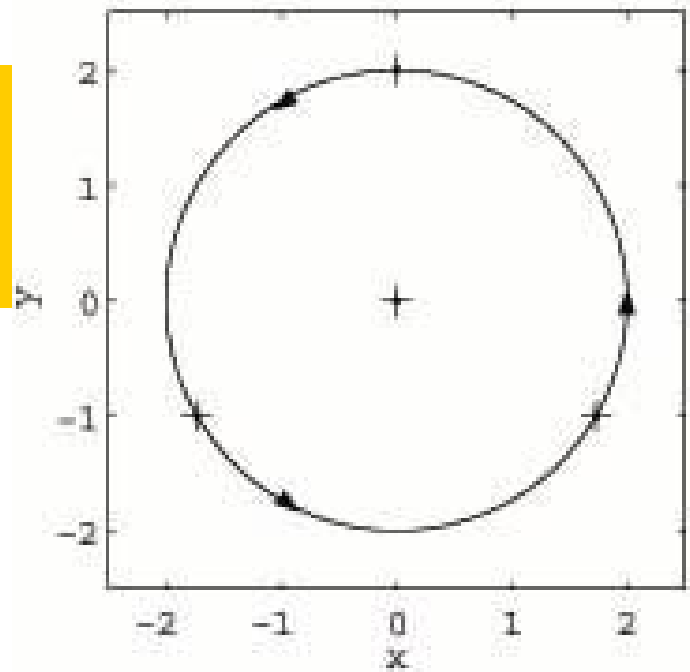
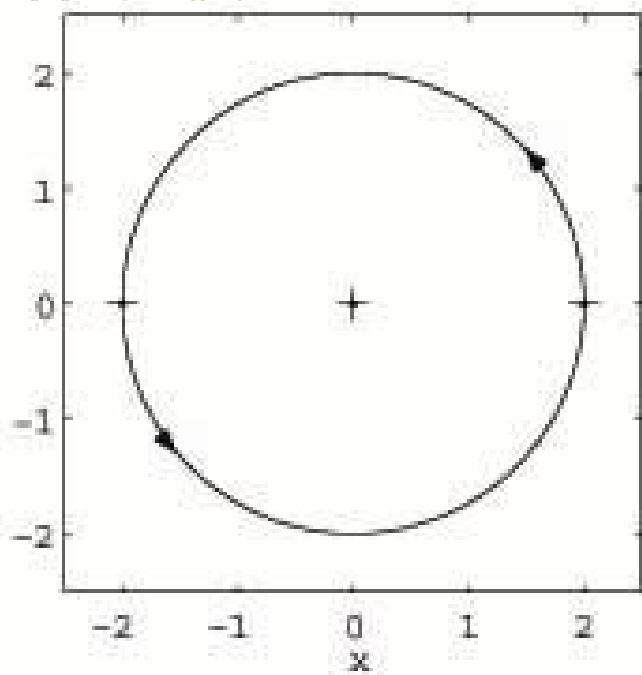
$$\vec{x}_N(t) = \vec{0}; \quad \vec{x}_j(t) = a \left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right), \sin\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right) \right)$$



N=3

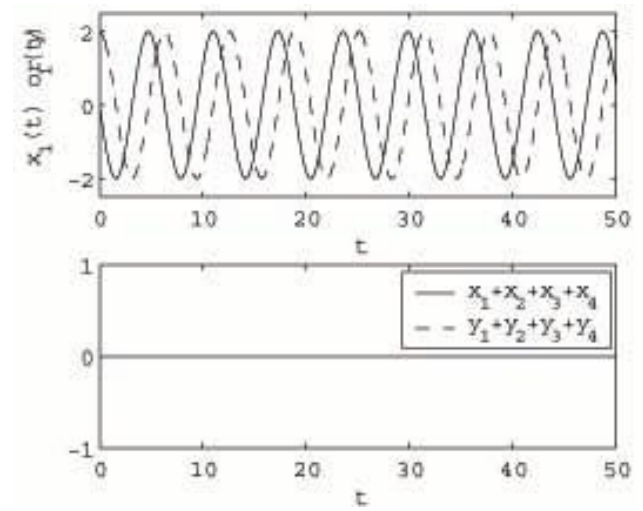
[back](#)

N=4



N=3

[back](#)



N=4

Analytical Solutions

- Two **opposite** vortices (Bao, Du & Zhang, SIAP, 07')

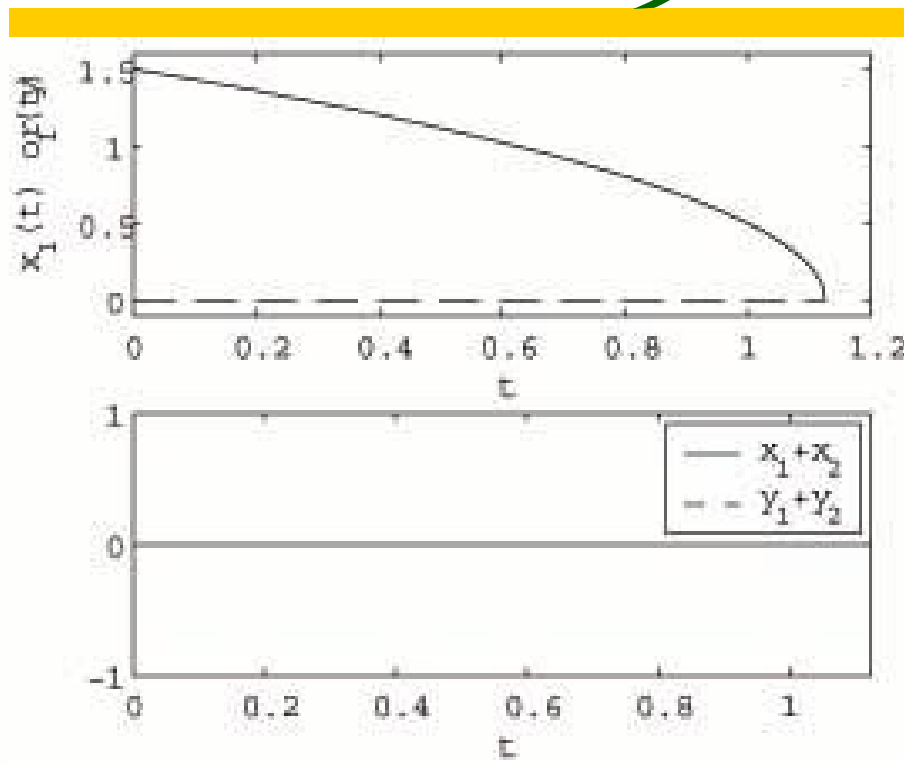
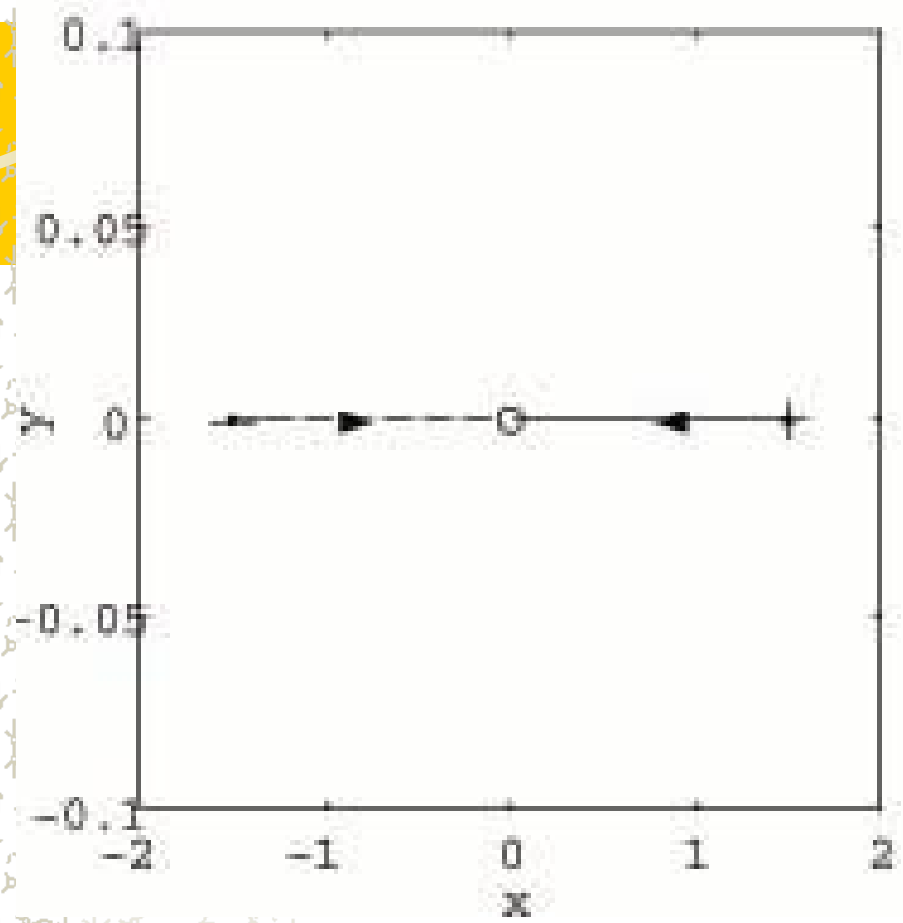
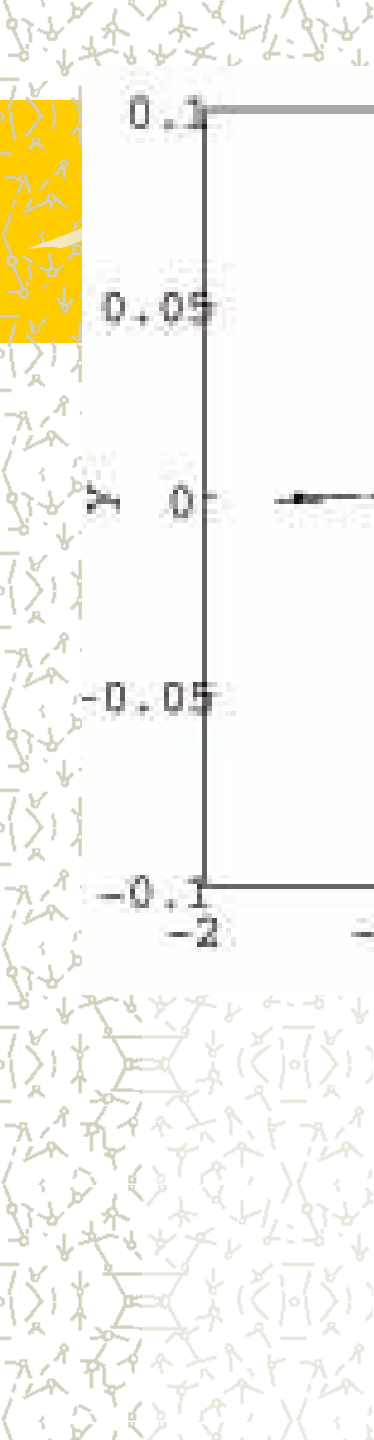
$$\vec{x}_1^0 = -\vec{x}_2^0 = a(\cos(\theta_0) \quad , \quad \sin(\theta_0)), \quad m_1 = -m_2 = 1$$

- Analytical solutions for **GLE** [figure](#)

