From Weyl Conjecture to Fundamental Gap Conjecture and Beyond

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Outline

- Eigenvalue problem with Dirichlet BC – Schroedinger/Laplacian operator (SO/LO)
- The Weyl's law and conjecture
  - Weyl's law for Laplacian operator in 1D/2D/3D
  - Weyl conjecture
  - Extension to cases with potentials and other BCs
  - Extension to fractional Schroedinger operator (FSO)
- The fundamental gap conjecture
  - For Laplacian/Schrodinger operator (LO/SO)
  - Extension to fractional Schroedinger operator (FSO)
  - Other gaps and gaps distribution statistics
- Fundamental gaps of constrained nonlinear eigenvalue problem (GPE/NLSE)
- Conclusions

Eigenvalue (Strum-Liouville) Problem

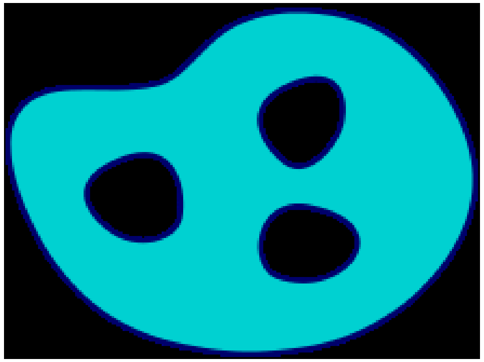
- Consider the eigenvalue problem

$$\left(-\Delta+V(\vec{x})\right)u(\vec{x})=\lambda\, u(\vec{x}),\qquad \vec{x}\in\Omega\subset\mathbb{R}^d$$

  - Ω is bounded, with Dirichlet BC:

$$u(\vec{x})=0,\qquad \vec{x}\in\Gamma=\partial\Omega$$

  - $u(\vec{x})$: eigenfunction (or state in physics)
  - λ : eigenvalue (spectrum or energy level)
  - $V(\vec{x})\in C^0(\Omega)$ or $L^\infty(\Omega)$: real-valued given potential



Hearing a shape of a drum

- Arising from wave equation

$$\partial_{tt}\phi(\vec{x},t)=\left(\Delta-V(\vec{x})\right)\phi(\vec{x},t)\quad\overset{\phi(\vec{x},t)=e^{-i\sqrt{\lambda}\,t}u(\vec{x})}{\Longrightarrow}\quad\left(-\Delta+V(\vec{x})\right)u(\vec{x})=\lambda u(\vec{x})$$

Time-independent Schrodinger Equation

- The Schrodinger equation —Ewin Schrodinger, 1925

$$i\hbar\partial_t\psi(\vec{x},t)=\left(-\frac{\hbar^2}{2m}\Delta+V(\vec{x})\right)\psi(\vec{x},t)$$

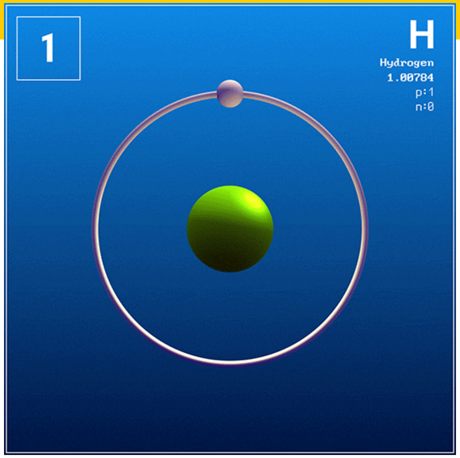
$$\hbar=m=1\Downarrow\psi(\vec{x},t)=e^{-iEt}u(\vec{x})$$

$$\left(-\frac{1}{2}\Delta+V(\vec{x})\right)u(\vec{x})=Eu(\vec{x})\Rightarrow L_{SO}u:=\left(-\frac{1}{2}\Delta+V(\vec{x})\right)u(\vec{x})=Eu(\vec{x})$$

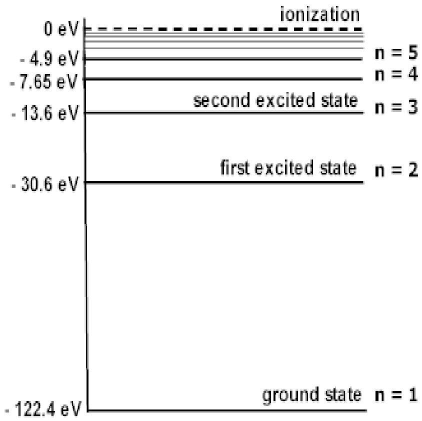
- Quantum Many-body Problem – Paul Dirac
  - Quantum physics/chemistry -- DFT
  - Materials science– electronic structure, materials simulation & design

$$\left[\sum_{j=1}^{N}\left(-\frac{\hbar^2}{2m}\Delta_j+V(\vec{r}_j)\right)+\sum_{1\le j<k\le N}V_{\text{int}}(\vec{r}_j-\vec{r}_k)\right]\Phi=E\,\Phi$$

- Many other applications, ……



Hydrogen atom



Spectrum of Hydrogen

Eigenvalues & Eigenfunctions

$$\left(-\Delta + V(\vec{x})\right)u(\vec{x}) = \lambda\, u(\vec{x}), \quad \vec{x}\in\Omega$$

$$u(\vec{x}) = 0, \quad \vec{x}\in\Gamma = \partial\Omega$$

- Eigenvalues

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots \quad \overset{V(\vec{x})\ge 0 \,\&\, \text{weakly convex}}{\Longrightarrow} \quad 0 < \lambda_1 < \lambda_2 \le \ldots \le \lambda_n \le \ldots$$

  – with the times an eigenvalue appear being its algebraic multiplicity

- Eigenfunctions can be taken as orthonormal

$$u_1, u_2, \ldots, u_n, \ldots \Rightarrow \|u_n\|_{L^2}^2 = \int_\Omega u_n(\vec{x})^2\, d\vec{x} = 1, \quad (u_n, u_l) = \delta_{nl}$$

- Rayleigh quotient – variational formulation : $E(v) = \int_\Omega \left(|\nabla v|^2 + V(\vec{x})v^2\right) d\vec{x}$

$$\lambda_1 = \min_{0\ne v\in H_0^1(\Omega)} \frac{E(v)}{\|v\|_{L^2}^2}, \quad \lambda_{n+1} = \min_{0\ne v\in U_n^\perp} \frac{E(v)}{\|v\|_{L^2}^2}, \quad U_n = \text{span}\{u_1, \ldots, u_n\},\, n\ge 1$$

Asymptotics of Eigenvalues

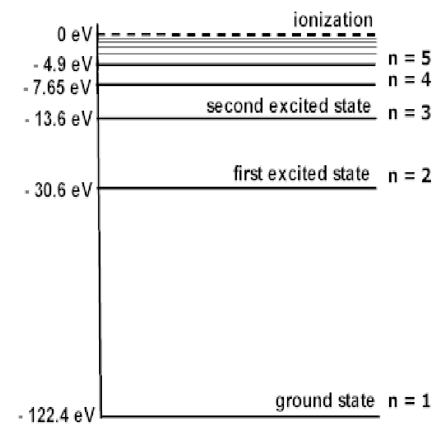
$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots$$

- The number of states below a given energy

$$\lambda_n = O\left(n^?\right),\, n \gg 1 \;\Rightarrow N(E) := \#\{n \in \mathbb{Z}^+ \mid \lambda_n \le E\} = O(E^{??}),\, E \gg 1$$

$$\Downarrow \exists \gamma > 0 \,\&\, C > 0$$

$$\lambda_n = Cn^\gamma + o(n^\gamma),\, n \gg 1 \;\Rightarrow N(E) = \left(\frac{E}{C}\right)^{1/\gamma} + o(E^{1/\gamma}), \quad E \gg 1$$



- An example in 1D:

$$-u''(x) = \lambda u(x), \quad x \in (0, S); \quad u(0) = u(S) = 0$$

  – All eigenvalues

$$\lambda_n = \frac{n^2 \pi^2}{S^2}, \quad n = 1, 2, \ldots \Rightarrow \lambda_n = \frac{\pi^2}{S^2} n^2,\, n \gg 1 \Leftrightarrow N(E) = \frac{S}{\pi} E^{1/2},\, E \gg 1$$

Asymptotics of Eigenvalues

$$0<\lambda_1<\lambda_2\le\ldots\le\lambda_n\le\ldots$$

- Another example in 2D:

$$-\Delta u(x,y)=\lambda\, u(x,y),\quad (x,y)\in\Omega=(0,L_1)\times(0,L_2)$$

$$u(x,y)=0,\quad (x,y)\in\Gamma=\partial\Omega$$

  - All eigenvalues:

$$\tilde{\lambda}_{lk}=\frac{l^2\pi^2}{L_1^2}+\frac{k^2\pi^2}{L_2^2},\quad l,k=1,2,\ldots$$

  - Re-arrange them and obtained by Hermann Weyl, 1911

$$0<\lambda_1<\lambda_2\le\ldots\le\lambda_n\le\ldots \overset{L_1=L_2=1}{\Rightarrow} \lambda_1=2\pi^2,\lambda_2=\lambda_3=5\pi^2,\lambda_4=8\pi^2,\ldots$$

$$N(E)=\frac{S}{4\pi}E+o(E),\ E\gg 1 \overset{S=|\Omega|=L_1L_2}{\Leftrightarrow} \lambda_n=\frac{4\pi}{S}n+o(n),\ n\gg 1$$

The Weyl's Law

$$-\Delta u(\vec{x}) = \lambda u(\vec{x}), \quad \vec{x} \in \Omega \subset \mathbb{R}^d$$

$$u(\vec{x}) = 0, \quad \vec{x} \in \Gamma = \partial\Omega$$

- All eigenvalues:

$$0 < \lambda_1 < \lambda_2 \le \ldots \le \lambda_n \le \ldots$$

- The number of states below an energy *E*

$$N(E) := \#\{n \in \mathbb{Z}^+ \mid \lambda_n \le E\}, \qquad E > 0$$

- Hermann Weyl proved in 1D&2D&3D (1911 by counting states & 1912 by variational method):

$$\lim_{E\to+\infty} \frac{N(E)}{E^{d/2}} = \frac{\omega_d S}{(2\pi)^d} \quad \Leftrightarrow \lambda_n = \frac{4\pi^2}{(\omega_d S)^{2/d}} n^{2/d} + o(n^{2/d}), \quad n \gg 1$$

$$S = |\Omega|, \quad \omega_d \text{ --the volume of the unit ball in } \mathbb{R}^d, \quad \omega_1 = 2, \omega_2 = \pi, \ldots$$

  - In 2D: $N(E) = \dfrac{S}{4\pi} E + o(E), E \gg 1 \Leftrightarrow \lambda_n = \dfrac{4\pi}{S} n + o(n), n \gg 1$

https://en.wikipedia.org/wiki/Weil_conjectures

The Weyl Conjecture

$$N(E) := \#\{n \in \mathbb{Z}^+ \mid \lambda_n \le E\}, \qquad E > 0$$

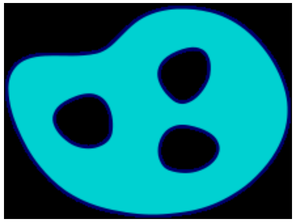
- Hermann Weyl made the conjecture in 1912:

$$N(E) = \frac{\omega_d S}{(2\pi)^d} E^{d/2} - \frac{\omega_{d-1} L}{4(2\pi)^{d-1}} E^{(d-1)/2} + o(E^{(d-1)/2}), \quad E \gg 1$$

$$\overset{L=|\partial\Omega|}{\Leftrightarrow} \lambda_n = \frac{4\pi^2}{(\omega_d S)^{2/d}} n^{2/d} + \frac{2\pi^2 \omega_{d-1} L}{d(\omega_d S)^{(d+1)/d}} n^{1/d} + o(n^{1/d}), \quad n \gg 1$$

  - In 2D:

$$N(E) = \frac{S}{4\pi} E - \frac{L}{4\pi} E^{1/2} + o(E^{1/2}), E \gg 1 \Leftrightarrow \lambda_n = \frac{4\pi}{S} n + \frac{2\sqrt{\pi} L}{S^{3/2}} n^{1/2} + o(n^{1/2}), n \gg 1$$



area & perimeter

- Some results:
  - Richard Courant in1922 proved a bound at : $O(E^{(d-1)/2} \log E)$
  - Boris Levitan in 1952 for compact closed domain: $O(E^{(d-1)/2})$
  - Robert Seeley in 1978 extend to general domain:
  - Hans Duistermaat & Victor Guilemin in 1975 under a strong condition: $o(E^{(d-1)/2})$
  - Victor Ivrii in 1980: $o(E^{(d-1)/2})$

https://en.wikipedia.org/wiki/Weil_conjectures

A Spectral Method

$$\left(-\Delta + V(\vec{x})\right)u(\vec{x}) = \lambda\, u(\vec{x}), \quad \vec{x} \in \Omega$$
$$u(\vec{x}) = 0, \quad \vec{x} \in \Gamma = \partial\Omega$$

- Variational Form:

  $$\text{Find } \ 0 \neq u \in H_0^1(\Omega) \quad \& \,\lambda \in \mathbb{R} \quad \text{s.t.}$$
  $$a(u,v) + b(u,v) = \lambda\,(u,v), \quad \forall v \in H_0^1(\Omega)$$

