Band-limited Wavelets and Framelets in Low Dimensions

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Abstract

In this paper, we study the problem of constructing non-separable band-limited wavelet tight frames, Riesz wavelets and orthonormal wavelets in $\mathbb{R}^2$ and $\mathbb{R}^3$. We first construct a class of non-separable band-limited refinable functions in low-dimensional Euclidean spaces by using univariate Meyer's refinable functions along multiple directions defined by classic box-spline direction matrices. These non-separable band-limited definable functions are then used to construct non-separable band-limited wavelet tight frames via the unitary and oblique extension principles. However, these refinable functions cannot be used for constructing Riesz wavelets and orthonormal wavelets in low dimensions as they are not stable. Another construction scheme is then developed to construct stable refinable functions in low dimensions by using a special class of direction matrices. The resulting stable refinable functions allow us to construct a class of MRA-based non-separable band-limited Riesz wavelets and particularly band-limited orthonormal wavelets in low dimensions with small frequency support.

Keywords: band-limited function, non-separable function, multiresolution analysis, wavelet, Riesz basis, tight frame, extension principle.

1. Introduction

Wavelet is a prominent mathematical tool in the field of signal processing. There has been abounding literature on the construction of univariate wavelets (see for instance [25, 11, 12, 10]) and wavelet tight frames (see for instance [31, 13, 1, 8, 9]). The construction of multivariate wavelets is usually handled via the tensor product of 1D wavelets, which brings about separable multivariate wavelets. Compared to separable wavelets, non-separable wavelets have some properties attractive to the applications of multi-dimensional signal processing. For instance, the non-separable wavelets allow true 2D processing of an image

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by treating the image as data on 2D areas instead of on 1D columns and rows. There has been a continuous research effort on the construction of multivariate non-separable wavelets or wavelet tight frames (see for instance [29, 4, 32, 20, 17, 7, 21, 22]). Most of these works focus on the construction of wavelets which are compactly supported in spatial domain.

In some applications, the targeted signals often have their frequency components restricted to certain bands, the so-called *band-limited* case. To efficiently analyze and process such band-limited signals, one desired type of wavelets is band-limited wavelet, i.e. the type of wavelets whose support in frequency domain is compact. The construction of orthonormal band-limited wavelets or band-limited wavelet tight frames has drawn a lot of attention of researchers; see for instance [3, 27, 19, 18, 16, 1, 15, 2, 6]. Some well-known band-limited wavelets include the orthonormal Shannon wavelets and the Meyer's wavelets [25]. Similarly, most of these studies concentrate on the 1D case. The 2D and higher-dimensional cases are handled via the tensor product of 1D band-limited wavelets (see for instance [24, 2]). 2D tensor band-limited wavelets have been used in various image processing tasks. For instance, the Meyer's wavelets are used in [14] for image de-blurring and used in [33] for image compression. To the best of our knowledge, the systematic construction of non-separable multivariate band-limited wavelets has not been well studied in the past. Compared to tensor multivariate band-limited wavelets, non-separable band-limited wavelets have more degrees of freedom, which is likely to result in better designs such as smaller frequency support with fast spatial decay.

This paper aims at constructing non-separable band-limited wavelet tight frames (framelets), Riesz wavelets and orthonormal wavelets in low-dimensional Euclidean spaces including both $\mathbb{R}^2$ and $\mathbb{R}^3$. The construction is based on *multiresolution analysis* (MRA) [23, 26]. Our MRA-based constructions of band-limited framelets or wavelets in low dimensions start with the construction of a class of non-separable band-limited refinable functions using the univariate Meyer's refinable functions along multiple directions, in a manner similar to the extension of univariate B-splines to box splines in several variables. The constructed non-separable refinable functions are then used to construct non-separable band-limited framelets via the *unitary extension principle* [31] or the *oblique extension principle* [13].