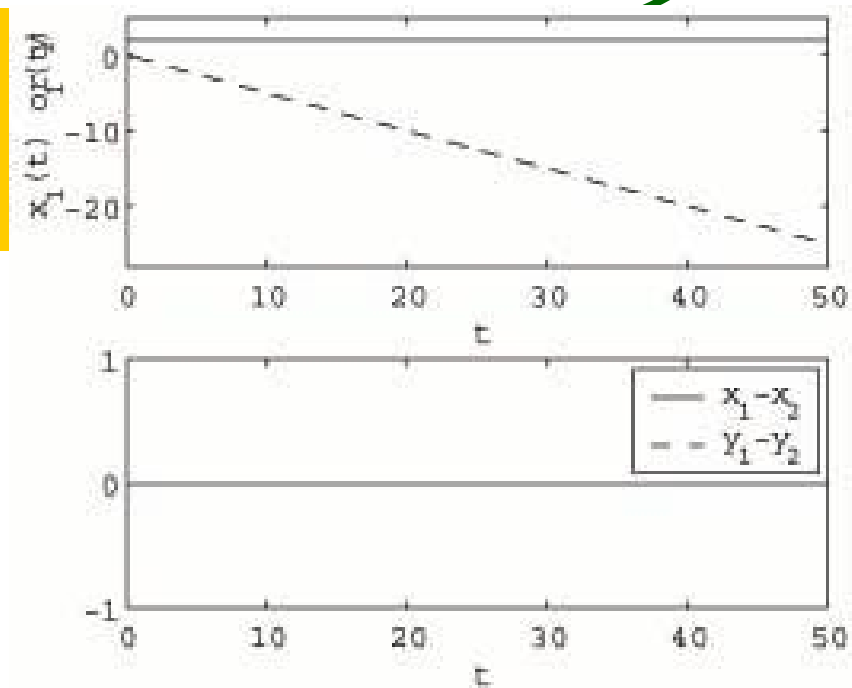
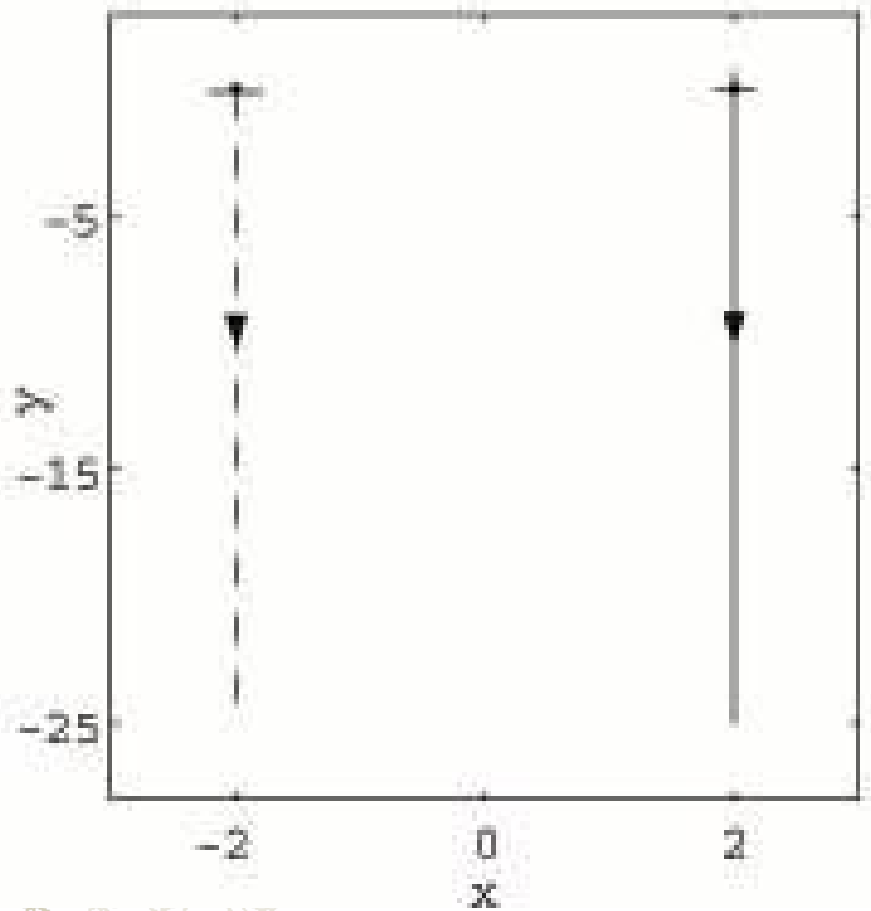
$$\vec{x}_1(t) = -\vec{x}_2(t) = \sqrt{a^2 - \frac{2}{\kappa}t} \quad (\cos(\theta_0) \quad , \quad \sin(\theta_0))$$

- Analytical solutions for **NLSE** [figure](#) [next](#)

$$\vec{x}_j(t) = \vec{x}_j^0 + \frac{t}{a}(-\sin(\theta_0) \quad , \quad \cos(\theta_0)), \quad j = 1, 2$$



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Analytical Solutions

- $N (\geq 3)$ opposite vortices on a **circle and its center**

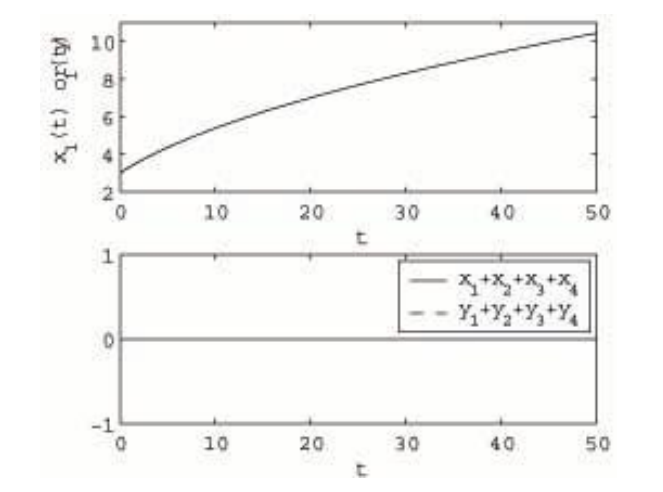
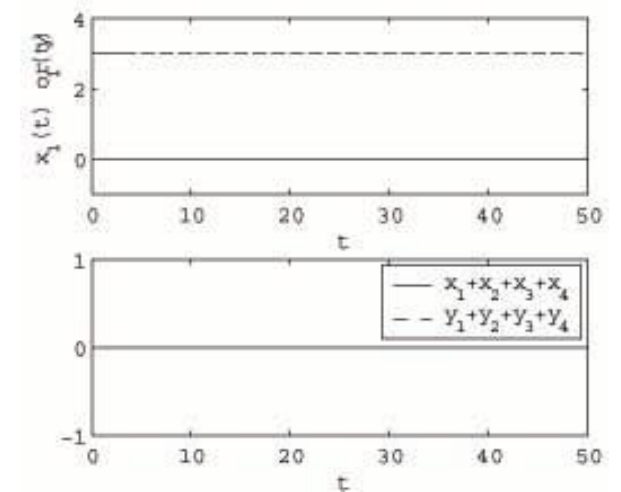
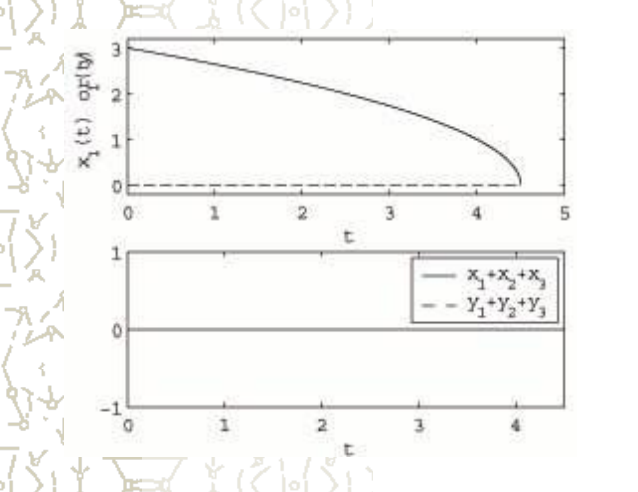
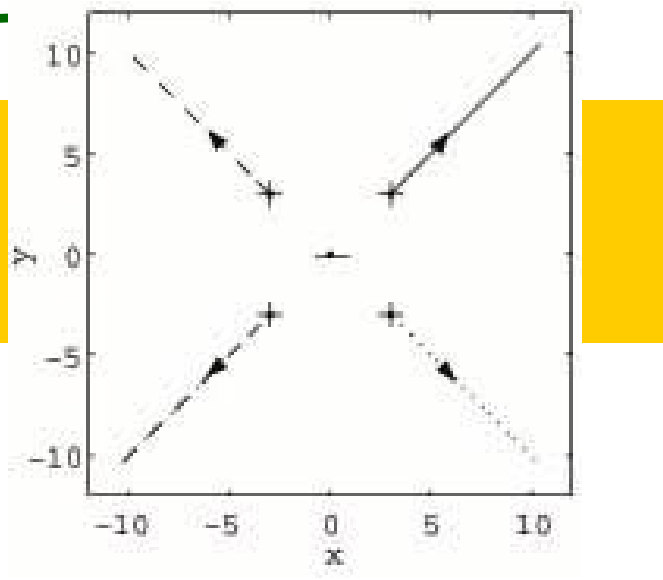
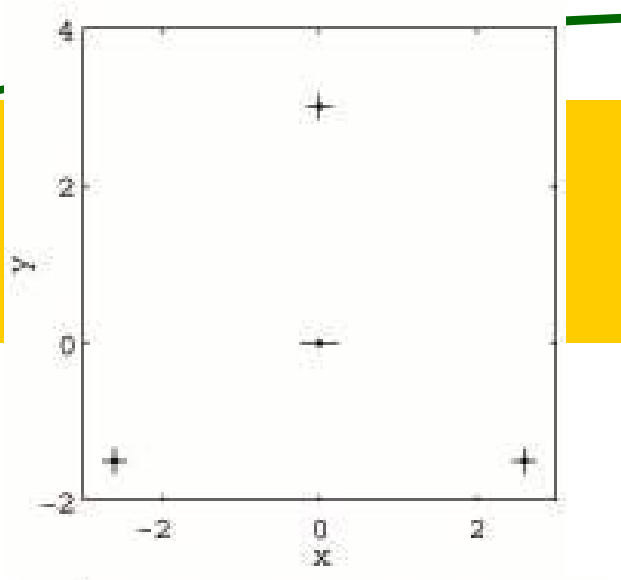
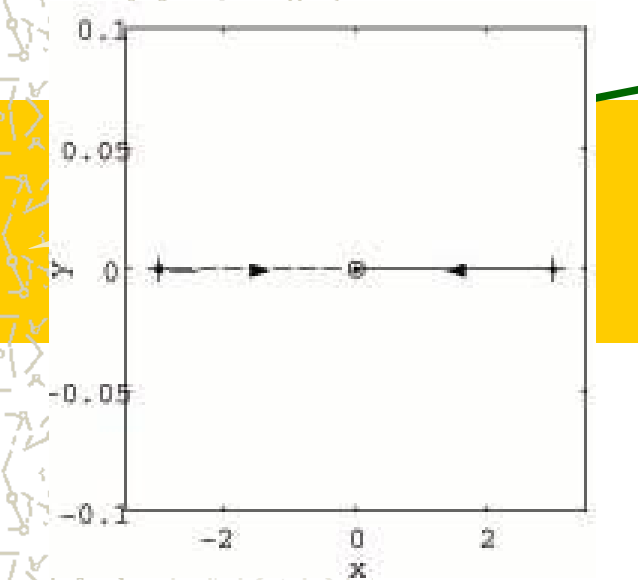
$$\vec{x}_N^0 = \vec{0} \quad (-); \quad \vec{x}_j^0 = a \left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right) \right) \quad (+), \quad 1 \leq j \leq N-1$$

- Analytical solutions for **GLE** [figure](#)

$$\vec{x}_N(t) = \vec{0}; \quad \vec{x}_j(t) = \sqrt{a^2 + \frac{2(N-4)}{\kappa}t} \left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right) \right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for **NLSE** [figure](#) [next](#)

$$\vec{x}_N(t) = \vec{0}; \quad \vec{x}_j(t) = a \left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right), \sin\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right) \right)$$

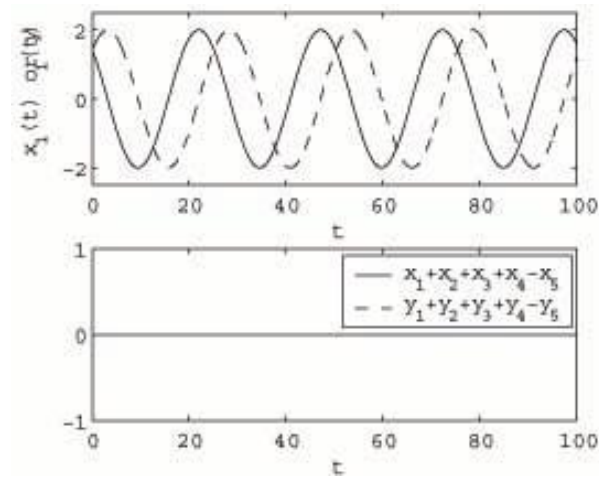
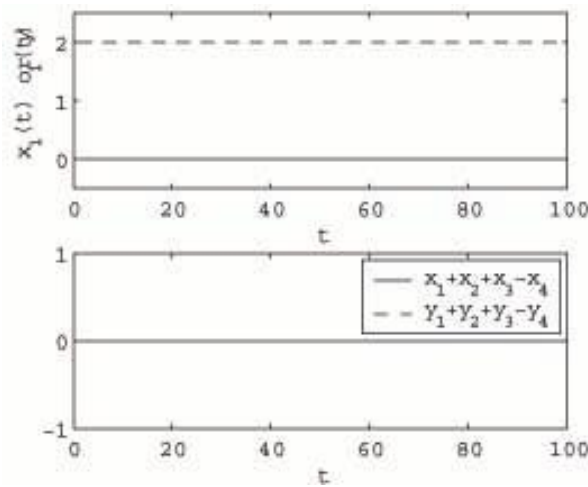
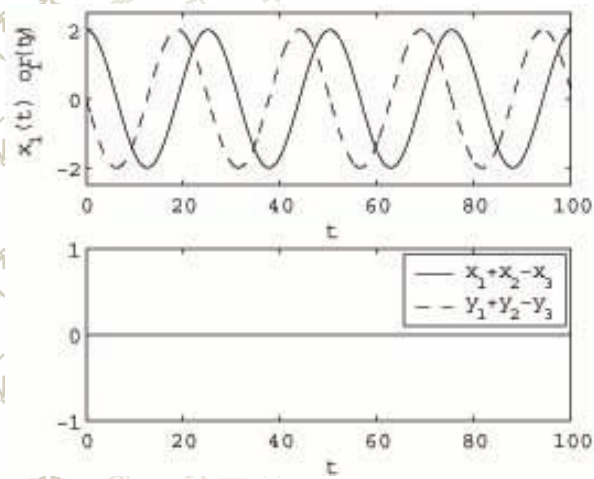
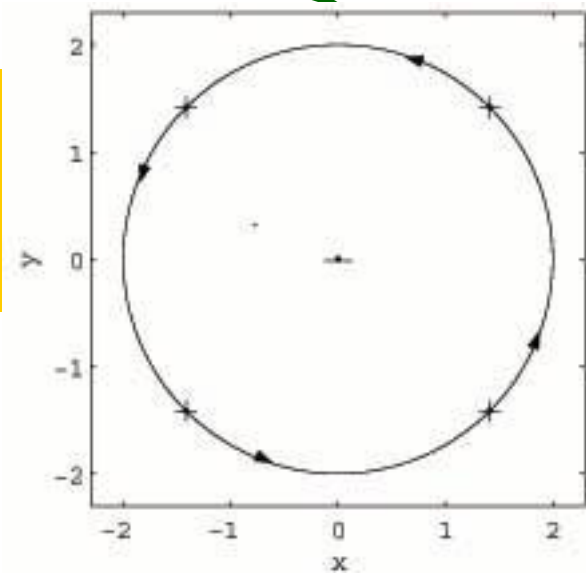
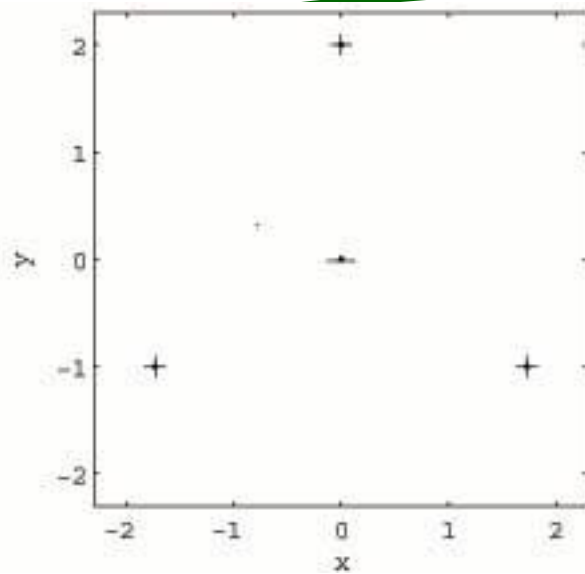
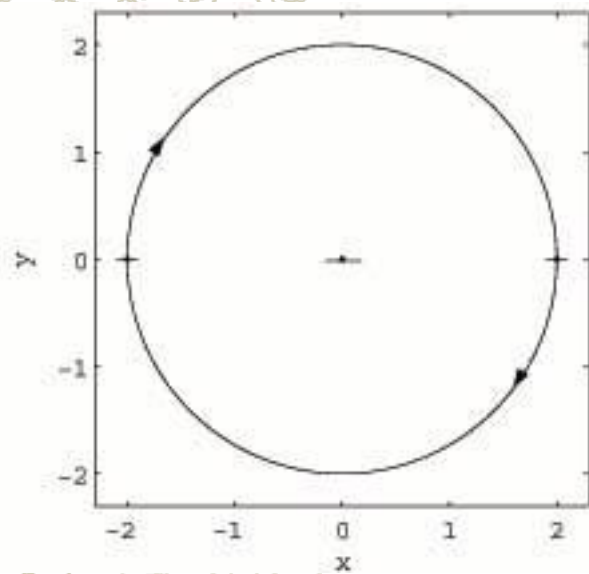