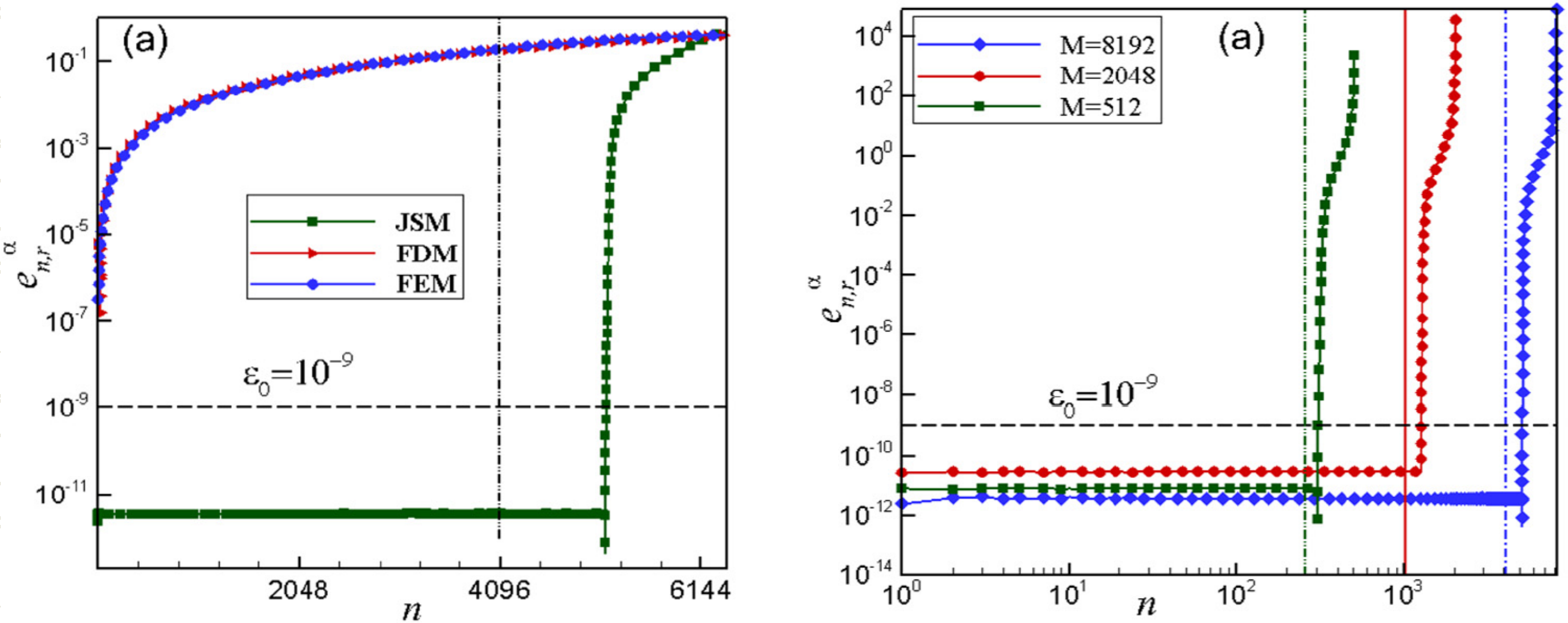
  – where

$$a(u,v) = \int_\Omega \nabla u(\vec{x}) \bullet \nabla v(\vec{x})\, d\vec{x}, \quad b(u,v) = \int_\Omega V(\vec{x})u(\vec{x})v(\vec{x})\, d\vec{x}, \quad (u,v) = \int_\Omega u(\vec{x})v(\vec{x})\, d\vec{x}$$

- A spectral method: $U_M := \text{span}\{\phi_1, \phi_2, \ldots, \phi_M\} \subset H_0^1(\Omega)$

$$\text{Find } \ 0 \neq u_M \in U_M \quad \& \,\lambda \in \mathbb{R} \quad \text{s.t.}$$
$$a(u_M, v) + b(u_M, v) = \lambda\,(u_M, v), \quad \forall v \in U_M \Leftrightarrow (D + A)U = \lambda BU$$

Comparison on Resolution



- Spectral method has much better resolution than FDM & FEM
- Spectral method can resolve # of eigenvalues proportional to DoF

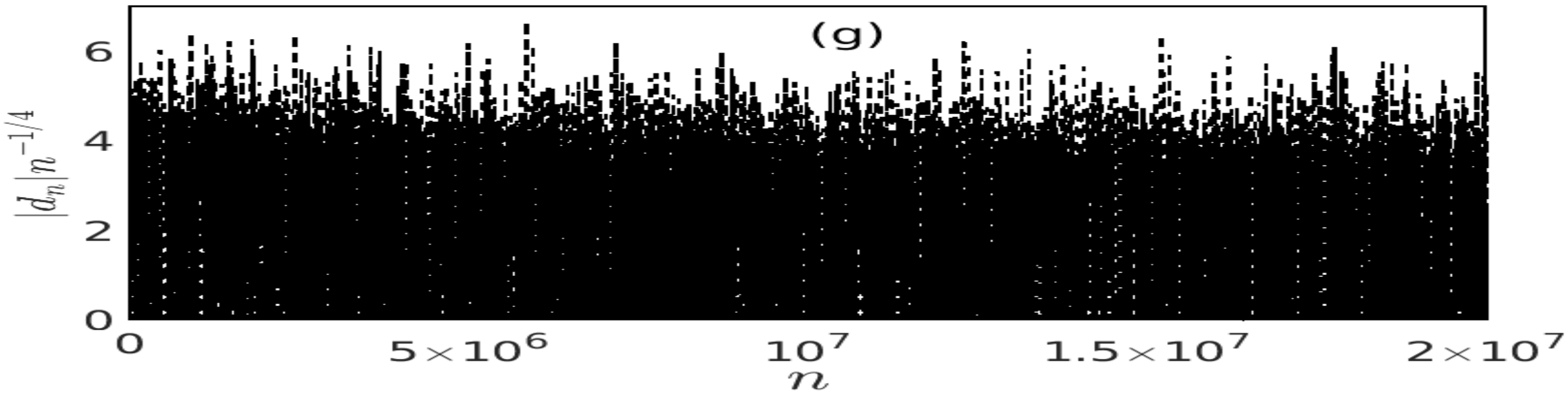
W. Bao, L. Chen, X. Jiang and Y. Ma, A Jacobi spectral method for computing eigenvalue gaps and their distribution statistics of the fractional Schrodinger operator, J. Comput. Phys., Vol. 421 (2020), article 109733.

Numerical Results in 2D

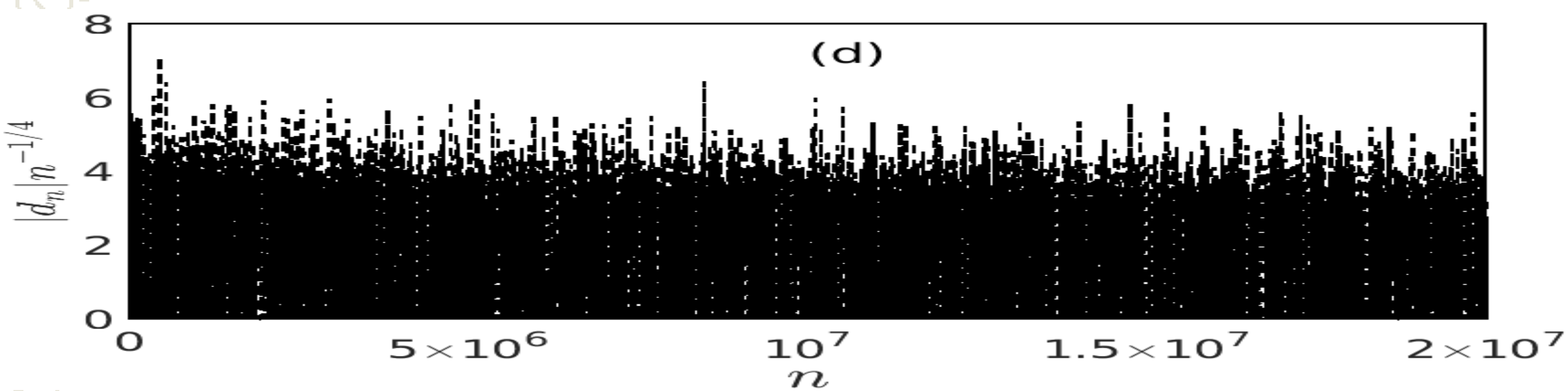
- Setup: $d=2, \quad V(\vec{x})\equiv 0, \quad \Omega=(-1,1)\times(-L_2,L_2), \quad 0<L_2\le 1, \quad S=4L_2, \quad L=4(1+L_2)$

$$d_n := \lambda_n - \frac{4\pi}{S}n - \frac{2\sqrt{\pi}L}{S^{3/2}}n^{1/2}, \quad d_n^{\text{ave}} := \frac{1}{n}\sum_{m=1}^{n} d_m, \quad n=1,2,\ldots$$

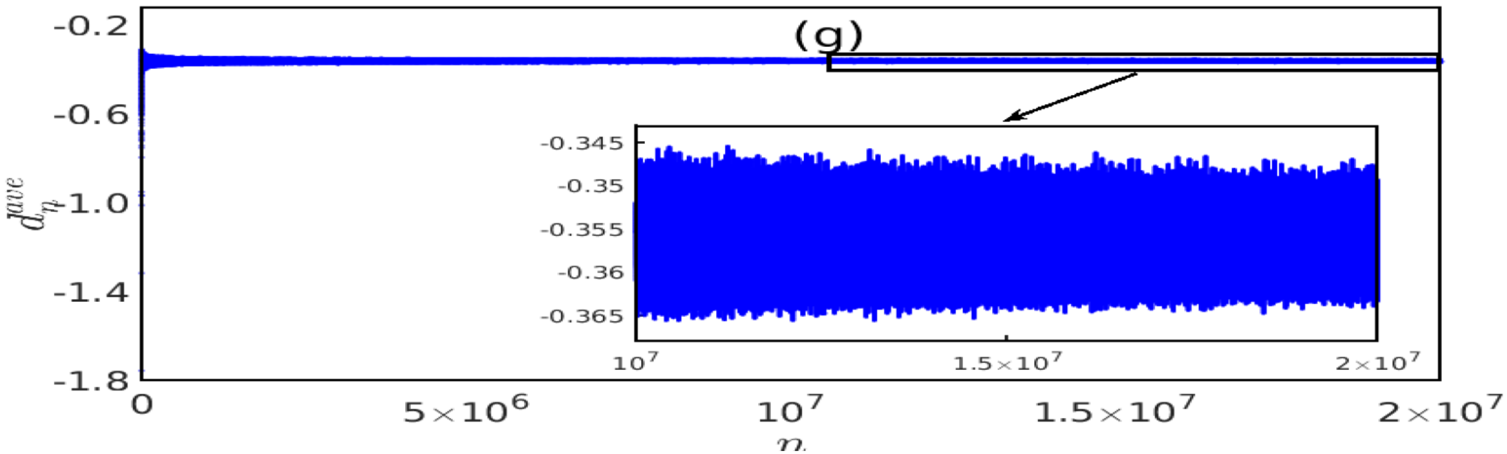
$L_2 = 1 \Rightarrow$



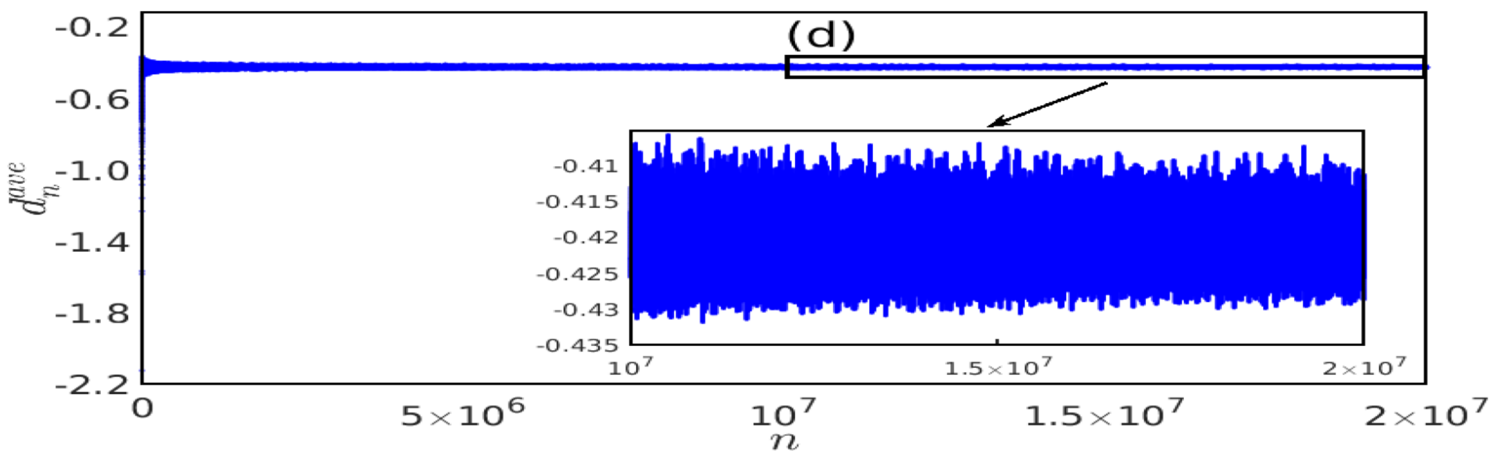
$L_2 = \frac{2}{e} \Rightarrow$



$L_2 = 1 \Rightarrow$



$L_2 = \frac{2}{e} \Rightarrow$



- Numerical observation – Bao & Chen 20'

Hardy, 1916 showed $N(E) - \frac{S}{4\pi}E + \frac{L}{4\pi}E^{1/2} \geq CE^{1/4}(\log E)^{1/4},\ E \gg 1$

$$\lambda_n = \frac{4\pi}{S}n + \frac{2\sqrt{\pi}L}{S^{3/2}}n^{1/2} + \alpha_n n^{1/4}, \quad n = 1, 2, \ldots$$

α_n is oscillatory $\lim_{n\to+\infty} \frac{\alpha_n}{n^{\varepsilon}} = 0, \forall \varepsilon > 0$

d_n^{ave} is bounded

Extension to Laplacian Operator with Neumann BC

- The Eigenvalue Problem:

$$-\Delta u(\vec{x}) = \lambda\, u(\vec{x}), \quad \vec{x} \in \Omega$$

$$\frac{\partial u(\vec{x})}{\partial \vec{n}} = 0, \quad \vec{x} \in \Gamma = \partial\Omega$$

  - Order of all eigenvalues:

$$0 = \lambda_1 < \lambda_2 \le \ldots \le \lambda_n \le \ldots$$

- In 1D with $\Omega = (0, S)$:

$$\lambda_n = \frac{(n-1)^2 \pi^2}{S^2}, \quad n = 1, 2, \ldots \Rightarrow \lambda_n = \frac{\pi^2}{S^2} n^2 - \frac{2\pi^2}{S^2} n + \frac{\pi^2}{S^2}, n \gg 1 \Leftrightarrow N(E) = \frac{S}{\pi} E^{1/2} + 1, E \gg 1$$

- In 2D:

$$N(E) = \frac{S}{4\pi} E + \frac{L}{4\pi} E^{1/2} + o(E^{1/2}), E \gg 1 \Leftrightarrow \lambda_n = \frac{4\pi}{S} n - \frac{2\sqrt{\pi} L}{S^{3/2}} n^{1/2} + o(n^{1/2}), n \gg 1$$

  - Our numerical results – Bao & Chen 20'

$$\lambda_n = \frac{4\pi}{S} n - \frac{2\sqrt{\pi} L}{S^{3/2}} n^{1/2} + \tilde{\alpha}_n n^{1/4}, \quad n = 1, 2, \ldots$$

$\tilde{\alpha}_n$ is oscillatory $\lim_{n\to+\infty} \frac{\tilde{\alpha}_n}{n^{\varepsilon}} = 0, \forall \varepsilon > 0$

$\tilde{d}_n^{\text{ave}}$ is bounded

Extension to Fractional Schrodinger Operator (FSO/FLO)

- The FSO in 1D:

$$L_{\mathrm{FSO}}u := \left(\left(-\partial_{xx}\right)^{\alpha/2} + V(x)\right)u(x) = \lambda\, u(x), \quad x \in \Omega = (0,S),$$

$$u(x) = 0, \quad x \in \Omega^c = \mathbb{R} \setminus \Omega; \qquad 0 < \alpha \le 2$$

