COMPACTLY SUPPORTED ORTHONORMAL COMPLEX WAVELETS WITH DILATION 4 AND SYMMETRY

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Abstract. In this paper, we provide a family of compactly supported orthonormal complex wavelets with dilation 4 such that the generating wavelet functions are symmetric/antisymmetric and have linearly increasing orders of vanishing moments and smoothness.

1. Introduction and Main Result

It is well known in Daubechies [5] that except the Haar wavelet which is discontinuous, compactly supported dyadic orthonormal real-valued wavelets cannot have symmetry. In order to achieve symmetry, wavelets with other dilations have been considered in the literature. Indeed, a few examples of compactly supported orthonormal real-valued wavelets with symmetry and dilations greater than two have been reported in [1, 3, 4, 6, 9]. All such examples available in the literature are obtained by solving systems of nonlinear quadratic algebraic equations. For example, by complicated calculation, only some examples of compactly supported C^1 symmetric orthonormal real-valued wavelets with dilation 4 have been obtained in [6]. To the best of our knowledge, no compactly supported C^2 symmetric orthonormal real-valued wavelets with dilation 4 have been known so far in the literature. On the other hand, it has been observed in the interesting work Lawton [10] that symmetry can also be achieved by considering orthonormal complex wavelets. Motivated by [8, 10], in this paper we consider symmetric orthonormal complex wavelets with dilation 4 and we provide a family of compactly supported arbitrarily smooth orthonormal complex wavelets with dilation 4 and symmetry.

In order to introduce our main result, let us recall some necessary notation. Throughout the paper, i denotes the imaginary unit such that $i^2 = -1$. The Fourier transform in this paper is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-i\xi x}\, dx$ for $f \in L_1(\mathbb{R})$ and is naturally extended to L_2 functions and distributions.

For a compactly supported complex-valued function $\phi : \mathbb{R} \mapsto \mathbb{C}$, we say that ϕ is *refinable* with dilation 4 if $\hat{\phi}(4\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ for a 2π-periodic trigonometric polynomial $\hat{a}$ with complex coefficients. Such a trigonometric polynomial $\hat{a}$ is called the *mask* for the refinable function ϕ. We say that ϕ is an *orthonormal refinable function* with dilation 4 and mask $\hat{a}$, if $\hat{\phi}(4\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ and the integer shifts of ϕ are orthonormal, that is,

$$\langle \phi(\cdot - k), \phi\rangle := \int_{\mathbb{R}} \phi(x-k)\overline{\phi(x)}\, dx = \delta_k, \qquad \forall\, k \in \mathbb{Z}, \tag{1.1}$$

where $\overline{\phi(x)}$ denotes the complex conjugate of $\phi(x)$ and δ denotes the Dirac sequence such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \neq 0$. If ϕ is an orthonormal refinable function with dilation 4 and mask $\hat{a}$, then it is well known that $\hat{a}$ must be an *orthogonal mask*, that is,

$$|\hat{a}(\xi)|^2 + |\hat{a}(\xi + \pi/2)|^2 + |\hat{a}(\xi+\pi)|^2 + |\hat{a}(\xi + 3\pi/2)|^2 = 1, \qquad \xi \in \mathbb{R}. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 42C40, 42C05.

Key words and phrases. Orthonormal complex wavelets, symmetry, smoothness, vanishing moments.

Research supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC Canada) under Grant RGP 228051. February 25, 2008.

For a function $f \in L_2(\mathbb{R})$, its smoothness is measured by the quantity

$$\nu_2(f) := \sup\Big\{\nu \in \mathbb{R} \;:\; \int_{\mathbb{R}} |\hat{f}(\xi)|^2(1+\xi^2)^\nu \, d\xi < \infty\Big\}. \tag{1.3}$$

For a smooth function f, we denote $f^{(j)}$ the jth derivative of f. For a compactly supported function $f : \mathbb{R} \mapsto \mathbb{C}$, we say that f has m *vanishing moments* if $\hat{f}^{(j)}(0) = 0$ for all $j = 0, 1, \dots, m-1$. The notion of vanishing moments plays an important role in wavelet analysis.

For each positive integer m, we define P_m to be a polynomial of degree $m-1$ such that

$$P_m(x) := [(1-x)(1-2x)^2]^{-m} + O(x^m), \qquad x \to 0. \tag{1.4}$$

That is, P_m is the $(m-1)$-th Taylor polynomial of the function $[(1-x)(1-2x)^2]^{-m}$ at $x = 0$. Now we state the main result of this paper.

Theorem 1. *Let m be a positive odd integer and denote P_m the polynomial defined in (1.4). Then there exist two polynomials P^r_m and P^i_m with real coefficients such that*

$$P_m(x) = [P^r_m(x)]^2 + [P^i_m(x)]^2, \qquad x \in \mathbb{R} \quad \textit{with} \quad P^r_m(0) = 1, P^i_m(0) = 0. \tag{1.5}$$

Define a 2π-periodic trigonometric polynomial with complex coefficients by

$$\widehat{a^m}(\xi) := 4^{-m} e^{i\xi(3m-1)/2}(1 + e^{-i\xi} + e^{-2i\xi} + e^{-3i\xi})^m [P^r_m(\sin^2(\xi/2)) + iP^i_m(\sin^2(\xi/2))] \tag{1.6}$$

and define

$$\widehat{\phi^m}(\xi) := \prod_{j=1}^{\infty} \widehat{a^m}(4^{-j}\xi), \qquad \xi \in \mathbb{R}. \tag{1.7}$$

Then ϕ^m is a compactly supported orthonormal refinable function with dilation 4 and mask $\widehat{a^m}$. Moreover, there exist three 2π-periodic trigonometric polynomials $\widehat{b^{m,1}}$, $\widehat{b^{m,2}}$ and $\widehat{b^{m,3}}$ such that