N=3

N=4

N=5

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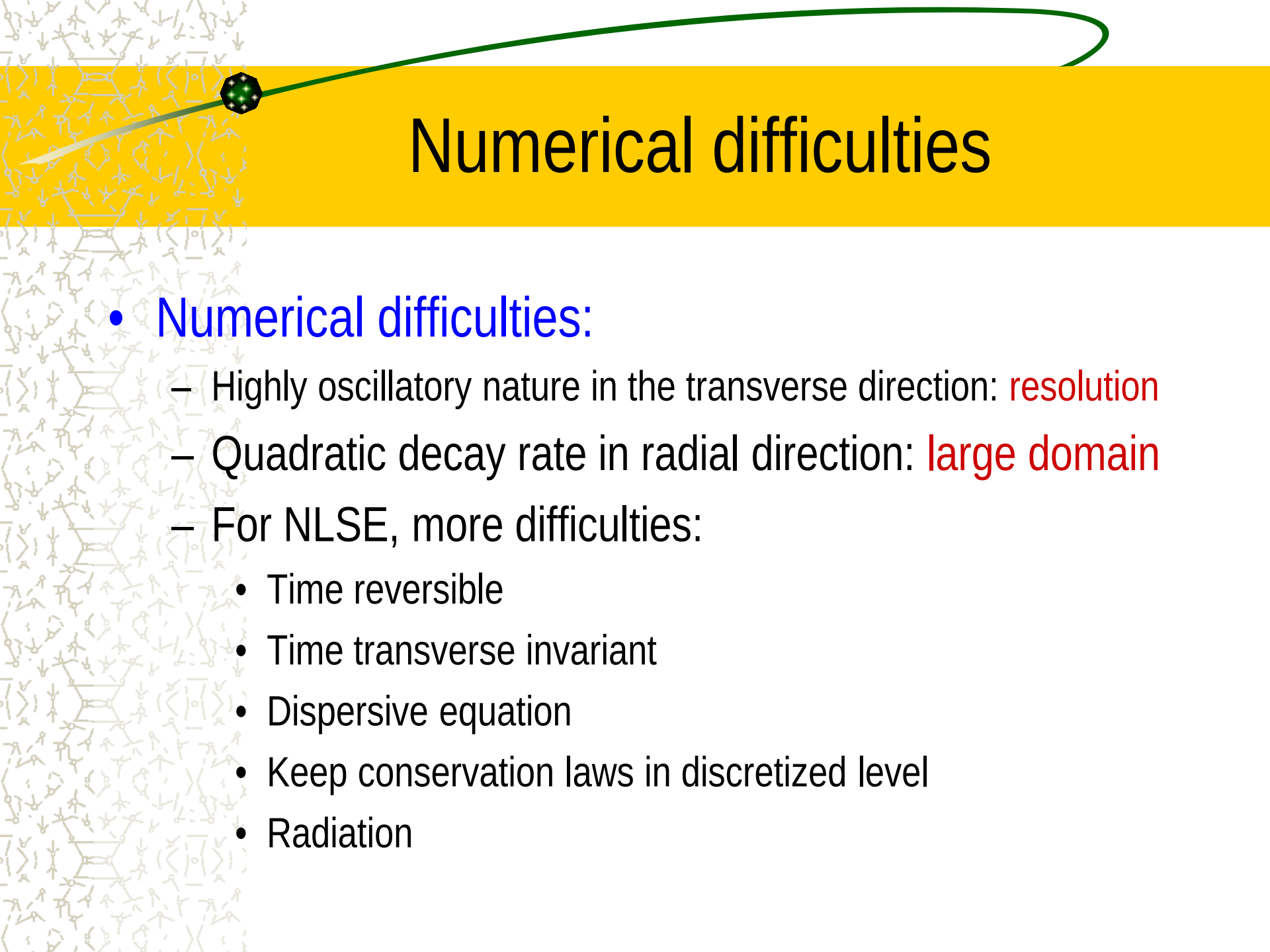


N=3

N=4

N=5

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Numerical difficulties

- Numerical difficulties:

- Highly oscillatory nature in the transverse direction: resolution
- Quadratic decay rate in radial direction: large domain
- For NLSE, more difficulties:
 - Time reversible
 - Time transverse invariant
 - Dispersive equation
 - Keep conservation laws in discretized level
 - Radiation

Numerical methods

- For 'bright-tail' vortex (Bao, Du & Zhang, Eur. J. Appl. Math., 07')
 - Apply a time-splitting technique: decouple nonlinearity
 - Adapt polar coordinate: resolve solution in transverse better
 - Use Fourier pseudo-spectral discretization in transverse
 - Use 2nd or 4th order FEM or FDM in radial direction
- For 'dark-tail' vortex (Bao & Zhang, M3AS, 05')
 - Apply a time-splitting technique: decouple nonlinearity
 - Use Fourier pseudo-spectral discretization
 - Use generalized Laguerre-Hermite pseudo-spectral method

Vortex dynamics & interaction

- Data chosen (Bao, Du & Zhang, SIAM, J. Appl. Math., 07')

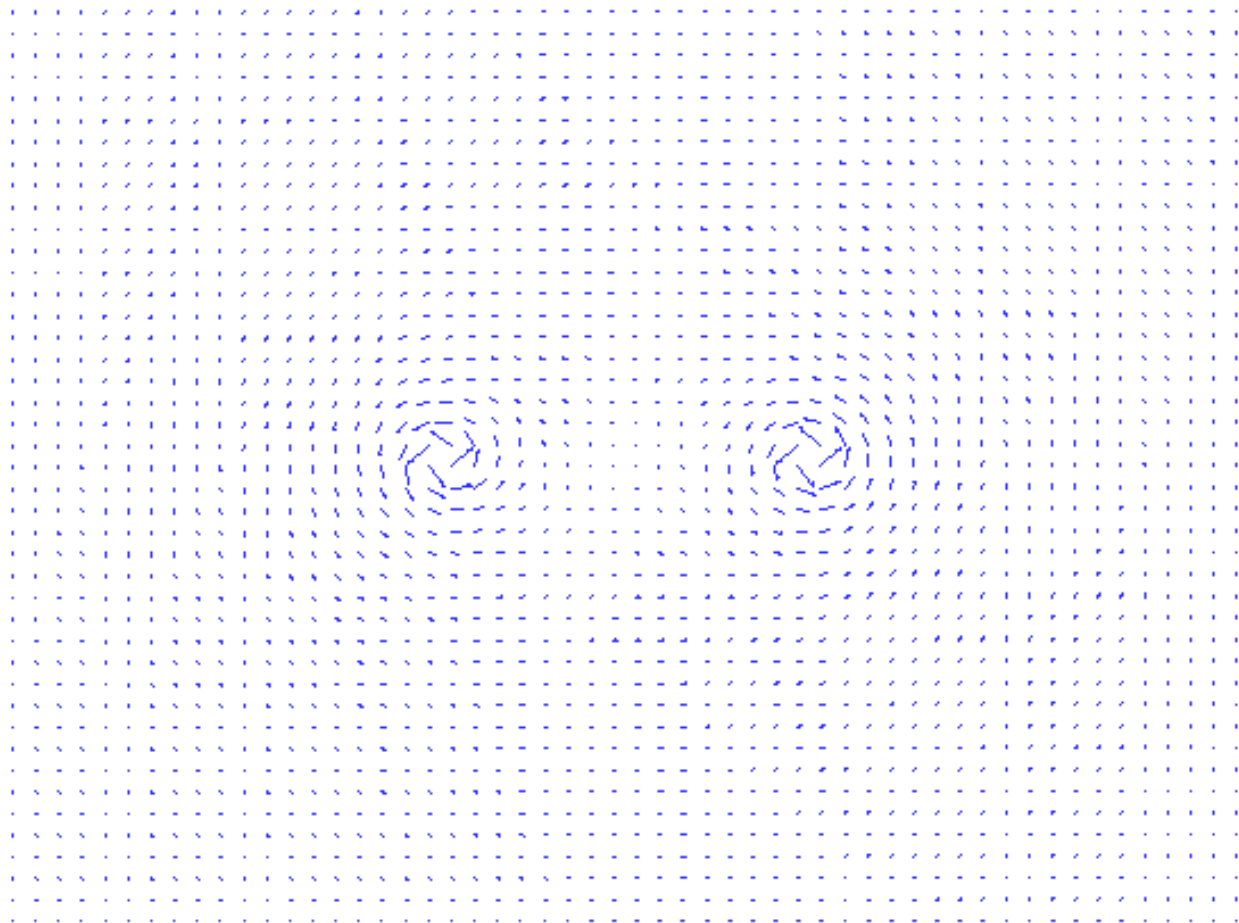
$$\varepsilon = 1, \quad V(\vec{x}, t) \equiv 1, \quad \psi_0(\vec{x}) = \prod_{j=1}^N \phi_{n_j}(\vec{x} - \vec{x}_j), \quad m = \sum_{j=1}^N n_j$$