  - Fractional derivative defined as

$$\left(-\partial_{xx}\right)^{\alpha/2} u(x) = F^{-1}(|\,\xi\,|^{\alpha}\ (Fu)(\xi)), \quad x,\xi \in \mathbb{R}$$

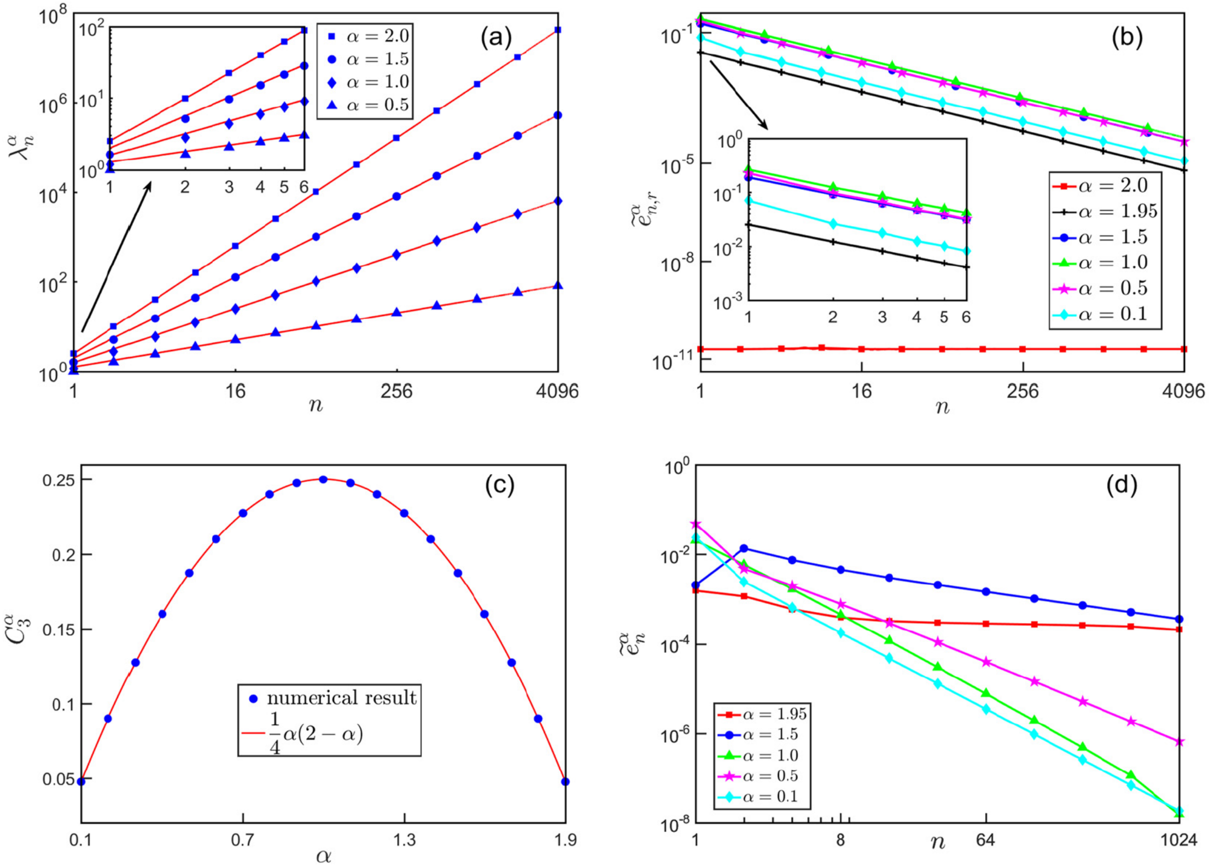
- Order of all eigenvalues:

$$\lambda_1^{\alpha} \le \lambda_2^{\alpha} \le \ldots \le \lambda_n^{\alpha} \le \ldots \Rightarrow u_1^{\alpha}, u_2^{\alpha}, \ldots$$

- A Jacobi spectral method:

W. Bao, L. Chen, X. Jiang and Y. Ma, A Jacobi spectral method for computing eigenvalue gaps and their distribution statistics of the fractional Schrodinger operator, J. Comput. Phys., Vol. 421 (2020), article 109733.

Asymptotics of Eigenvalues of FLO



Conjecture:

$$\lambda_n^\alpha = \left(\frac{n\pi}{S}\right)^\alpha - \left(\frac{\pi}{S}\right)^\alpha \frac{\alpha(2-\alpha)}{4} n^{\alpha-1} + O(n^{\alpha-2}), \quad n = 1, 2, \ldots \qquad u_n^\alpha(x) = x^{\alpha/2}(S-x)^{\alpha/2} v_n^\alpha(x), \quad 0 \le x \le S$$

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Asymptotics of FSO in 1D

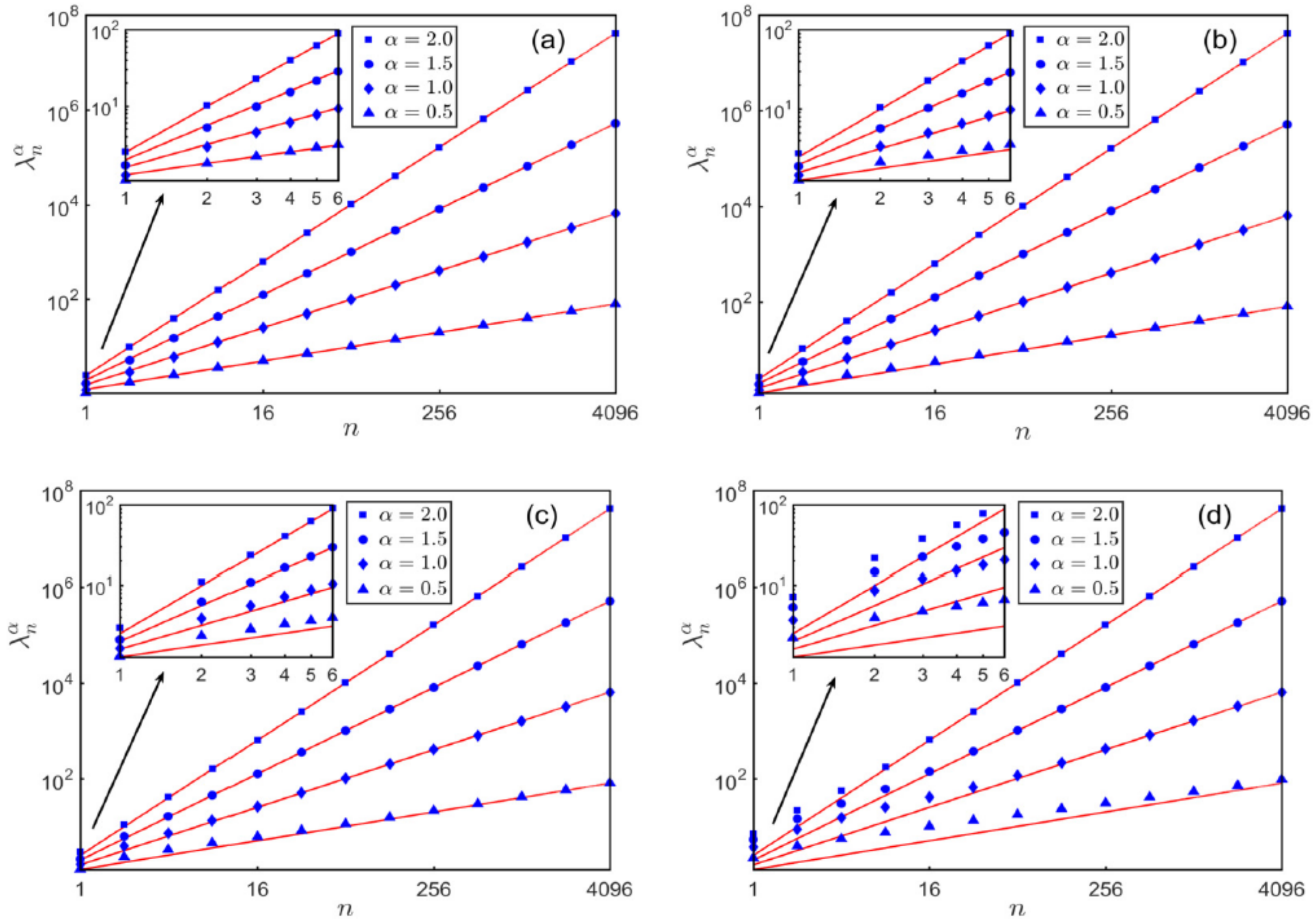

Fig. 12. Eigenvalues λ_n^α ($n = 1, 2, \ldots, 4096$) of (1.1) with $\Omega = (-1, 1)$ and different α for differential external potentials (symbols denote numerical results and solid lines are from fitting formulas when $n \gg 1$): (a) Case I, (b) Case II, (c) Case III, and (d) Case IV.

Asymptotics of Eigenvalues of FSO/FLO in 1D

$$\left[(-\partial_{xx})^{\alpha/2}+V(x)\right]u(x)=\lambda u(x), x\in\Omega=(0,S); \quad u(x)=0, x\in\Omega^c=\mathbb{R}\setminus\Omega; \quad 0<\alpha\le 2$$

- Denote eigenvalues without/with potentials:

$$0<\lambda_1<\lambda_2\le\ldots\le\lambda_n\le\ldots$$

$$\lambda_1^V\le\lambda_2^V\le\ldots\le\lambda_n^V\le\ldots$$

- Define

$$\widehat{d}_n:=\lambda_n^V-\lambda_n, \quad n=1,2,\ldots$$

$$\widehat{d}_n^{\text{ave}}:=\frac{1}{n}\sum_{m=1}^{n}\widehat{d}_m,$$

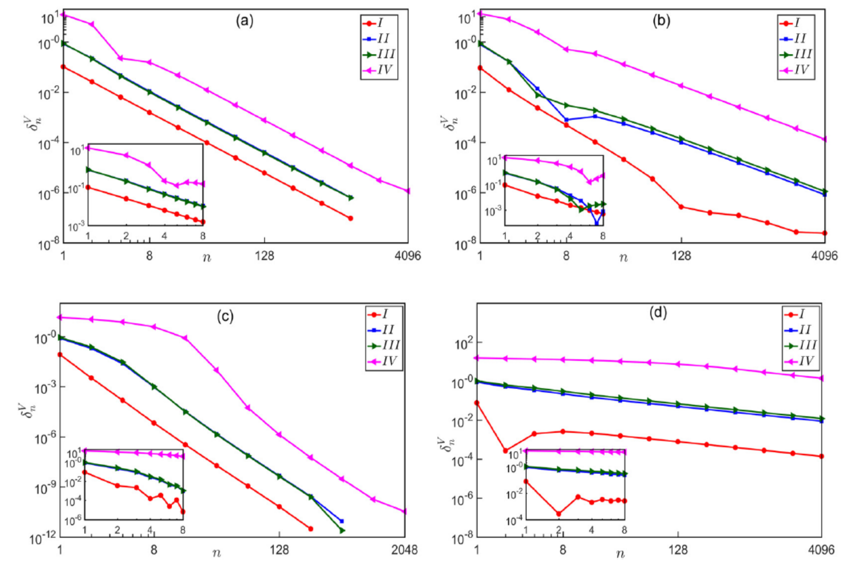

Fig. 15. Differences of the eigenvalues of (1.1) with potential V and without potential, i.e. $\delta_n^V:=\lambda_n^\alpha-\lambda_n^{\alpha,0}-C_V$ $(1\le n\le N=4096)$ for different potentials $V(x)$ and α: (a) $\alpha=2$, (b) $\alpha=\sqrt{2}$, (c) $\alpha=1$, and (d) $\alpha=0.5$.

- Numerical results – Bao, Chen, Jiang & Ma, JCP, 20' – H. Hochstadt, CPAM 1961' for $\alpha=2$

$$\lambda_n^V=\lambda_n+C_V+O(n^{-\tau_1(\alpha)}), \quad n\gg 1$$

$$\tau_1(\alpha)=\begin{cases}\alpha, & 0<\alpha\le 2\,\&\,\alpha\ne 1\\ \approx 4.5, & \alpha=1,\end{cases}$$

$$\widehat{d}_n^{\text{ave}}=C_V+O(n^{-\alpha/2}), \quad C_V=\frac{1}{|\Omega|}\int_\Omega V(x)dx$$

$$0<\alpha<2 \quad ????$$

Extension to Directional FSO

- The directional FSO in high dimensions:

$$L_{\text{D-FSO}}u := \left(D_{\vec{x}}^{\alpha} + V(\vec{x})\right)u(\vec{x}) = \lambda\, u(\vec{x}), \quad \vec{x} \in \Omega \subset \mathbb{R}^d$$

$$u(\vec{x}) = 0, \quad \vec{x} \in \Omega^c = \mathbb{R}^d \setminus \Omega; \qquad 0 < \alpha \le 2, \quad d \ge 2$$

  - Fractional derivative defined as

$$D_{\vec{x}}^{\alpha} u(\vec{x}) = F^{-1}\Big(\sum_{j=1}^{d} |\,\xi_j\,|^{\alpha}\,(Fu)(\vec{\xi})\Big), \quad \vec{x}, \vec{\xi} \in \mathbb{R}^d$$

- Order of all eigenvalues:

$$\lambda_1^{\alpha} \le \lambda_2^{\alpha} \le \ldots \le \lambda_n^{\alpha} \le \ldots \Rightarrow u_1^{\alpha}, u_2^{\alpha}, \ldots$$

- A Jacobi spectral method:

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Asymptotics of Eigenvalues of D-FLO in 2D

- **Setup**: $d=2, \quad V(\vec{x})\equiv 0, \quad \Omega=(-1,1)\times(-L_2,L_2), \quad 0<L_2\le 1, \quad S=4L_2, \quad L=4(1+L_2)$

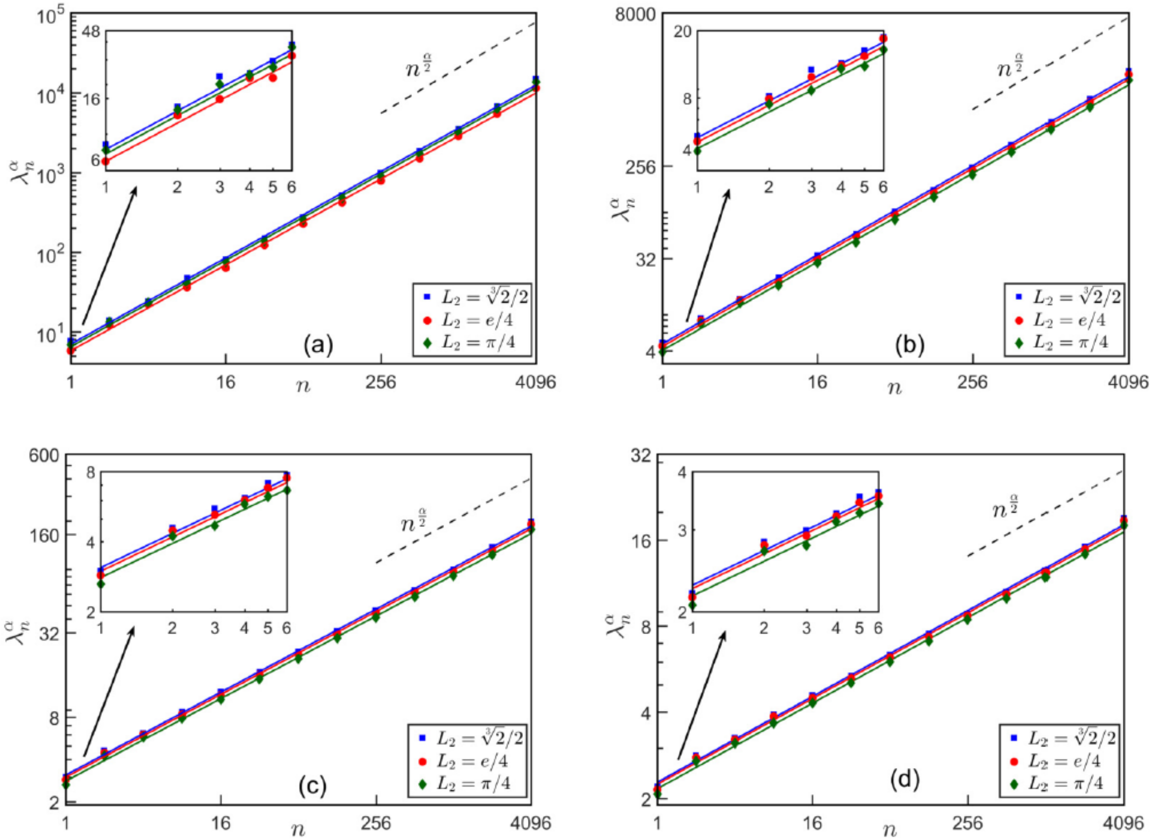

Fig. 17. Eigenvalues of (6.1) with $d=2$, $L_1=1$, $V(\mathbf{x})\equiv 0$ and different L_2 and α (symbols denote numerical results and solid lines are from the fitting formula $C_2^\alpha n^{\alpha/2}$ when $n\gg 1$): (a) $\alpha=1.9$, (b) $\alpha=1.5$, (c) $\alpha=1.0$, and (d) $\alpha=0.5$.