(1) *$\{2^j\psi^{m,1}(4^j \cdot -k),\ 2^j\psi^{m,2}(4^j \cdot -k),\ 2^j\psi^{m,3}(4^j \cdot -k) \;:\; j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L_2(\mathbb{R})$, where*

$$\widehat{\psi^{m,1}}(4\xi) := \widehat{b^{m,1}}(\xi)\widehat{\phi^m}(\xi), \quad \widehat{\psi^{m,2}}(4\xi) := \widehat{b^{m,2}}(\xi)\widehat{\phi^m}(\xi), \quad \widehat{\psi^{m,3}}(4\xi) := \widehat{b^{m,3}}(\xi)\widehat{\phi^m}(\xi). \tag{1.8}$$

(2) *$\phi(1-\cdot) = \phi$ and $\psi^{m,1}(1-\cdot) = \psi^{m,1}$, $\psi^{m,2}(1-\cdot) = -\psi^{m,2}$, $\psi^{m,3}(1-\cdot) = -\psi^{m,3}$.*

(3) *All $\psi^{m,1}, \psi^{m,2}, \psi^{m,3}$ have m vanishing moments and* $\liminf_{m\to\infty} \frac{\min_{1\leqslant \ell \leqslant 3} \nu_2(\psi^{m,\ell})}{m} > 0$.

The polynomials P^r_m and P^i_m can be constructed from P_m by [8, Proposition 9]. In Section 2, we shall propose an algorithm to construct the high-pass wavelet masks $\widehat{b^{m,1}}$, $\widehat{b^{m,2}}$ and $\widehat{b^{m,3}}$ in Theorem 1 from the low-pass mask $\widehat{a^m}$. In Section 3 we shall present several examples of orthonormal complex wavelets with dilation 4 and symmetry to illustrate the main result in Theorem 1 and Algorithm 3 of this paper. The proof of Theorem 1 will be provided in Section 4.

2. An Algorithm for Constructing High-pass Wavelet Masks

In this section, in order to obtain the wavelet masks $\widehat{b^{m,1}}, \widehat{b^{m,2}}, \widehat{b^{m,3}}$ in Theorem 1, we shall propose an algorithm to construct high-pass wavelet masks with symmetry from a given symmetric low-pass mask with complex coefficients.

Throughout the paper, we shall use $\mathrm{Re}(c)$, $\mathrm{Im}(c)$, and $\overline{c}$ to denote the real part, the imaginary part, and the complex conjugate of a complex number $c \in \mathbb{C}$. For a nonzero 2π-periodic trigonometric polynomial $\hat{a}(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}$, we denote $\deg(\hat{a}) := \min\{N \in \mathbb{Z} \;:\; a_k = 0 \quad \text{for all } k > N\}$.

In order to state the algorithm for deriving high-pass wavelet masks from low-pass masks, we need the following result.

Lemma 2. *Let $A(\xi) := [A_1(\xi), A_2(\xi), A_3(\xi), A_4(\xi)]^T$ be a column vector of 2π-periodic trigonometric polynomials with complex coefficients such that A satisfies*

$$A_1(-\xi) = A_1(\xi), \quad A_2(-\xi) = A_2(\xi), \quad A_3(-\xi) = -A_3(\xi), \quad A_4(-\xi) = -A_4(\xi) \tag{2.1}$$

and

$$|A_1(\xi)|^2 + |A_2(\xi)|^2 + |A_3(\xi)|^2 + |A_4(\xi)|^2 = 1 \qquad \forall\, \xi \in \mathbb{R}. \tag{2.2}$$

Denote $\deg(A) := \max(\deg(A_1), \deg(A_2), \deg(A_3), \deg(A_4))$. *If* $\deg(A) > 0$, *then by* (2.1) *we can write*

$$A(\xi) = \begin{bmatrix} f_1 \\ f_2 \\ -g_1 \\ -g_2 \end{bmatrix} e^{ik\xi} + \begin{bmatrix} f_3 \\ f_4 \\ -g_3 \\ -g_4 \end{bmatrix} e^{i(k-1)\xi} + \cdots + \begin{bmatrix} f_3 \\ f_4 \\ g_3 \\ g_4 \end{bmatrix} e^{-i(k-1)\xi} + \begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} e^{-ik\xi}, \tag{2.3}$$

where $k = \deg(A)$ and $f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4$ are some complex numbers (when $\deg(A) = 1$, we replace f_3, f_4, g_3, g_4 in (2.3) *by $f_3/2$, $f_4/2, 0, 0$ respectively). Now we define*

$$B(\xi) := [B_1(\xi), B_2(\xi), B_3(\xi), B_4(\xi)]^T := U_A(\xi)A(\xi), \tag{2.4}$$

where

$$U_A(\xi) := \begin{bmatrix} c_0\overline{f_1}(\cos\xi - i\frac{h}{c}) & c_0\overline{f_2}(\cos\xi - i\frac{h}{c}) & ic_0\overline{g_1}\sin\xi & ic_0\overline{g_2}\sin\xi \\ -\sqrt{\frac{2}{c}}f_2 & \sqrt{\frac{2}{c}}f_1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{c}}g_2 & -\sqrt{\frac{2}{c}}g_1 \\ ic_0\overline{f_1}\sin\xi & ic_0\overline{f_2}\sin\xi & c_0\overline{g_1}(\cos\xi + i\frac{h}{c}) & c_0\overline{g_2}(\cos\xi + i\frac{h}{c}) \end{bmatrix} \tag{2.5}$$

with

$$c := |f_1|^2 + |f_2|^2 + |g_1|^2 + |g_2|^2, \qquad h := Im(\overline{f_1}f_3 + \overline{f_2}f_4 - \overline{g_1}g_3 - \overline{g_2}g_4), \quad c_0 := \sqrt{\frac{2c}{c^2 + h^2}}. \tag{2.6}$$

Then (2.1) *and* (2.2) *hold with A being replaced by B, $\deg(B) < \deg(A)$, and $U_A(\xi)\overline{U_A(\xi)}^T = I_4$.*