- Two vortices with like winding numbers

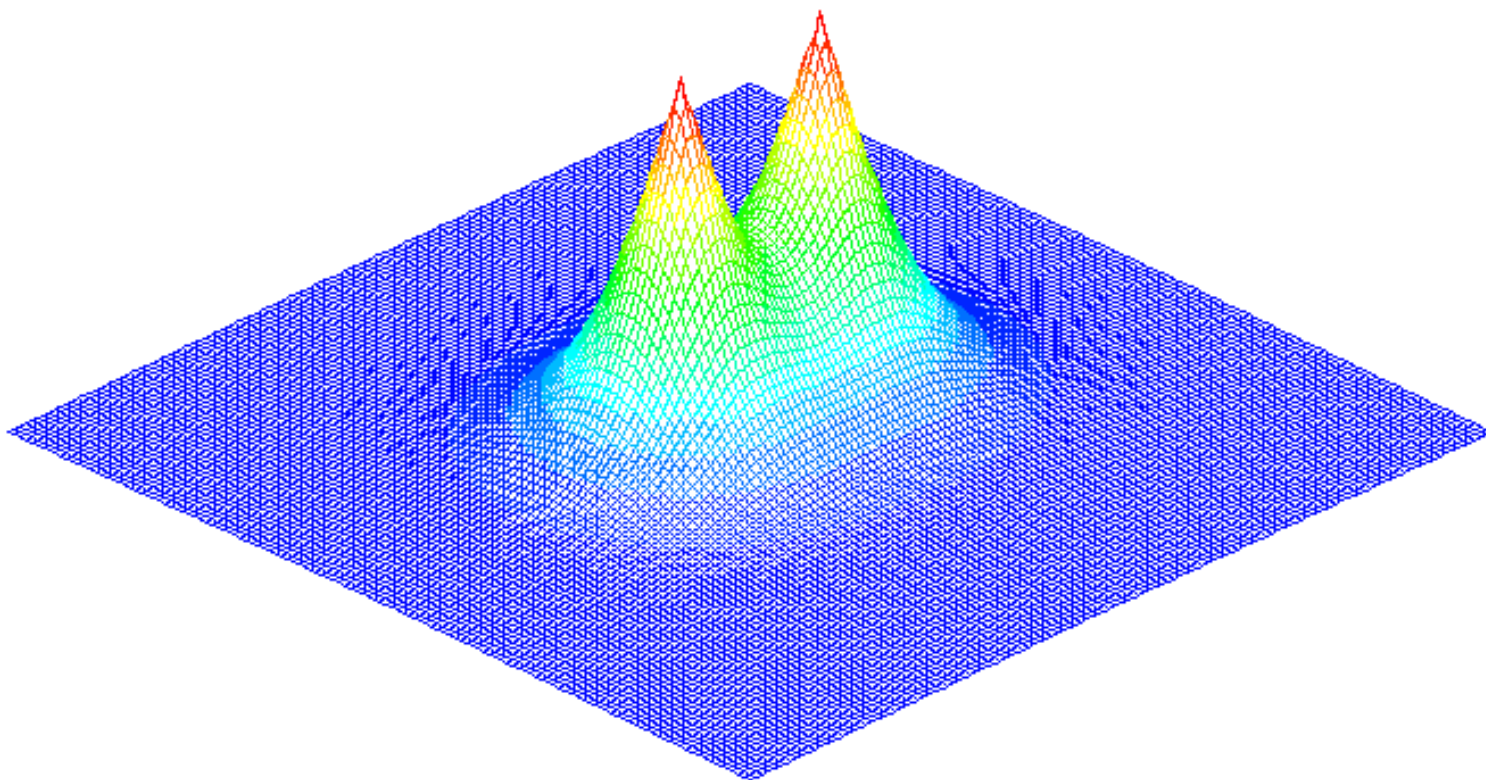
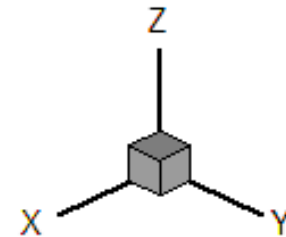
$$N = 2, \quad n_1 = 1, \quad n_2 = 1, \quad \vec{x}_1 = (a, 0), \quad \vec{x}_2 = (-a, 0)$$

- For GLE: velocity density
- For GPE: velocity density
- Trajectory

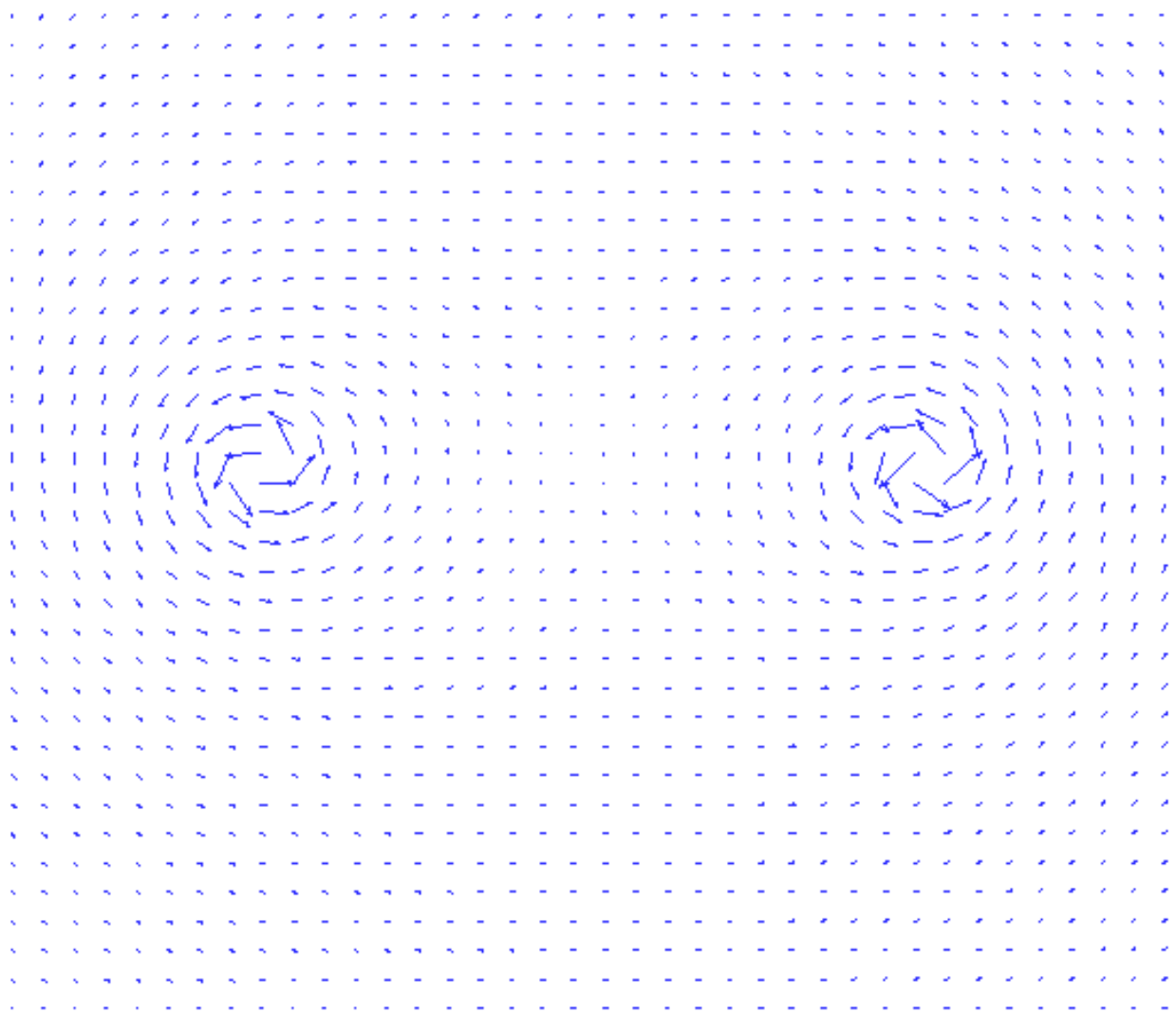
- Summary



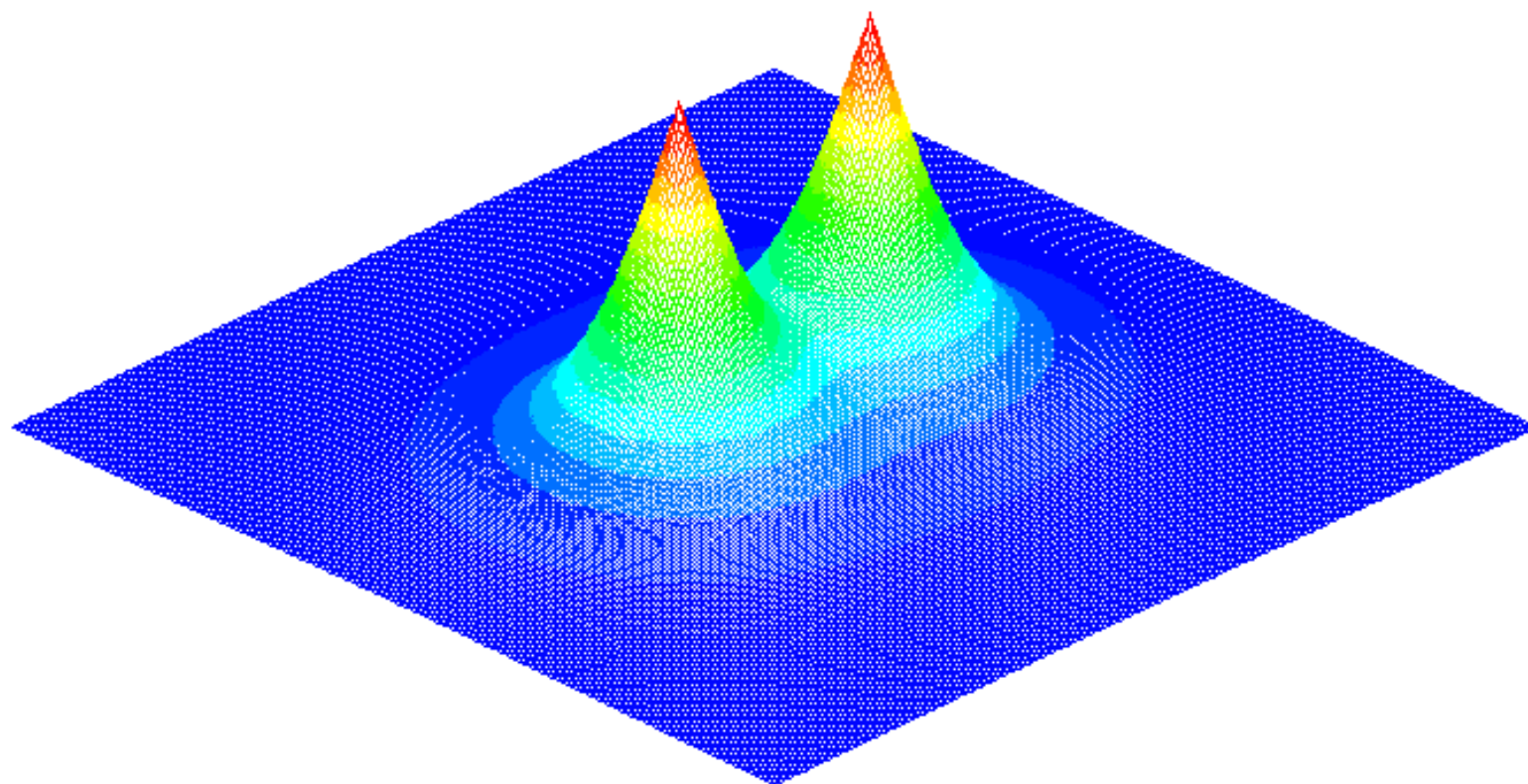
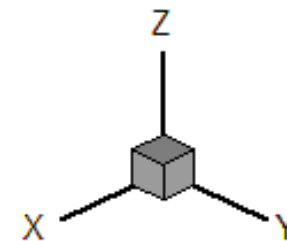
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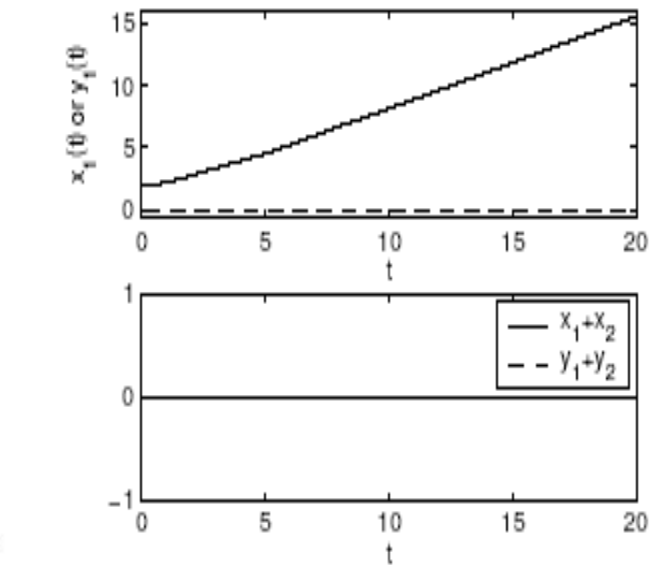
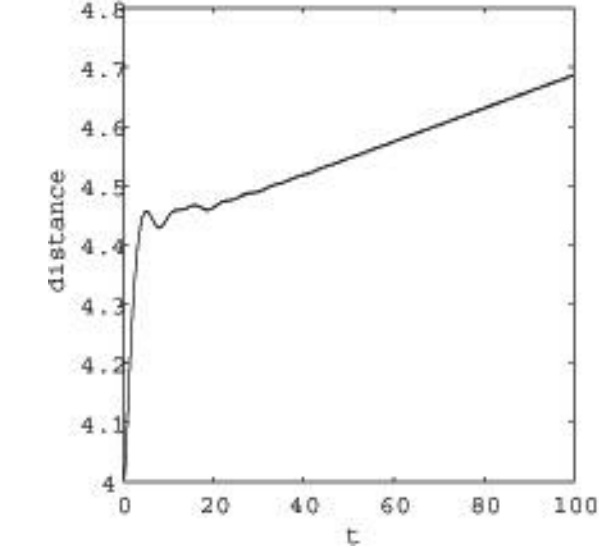
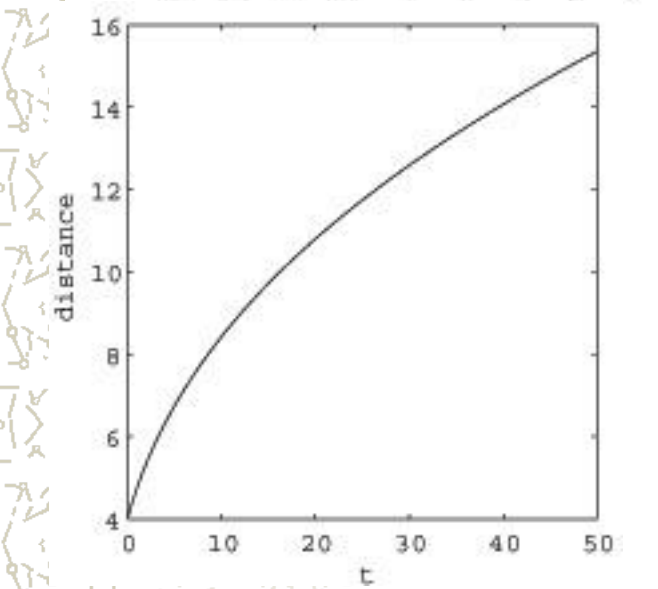
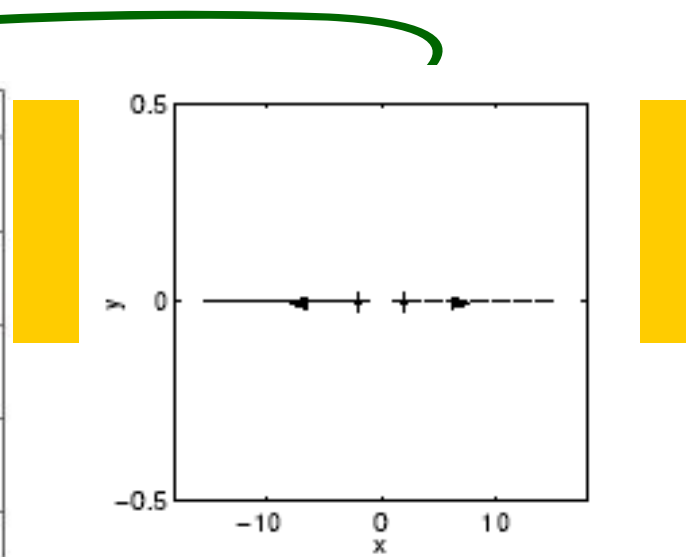
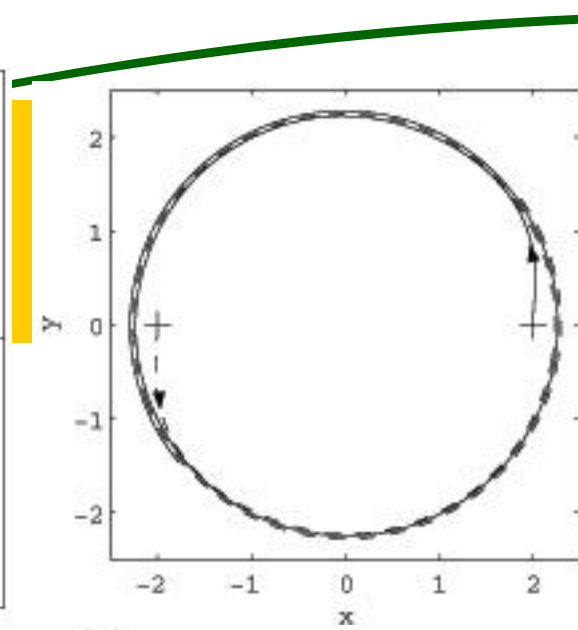
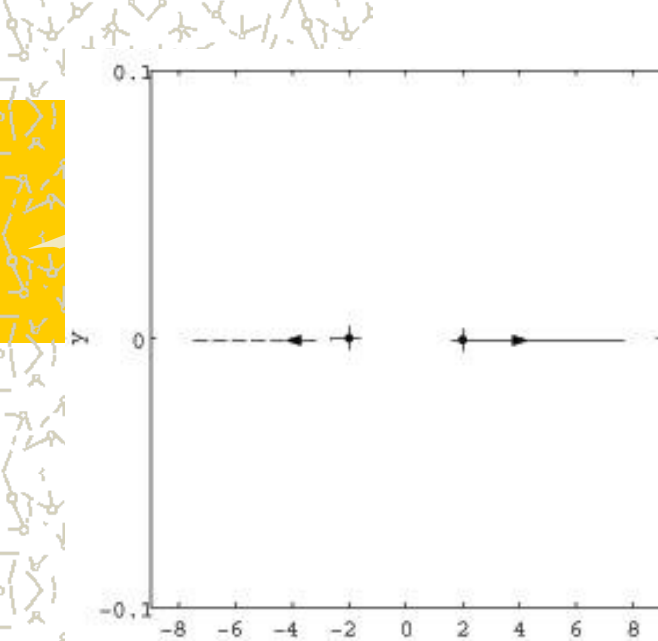
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GLE