The Generalized Weyl's Law of D-FSO

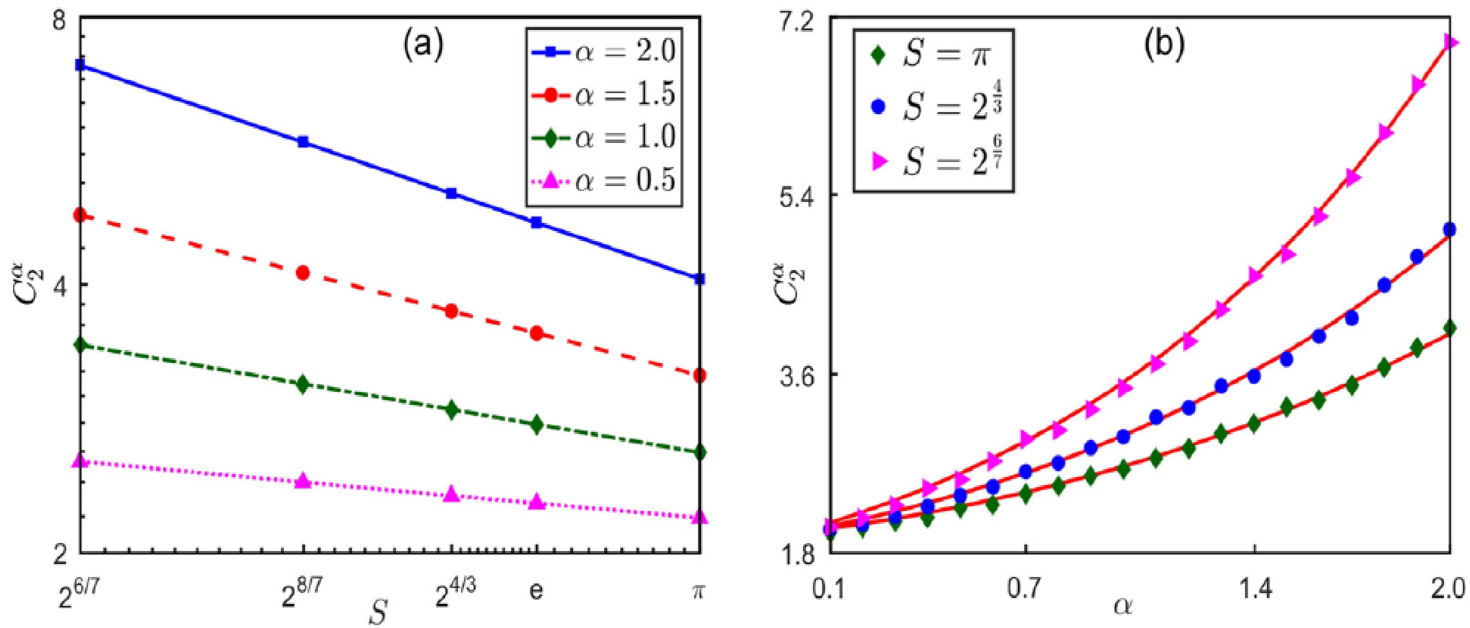

Fig. 18. Numerical results of C_2^α (symbols denote numerical results and solid lines are from the fitting formula (6.7)) for different areas $S = |\Omega| = 4L_2$ and α: (a) plots of C_2^α as a function of S for different α, and (b) plots of C_2^α as a function of α for different S.

- **Conjecture:** $$\lambda_n^\alpha = \frac{4}{2+\alpha}\left(\frac{4\pi}{S}\right)^{\alpha/2} n^{\alpha/2} + o(n^{\alpha/2}), \quad n \gg 1$$

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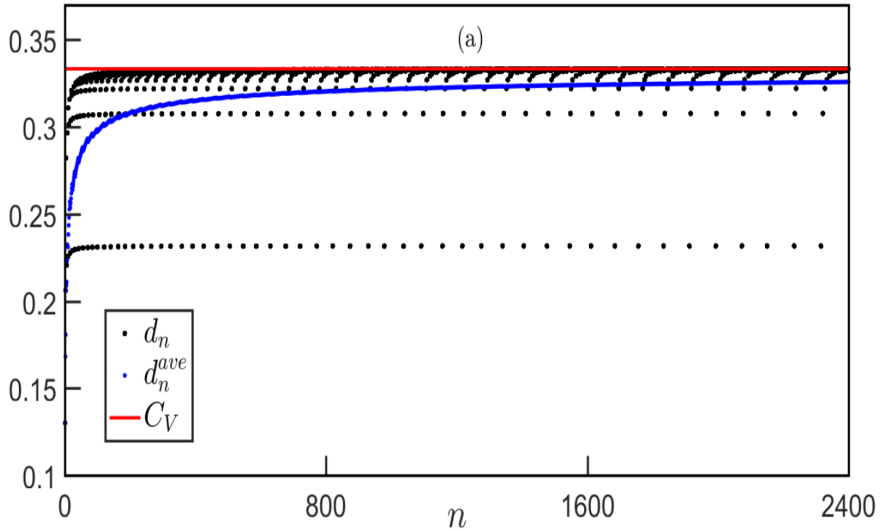
Asymptotics of Eigenvalues with Potentials in 2D

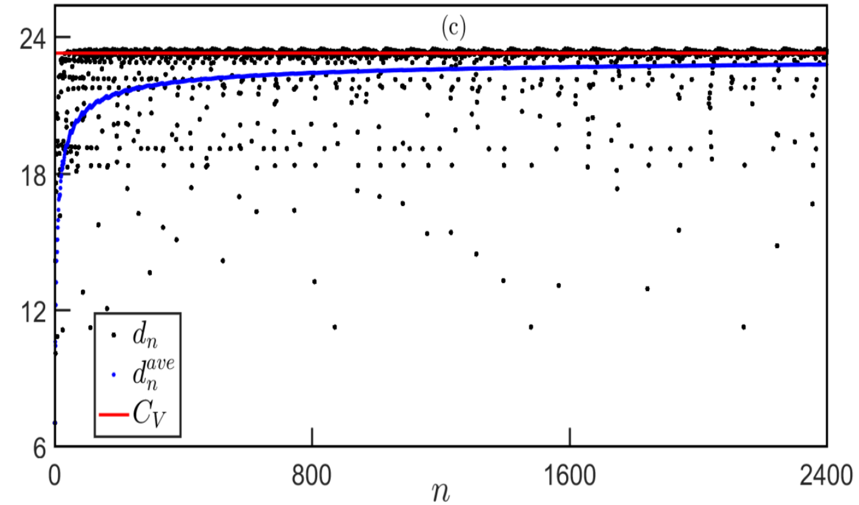
$$\left(-\Delta+V(\vec{x})\right)u(\vec{x})=\lambda u(\vec{x}), \quad \vec{x}\in\Omega$$
$$u(\vec{x})=0, \quad \vec{x}\in\Gamma=\partial\Omega$$

- Denote eigenvalues without/with potentials:

$$0<\lambda_1<\lambda_2\le\ldots\le\lambda_n\le\ldots; \qquad \lambda_1^V\le\lambda_2^V\le\ldots\le\lambda_n^V\le\ldots$$

- Define $\widehat{d}_n:=\lambda_n^V-\lambda_n, \quad n=1,2,\ldots \qquad \widehat{d}_n^{\text{ave}}:=\frac{1}{n}\sum_{m=1}^{n}\widehat{d}_m,$





Asymptotics of Eigenvalues with Potentials in 2D

$$\widehat{d}_n := \lambda_n^V - \lambda_n, \quad n = 1, 2, \dots \qquad \widehat{d}_n^{\text{ave}} := \frac{1}{n}\sum_{m=1}^{n} d_m,$$

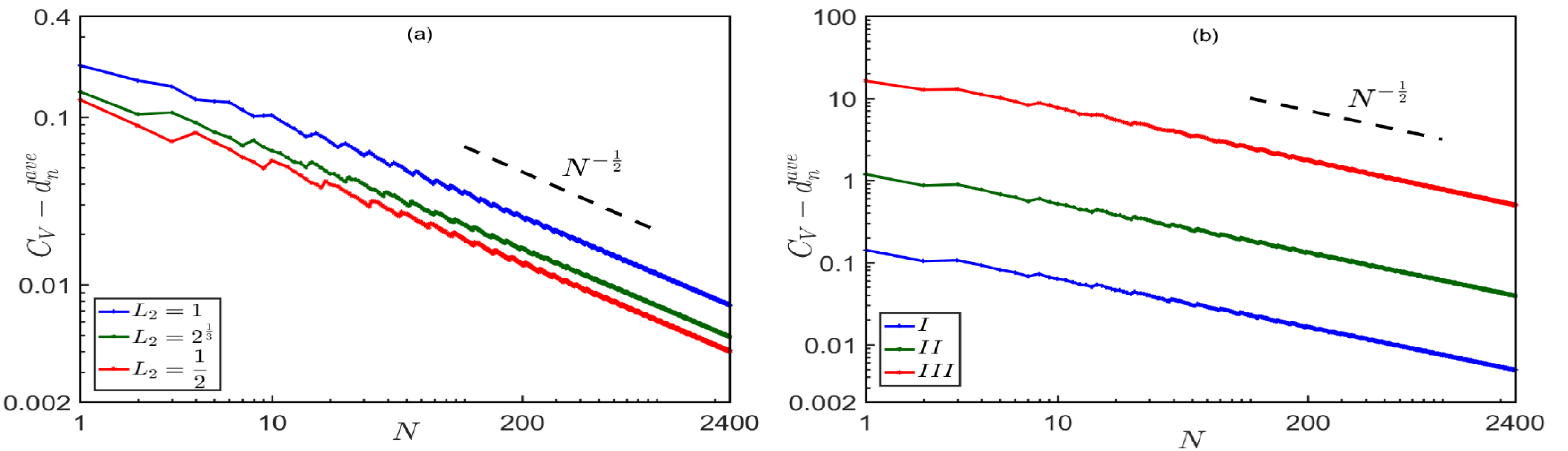


Figure 3.3: Plots of $C_V - d_n^{\text{ave}}$ for different L_2 and $V(x,y)$: (a) with $V(x,y) = \frac{x^2+y^2}{2}$ for different L_2, and (b) with $L_2 = \frac{2^{1/3}}{2}$ for different potentials (I) $V(x,y) = \frac{x^2+y^2}{2}$, (II) $V(x,y) = 4x^2 + 4y^2 + \sin(\frac{\pi x}{2}) + \sin(\frac{\pi y}{2})$, and (III) $V(x,y) = 50(x^2+y^2) + \sin(2\pi x) + \sin(2\pi y)$.