Nevertheless, the two extension principles cannot be used to construct band-limited Riesz wavelets or orthonormal wavelets from a given refinable function. An alternative scheme is provided in [29] to construct Riesz wavelets and orthonormal wavelets in low dimensions, but it requires the refinable function for generating MRA to be stable. Unfortunately, the refinable functions derived using standard box-spline direction matrices are not stable. Thus, we construct a class of stable refinable band-limited functions by using a special class of direction matrices. Moreover, compared to the construction of non-separable orthonormal refinable functions compactly supported in spatial domain (see for instance [20, 21]), the construction of band-limited orthonormal refinable functions is indeed an easier task, as the standard normalization technique ([12, 24]) can be directly applied on a band-limited stable refinable function to generate an orthonormal refinable function without destroying the compactness of its support in frequency domain. These stable or orthonormal refinable functions can be used to construct the non-separable band-limited Riesz wavelets and orthonormal wavelets via the scheme proposed in [29]. The non-separable band-limited wavelets constructed in this paper have smaller frequency support size than tensor product wavelets, while keeping the fast decay in spatial domain.

The rest of this paper is organized as follows. In Section 2, we give a brief review on basic concepts, definitions and existing results which will be used in this paper. In Section 3, we present the construction scheme of non-stable band-limited refinable functions using standard box-spline direction matrices, followed by the derivation of MRA-based wavelet tight frames. Section 4 is devoted to the construction of band-limited stable refinable functions using a class of specially designed direction matrices, which are then used to construct non-separable band-limited Riesz wavelets and orthonormal wavelets. Several examples of the constructed wavelet tight frames and wavelets are given in Section 5.

2. Notations and Preliminaries

In this paper, we use $\mathbb{Z}$, $\mathbb{Z}^*$, $\mathbb{Z}^+$, $\mathbb{R}$ to denote the set of integers, nonnegative integers, positive integers, and real numbers, respectively. For any $d \in \mathbb{Z}^+$, let $\mathbb{Z}_2^d$ denote the set $\{0,1\}^d$, and $\mathbb{R}^d$ denote the d-dimensional Euclidean space with the inner product given by

$$x \cdot y = \sum_{j=1}^{d} x_j y_j, \text{ for } x = (x_1, \ldots, x_d) \text{ and } y = (y_1, \ldots, y_d) \in \mathbb{R}^d.$$

For any $n \in \mathbb{Z}^*$, $C^n(\mathbb{R}^d)$ denotes the set of functions on $\mathbb{R}^d$ that have continuous derivatives up to order n. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the usual inner product and norm of the Hilbert space $L_2(\mathbb{R}^d)$. For any function $f \in L_2(\mathbb{R}^d)$, its Fourier transform $\widehat{f}$ is defined formally as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \exp(-i\xi \cdot x) dx, \quad \xi \in \mathbb{R}^d.$$

Let I be a countable index set, then a sequence $\{v_n,\, n \in I\} \subset L_2(\mathbb{R}^d)$ is called a *frame* of $L_2(\mathbb{R}^d)$ if there exist two positive constants A and B such that

$$A\|f\|_2^2 \leqslant \sum_{n \in I} |\langle v_n, f \rangle|^2 \leqslant B\|f\|_2^2, \quad \forall f \in L_2(\mathbb{R}^d).$$

A frame $\{v_n\}_{n \in I}$ is called a *tight frame* when $A = B = 1$. In addition, $\{v_n,\, n \in I\}$ is called a *Riesz basis* of $L_2(\mathbb{R}^d)$ if the linear span of $\{v_n,\, n \in I\}$ is dense in $L_2(\mathbb{R}^d)$ and there also exist $A, B > 0$ such that

$$A \sum_{n \in I} |c_n|^2 \leqslant \| \sum_{n \in I} c_n v_n \|_2^2 \leqslant B \sum_{n \in I} |c_n|^2, \quad \forall \{c_n\}_{n \in I} \in \ell_2(I).$$

When $A = B = 1$, the Riesz basis $\{v_n,\, n \in I\}$ becomes an *orthonormal basis* for $L_2(\mathbb{R}^d)$. The same definition of tight frame, Riesz basis and orthonormal basis is also applicable to the subspaces of $L_2(\mathbb{R}^d)$.