GPE

NLWE

back

Summary of interaction

- **For GLE** (Bao, Du & Zhang, SIAP, 07')
 - Undergo a repulsive interaction
 - Core size of the two vortices will well separated
 - Energy decreasing: $E(\psi(\vec{x}, t)) \rightarrow 2E(\psi_1(\vec{x})), \quad t \rightarrow \infty$
- **For NLSE** (Bao, Du & Zhang, SIAP, 07')
 - Behave like point vortices in ideal fluid
 - The two vortex centers move almost along a circle
 - Energy conservation & radiation
- **For NLWE** (Bao & Zhang, Physica D 07')
 - Similar as GLE but with different speed

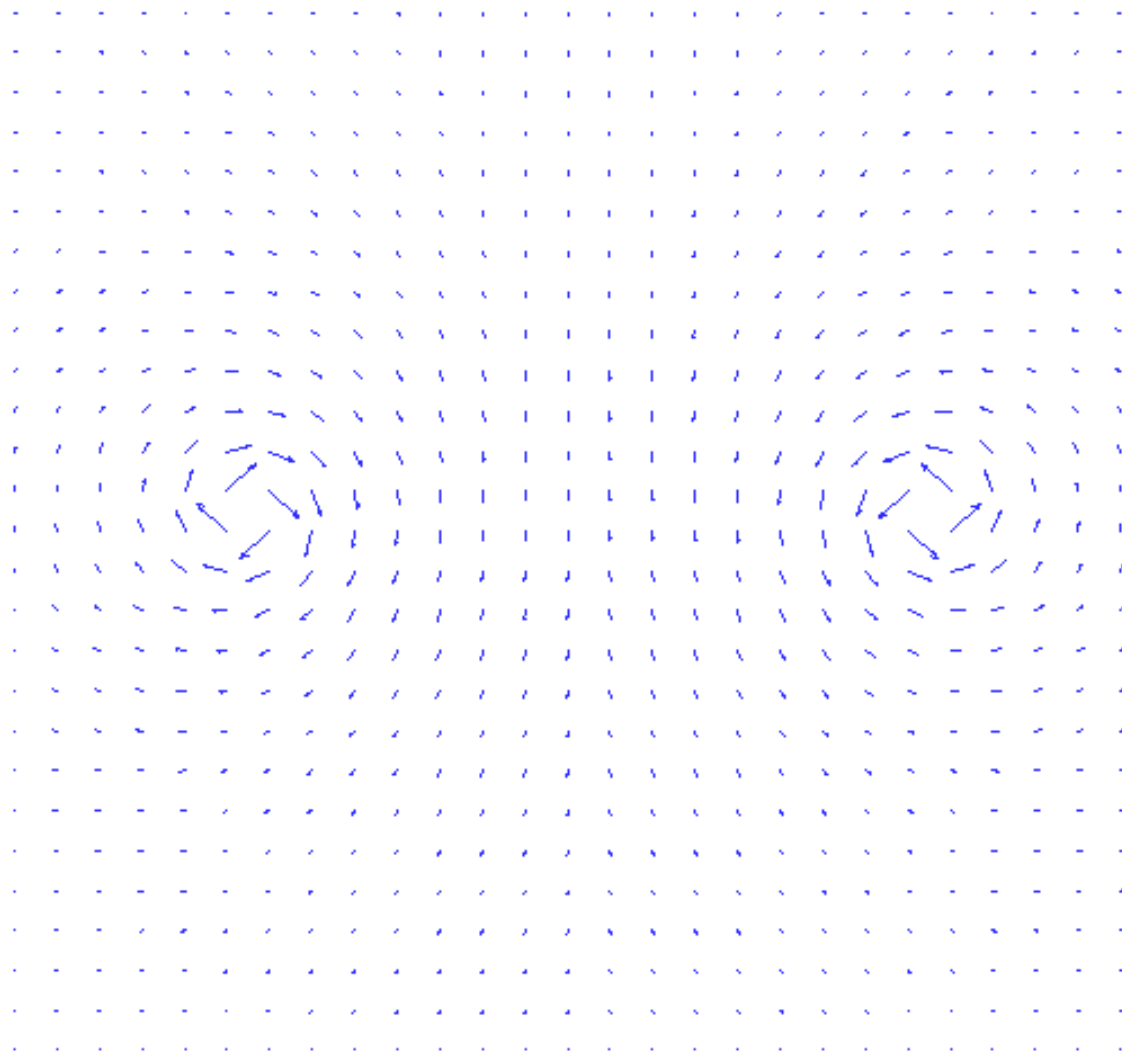
Vortex dynamics & interaction

- Two vortices with opposite winding numbers

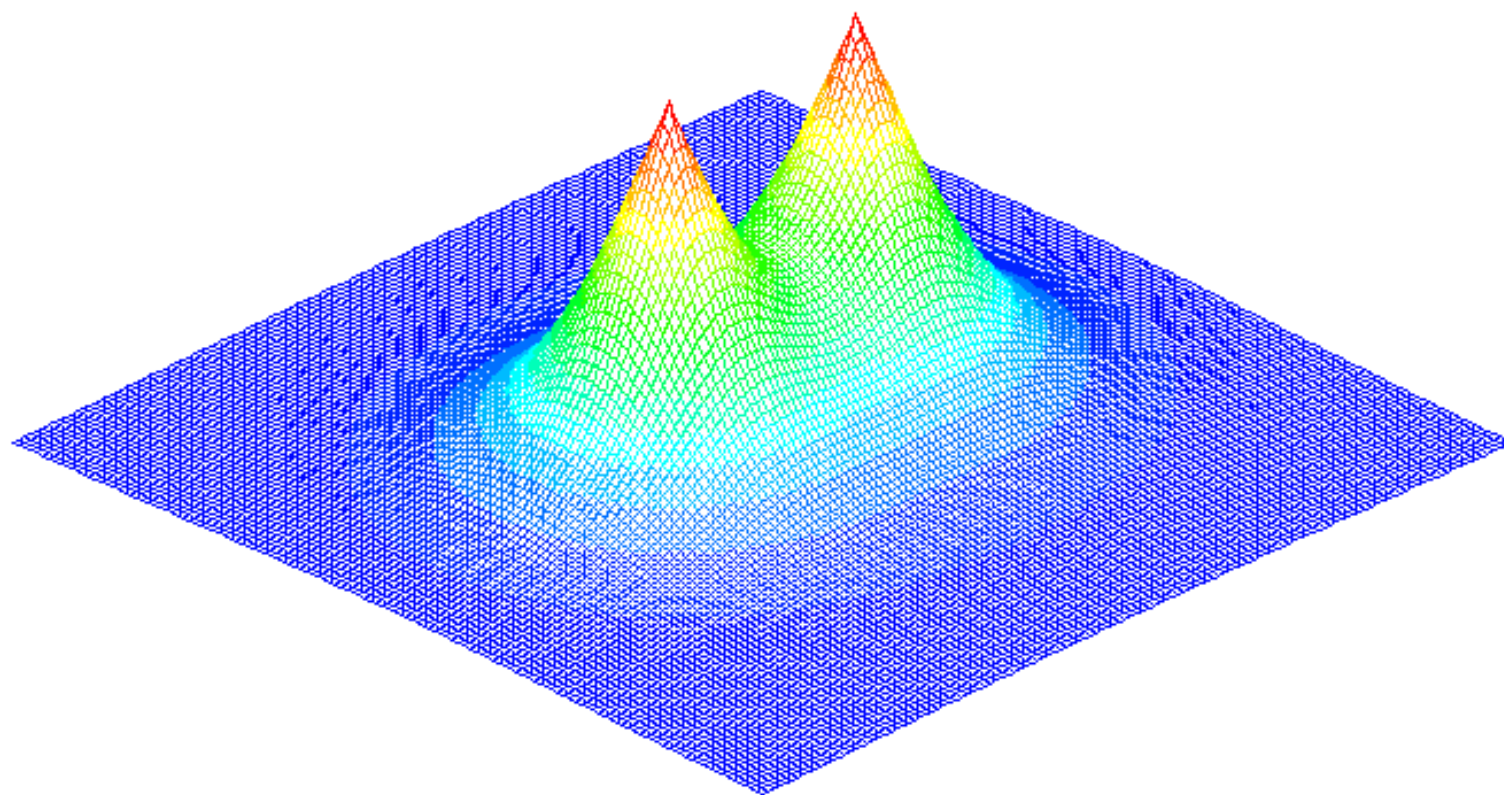
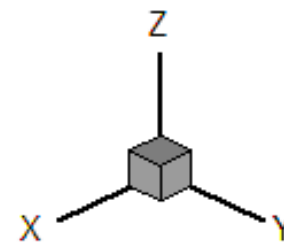
$$N = 2, \quad n_1 = 1, \quad n_2 = -1, \quad \vec{x}_1 = (a, 0), \quad \vec{x}_2 = (-a, 0)$$