Proof. By calculation, it follows from the definition of the vector $B(\xi)$ in (2.4) that we have

$$\begin{aligned} B_1(\xi) &:= c_0\Big((\overline{f_1}A_1(\xi) + \overline{f_2}A_2(\xi))(\cos\xi - i\frac{h}{c}) + i(\overline{g_1}A_3(\xi) + \overline{g_2}A_4(\xi))\sin\xi\Big), \\ B_2(\xi) &:= \sqrt{2/c}\big(f_1A_2(\xi) - f_2A_1(\xi)\big), \\ B_3(\xi) &:= \sqrt{2/c}\big(g_2A_3(\xi) - g_1A_4(\xi)\big), \\ B_4(\xi) &:= c_0\Big(i(\overline{f_1}A_1(\xi) + \overline{f_2}A_2(\xi))\sin\xi + (\overline{g_1}A_3(\xi) + \overline{g_2}A_4(\xi))(\cos\xi + i\frac{h}{c})\Big). \end{aligned} \tag{2.7}$$

By (2.1), now it is straightforward to see that $B_1(-\xi) = B_1(\xi)$, $B_2(-\xi) = B_2(\xi)$, $B_3(-\xi) = -B_3(\xi)$ and $B_4(-\xi) = -B_4(\xi)$.

By (2.3) and (2.7), it is easy to see that $\deg(B_2) < k$, $\deg(B_3) < k$, and the degrees of B_1 and B_4 are no greater than $k+1$. By a direct calculation, it follows from (2.3) and (2.7) that

$$\begin{aligned} B_1(\xi) &= c_1e^{i(k+1)\xi} + c_2e^{ik\xi} + \cdots + c_2e^{-ik\xi} + c_1e^{-i(k+1)\xi}, \\ B_4(\xi) &= -c_3e^{i(k+1)\xi} - c_4e^{ik\xi} + \cdots + c_4e^{-ik\xi} + c_3e^{-i(k+1)\xi}, \end{aligned}$$

where

$$\begin{aligned}
c_1 &:= c_0\big(|f_1|^2+|f_2|^2-|g_1|^2-|g_2|^2\big)/2,\\
c_2 &:= c_0\big((\overline{f_1}f_3+\overline{f_2}f_4-\overline{g_1}g_3-\overline{g_2}g_4)-i2(|f_1|^2+|f_2|^2)h/c\big)/2,\\
c_3 &:= c_0\big(|g_1|^2+|g_2|^2-|f_1|^2-|f_2|^2\big)/2,\\
c_4 &:= c_0\big((\overline{g_1}g_3+\overline{g_2}g_4-\overline{f_1}f_3-\overline{f_2}f_4)+i2(|g_1|^2+|g_2|^2)h/c\big)/2.
\end{aligned}$$

To show that $\deg(B)<\deg(A)=k$, we have to prove that $\deg(B_1)<k$ and $\deg(B_4)<k$. In other words, we need to prove that $c_1=c_2=c_3=c_4=0$. From (2.2), we observe that

$$|f_1|^2+|f_2|^2=|g_1|^2+|g_2|^2 \quad \text{and} \quad \mathrm{Re}(\overline{f_1}f_3+\overline{f_2}f_4)=\mathrm{Re}(\overline{g_1}g_3+\overline{g_2}g_4). \tag{2.8}$$

It follows directly from the first identity in (2.8) that $c_1=c_3=0$. By the second identity in (2.8), we deduce that $\mathrm{Re}(c_2)=\mathrm{Re}(c_4)=0$. To show that $c_2=c_4=0$, now it suffices to show that

$$\mathrm{Im}(\overline{f_1}f_3+\overline{f_2}f_4-\overline{g_1}g_3-\overline{g_2}g_4)=2(|f_1|^2+|f_2|^2)h/c=2(|g_1|^2+|g_2|^2)h/c.$$

But the above identity is obviously true by the definition of h and c in (2.6). Consequently, we verified that $\deg(B)<\deg(A)$.

Now we prove $U_A(\xi)\overline{U_A(\xi)}^T=I_4$. By a direct calculation, we have

$$U_A(\xi)\overline{U_A(\xi)}^T=\begin{bmatrix} U_{1,1} & 0 & 0 & U_{1,4}\\ 0 & 2(|f_1|^2+|f_2|^2)/c & 0 & 0\\ 0 & 0 & 2(|g_1|^2+|g_2|^2)/c & 0\\ U_{4,1} & 0 & 0 & U_{4,4}\end{bmatrix}.$$

with

$$\begin{aligned}
U_{1,1} &:= c_0^2\Big((|f_1|^2+|f_2|^2)(\cos^2\xi+h^2/c^2)+(|g_1|^2+|g_2|^2)\sin^2\xi\Big),\\
U_{1,4} &:= ic_0^2(|g_1|^2+|g_2|^2-|f_1|^2-|f_2|^2)(\cos\xi-ih/c)\sin\xi,\\
U_{4,1} &:= ic_0^2\Big((|f_1|^2+|f_2|^2-|g_1|^2-|g_2|^2)(\cos\xi+ih/c)\sin\xi\Big),\\
U_{4,4} &:= c_0^2\Big((|f_1|^2+|f_2|^2)\sin^2\xi+(|g_1|^2+|g_2|^2)(\cos^2\xi+h^2/c^2)\Big).
\end{aligned}$$

Note that by (2.8), we have $|f_1|^2+|f_2|^2=c/2=|g_1|^2+|g_2|^2$. Now $U_A(\xi)\overline{U_A(\xi)}^T=I_4$ can easily verified by the definition of c_0,c,h in (2.6). Consequently, we conclude that $\overline{B(\xi)}^TB(\xi)=\overline{A(\xi)}^T\overline{U_A(\xi)}^TU_A(\xi)A(\xi)=\overline{A(\xi)}^TA(\xi)=1$, which completes the proof. ∎

Now we state the algorithm for deriving the high-pass wavelet masks $\widehat{b^{m,1}}$, $\widehat{b^{m,2}}$ and $\widehat{b^{m,3}}$ from the low-pass mask $\widehat{a^m}$. Note that $\widehat{a^m}(-\xi)=e^{i\xi}\widehat{a^m}(\xi)$, where $\widehat{a^m}$ is defined in (1.6).