- Numerical & analytical results – Bao, Chen & Rudnick, 20'

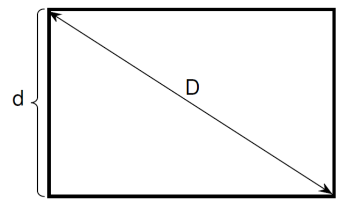
$$\widehat{d}_n \text{ is oscillatory \& } \widehat{d}_n^{\text{ave}} = C_V + O(n^{-1/2}), \quad n \gg 1, \quad C_V = \frac{1}{|\Omega|}\int_\Omega V(\vec{x})\,d\vec{x}$$

The Fundamental Gap of SO

$$\left(-\Delta+V(\vec{x})\right)u(\vec{x})=\lambda\, u(\vec{x}),\quad \vec{x}\in\Omega$$

$$u(\vec{x})=0,\quad \vec{x}\in\Gamma=\partial\Omega$$

- Order of all eigenvalues:



$$V(\vec{x})\geq 0\ \&\ \text{weakly convex}\Rightarrow\quad 0<\lambda_1<\lambda_2\leq\ldots\leq\lambda_n\leq\ldots$$

- Fundamental gap: $\delta_{\text{fd}}:=\lambda_2-\lambda_1>0$

- In 1D with no potential: $-u''(x)=\lambda\, u(x),\quad x\in(0,S);\quad u(0)=u(S)=0$

$$\lambda_n=\frac{n^2\pi^2}{S^2},\quad n=1,2,\ldots\Rightarrow\delta_{\text{fd}}:=\lambda_2-\lambda_1=\frac{2^2\pi^2}{S^2}-\frac{\pi^2}{S^2}=\frac{3\pi^2}{D^2},\quad D-\text{diameter of }\Omega$$

- In 2D on a rectangular with no potential

$$d=2,\ V(\vec{x})\equiv 0,\ \Omega=(-L_1,L_1)\times(-L_2,L_2)$$

$$0<L_2\leq L_1,\quad D=\sqrt{L_1^2+L_2^2}\xrightarrow{L_2\to 0}L_1$$

$$\tilde{\lambda}_{lk}=\frac{l^2\pi^2}{L_1^2}+\frac{k^2\pi^2}{L_2^2},\quad l,k=1,2,\ldots\Rightarrow$$

$$\delta_{\text{fd}}:=\tilde{\lambda}_{21}-\tilde{\lambda}_{11}=\frac{2^2\pi^2}{L_1^2}+\frac{\pi^2}{L_2^2}-\frac{\pi^2}{L_1^2}-\frac{\pi^2}{L_2^2}=\frac{3\pi^2}{L_1^2}\geq\frac{3\pi^2}{D^2}\xrightarrow{L_2\to 0}\frac{3\pi^2}{L_1^2}$$

Fundamental Gap Conjecture of SO

$$\left(-\Delta + V(\vec{x})\right)u(\vec{x}) = \lambda u(\vec{x}), \quad \vec{x} \in \Omega$$

$$u(\vec{x}) = 0, \quad \vec{x} \in \Gamma = \partial\Omega$$

- Fundamental Gap conjecture – convex domain & weak convex potential

$$\delta_{\text{fd}} := \lambda_2 - \lambda_1 \geq \frac{3\pi^2}{D^2}$$

$$D := \max_{\vec{x},\vec{y} \in \bar{\Omega}} |\vec{x} - \vec{y}|$$

$$r_0 := \max_{\vec{x} \in \bar{\Omega}} \sup \left\{ r > 0 \mid B_r(\vec{x}) \subseteq \bar{\Omega} \right\}$$

  - Observed by Michiel van Den Berg, J. Stat. Phys. 1983
  - Suggested by S. T. Yau, 86'
  - For *d*=*1*,partial result by Ashbaugh & Benguria 89';& proved by Lavine 94'
  - For *d*>*1*, partial results by Qi Huang Yu & JiaQing Zhong 86',Li & Yau,...
  - Completely proved by Ben Andrews & Julie Clutterbuck, JAMS 2011!!!!
    - Gradient flow, geometric analysis, sharp estimates of ODE, etc.
    - Results for whole space with sub-harmonic potential

$$\delta_{\text{fd}} := \lambda_2 - \lambda_1 \geq \frac{3\pi^2}{D^2}\left(1 + \frac{(d-1)r_0^2}{D^2}\right) ???$$

Extension to FSO

- Fractional Schrodinger operator (FSO)

$$L_{\mathrm{FSO}} u := \left((-\Delta)^{\alpha/2} + V(\vec{x})\right) u(\vec{x}) = \lambda\, u(\vec{x}), \quad \vec{x} \in \Omega \subset \mathbb{R}^d$$

$$u(\vec{x}) = 0, \quad \vec{x} \in \Omega^c = \mathbb{R}^d \setminus \Omega; \qquad 0 < \alpha \le 2$$

  - Fractional derivative

$$(-\Delta)^{\alpha/2} u(\vec{x}) = F^{-1}(|\vec{\xi}\,|^{\alpha}\,(Fu)(\vec{\xi}\,)), \quad \vec{x}, \vec{\xi} \in \mathbb{R}^d$$

  - Order of all eigenvalues: $0 < \lambda_1^{\alpha} < \lambda_2^{\alpha} \le \ldots \le \lambda_n^{\alpha} \le \ldots$

- Fundamental gap conjecture:

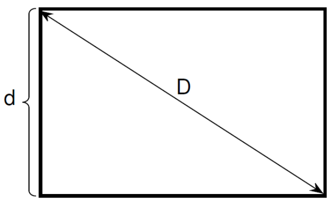
$$D := \max_{\vec{x},\vec{y} \in \bar{\Omega}} |\vec{x} - \vec{y}|$$

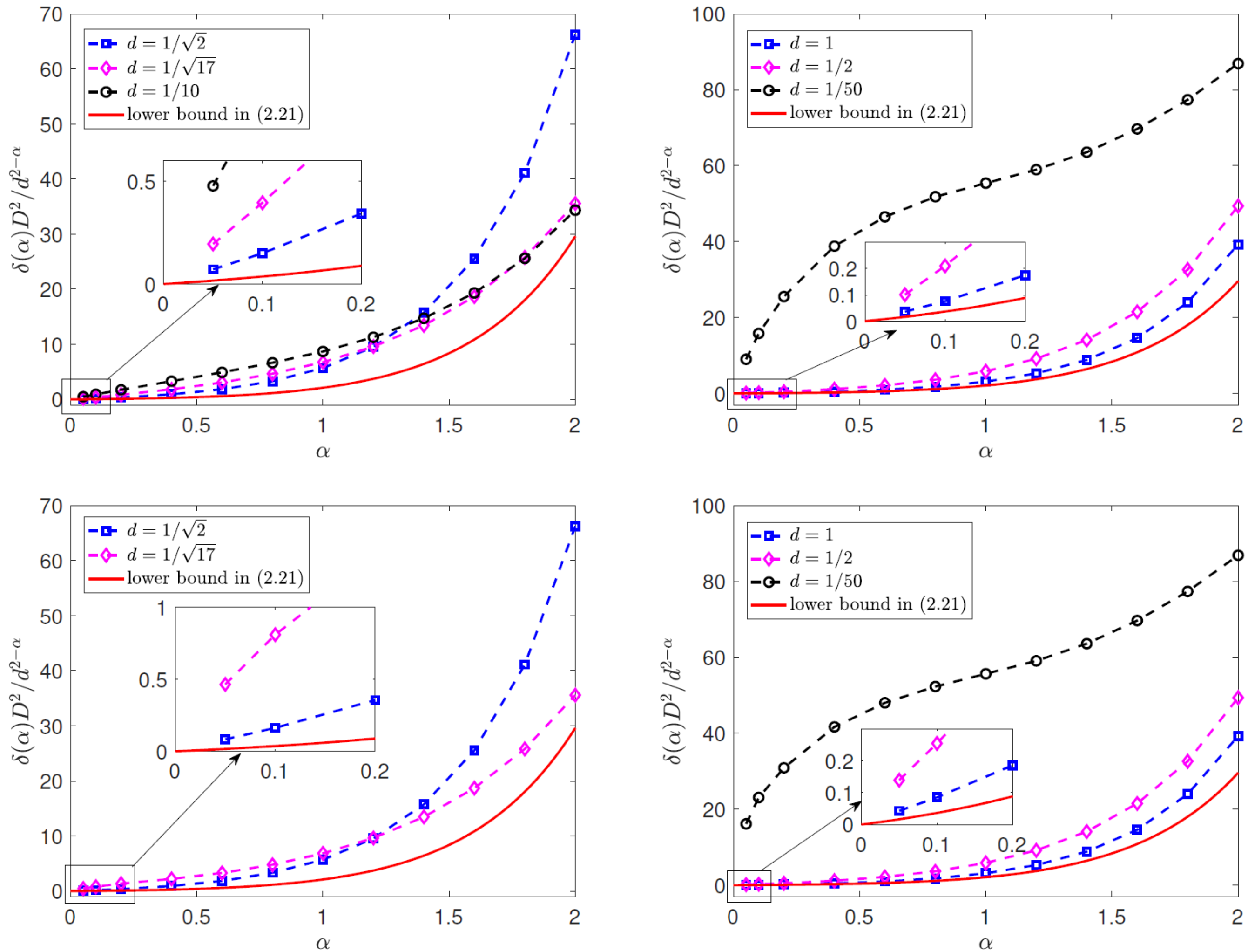
$$r_0 := \max_{\vec{x} \in \bar{\Omega}} \sup \{ r > 0 \mid B_r(\vec{x}) \subseteq \bar{\Omega} \}$$

$$\delta_{\mathrm{fd}}(\alpha) := \lambda_2^{\alpha} - \lambda_1^{\alpha} \ge \frac{3\alpha\pi^{\alpha}}{2(d+3)^{1-\alpha/2}} \frac{r_0^{2-\alpha}}{D^2}$$

W. Bao, X. Ruan, J. Shen and C. Sheng, Fundamental gaps of the fractional Schrodinger operator, Commun. Math. Sci., Vol. 17 (2019), pp. 447-471.

Numerical Results





W. Bao, X. Ruan, J. Shen and C. Sheng, Fundamental gaps of the fractional Schrodinger operator, Commun. Math. Sci., Vol. 17 (2019), pp. 447-471.

Other Gaps and their Statistics

$$L_{\mathrm{FSO}}u := \left((-\Delta)^{\alpha/2} + V(\vec{x})\right)(\vec{x}) = \lambda\, u(\vec{x}), \quad \vec{x} \in \Omega \subset \mathbb{R}^d$$

$$u(\vec{x}) = 0, \quad \vec{x} \in \Omega^c = \mathbb{R}^d \setminus \Omega; \qquad 0 < \alpha \le 2$$

- Order of eigenvalues:

$$0 < \lambda_1^\alpha < \lambda_2^\alpha \le \ldots \le \lambda_n^\alpha \le \ldots$$

- Ratio of repeated eigenvalues: $R(n) := \dfrac{\#\{1 \le l \le n \mid \lambda_l^\alpha = \lambda_{l+1}^\alpha\}}{n}, \quad n = 1, 2, \ldots$
- Nearest neighbor gaps $\delta_{\mathrm{nn}}(n) := \lambda_{n+1} - \lambda_n, \quad n = 1, 2, \ldots$
- Average gaps $\delta_{\mathrm{ave}}(n) := \dfrac{1}{n}\sum_{l=1}^{n} \delta_{\mathrm{nn}}(l) = \dfrac{1}{n}\left(\lambda_{n+1} - \lambda_1\right), \quad n = 1, 2, \ldots$
- Minimal gaps: $\delta_{\min}(n) := \min_{1 \le l \le n} \delta_{\mathrm{nn}}(l), \quad n = 1, 2, \ldots$
- Normalized gaps (or ``unfolding'' local statistics in physics)

$$\text{If } \lim_{n \to +\infty} \frac{\lambda_n}{n^\gamma} = C > 0 \quad \overset{\tilde{\lambda}_n = (\lambda_n / C)^{1/\gamma}}{\Longrightarrow} \quad \delta_{\mathrm{norm}}(n) := \tilde{\lambda}_{n+1} - \tilde{\lambda}_n, \quad n = 1, 2, \ldots$$

- Gaps distribution statistics $\dfrac{\#\{1 \le l \le n \mid \delta_{\mathrm{norm}}(l) < x\}}{n} \overset{n \to +\infty}{\to} \displaystyle\int_0^x P(s)\,ds, \quad x \ge 0$

For Laplacian Operator (LO) in 1D

$$-u''(x) = \lambda u(x), \quad x \in (0, S);$$
$$u(0) = u(S) = 0$$

- All eigenvalues:

$$\lambda_n = \frac{n^2 \pi^2}{S^2}, \quad n = 1, 2, \ldots$$

- Different gaps and statistics:
  - Ratio of repeated eigenvalues: $R(n) \equiv 0, \quad n = 1, 2, \ldots$
  - Nearest neighbour gaps: $\delta_{\text{nn}}(n) := \frac{\pi^2 (2n+1)}{S^2}, \quad n = 1, 2, \ldots$
  - Averages gaps: $\delta_{\text{ave}}(n) = \frac{\pi^2 (n+2)}{S^2}, \quad n = 1, 2, \ldots$
  - Minimal gaps: $\delta_{\min}(n) \equiv \frac{3\pi^2}{S^2}, \quad n = 1, 2, \ldots$
  - Normalized gaps: $\delta_{\text{norm}}(n) := \tilde{\lambda}_{n+1} - \tilde{\lambda}_n \overset{\tilde{\lambda}_n = n}{=} n + 1 - n = 1, \quad n = 1, 2, \ldots$
  - Gaps distribution statistics:

$$P(s) = \delta(s-1), \quad s \geq 0$$

For Fractional Laplacian Operator (FLO) in 1D

$$(-\partial_{xx})^{\alpha/2} u(x) = \lambda u(x), x \in \Omega = (0, S);$$

$$u(x) = 0, x \in \Omega^c = \mathbb{R} \setminus \Omega; \quad 0 < \alpha \le 2$$

- All eigenvalues:

$$\lambda_n^\alpha = \left(\frac{n\pi}{S}\right)^\alpha - \left(\frac{\pi}{S}\right)^\alpha \frac{\alpha(2-\alpha)}{4} n^{\alpha-1} + O(n^{\alpha-2}), \quad n = 1, 2, \dots$$

- Different gaps and statistics:
  - Ratio of repeated eigenvalues: $R(n) \equiv 0, \quad n = 1, 2, \dots$
  - Nearest neighbour gaps: $\delta_{\text{nn}}(n) := \left(\frac{\pi}{S}\right)^\alpha \left[\alpha n^{\alpha-1} + \frac{\alpha(\alpha-1)(\alpha+2)}{4} n^{\alpha-2} + O(n^{\alpha-3})\right] + O(n^{\alpha-2}), \quad n = 1, 2, \dots$
  - Averages gaps: $\delta_{\text{ave}}(n) = \left(\frac{\pi}{S}\right)^\alpha n^{\alpha-1} + O(n^{\alpha-2}), \quad n = 1, 2, \dots$
  - Normalized gaps: $\delta_{\text{norm}}(n) := 1 + O(n^{-2}), \quad n = 1, 2, \dots$
  - Gaps distribution statistics: $P(s) = \delta(s-1), \quad s \ge 0$

W. Bao, L. Chen, X. Jiang and Y. Ma, A Jacobi spectral method for computing eigenvalue gaps and their distribution statistics of the fractional Schrodinger operator, J. Comput. Phys., Vol. 421 (2020), article 109733.