- For GLE: velocity density
- For NLSE:
 - Case 1: velocity density
 - Case 2: velocity density
- Trajectory

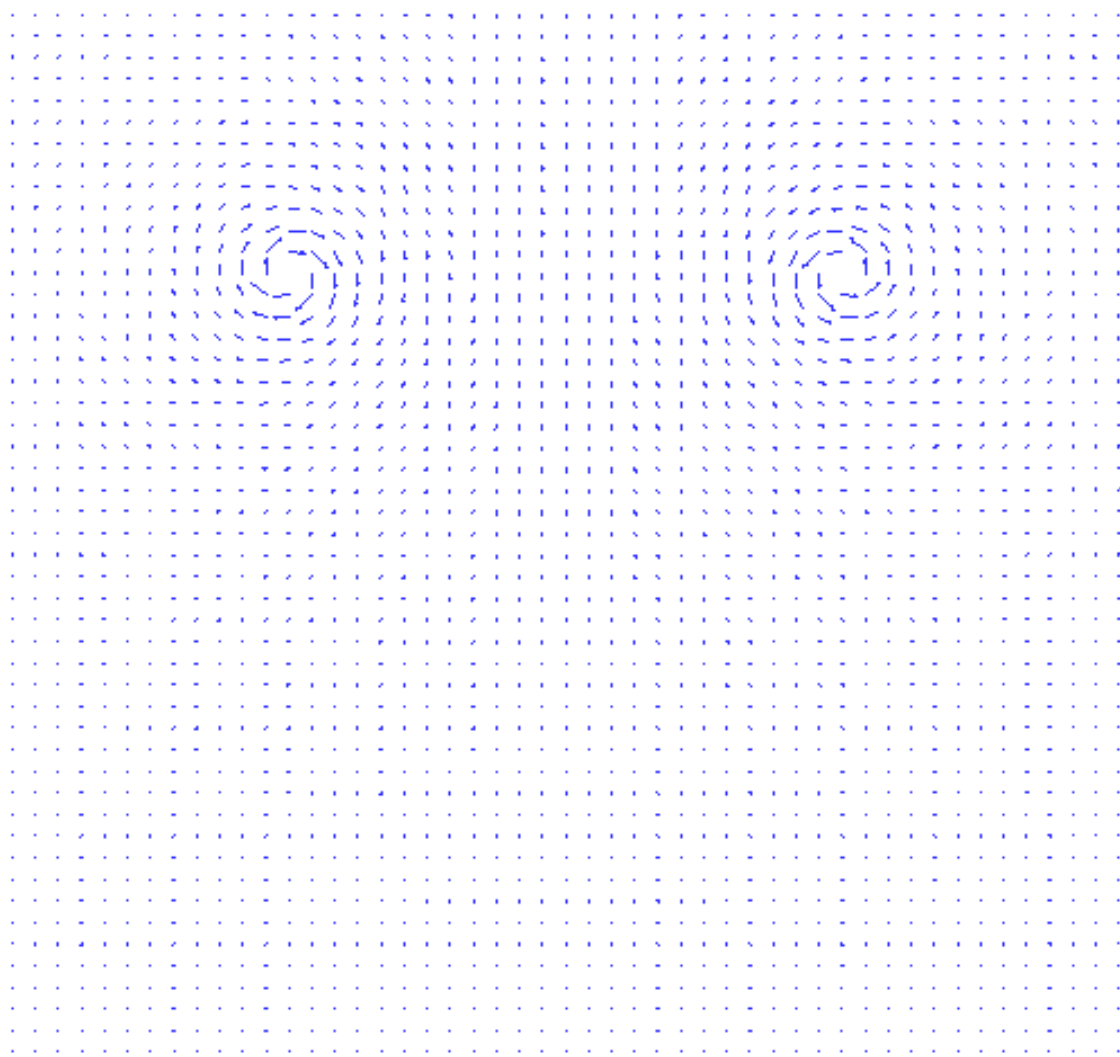
- Summary



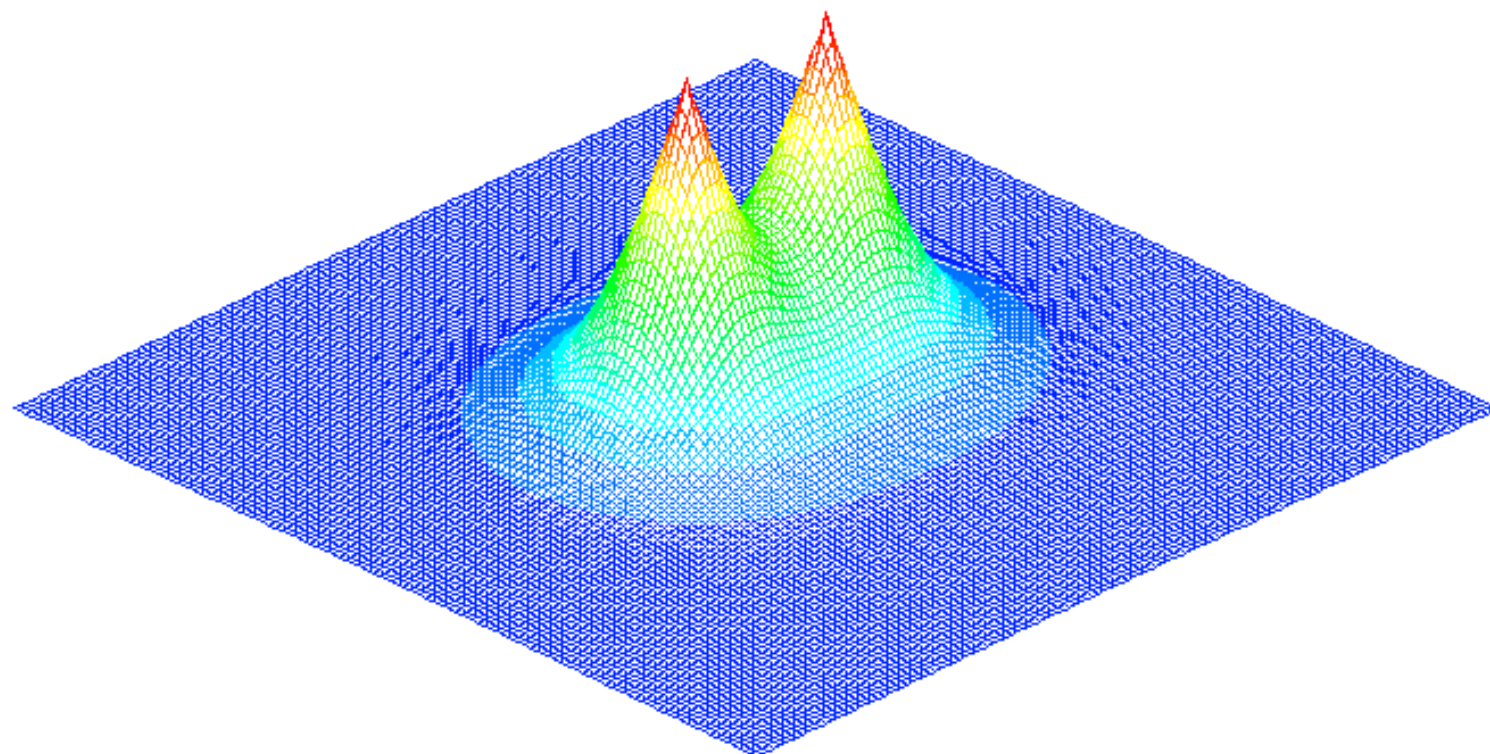
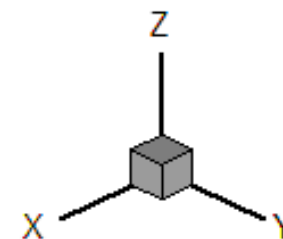
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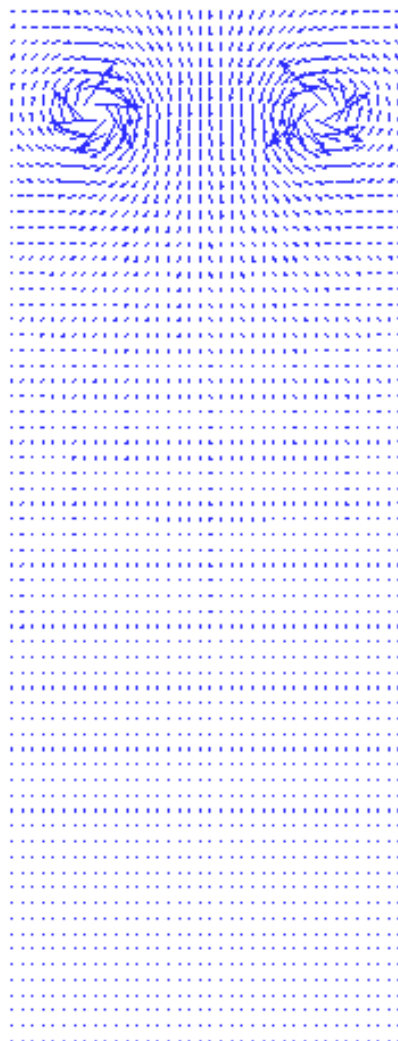
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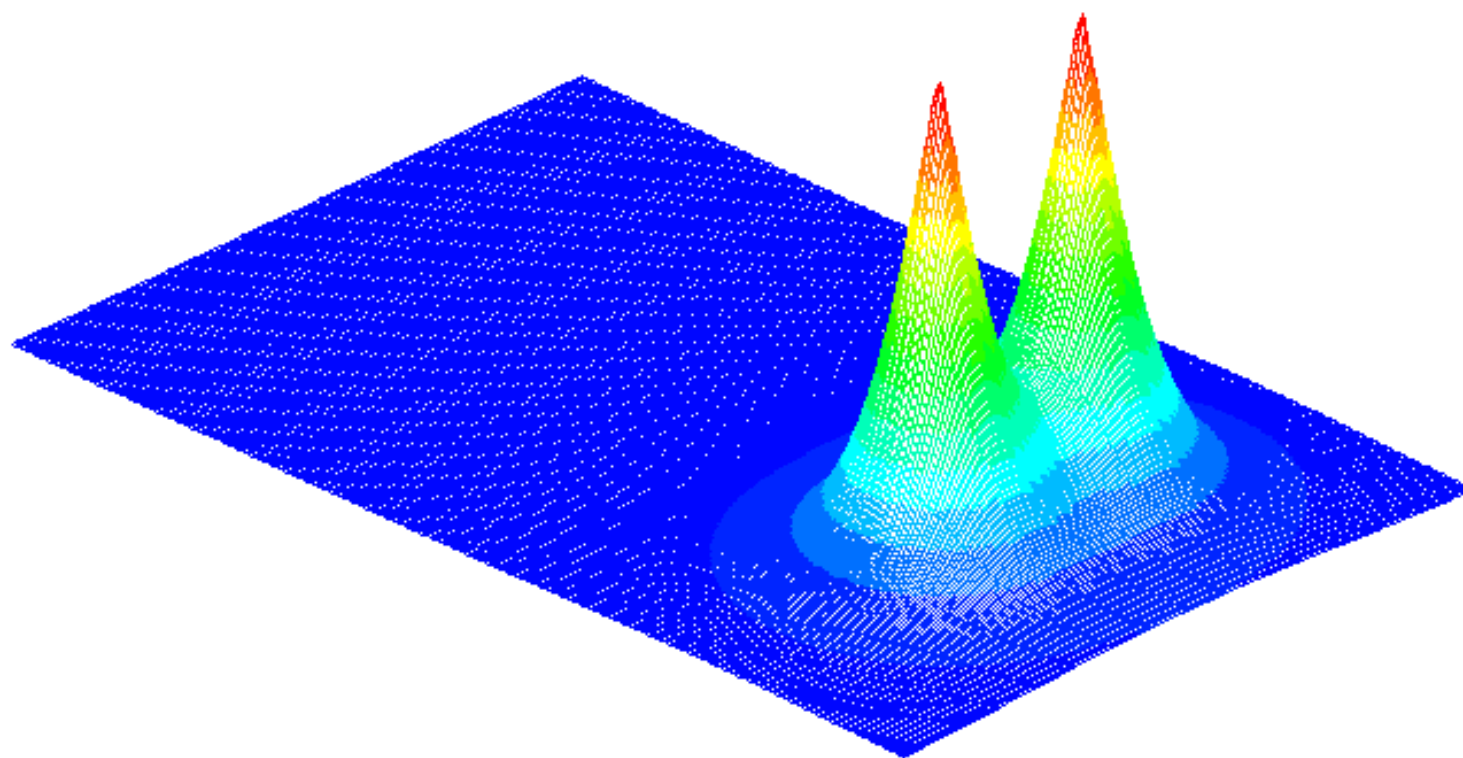
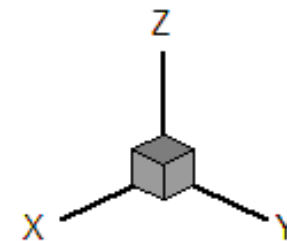