Algorithm 3. *Let $\hat{a}(\xi)=\sum_{k\in\mathbb{Z}}a_ke^{-ik\xi}$ be a 2π-periodic trigonometric polynomial with complex coefficients such that $\hat{a}(0)=1$, $\hat{a}$ satisfies (1.2) and $\hat{a}(-\xi)=e^{i\xi}\hat{a}(\xi)$. Denote*

$$A^0(\xi):=\sqrt{2}[\widehat{\mathbf{a}^0}(\xi)+\widehat{\mathbf{a}^0}(-\xi),\widehat{\mathbf{a}^2}(\xi)+\widehat{\mathbf{a}^2}(-\xi),\widehat{\mathbf{a}^0}(\xi)-\widehat{\mathbf{a}^0}(-\xi),\widehat{\mathbf{a}^2}(\xi)-\widehat{\mathbf{a}^2}(-\xi)]^T, \tag{2.9}$$

where

$$\widehat{\mathbf{a}^j}(\xi):=\sum_{k\in\mathbb{Z}}a_{j+4k}e^{-ik\xi}, \qquad \xi\in\mathbb{R}, j\in\mathbb{Z}. \tag{2.10}$$

Then A^0 satisfies all the conditions in (2.1) and (2.2) with A being replaced by A^0.

(1) *Recursively apply Lemma 2 and define*

$$A^j(\xi) := U_{A^{j-1}}(\xi)U_{A^{j-2}}(\xi)\cdots U_{A^1}(\xi)U_{A^0}(\xi)A^0(\xi) \tag{2.11}$$

until $\deg(A^j) = 0$ *for some nonnegative integer* $j \leqslant \deg(A^0)$*, where all* $U_{A^0}, \dots, U_{A^{j-1}}$ *are given in Lemma 2.*

(2) *By* $\deg(A^j) = 0$*, it follows from* (2.1) *and* (2.2) *that* $A^j = [h_1, h_2, 0, 0]^T$ *for some complex numbers* $h_1, h_2 \in \mathbb{C}$ *such that* $|h_1|^2 + |h_2|^2 = 1$*. Define a matrix* $U(\xi)$ *by*

$$U(\xi) := \overline{U_{A^0}(\xi)}^T \cdots \overline{U_{A^{j-1}}(\xi)}^T \begin{bmatrix} h_1 & -\overline{h_2} & 0 & 0 \\ h_2 & \overline{h_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.12}$$

(3) *Obtain the high-pass wavelet masks* $\widehat{b^1}, \widehat{b^2}$ *and* $\widehat{b^3}$ *by*

$$\begin{aligned} \widehat{b^{\ell-1}}(\xi) := \frac{\sqrt{2}}{4}\Big(&(1+e^{-i\xi})U_{1,\ell}(4\xi) + (e^{i\xi} + e^{-i2\xi})U_{2,\ell}(4\xi) \\ &+ (1-e^{-i\xi})U_{3,\ell}(4\xi) + (e^{-i2\xi} - e^{i\xi})U_{4,\ell}(4\xi)\Big), \qquad \ell = 2,3,4, \end{aligned} \tag{2.13}$$

where $U_{j,k}(\xi)$ *denotes the* (j,k)*-entry of the* 4×4 *matrix* $U(\xi)$*.*

Then

$$\widehat{b^1}(-\xi) = e^{i\xi}\widehat{b^1}(\xi), \quad \widehat{b^2}(-\xi) = -e^{i\xi}\widehat{b^2}(\xi), \quad \widehat{b^3}(-\xi) = -e^{i\xi}\widehat{b^3}(\xi), \tag{2.14}$$

and $P_{[\hat{a},\widehat{b^1},\widehat{b^2},\widehat{b^3}]}(\xi)\overline{P_{[\hat{a},\widehat{b^1},\widehat{b^2},\widehat{b^3}]}(\xi)}^T = I_4$*, that is,* $P_{[\hat{a},\widehat{b^1},\widehat{b^2},\widehat{b^3}]}(\xi)$ *is a unitary matrix, where*

$$P_{[\hat{a},\widehat{b^1},\widehat{b^2},\widehat{b^3}]}(\xi) := \begin{bmatrix} \hat{a}(\xi) & \hat{a}(\xi+\pi/2) & \hat{a}(\xi+\pi) & \hat{a}(\xi+3\pi/2) \\ \widehat{b^1}(\xi) & \widehat{b^1}(\xi+\pi/2) & \widehat{b^1}(\xi+\pi) & \widehat{b^1}(\xi+3\pi/2) \\ \widehat{b^2}(\xi) & \widehat{b^2}(\xi+\pi/2) & \widehat{b^2}(\xi+\pi) & \widehat{b^2}(\xi+3\pi/2) \\ \widehat{b^3}(\xi) & \widehat{b^3}(\xi+\pi/2) & \widehat{b^3}(\xi+\pi) & \widehat{b^3}(\xi+3\pi/2) \end{bmatrix}. \tag{2.15}$$

Proof. Since $\hat{a}(-\xi) = e^{i\xi}\hat{a}(\xi)$, by the definition of $\widehat{\mathbf{a^j}}$ in (2.10), we have

$$\widehat{\mathbf{a}^1}(\xi) = \widehat{\mathbf{a}^0}(-\xi) \quad \text{and} \quad \widehat{\mathbf{a}^{-1}}(\xi) = \widehat{\mathbf{a}^2}(-\xi), \qquad \xi \in \mathbb{R}.$$

Now by (1.2), we see that (2.2) holds with A being replaced by A^0. It is evident that (2.1) holds with A being replaced by A^0.

By Lemma 2, we have $U(\xi)\overline{U(\xi)}^T = I_4$ and it is not difficult to deduce that the symmetry pattern of $U(\xi)$ is the same as $\overline{U_A(\xi)}^T$ in (2.5). More precisely, we have

$$U_{j,k}(-\xi) = \begin{cases} U_{j,k}(\xi), & j,k \in \{1,2\} \text{ or } j,k \in \{3,4\}, \\ -U_{j,k}(\xi), & \text{otherwise.} \end{cases}$$

Now the symmetry of $\widehat{b^1}, \widehat{b^2}$ and $\widehat{b^3}$ in (2.14) can be checked easily using (2.13). Note that $\widehat{b^0}$ in (2.13) is just $\hat{a}$. ∎

3. Some Examples of Symmetric Orthonormal Complex Wavelets

In this section, using Theorem 1 and Algorithm 3, we present several examples of symmetric orthonormal complex wavelets with dilation 4.