Gaps of FLO in 1D

$\Omega = (0, S);$

$V(x) \equiv 0$

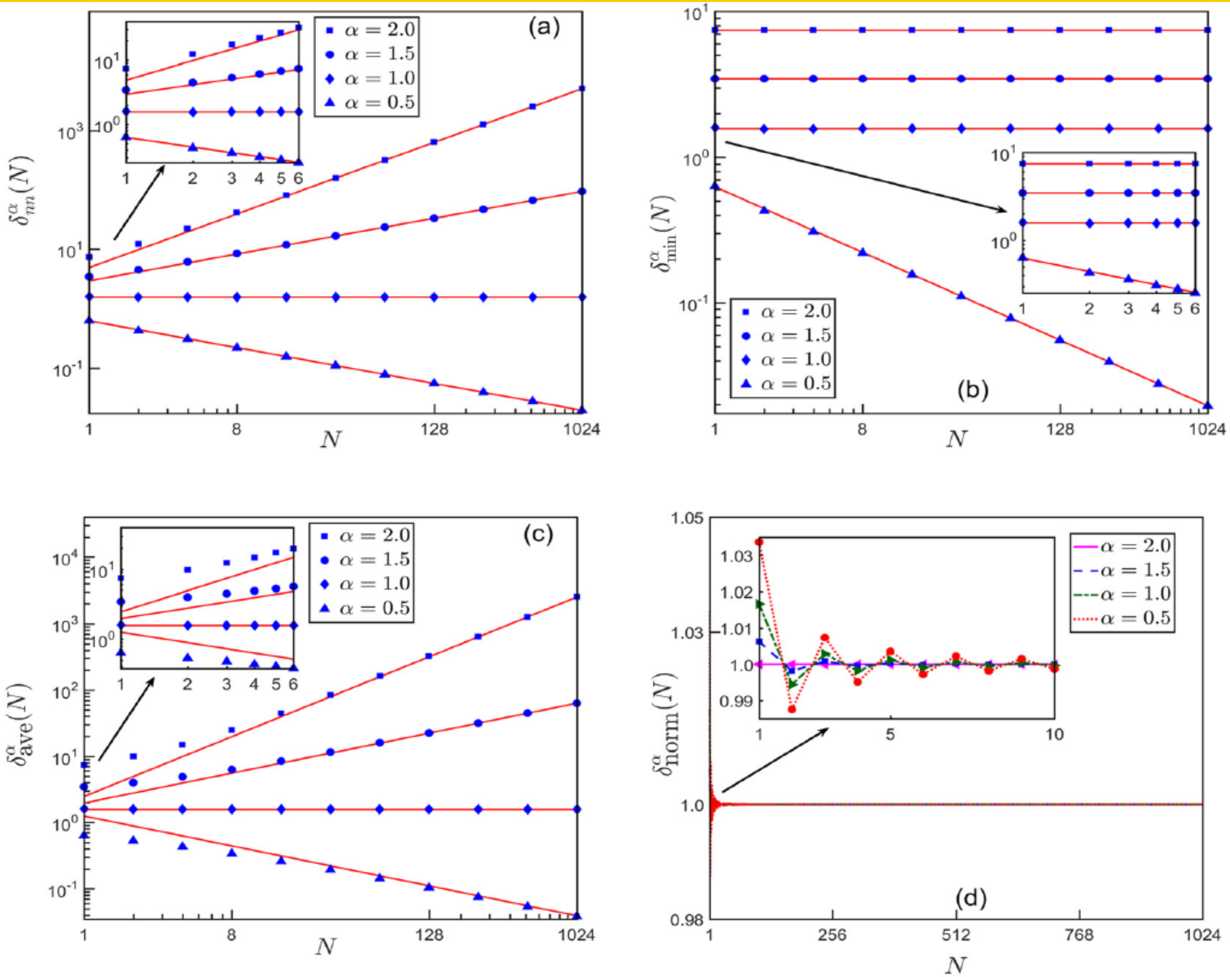


Fig. 7. Different eigenvalue gaps of (1.1) with $\Omega = (-1, 1)$, $V(x) \equiv 0$ and different α for (symbols denote numerical results and solid lines are from fitting formulas when $N \gg 1$): (a) the nearest neighbour gaps $\delta_{\text{nn}}^{\alpha}(N)$, (b) the minimum gaps $\delta_{\min}^{\alpha}(N)$, (c) the average gaps $\delta_{\text{ave}}^{\alpha}(N)$, and (d) the normalized gaps $\delta_{\text{norm}}^{\alpha}(N)$.

Gaps Statistics of FLO in 1D

$\Omega = (0, S);$

$V(x) \equiv 0$

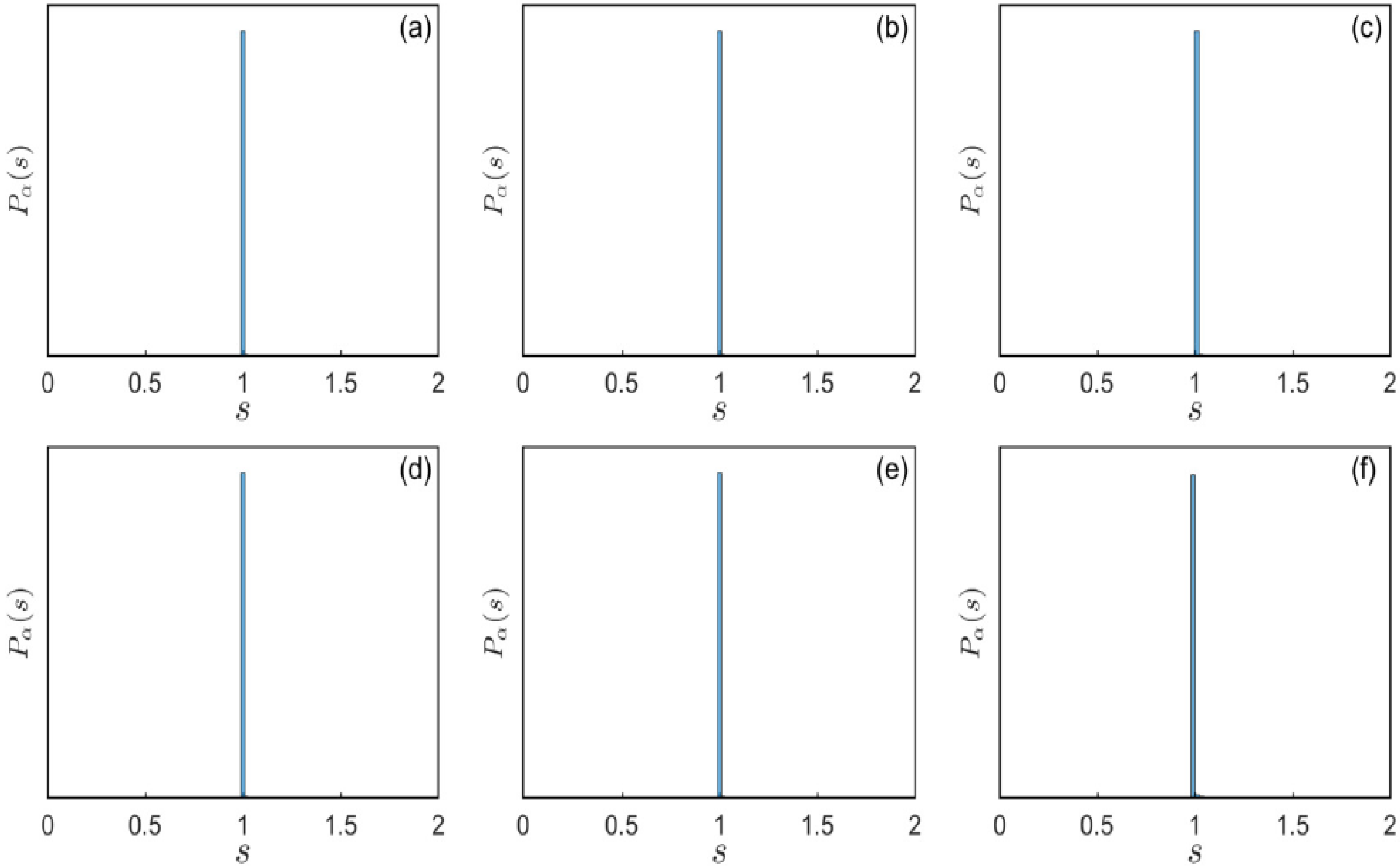


Fig. 8. The histogram of the normalized gaps $\{\delta^{\alpha}_{\text{norm}}(n) \mid 1 \le n \le N = 4096\}$ of (1.1) with $\Omega = (-1, 1)$ and $V(x) \equiv 0$ for different α: (a) $\alpha = 2.0$, (b) $\alpha = 1.9$, (c) $\alpha = \sqrt{3}$, (d) $\alpha = 1.5$, (e) $\alpha = 1.0$, and (f) $\alpha = 0.5$.

Gaps of FSO in 1D

$\Omega = (0, S);$

$V(x) \neq 0$

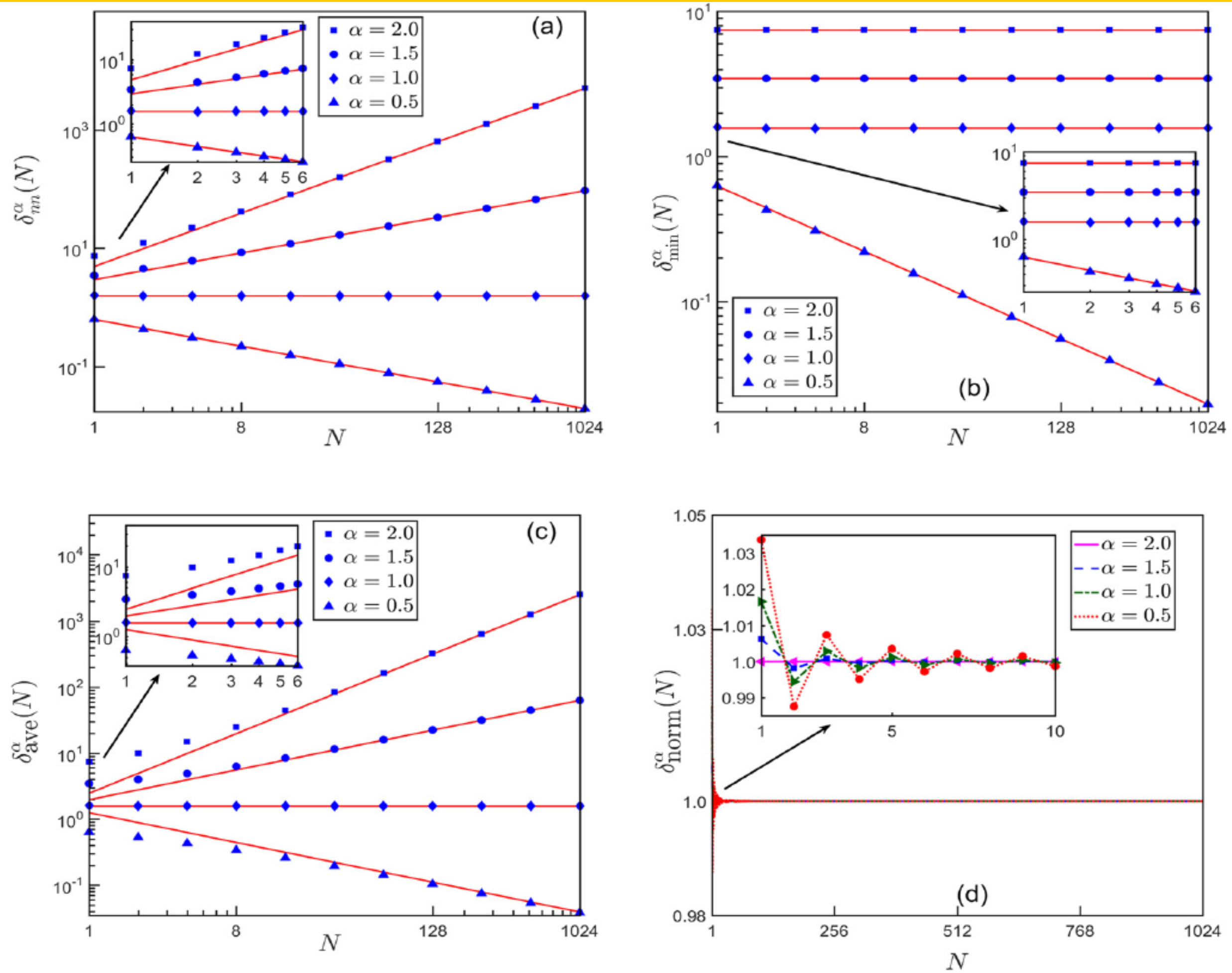


Fig. 7. Different eigenvalue gaps of (1.1) with $\Omega = (-1, 1)$, $V(x) \equiv 0$ and different α for (symbols denote numerical results and solid lines are from fitting formulas when $N \gg 1$): (a) the nearest neighbour gaps $\delta^{\alpha}_{\mathrm{nn}}(N)$, (b) the minimum gaps $\delta^{\alpha}_{\min}(N)$, (c) the average gaps $\delta^{\alpha}_{\mathrm{ave}}(N)$, and (d) the normalized gaps $\delta^{\alpha}_{\mathrm{norm}}(N)$.

Gaps Statistics of FSO in 1D

$$\Omega = (0, S);$$
$$V(x) \neq 0$$

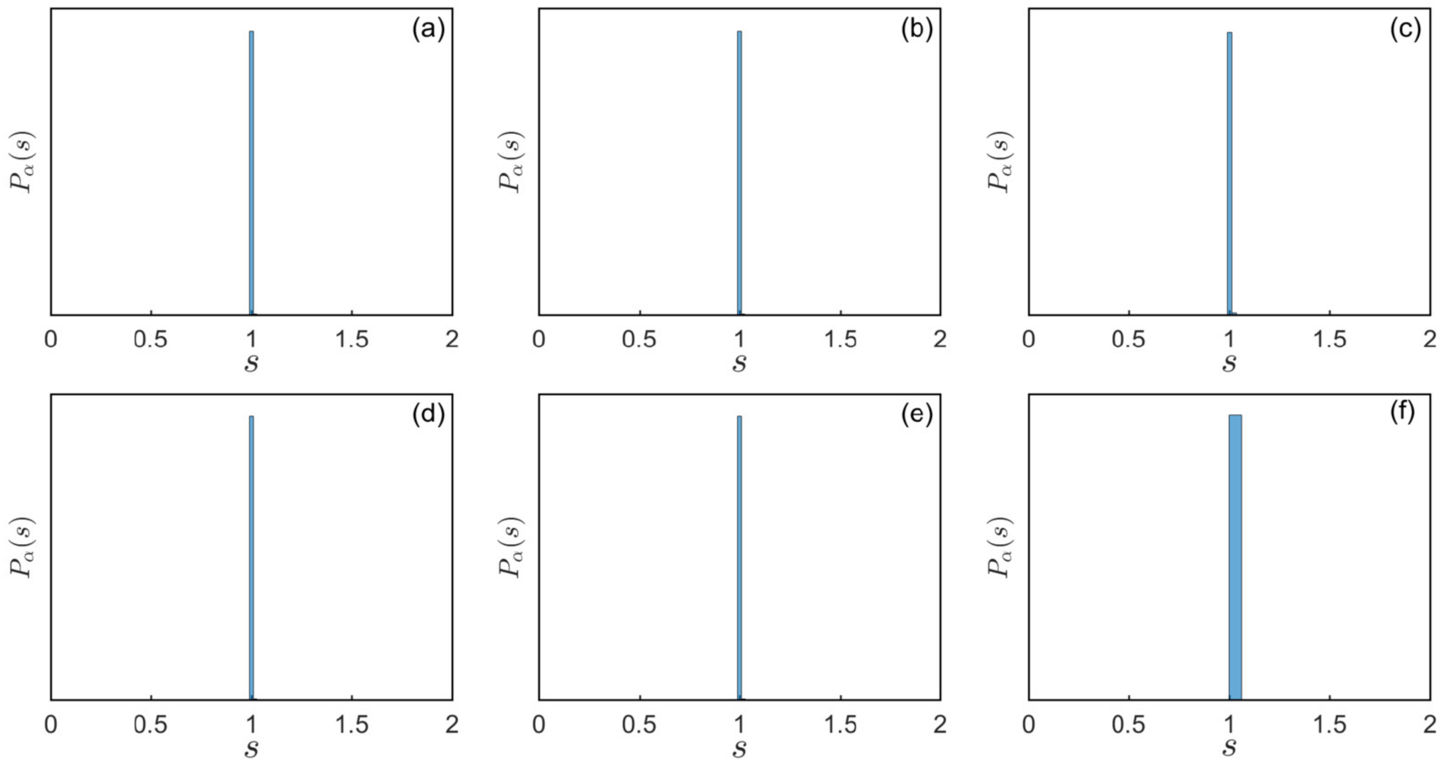


Fig. 14. The histogram of the normalized gaps $\{\delta^{\alpha}_{\text{norm}}(n) \mid 1 \leq n \leq N = 4096\}$ of (1.1) with $\Omega = (-1, 1)$ and $V(x) = \frac{x^2}{2}$ for different α: (a) $\alpha = 2.0$, (b) $\alpha = 1.9$, (c) $\alpha = \sqrt{3}$, (d) $\alpha = 1.5$, (e) $\alpha = 1.0$, and (f) $\alpha = 0.5$.