An affine system is generated by a finite set of generators $\Psi = \{\psi_\ell,\, \ell = 1, \ldots, L\} \subset L_2(\mathbb{R}^d)$ in the following manner:

$$X(\Psi) = \{2^{jd/2}\psi(2^j \cdot -k) : \psi \in \Psi, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}. \tag{1}$$

If $X(\Psi)$ forms a Riesz (orthonormal) basis of $L_2(\mathbb{R}^d)$, it is called a Riesz (orthornormal) wavelet system. If $X(\Psi)$ forms a tight frame of $L_2(\mathbb{R}^d)$, then it is called a wavelet tight frame or framelet system. The generators $\psi_1, \psi_2, \ldots, \psi_L$ are often referred as *wavelets* or *framelets*. The construction of wavelet or framelet systems often starts with the construction of MRA, which is built on refinable functions. A function $\phi \in L_2(\mathbb{R}^d)$ is called *refinable* if it satisfies the following refinement equation:

$$\widehat{\phi}(2\xi) = \tau_0(\xi)\widehat{\phi}(\xi) \quad \text{a.e. on} \quad \mathbb{R}^d,$$

where τ_0 is a 2π-periodic measurable function known as the *refinement mask*. In this paper, we follow [4] for the definition of MRA. Given a refinable function $\phi \in L_2(\mathbb{R}^d)$ with $\widehat{\phi}(0) \neq 0$, the sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ defined by

$$V_j = \overline{\text{span}\{\phi(2^j \cdot -k), \quad k \in \mathbb{Z}^d\}}, \quad j \in \mathbb{Z} \tag{2}$$

will form an MRA for $L_2(\mathbb{R}^d)$. Recall that $\{V_j\}_{j\in\mathbb{Z}}$ is called an MRA if it satisfies (i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$; (ii) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R}^d)$; and (iii) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$ In this paper, we only consider the refinable function ϕ satisfying the following properties:

$$\lim_{\xi \longrightarrow 0} \widehat{\phi}(\xi) = 1 \text{ and } [\widehat{\phi}, \widehat{\phi}] = \sum_{k\in\mathbb{Z}^d} |\widehat{\phi}|^2(\cdot + 2\pi k) \text{ is essentially bounded.} \tag{3}$$

Given an MRA generated by the refinable function ϕ, we can construct a set of MRA-based framelets $\Psi = \{\psi_\ell,\, \ell = 1, \ldots, L\} \subset V_1$ which is defined by

$$\widehat{\psi_\ell}(2\xi) = \tau_\ell(\xi)\widehat{\phi}(\xi), \quad \ell = 1, \ldots, L, \tag{4}$$

where $\{\tau_\ell,\, \ell = 1, \ldots, L\}$ is a set of 2π-periodic measurable functions called *wavelet masks*. The so-called *unitary extension principle* (UEP) provides a sufficient condition on Ψ such that the resulting affine system $X(\Psi)$ forms a tight frame of $L_2(\mathbb{R}^d)$.

Proposition 1. Unitary Extension Principle (UEP) [31]. *Let ϕ be a refinable function with mask τ_0 and $\{\tau_\ell,\, \ell = 1, \ldots, L\}$ be a set of 2π-periodic functions. Assume that ϕ satisfies* (3) *and the masks $\{\tau_0, \tau_1, \ldots, \tau_L\}$ are essentially bounded and measurable. For a given $\Psi = \{\psi_\ell,\, \ell = 1, \ldots, L\}$ defined by* (4)*, the associated affine system $X(\Psi)$ forms a tight frame of $L_2(\mathbb{R}^d)$ provided that the masks $\{\tau_0, \tau_1, \ldots, \tau_L\}$ satisfy the following equalities:*

$$\sum_{\ell=0}^{L} \tau_\ell(\xi)\overline{\tau_\ell(\xi + \nu\pi)} = \delta_{\nu,\{0\}^d}, \quad \forall\, \nu \in \mathbb{Z}_2^d,$$

for almost all $\xi \in [-\pi, \pi]^d$.

When using the UEP to construct wavelet tight frames, there are certain limitations on the vanishing moments of the resulting framelets. As an extension of the UEP, the so-called *oblique extension principle* (OEP) provides another construction scheme of wavelet tight frames with better approximation order.