Before presenting some examples, let us recall a quantity $\nu_2(\hat{a}, 4)$ from [7]. Let $\hat{a}$ be a 2π-periodic trigonometric polynomial with $\hat{a}(0) = 1$. Write $\hat{a}(\xi) = (1 + e^{-i\xi} + e^{-i2\xi} + e^{-i3\xi})^m \hat{c}(\xi)$ for some nonnegative integer m and some 2π-periodic trigonometric polynomial $\hat{c}(\xi)$ with $|\hat{c}(\pi/2)|^2 + |\hat{c}(\pi)|^2 + |\hat{c}(3\pi/2)|^2 \neq 0$. Write $|\hat{c}(\xi)|^2 = \sum_{k=-K}^{K} c_k e^{-ik\xi}$, where K is some nonnegative integer.

Denote $\rho(\hat{a},4)$ the spectral radius of the square matrix $(c_{4j-k})_{-K/3\leqslant j,k\leqslant K/3}$ and define $\nu_2(\hat{a},4) := -1/2 - \log_4\sqrt{\rho(\hat{a},4)}$ (see [6, Theorem2.1] or [7, page 61]). The quantity $\nu_2(\hat{a},4)$ plays an important role in wavelet analysis and subdivision schemes. Define $\hat{\phi}(\xi) := \prod_{j=1}^{\infty}\hat{a}(4^{-j}\xi)$. Then ϕ is an orthonormal refinable function with dilation 4 and mask $\hat{a}$, if and only if, (1.2) holds and $\nu_2(\hat{a},4) > 0$, which in addition imply that $\nu_2(\phi) = \nu_2(\hat{a},4)$. See [7] for more details on the quantity $\nu_2(\hat{a},4)$ and related references.

Example 1. Let $m = 3$. Then

$$P_3(x) = 1 + 15\,x + 126\,x^2, \quad P_3^r(x) = 1 + 15/2\,x, \quad P_3^i(x) = 3\sqrt{31}/2\,x.$$

By Theorem 1 and Algorithm 3, we have

$$\widehat{a^3}(\xi) = \frac{e^{i4\xi}}{512}(1 + e^{-i\xi} + e^{-i2\xi} + e^{-i3\xi})^3\big[(38 + i6\sqrt{31}) - (15 + i3\sqrt{31})(e^{i\xi} + e^{-i\xi})\big]$$

and

$$\begin{aligned}
\widehat{b^{3,1}}(\xi) &= \frac{(e^{-i\xi}-1)^3}{512\sqrt{19}}(1 - e^{i2\xi})\big[8(9\sqrt{31} - i23) + 3(11\sqrt{31} - i39)(e^{-i\xi} + e^{i\xi}) \\
&\quad + 4(3\sqrt{31} - i13)(e^{-i2\xi} + e^{i2\xi}) + 3(\sqrt{31} - i5)(e^{-i3\xi} + e^{i3\xi})\big], \\
\widehat{b^{3,2}}(\xi) &= -e^{i\xi}\frac{(e^{-i\xi}-1)^3}{2\sqrt{5}}, \\
\widehat{b^{3,3}}(\xi) &= -e^{i\xi}\frac{(e^{-i\xi}-1)^3}{128\sqrt{95}}\big[(366\sqrt{31} - i710) + 60(5\sqrt{31} - i11)(e^{-i\xi} + e^{i\xi}) + 30(5\sqrt{31} - i17) \\
&\quad (e^{-i2\xi} + e^{i2\xi}) + 20(3\sqrt{31} - i13)(e^{-i3\xi} + e^{i3\xi}) + 15(\sqrt{31} - i5)(e^{-i4\xi} + e^{i4\xi})\big].
\end{aligned}$$

By calculation, $\nu_2(\widehat{a^3},4) = 3 - \log_{16}223 \approx 1.049775$. In fact, by [6, Corollary 2.2], we have $\nu_\infty(\widehat{a^3},\infty) = 5/4 - \log_{16}5 \approx 0.669517$ (see [6, 7]) and $\phi^3 \in C^{0.669517}(\mathbb{R})$. See Figure 1 for the graphs of ϕ^3, $\psi^{3,1}$, $\psi^{3,2}$, and $\psi^{3,3}$.

Example 2. Let $m = 5$. Then $P_5(x) = 1 + 25\,x + 335\,x^2 + 3195\,x^3 + 24310\,x^4$ and

$$P_5^r(x) = 1 + 25/2\,x + (715 - t^2)/8\,x^2, \quad P_5^i(x) = tx(1/2 + x(852490 + 805t^2 - t^4)/61480),$$

where $t \approx 9.05881$ is a real root satisfying $t^6 - 805t^4 - 660365t^2 + 59059225 = 0$. By Theorem 1 and Algorithm 3, we have

$$\widehat{a^5}(\xi) = \sum_{k=-9}^{0} a_k^5(e^{-ik\xi} + e^{i(k-1)\xi}), \qquad \widehat{b^{5,1}}(\xi) = \sum_{k=-9}^{0} b_k^{5,1}(e^{-ik\xi} + e^{i(k-1)\xi}),$$

$$\widehat{b^{5,2}}(\xi) = \sum_{k=-9}^{0} b_k^{5,2}(-e^{-ik\xi} + e^{i(k-1)\xi}), \qquad \widehat{b^{5,3}}(\xi) = \sum_{k=-9}^{0} b_k^{5,3}(-e^{-ik\xi} + e^{i(k-1)\xi}),$$