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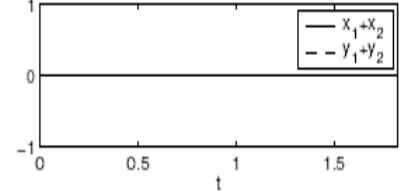
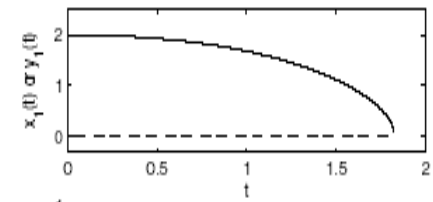
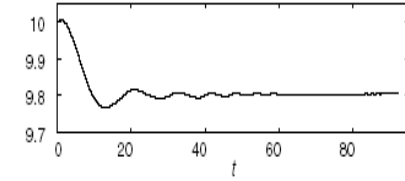
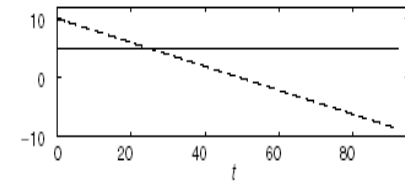
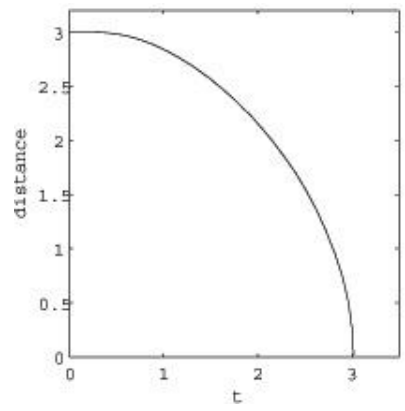
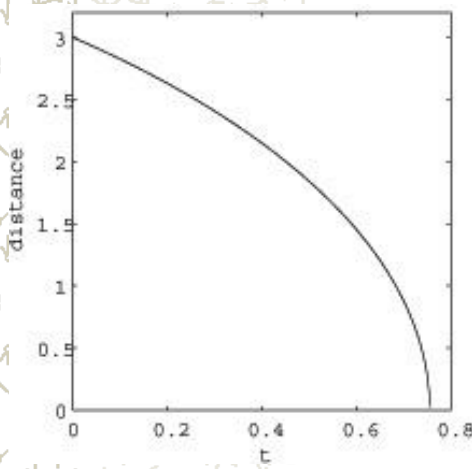
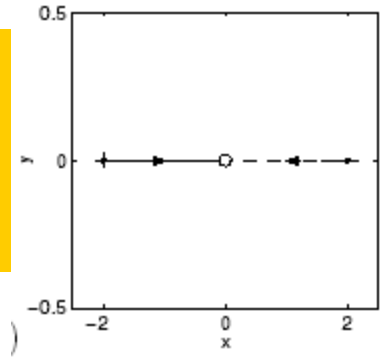
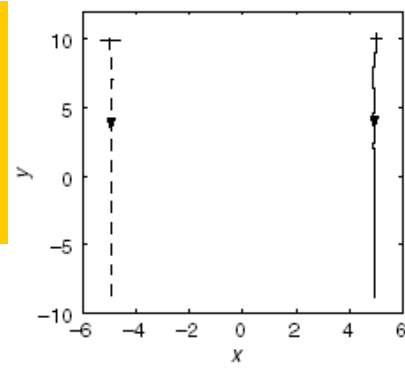
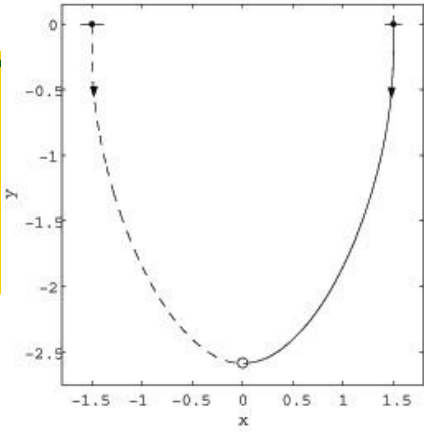
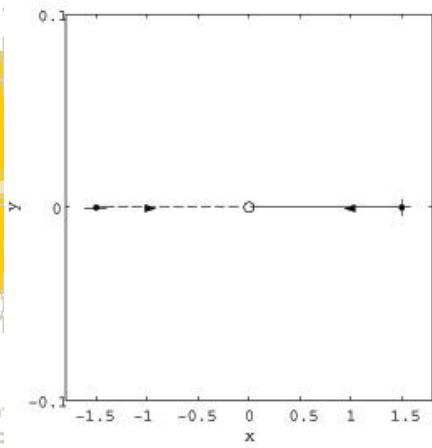


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GLE

NLSE

NLSE

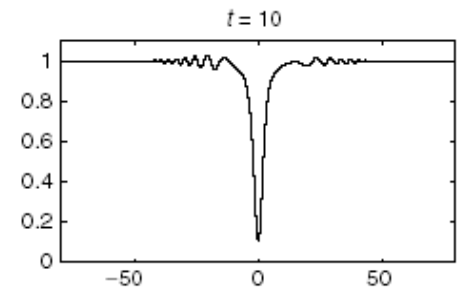
NLWE

back

Summary of interaction

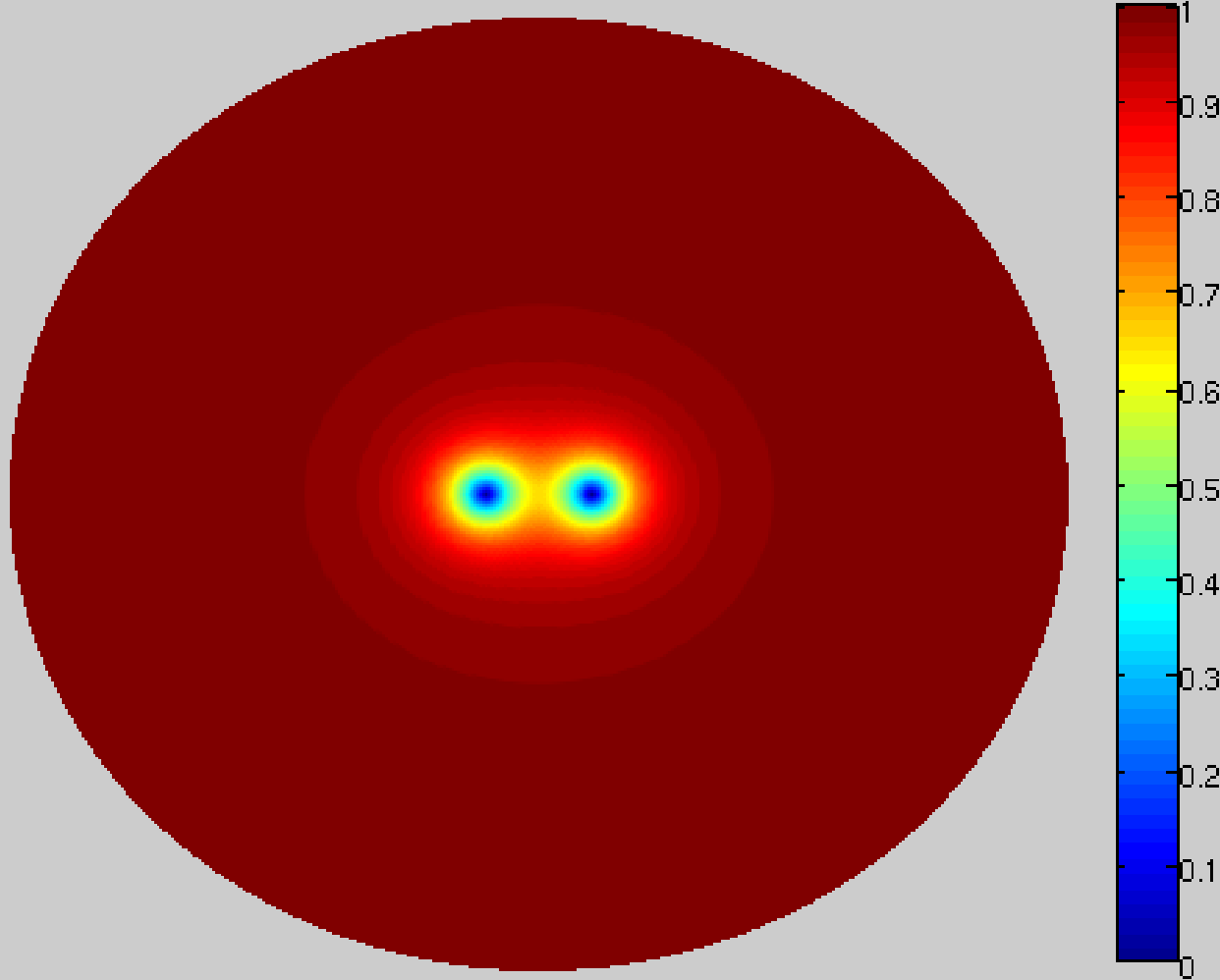
- **For GLE** (Bao, Du & Zhang, SIAP, 07')
 - Undergo an attractive interaction & merge at finite time
 - Energy decreasing: $E(\psi(\vec{x}, t)) \rightarrow 0, \quad t \rightarrow \infty$

- **For GPE** (Bao, Du & Zhang, SIAP, 07')
 - depends on initial distance of the two centers
 - Small: merge at finite time & generate shock wave
 - Large: move almost paralleling & don't merge, solitary wave
 - Energy conservation & radiation
 - Sound wave generation

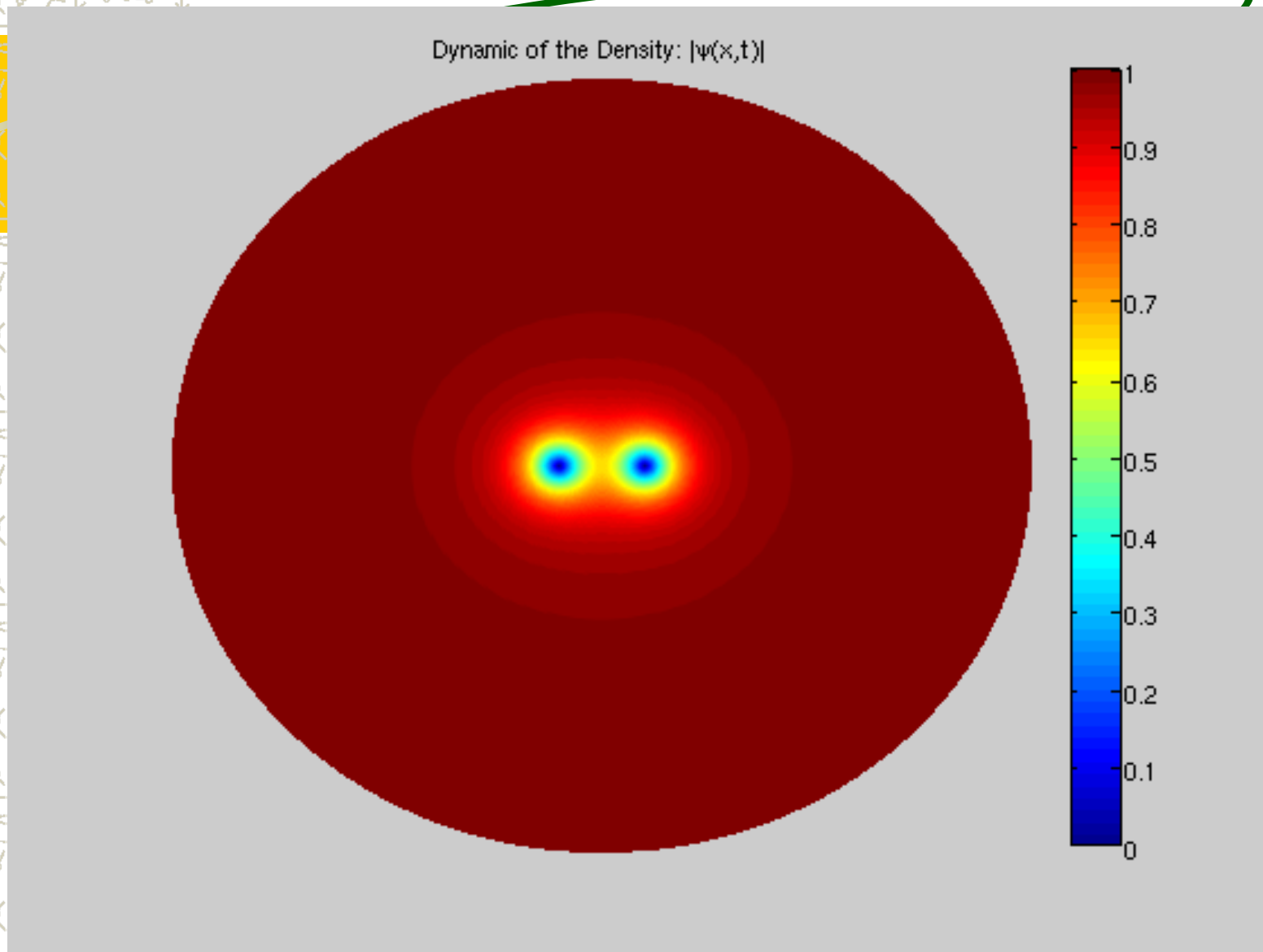


- **For NLWE** (Bao & Zhang, Physica D 07')
 - Undergo an attractive interaction & merge at finite time