Gaps of D-FLO in 2D

$\Omega = (-1,1)\times(-L_2, L_2)$
$V(x,y)\equiv 0$

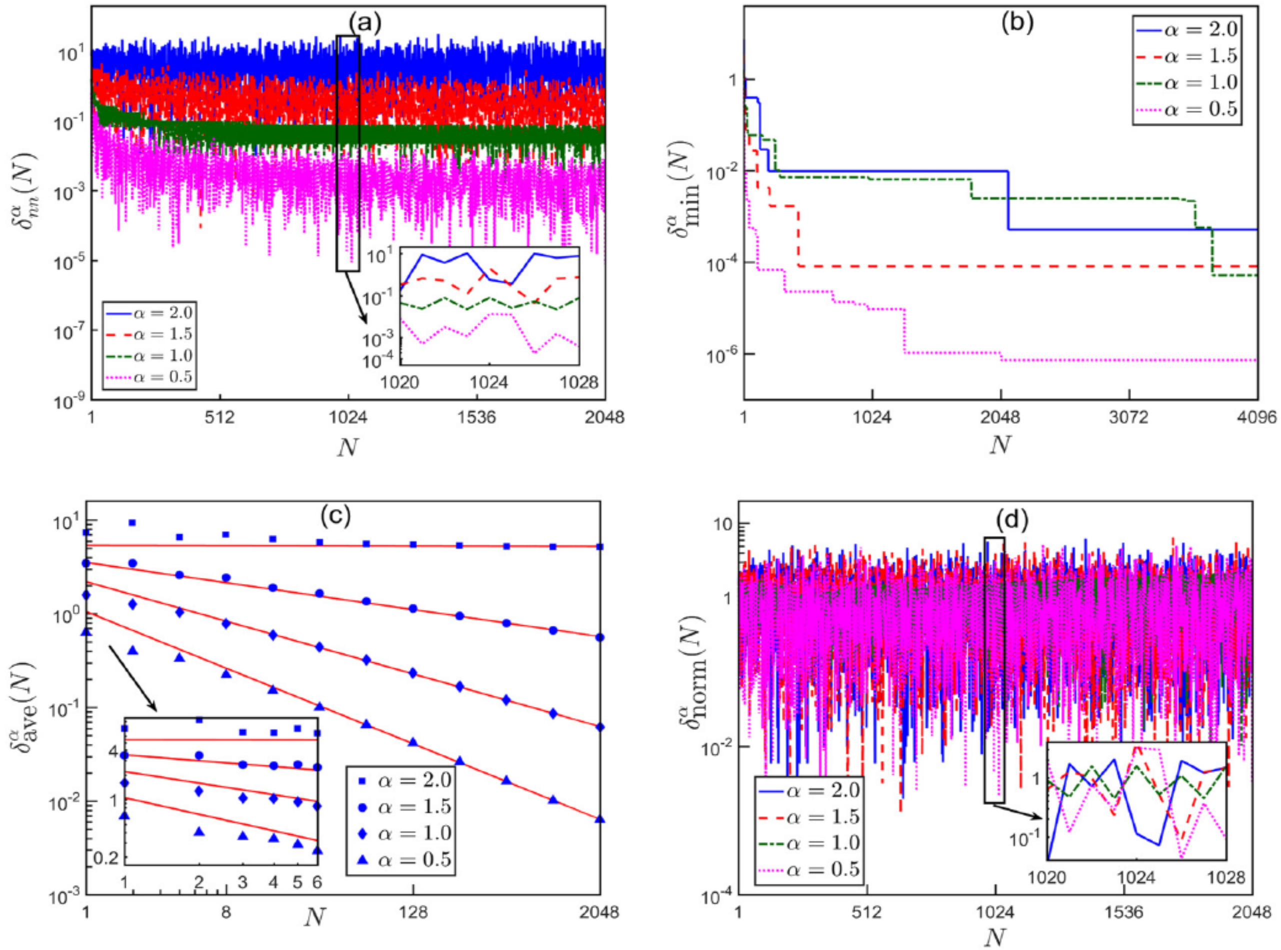


Fig. 19. Different eigenvalue gaps of (6.1) with $d = 2$, $L_1 = 1$, $V(\mathbf{x}) \equiv 0$, $L_2 = \frac{3\sqrt{2}}{2}$ and different α for: (a) the nearest neighbour gaps $\delta^{\alpha}_{\mathrm{nn}}(N)$, (b) the minimum gaps $\delta^{\alpha}_{\min}(N)$, (c) the average gaps $\delta^{\alpha}_{\mathrm{ave}}(N)$ (symbols denote numerical results and solid lines are from fitting formulas when $N \gg 1$), and (d) the normalized gaps $\delta^{\alpha}_{\mathrm{norm}}(N)$.

Gaps Statistics of D-FLO in 2D

$$\Omega = (-1,1)\times(-L_2, L_2)$$
$$V(x,y) \equiv 0$$

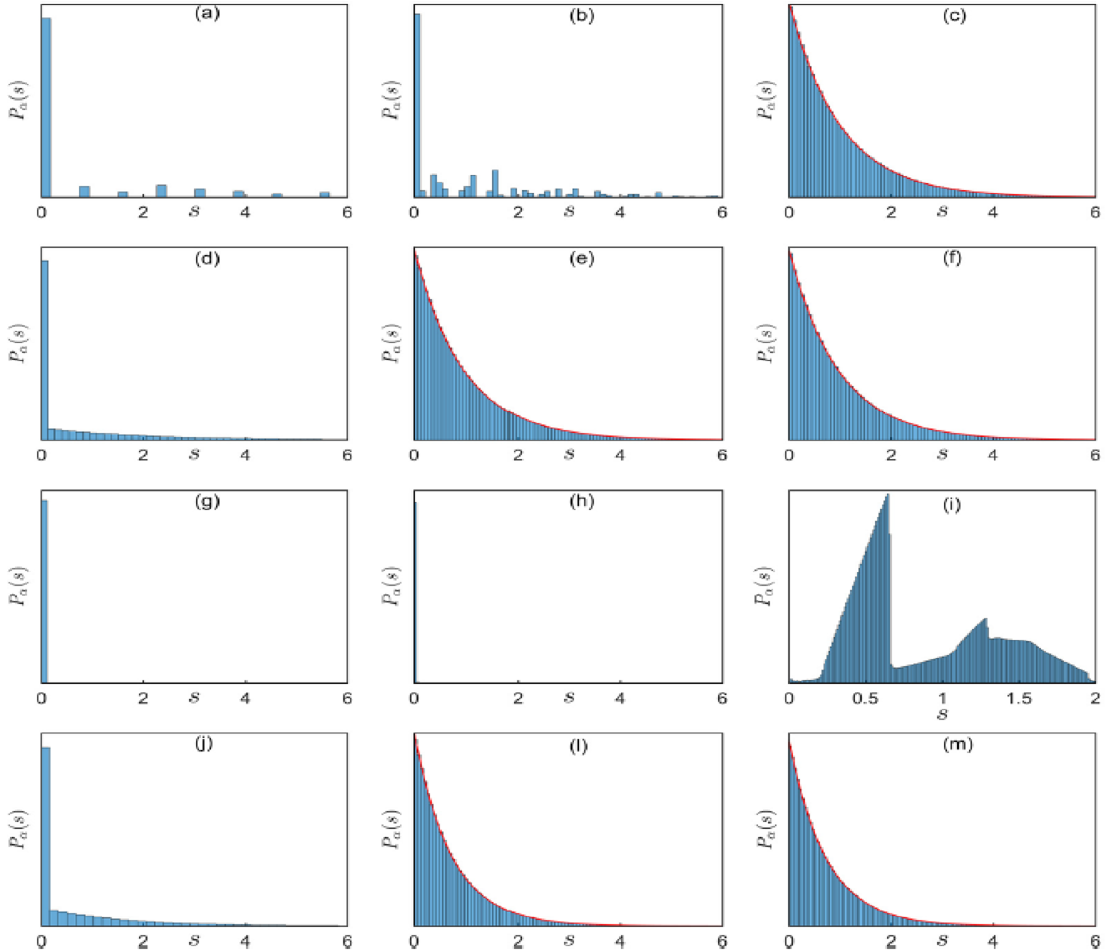


Fig. 20. The histogram of the normalized gaps $\{\delta^{\alpha}_{\text{norm}}(n) \mid 1 \le n \le N = 4000000\}$ of (6.1) with $d = 2$, $L_1 = 1$ and $V(\mathbf{x}) \equiv 0$ for different $0 < \alpha \le 2$ and $0 < L_2 \le 1$: (a) $\alpha = 2.0$ and $L_2 = 1$, (b) $\alpha = 2.0$ and $L_2 = 2/3$, (c) $\alpha = 2.0$ and $L_2 = \frac{\sqrt[3]{2}}{2}$; (d) $\alpha = 1.5$ and $L_2 = 1$, (e) $\alpha = 1.5$ and $L_2 = 2/3$, (f) $\alpha = 1.5$ and $L_2 = \frac{\sqrt[3]{2}}{2}$; (g) $\alpha = 1.0$ and $L_2 = 1$, (h) $\alpha = 1.0$ and $L_2 = 2/3$, (i) $\alpha = 1.0$ and $L_2 = \frac{\sqrt[3]{2}}{2}$; (j) $\alpha = 0.5$ and $L_2 = 1$, (l) $\alpha = 0.5$ and $L_2 = 2/3$, (m) $\alpha = 0.5$ and $L_2 = \frac{\sqrt[3]{2}}{2}$. Solid lines are fitting curves for the gaps distribution statistics $P_\alpha(s)$.

Ratio of Repeated Eigenvalues

$$\Omega = (-1,1)\times(-L_2, L_2), \quad V(x,y)\equiv 0, \quad L_2^2 \in \mathbb{Q}$$

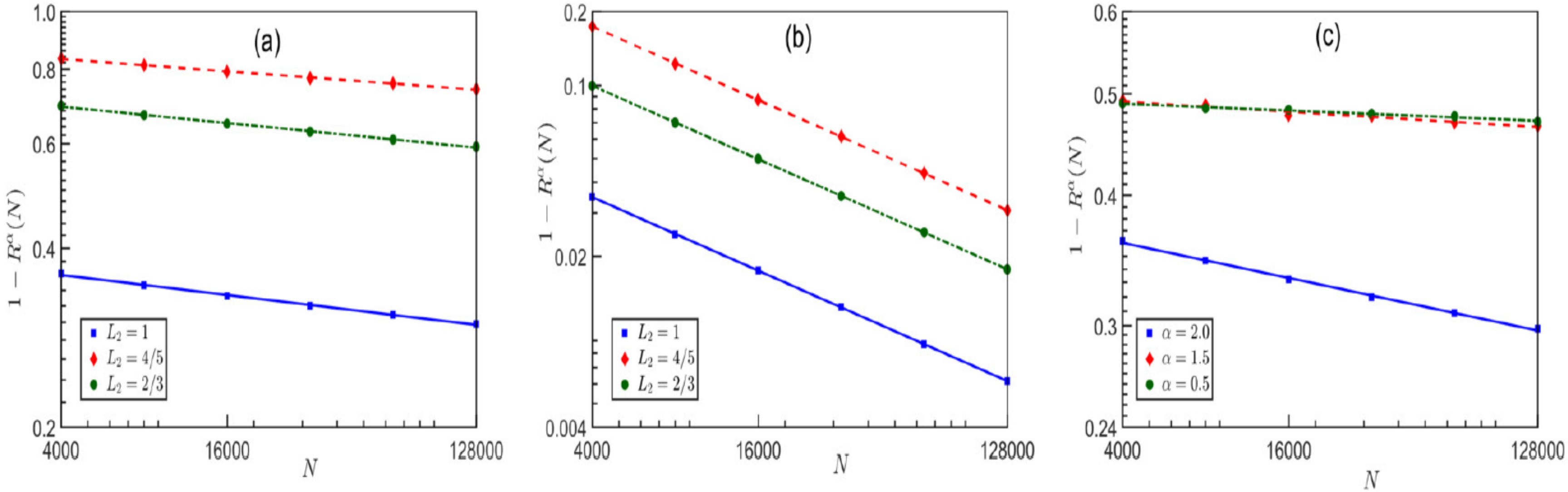


Fig. 21. Plots of $1 - R^{\alpha}(N)$ vs N ($N \gg 1$) for different α and L_2: (a) $\alpha = 2$ for different $L_2 \in \mathbb{Q}$; (b) $\alpha = 1$ for different $L_2 \in \mathbb{Q}$; and (c) $L_2 = 1$ for different $0 < \alpha \le 2$.

$$R(n) := \frac{\#\{1 \le l \le n \mid \lambda_l^{\alpha} = \lambda_{l+1}^{\alpha}\}}{n}, \quad n = 1, 2, \ldots \Rightarrow \quad R(n) \overset{n\to+\infty}{\rightarrow} 1$$

W. Bao, L. Chen, X. Jiang and Y. Ma, A Jacobi spectral method for computing eigenvalue gaps and their distribution statistics of the fractional Schrodinger operator, J. Comput. Phys., Vol. 421 (2020), article 109733.