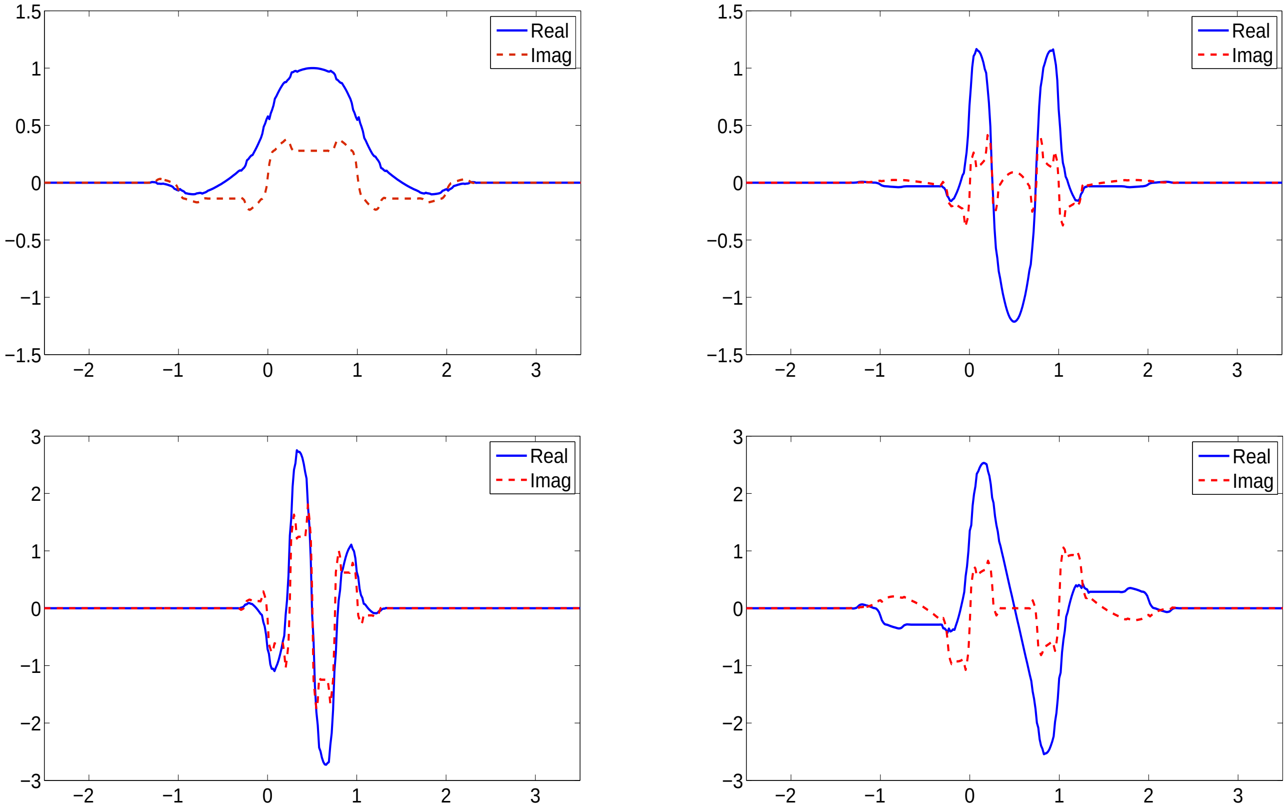

FIGURE 1. The graphs of the orthonormal refinable function ϕ^3 and its associated three wavelets $\psi^{3,1}, \psi^{3,2}, \psi^{3,3}$ in Example 1.

where $(a^5) := [a^5_{-9}, a^5_{-8}, \dots, a^5_0]^T$ and $(b^{5,\ell}) := [b^{5,\ell}_{-9}, \dots, b^{5,\ell}_0]^T, \ell = 1, 2, 3$ are given by

$$(a^5) = \begin{bmatrix} 0.00482893 \\ 0.00177718 \\ -0.00334978 \\ -0.0085988 \\ -0.0351849 \\ -0.0156536 \\ 0.0258503 \\ 0.0854206 \\ 0.203534 \\ 0.241376 \end{bmatrix} + i \begin{bmatrix} -0.0082002 \\ -0.00709438 \\ -0.00488276 \\ -0.00156533 \\ 0.0438589 \\ 0.0438589 \\ 0.0394356 \\ 0.0305892 \\ -0.0646825 \\ -0.0713174 \end{bmatrix}, \quad (b^{5,1}) = \begin{bmatrix} 0.000822313 \\ 0.000839913 \\ 0.000822842 \\ 0.00066656 \\ -0.0108083 \\ -0.0106888 \\ -0.0100922 \\ -0.00899899 \\ 0.252284 \\ -0.214848 \end{bmatrix} + i \begin{bmatrix} 0.00107537 \\ 0.000613957 \\ -0.00018015 \\ -0.00104953 \\ -0.0099828 \\ -0.00511836 \\ 0.00440185 \\ 0.0165285 \\ 0.0815194 \\ -0.0878082 \end{bmatrix},$$

$$(b^{5,2}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.00127564 \\ 0.00110362 \\ 0.000759572 \\ 0.000243505 \\ -0.11512 \\ 0.314861 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.000751198 \\ 0.000276461 \\ -0.000521098 \\ -0.00133765 \\ -0.0356261 \\ 0.106463 \end{bmatrix}, \quad (b^{5,3}) = \begin{bmatrix} 0.0828275 \\ 0.00716581 \\ 0.00493192 \\ 0.00158109 \\ -0.0534214 \\ -0.0480541 \\ -0.0345348 \\ -0.0162297 \\ 0.287684 \\ 0.106239 \end{bmatrix} + i \begin{bmatrix} 0.00487755 \\ 0.00179507 \\ -0.0033835 \\ -0.00868537 \\ -0.0218765 \\ -0.00375728 \\ 0.034852 \\ 0.0898156 \\ -0.113319 \\ -0.0381561 \end{bmatrix}.$$

By calculation, $\nu_2(\widehat{a^5}, 4) \approx 1.274852$. See Figure 2 for the graphs of ϕ^5, $\psi^{5,1}$, $\psi^{5,2}$, and $\psi^{5,3}$.