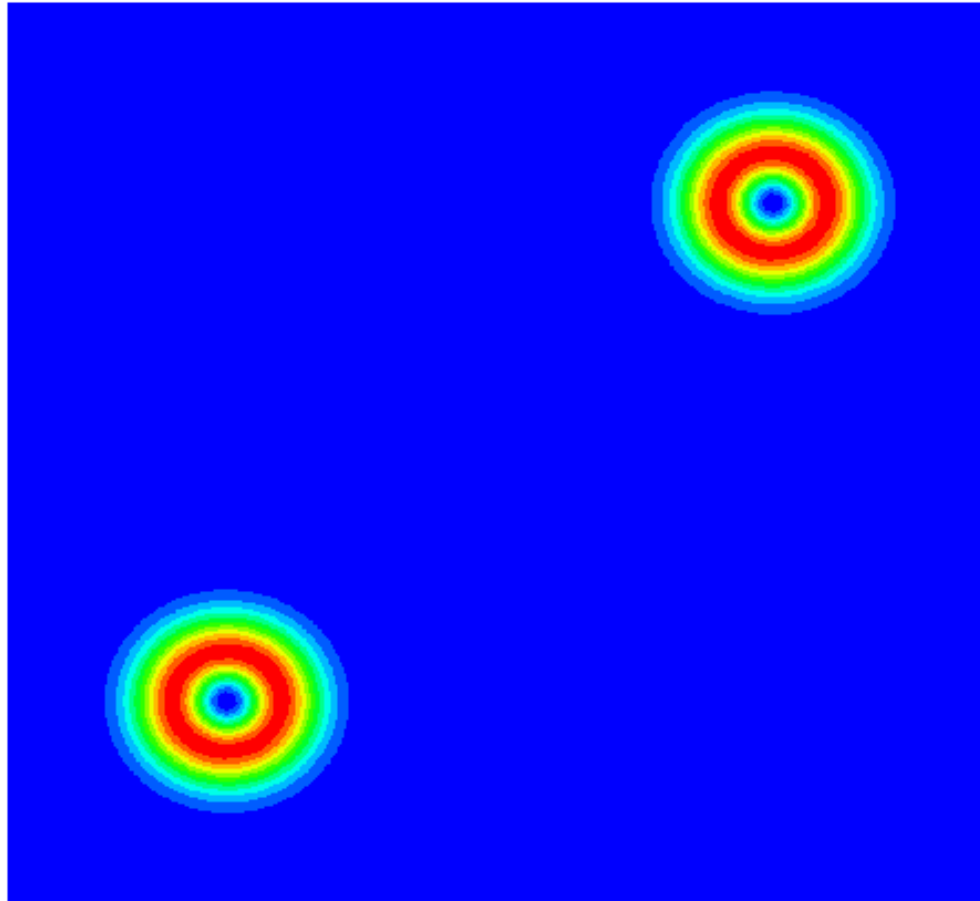
Dynamic of the Density: $|\psi(x,t)|$



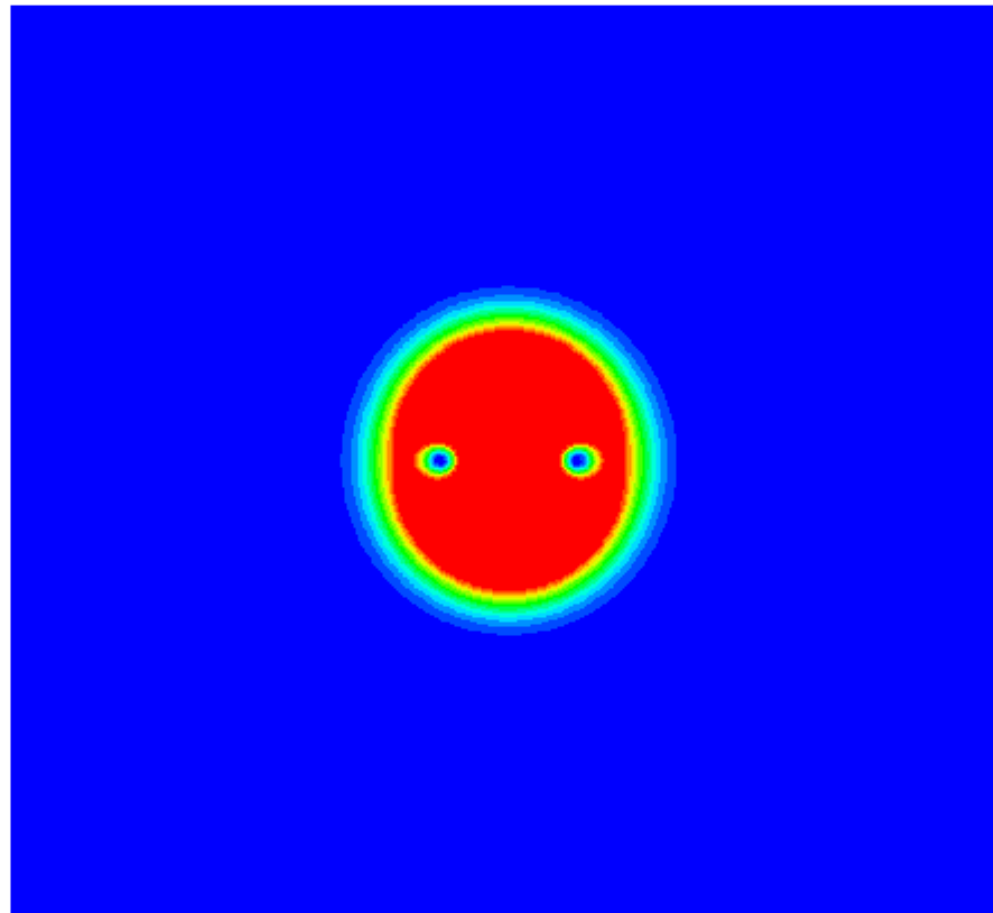
Vortex-pair on bounded domain



Vortex-dipole on bounded domain

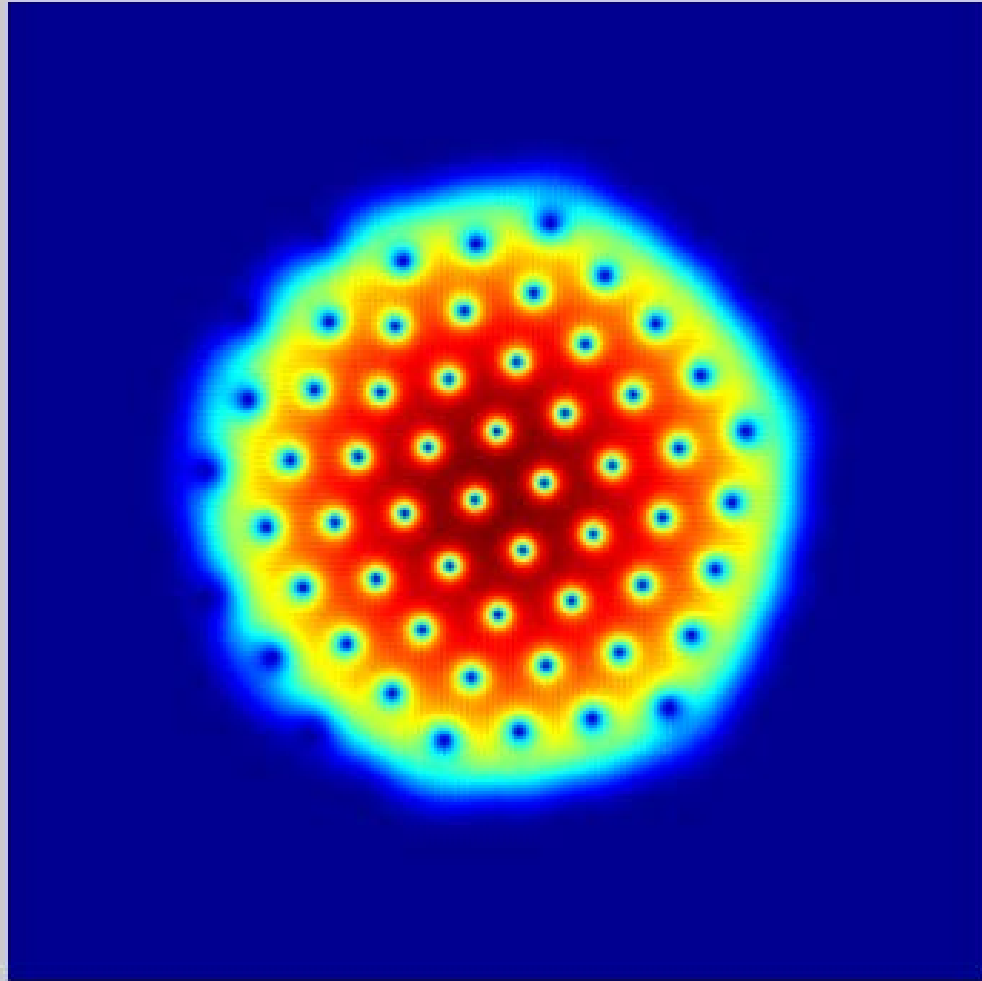


`dark-tail' vortex-pair in BEC – well-separate

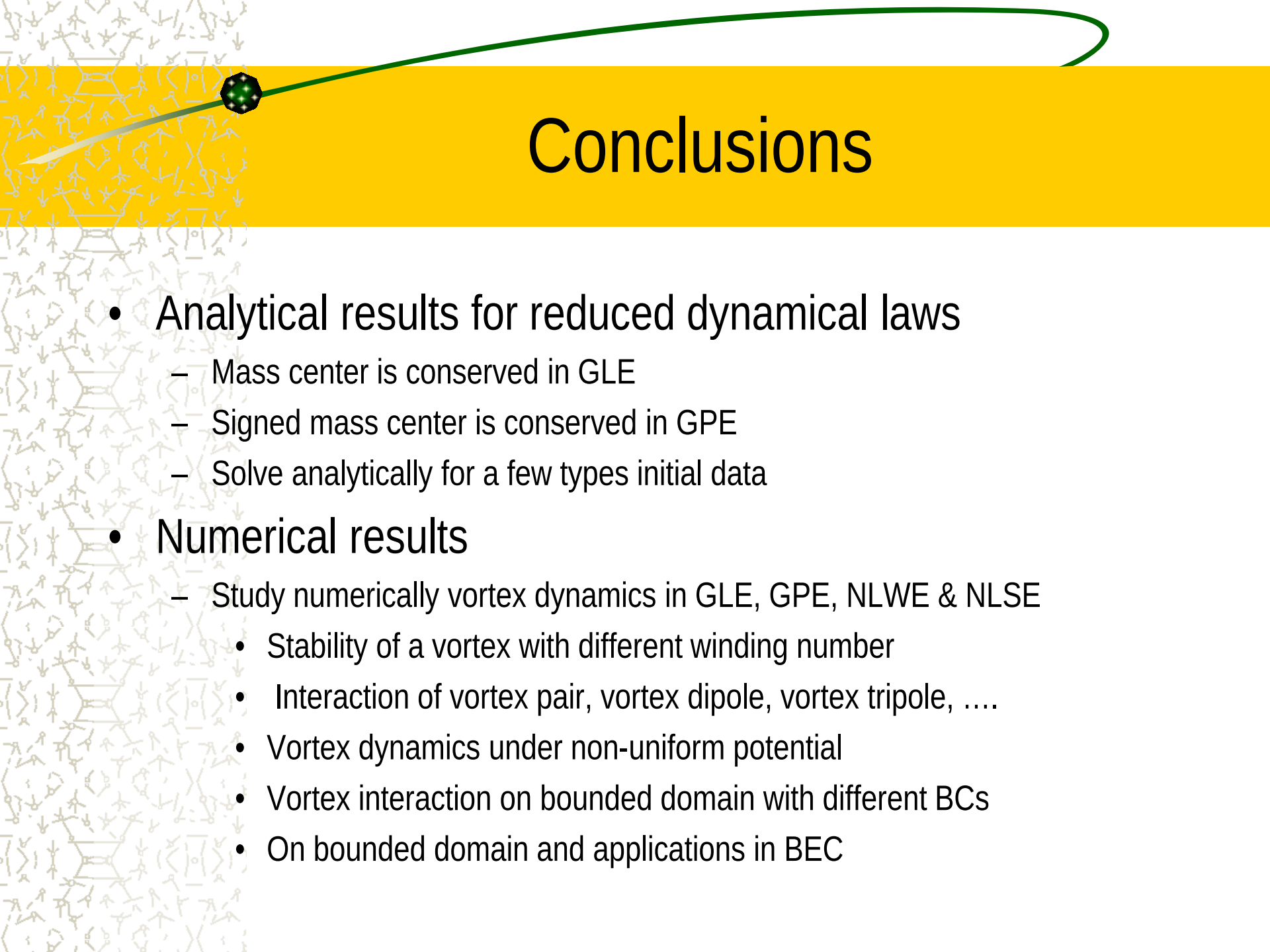


`dark-tail' vortex-dipole in BEC

$t=0$



‘dark-tail’ vortex lattice in BEC



Conclusions

- Analytical results for reduced dynamical laws
 - Mass center is conserved in GLE
 - Signed mass center is conserved in GPE
 - Solve analytically for a few types initial data
- Numerical results
 - Study numerically vortex dynamics in GLE, GPE, NLWE & NLSE
 - Stability of a vortex with different winding number
 - Interaction of vortex pair, vortex dipole, vortex tripole,
 - Vortex dynamics under non-uniform potential
 - Vortex interaction on bounded domain with different BCs
 - On bounded domain and applications in BEC