Proposition 2. Oblique Extension Principle (OEP) [13]. *Let* ϕ *be a refinable function with mask* τ_0 *and* $\{\tau_\ell,\ \ell = 1, \ldots, L\}$ *be a set of* 2π*-periodic functions. Assume that* ϕ *satisfies* (3) *and the masks* $\{\tau_0, \tau_1, \ldots, \tau_L\}$ *are essentially bounded and measurable. For a given set* $\Psi = \{\psi_\ell,\ \ell = 1, \ldots, L\}$ *defined as* (4)*, if there exists a* 2π*-periodic function* Θ *which is non-negative, essentially bounded, continuous at the origin with* $\Theta(0) = 1$ *such that*

$$\tau_0(\xi)\overline{\tau_0}(\xi + \nu\pi)\Theta(2\xi) + \sum_{\ell=1}^{L} \tau_\ell(\xi)\overline{\tau_\ell}(\xi + \nu\pi) = \delta_{\nu,\{0\}^d}\Theta(\xi) \quad \forall \nu \in \mathbb{Z}_2^d, \tag{5}$$

for almost all $\xi \in [-\pi, \pi]^d$*, then the resulting affine system* $X(\Psi)$ *forms a wavelet tight frame of* $L_2(\mathbb{R}^d)$.

The UEP/OEP is only for the construction of MRA-based wavelet tight frames and it cannot be used for the construction of MRA-based Riesz wavelets or orthonormal wavelets in $L_2(\mathbb{R}^d)$. An explicit construction scheme is provided in [29] for the construction of Riesz wavelets and orthonormal wavelets in low-dimensional spaces, $L_2(\mathbb{R}^2)$ and $L_2(\mathbb{R}^3)$, which starts with the construction of a set of pre-wavelets Ψ^d. Suppose that the refinable function $\phi \in L_2(\mathbb{R}^d)$ has its Fourier transform $\widehat{\phi}$ being real-valued. Define

$$\Psi^d = \{\psi_\mu : \widehat{\psi_\mu}(\xi) = \exp\big(i\eta(\mu)\frac{\xi}{2}\big)\tau_0(\frac{\xi}{2} + \mu\pi)[\widehat{\phi}, \widehat{\phi}](\frac{\xi}{2} + \mu\pi)\widehat{\phi}(\frac{\xi}{2}),\ \mu \in \mathbb{Z}_2^d \backslash \{0\}^d\}, \tag{6}$$

where τ_0 is the refinement mask of ϕ, and η is a mapping of $\mathbb{Z}_2^d \mapsto \mathbb{Z}_2^d$ defined as follows:

(i) for $d = 2$, the map η is

$$(0,0) \mapsto (0,0), (1,0) \mapsto (1,1), (0,1) \mapsto (0,1), (1,1) \mapsto (1,0).$$

(ii) for $d = 3$, the map η is

$$\begin{aligned}
&(0,0,0) \mapsto (0,0,0), \quad (1,0,0) \mapsto (1,1,0), \quad (0,1,0) \mapsto (0,1,1),\\
&(1,1,0) \mapsto (1,0,0), \quad (0,0,1) \mapsto (1,0,1), \quad (1,0,1) \mapsto (0,0,1),\\
&(0,1,1) \mapsto (0,1,0), \quad (1,1,1) \mapsto (1,1,1).
\end{aligned}$$

It is shown in [29] that $X(\Psi^d)$ forms a Riesz basis for $L_2(\mathbb{R}^d)$ with $d \leqslant 3$, if (i) the refinable ϕ generates an MRA and (ii) ϕ is stable. Recall that a function $\phi \in L_2(\mathbb{R}^d)$ is called *stable* if the sequence generated by its integer shifts $\{\phi(\cdot - k), k \in \mathbb{Z}^d\}$ is a Riesz basis for V_0 defined as (2). It is shown in [1, 30] that ϕ is stable if there exist two constants A and B such that

$$0 < A \leqslant [\widehat{\phi}, \widehat{\phi}] \leqslant B < \infty \text{ a.e. on } \mathbb{R}^d. \tag{7}$$

In addition, if $A = B = 1$, then ϕ is *orthonormal*, which indicates that the sequence $\{\phi(\cdot - k), k \in \mathbb{Z}^d\}$ is an orthonormal basis of V_0 defined as (2). It is shown in [29] that the affine system $X(\Psi^d)$ forms an orthonormal basis of $L_2(\mathbb{R}^d)$ for $d = 2, 3$, provided that the refinable function $\phi \in L_2(\mathbb{R}^d)$ is orthonormal.