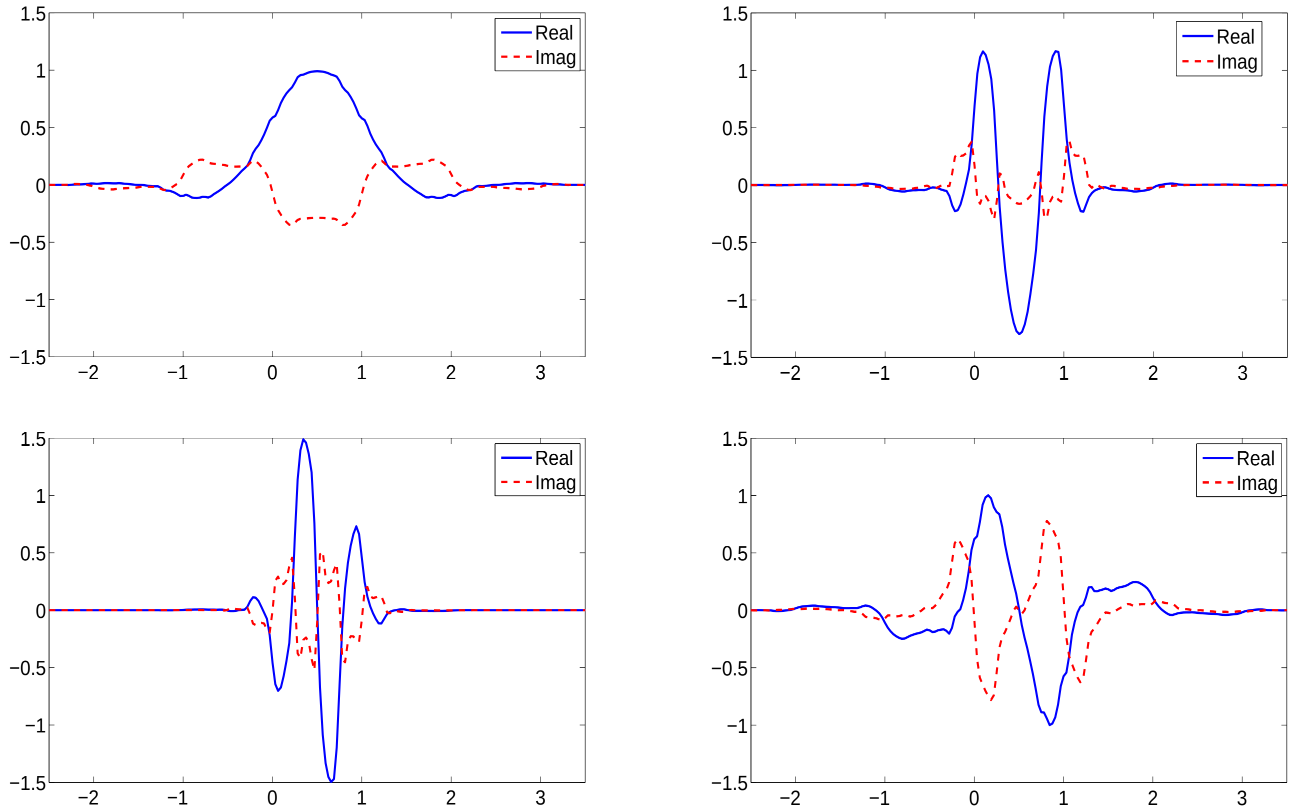

FIGURE 2. The graphs of the orthonormal refinable function ϕ^5 and its associated three wavelets $\psi^{5,1}, \psi^{5,2}, \psi^{5,3}$ in Example 2.

4. Proof of Theorem 1

First, we prove that there are two polynomials P^r_m and P^i_m with real coefficients satisfying (1.5). By [8, Proposition 9], (1.5) holds if and only if $P_m(0) = 1$ and

$$P_m(x) \geqslant 0 \qquad \forall\, x \in \mathbb{R}. \tag{4.1}$$

By the definition of P_m in (1.4), it is evident that $P_m(0) = 1$ and all the coefficients of P_m are nonnegative. Therefore, it is straightforward to see that $P_m(x) \geqslant 0$ for all $x \geqslant 0$. To prove (4.1), we have to show $P_m(x) \geqslant 0$ for all $x < 0$, which is the major part of this proof. Denote

$$A(\xi) := \cos^{2m}(\xi/2)\cos^{2m}(\xi)P_m(\sin^2(\xi/2)) \tag{4.2}$$

and

$$H(\xi) := A(\xi/4) + A(\xi/4 + \pi/2) + A(\xi/4 + \pi) + A(\xi/4 + 3\pi/2). \tag{4.3}$$

By (1.4), both A and H are 2π-periodic trigonometric polynomials such that

$$A(-\xi) = A(\xi), \quad H(-\xi) = H(\xi), \quad \deg(A) < 4m, \quad \deg(H) < m. \tag{4.4}$$

By the definition of A in (4.2) and H in (4.3), it follows from (1.4) that

$$\begin{aligned} H(\xi) &= A(\xi/4) + O(|\xi|^{2m}) = (1 - \sin^2(\xi/8))^m (1 - 2\sin^2(\xi/8))^{2m} P_m(\sin^2(\xi/8)) + O(|\xi|^{2m}) \\ &= 1 + O(|\xi|^{2m}), \qquad \xi \to 0. \end{aligned}$$

That is, $[H-1]^{(j)}(0) = 0$ for all $j = 0, \ldots, 2m-1$. Since $H(-\xi) = H(\xi)$ and $\deg(H) < m$, from the above relation, we must have $H(\xi) \equiv 1$. That is, we have

$$A(\xi) + A(\xi + \pi/2) + A(\xi + \pi) + A(\xi + 3\pi/2) = 1. \tag{4.5}$$

Note that there is a unique polynomial Q with real coefficients and $\deg(Q) < m$ such that

$$Q(\sin^2(\xi)) = \cos^{2m}(\xi/2)P_m(\sin^2(\xi/2)) + \sin^{2m}(\xi/2)P_m(\cos^2(\xi/2)).$$

By $\sin^2(\xi) = 4\sin^2(\xi/2)(1-\sin^2(\xi/2))$, the above identity can be rewritten as

$$Q(4y(1-y)) = (1-y)^m P_m(y) + y^m P_m(1-y) \quad \text{with} \quad y := \sin^2(\xi/2). \tag{4.6}$$