Gaps of D-FSO in 2D

$\Omega = (-1,1) \times (-L_2, L_2)$

$V(x,y) \neq 0$

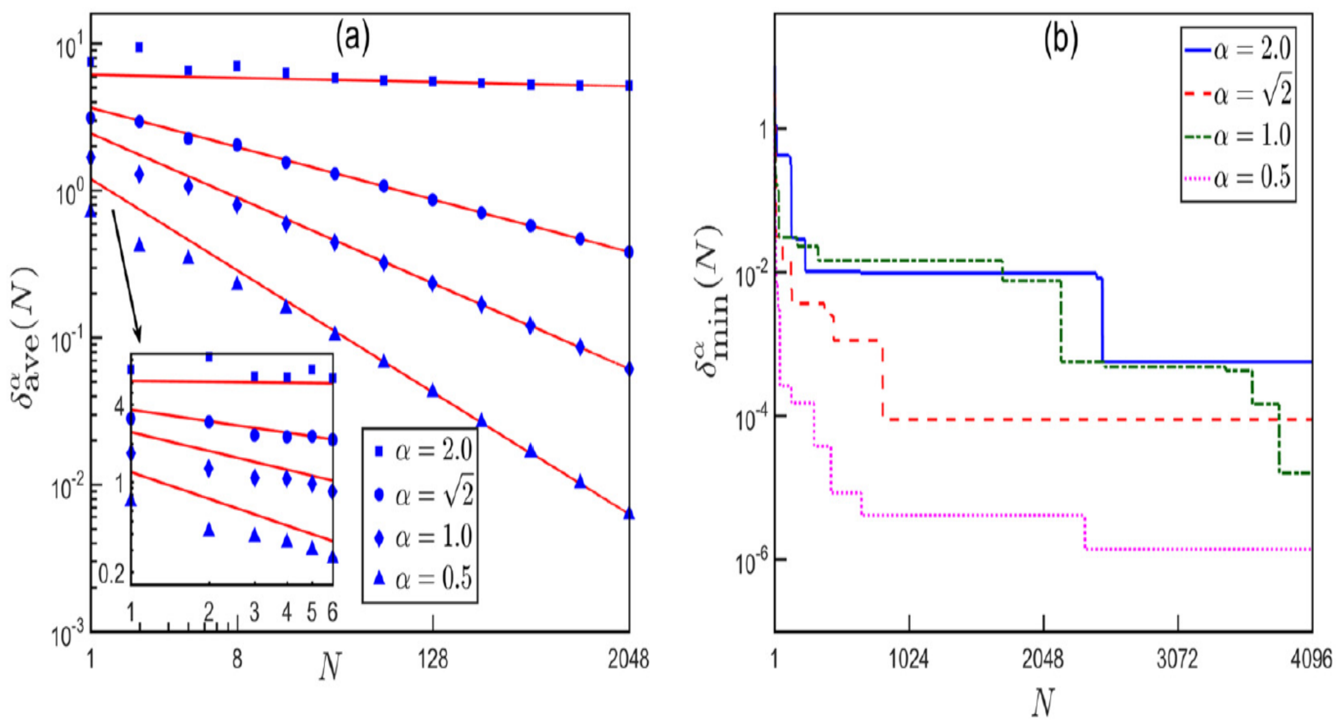


Fig. 24. Different gaps of (6.1) with $d = 2$, $L_1 = 1$, $L_2 = \sqrt[3]{2}/2$ and $V(x, y) = \frac{x^2+y^2}{2}$: (a) the average gaps $\delta^{\alpha}_{\text{ave}}(N)$ (symbols denote numerical results and solid lines are from fitting formulas when $N \gg 1$), and (b) the minimum gaps $\delta^{\alpha}_{\min}(N)$.

W. Bao, L. Chen, X. Jiang and Y. Ma, A Jacobi spectral method for computing eigenvalue gaps and their distribution statistics of the fractional Schrodinger operator, J. Comput. Phys., Vol. 421 (2020), article 109733.

Gaps Statistics of D-FSO in 2D

$$\Omega = (-1,1)\times(-L_2, L_2)$$

$$V(x,y) \neq 0$$

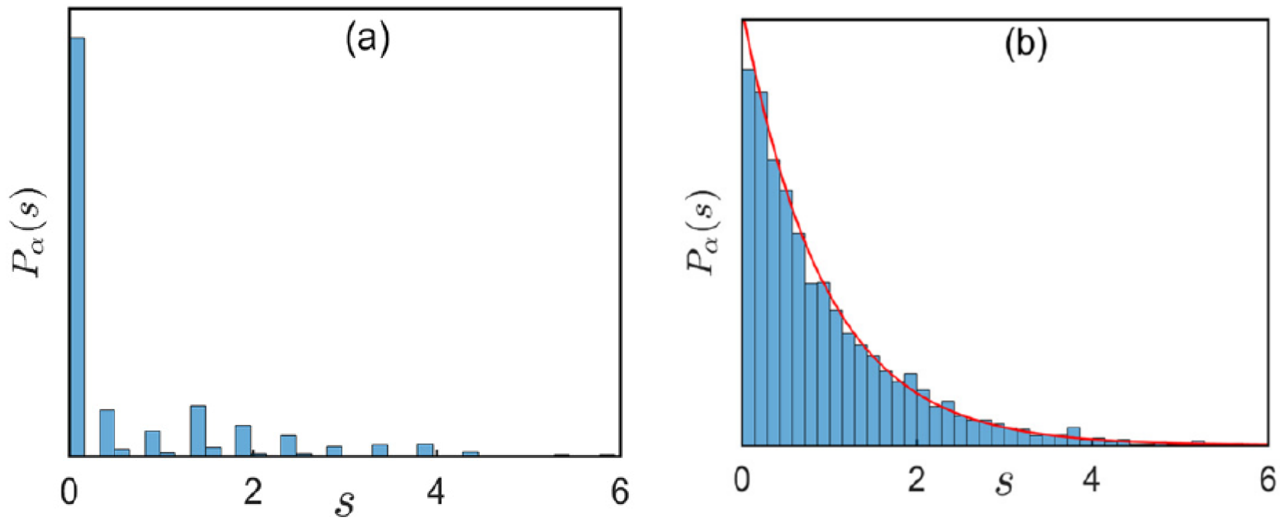


Fig. 25. The histogram of the normalized gaps $\{\delta^{\alpha}_{\text{norm}}(n) \mid 1 \le n \le N = 4096\}$ of (6.1) with $d = 2$ and $V(x, y) = \frac{x^2+y^2}{2}$: (a) $\alpha = 2$ and $L_2 = 1$; and (b) $\alpha = \sqrt{2}$ and $L_2 = \sqrt[3]{2}/2$ (the solid line is a fitting curve by the Poisson distribution).

W. Bao, L. Chen, X. Jiang and Y. Ma, A Jacobi spectral method for computing eigenvalue gaps and their distribution statistics of the fractional Schrodinger operator, J. Comput. Phys., Vol. 421 (2020), article 109733.

Summary of Gaps Statistics of D-FSO in 2D

$$d = 2, \quad \Omega = (-L_1, L_1) \times (-L_2, L_2)$$

	$\frac{L_2}{L_1} = 1$	$1 \neq \frac{L_2}{L_1} \in \mathbb{Q}$	$\frac{L_2}{L_1} \notin \mathbb{Q}$ & $\frac{L_2^2}{L_1^2} \in \mathbb{Q}$	$\frac{L_2^2}{L_1^2} \notin \mathbb{Q}$
$\alpha = 2$	$\delta(s)$	$\delta(s)$	$\delta(s)$	Poisson
$1 < \alpha < 2$	$\delta(s)$	Poisson	Poisson	Poisson
$\alpha = 1$	$\delta(s)$	$\delta(s)$	Bimodal distribution	Bimodal distribution
$0 < \alpha < 1$	$\delta(s)$	Poisson	Poisson	Poisson

W. Bao, L. Chen, X. Jiang and Y. Ma, A Jacobi spectral method for computing eigenvalue gaps and their distribution statistics of the fractional Schrodinger operator, J. Comput. Phys., Vol. 421 (2020), article 109733.

Time-independent GPE/NLSE

$$H\phi := -\frac{1}{2}\nabla^2\phi(\vec{x}) + V(\vec{x})\phi(\vec{x}), \quad H\phi + \beta|\phi(\vec{x})|^2\phi(\vec{x}) = \mu\,\phi(\vec{x}), \quad \vec{x}\in\mathbb{R}^d$$

- Eigenfunctions are with $\|\phi\|^2 := \int_{\mathbb{R}^d} |\phi(\vec{x})|^2\, d\vec{x} = 1$
  - Orthogonal in linear case & Superposition is valid for dynamics!!
  - Not orthogonal in nonlinear case !!!! No superposition for dynamics!!!
- The eigenvalue is also called as chemical potential

$$\mu := \mu(\phi) = E(\phi) + \frac{\beta}{2}\int_{\mathbb{R}^d} |\phi(\vec{x})|^4\, d\vec{x}$$

  - With energy $E(\phi) = \int_{\mathbb{R}^d} [\frac{1}{2}|\nabla\phi(\vec{x})|^2 + V(\vec{x})|\phi(\vec{x})|^2 + \frac{\beta}{2}|\phi(\vec{x})|^4]\, d\vec{x}$
- Ground states -- nonconvex minimization problem

$$E_g^\beta := E(\phi_g^\beta) = \min_{\phi\in S} E(\phi), \quad S = \left\{\phi \mid \|\phi\| = 1, \quad E(\phi) < \infty\right\}, \quad \mu_g^\beta = \mu(\phi_g^\beta)$$

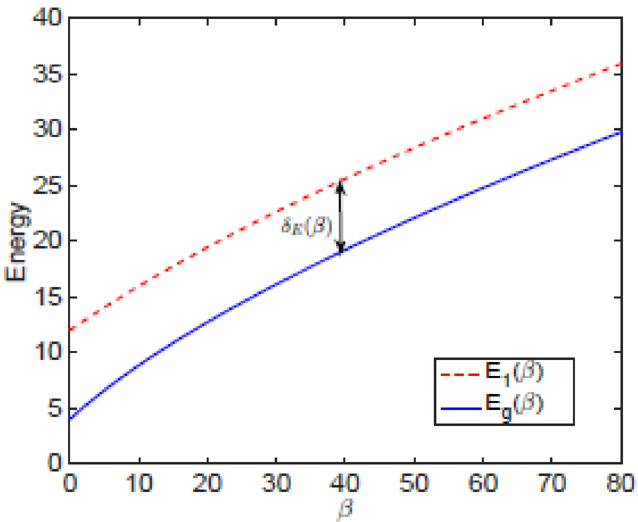
- First excited state: $\phi_1^\beta, \quad E_1^\beta := E(\phi_1^\beta) > E_g^\beta, \quad \mu_1^\beta := \mu(\phi_1^\beta) > \mu_1^\beta,$

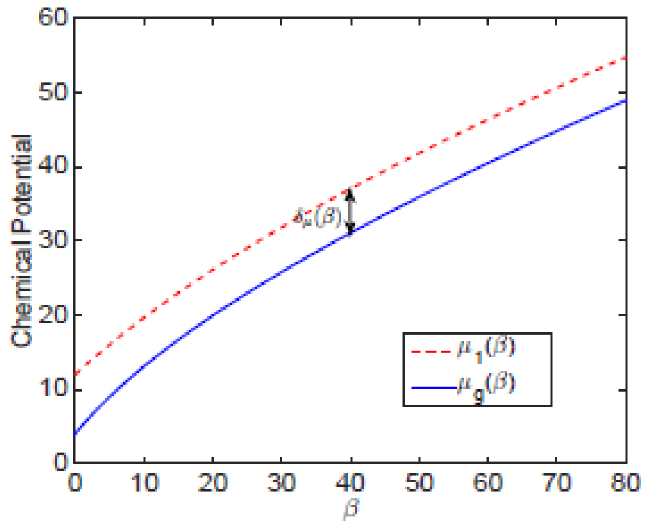
Fundamental gaps of GPE

- Gaps:

$$\delta_E(\beta) := E(\phi_1^\beta) - E(\phi_g^\beta) > 0, \quad \delta_\mu(\beta) := \mu(\phi_1^\beta) - \mu(\phi_g^\beta) > 0, \quad \beta \ge 0$$

$$\delta_E^\infty := \inf_{\beta\ge 0} \delta_E(\beta), \qquad \delta_\mu^\infty := \inf_{\beta\ge 0} \delta_\mu(\beta),$$





- Eigenspace of the first excited space of SO: $H := -\frac{1}{2}\nabla^2 + V(\vec{x})$

$$W_1 = \left\{\phi(\mathbf{x}) : \Omega \to \mathbb{C} \,|\, H\phi = E_1\phi,\, \phi|_{\partial\Omega} = 0 \text{ if } \Omega \text{ is bounded}\right\}$$

Fundamental Gaps of GPE

$$H\phi := -\frac{1}{2}\nabla^2\phi(\vec{x}) + V(\vec{x})\phi(\vec{x}), \quad H\phi + \beta\,|\phi(\vec{x})|^2\,\phi(\vec{x}) = \mu\,\phi(\vec{x}), \quad \vec{x}\in\Omega\subset\mathbb{R}^d$$

$$\phi|_{\partial\Omega} = 0; \quad \|\phi\|^2 := \int_\Omega |\phi(\vec{x})|^2\, d\vec{x} = 1$$

- **Fundamental gaps** – Bao & Ruan, Asymp. Anal. 18'
  - In the non-degenerate case, i.e. $\dim(W_1) = 1$

$$\delta_E(\beta) = \begin{cases} \frac{3\pi^2}{2L_1^2} + o(\beta), \\ \frac{4A_0}{3L_1}\beta^{\frac{1}{2}} + A_1 + o(1), \end{cases} \qquad \delta_\mu(\beta) = \begin{cases} \frac{3\pi^2}{2L_1^2} + o(\beta), & 0 \leqslant \beta \ll 1 \\ \frac{2A_0}{L_1}\beta^{\frac{1}{2}} + \frac{6}{L_1^2} + o(1), & \beta \gg 1. \end{cases}$$

$$\delta_E^\infty := \inf_{\beta\geqslant 0} \delta_E(\beta) \geqslant \frac{3\pi^2}{2D^2}, \qquad \delta_\mu^\infty := \inf_{\beta\geqslant 0} \delta_\mu(\beta) \geqslant \frac{3\pi^2}{2D^2}$$

  - In the degenerate case, i.e. $\dim(W_1) \geq 2$
    - Weak nonlinearity $\quad \delta_E(\beta) = \frac{3\pi^2}{2L^2} - \frac{5dA_0^2}{32}\beta + o(\beta), \qquad \delta_\mu(\beta) = \frac{3\pi^2}{2L^2} - \frac{5dA_0^2}{16}\beta + o(\beta)$
    - Strong nonlinearity $\quad \delta_E(\beta) = \frac{\pi}{2L^2}\ln(\beta) + \mathcal{O}(1), \qquad \delta_\mu(\beta) = \frac{\pi}{2L^2}\ln(\beta) + \mathcal{O}(1)$

$$\delta_E^\infty := \inf_{\beta\geqslant 0} \delta_E(\beta) \geqslant \frac{\pi^2}{2D^2}, \qquad \delta_\mu^\infty := \inf_{\beta\geqslant 0} \delta_\mu(\beta) \geqslant \frac{3\pi^2}{8D^2}$$

W. Bao and X. Rua, Fundamental gaps of the Gross-Pitaevskii equation with repulsive interaction, Asymptotic Analysis, Vol. 110 (2018), pp. 53-82.

Conclusions

- The Weyl's law and conjecture
  - Laplacian operator in 1D/2D/3D
  - Schrodinger operator in 1D/2D/3D
  - Extension to Fractional Laplacian/directional Schrodinger operator
- Fundamental gap conjecture
  - For Laplacian/Schrodinger operator
  - Extension to fractional Schrodinger operator
  - Different gaps and gaps statistics
- Future challenges
  - Other BCs, singular/random potentials, on manifold/graph (data science)
  - Other type operators – Dirac operator, relativistic Schrodinger operator, …..