Consequently, by the definition of Q, we have

$$A(\xi) + A(\xi+\pi) = \cos^{2m}(\xi)Q(\sin^2(\xi)) = (1-2y)^{2m}Q(4y(1-y))$$

and

$$A(\xi+\pi/2) + A(\xi+3\pi/2) = \sin^{2m}(\xi)Q(\cos^2(\xi)) = (4y(1-y))^m Q((1-2y)^2).$$

Now by (4.5) and the above two identities, we conclude that

$$(1-2y)^{2m}Q(4y(1-y)) + (4y(1-y))^m Q((1-2y)^2) = 1 \qquad \forall\, 0 \leqslant y \leqslant 1. \tag{4.7}$$

Taking $y = (1-\sqrt{x})/2$ with $0 \leqslant x \leqslant 1$ in (4.7), we get

$$x^m Q(1-x) + (1-x)^m Q(x) = 1 \qquad \forall\, 0 \leqslant x \leqslant 1. \tag{4.8}$$

From the above identity, now it is straightforward to see that

$$Q(x) = (1-x)^{-m} + O(x^m), \qquad x \to 0.$$

Since $\deg(Q) < m$, the above relation implies that Q must be the $(m-1)$th Taylor polynomial of $(1-x)^{-m}$ at $x=0$. So, $Q(x) = \sum_{j=0}^{m-1}\binom{m+j}{j}x^j$ and all the coefficients of Q are nonnegative.

Now we are ready to prove $P_m(x) \geqslant 0$ for all $x < 0$. By (4.6) and (4.7), we have

$$(1-2x)^{2m}(1-x)^m P_m(x) = 1-(4x(1-x))^m Q((1-2x)^2) - (1-2x)^{2m}x^m P_m(1-x) \qquad \forall x \in \mathbb{R}. \tag{4.9}$$

Since all the coefficients of P_m and Q are nonnegative, noting that m is an odd integer, we have

$$(4x(1-x))^m < 0, \quad Q((1-2x)^2) \geqslant 0, \quad x^m < 0, \quad (1-2x)^{2m}P_m(1-x) \geqslant 0, \qquad \forall\, x < 0.$$

Consequently, it follows from (4.9) that

$$(1-2x)^{2m}(1-x)^m P_m(x) \geqslant 1, \qquad \forall\, x < 0.$$

That is, we must have $P_m(x) > 0$ for all $x < 0$. Hence, (4.1) is verified and by [8, Proposition 9] (1.5) holds.

Next we prove that $\widehat{a^m}$ is an orthogonal mask. By (1.6) and (1.5), we have

$$|\widehat{a^m}(\xi)|^2 = \cos^{2m}(\xi/2)\cos^{2m}(\xi)P_m(\sin^2(\xi/2)) = A(\xi).$$

Now by (4.5), we see that $\widehat{a^m}$ is an orthogonal mask.

Since $P_m(\sin^2(\xi/2)) > 0$ for all $\xi \in \mathbb{R}$, it is easy to see that $\widehat{\phi^m}(\xi) = 0$ if and only if $\xi \in 2\pi\mathbb{Z}\backslash\{0\}$. Now it is a standard argument in wavelet analysis to check that $\phi^m \in L_2(\mathbb{R})$ is an orthonormal refinable function with dilation 4 and mask $\widehat{a^m}$.

Note that $\widehat{a^m}(-\xi) = e^{i\xi}\widehat{a^m}(\xi)$. By Algorithm 3, there exist 2π-periodic trigonometric polynomials $\widehat{b^{m,1}}$, $\widehat{b^{m,2}}$ and $\widehat{b^{m,3}}$ such that (2.14) holds with $\hat{a}, \widehat{b^1}, \widehat{b^2}, \widehat{b^3}$ being replaced by $\widehat{a^m}, \widehat{b^{m,1}}, \widehat{b^{m,2}}, \widehat{b^{m,3}}$. Moreover, $P_{[\widehat{a^m},\widehat{b^{m,1}},\widehat{b^{m,2}},\widehat{b^{m,3}}]}$ is a unitary matrix, where $P_{[\widehat{a^m},\widehat{b^{m,1}},\widehat{b^{m,2}},\widehat{b^{m,3}}]}$ is defined in (2.15). Now it is easy to verify that Item (2) holds. By the standard argument in wavelet analysis, Item (1) is true and all $\psi^{m,1}, \psi^{m,2}, \psi^{m,3}$ have m vanishing moments.

Since $\nu_2(\psi^{m,\ell}) \geqslant \nu_2(\phi^m)$ for all $\ell = 1,2,3$, to complete the proof, it suffices to prove that $\liminf_{m\to\infty}\frac{\nu_2(\phi^m)}{m} > 0$. Let $\widehat{c^m}$ be a 2π-periodic trigonometric polynomial with real coefficients such that

$$\widehat{c^m}(0) = 1 \quad \text{and} \quad |\widehat{c^m}(\xi)|^2 = \cos^{2m}(\xi/2)\cos^{2m}(\xi)P_m(\sin^2(\xi/2)).$$

Define ϕ^{c_m} by $\widehat{\phi^{c_m}}(\xi) := \prod_{j=1}^{\infty} \widehat{c^m}(4^{-j}\xi)$. Then ϕ^{c_m} is a refinable function with dilation 4 and mask $\widehat{c^m}$. It is known in [2] that $\lim_{m\to\infty} \frac{\nu_2(\phi^{c_m})}{m} = -m\log_4(\sin\frac{2\pi}{5}) > 0$. On the other hand, it is easy to see that $|\widehat{\phi^m}(\xi)|^2 = |\widehat{\phi^{c_m}}(\xi)|^2$. Consequently, $\nu_2(\phi^m) = \nu_2(\phi^{c_m})$ and therefore, $\lim_{m\to\infty} \frac{\nu_2(\phi^m)}{m} = -m\log_4(\sin\frac{2\pi}{5}) > 0$. This completes the proof. ∎

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