The tight frames and wavelets we are interested in this paper are band-limited wavelets and framelets. A function $\phi \in L_2(\mathbb{R})$ is said to be *band-limited* if $\mathrm{supp}(\widehat{\phi}) \subset [-\Omega_0, \Omega_0]$ for some constant $\Omega_0 > 0$. There have been various types of band-limited refinable functions studied in the past. One well-known band-limited univariate refinable function is the Meyer's refinable function Q_Ω (see for instance [25, 24, 12]) whose mask τ_Ω, when restricted to the single 2π-period $[-\pi, \pi]$, is defined by

$$\tau_\Omega(\xi) = \begin{cases} 1, & \text{if } |\xi| < \pi - \Omega; \\ h(\xi), & \text{if } \pi - \Omega \leq |\xi| \leq \Omega; \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

for some $\frac{\pi}{2} < \Omega \leqslant \frac{2\pi}{3}$. The function h is chosen such that $\tau_\Omega \in C^n(\mathbb{R})$ for some $n \in \mathbb{Z}^*$ and τ_Ω satisfies the quadrature mirror filter condition:

$$|\tau_\Omega(\xi)|^2 + |\tau_\Omega(\xi + \pi)|^2 = 1, \forall \xi \in \mathbb{R}.$$

Then it is seen that

$$\widehat{Q_\Omega}(\xi) = \begin{cases} h(\frac{\xi}{2}), & \text{if } |\xi| \in [2\pi - 2\Omega, 2\Omega]; \\ 1, & \text{if } |\xi| < 2\pi - 2\Omega; \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

and consequently $\mathrm{supp}(\widehat{Q_\Omega}) \in [-2\Omega, 2\Omega]$. Some basic facts of the Meyer's refinable function Q_Ω are listed as follows,

- (i) $\widehat{Q_\Omega}$ is continuous, nonnegative and symmetric about the origin;
- (ii) $\widehat{Q_\Omega}(\xi) = 1$ if and only if $\xi \in [-2\pi + 2\Omega, 2\pi - 2\Omega]$, and $\widehat{Q_\Omega}(\xi) > 0$ if and only if $\xi \in (-2\Omega, 2\Omega)$, where $\Omega \in (\frac{\pi}{2}, \frac{2\pi}{3}]$;
- (iii) $\widehat{Q_\Omega} \in C^n(\mathbb{R})$ for some $n \in \mathbb{Z}^*$ if and only if $\tau_\Omega \in C^n(\mathbb{R})$.

3. Band-limited Wavelet Tight Frames

In this section, we present a construction scheme of non-separable band-limited framelets. We start with the construction of non-separable refinable functions, and then construct non-separable framelets using the two extension principles, i.e. UEP and OEP. Our construction of non-separable band-limited refinable functions is motivated by the construction scheme of *box splines* [5, 28]. The basic idea is to multiply univariate Meyer's refinable functions along multiple directions given by the standard box-spline direction matrices. More specifically, let $\varphi \in L_2(\mathbb{R})$ be a univariate refinable function with the mask τ. Let Ξ be a $d \times n$ *full row rank* direction matrix with integer entries and $n \geqslant d$. Then we can define a refinable function $\phi_\Xi \in L_2(\mathbb{R}^d)$ as follows:

$$\widehat{\phi_\Xi}(\xi) = \prod_{r \in \mathrm{col}(\Xi)} \widehat{\varphi}(r^\top \cdot \xi), \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \tag{10}$$

where $\mathrm{col}(\Xi)$ enumerates all columns of the direction matrix Ξ. The multivariate function ϕ_Ξ in (10) also can be defined in terms of tempered distribution; see [34] for more details. Briefly, for a function $\varphi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, let f denotes its n-dimensional tensor function defined by

$$f(u) := \prod_{j=1}^{n} \varphi(u_j), \quad u = (u_1, \cdots, u_n).$$

Then, define the *multivariate F-truncated powers* $T_f(\cdot \mid \Xi)$ by

$$T_f(\cdot \mid \Xi)(h) = \int_{\mathbb{R}^n} f(u) h(u\Xi^\top) du \tag{11}$$

for any $h \in \mathcal{S}(\mathbb{R}^d)$. It is shown in [34] that $T_f(\cdot \mid \Xi) = \phi_\Xi$. It is noted that the condition that the matrix Ξ is a full row rank matrix cannot be dropped in general, otherwise the resulting function ϕ_Ξ may not be in $L_2(\mathbb{R}^d)$.

Clearly, the function ϕ_Ξ defined in (10) is still refinable as

$$\widehat{\phi_\Xi}(2\xi) = \prod_{r \in \mathrm{col}(\Xi)} \widehat{\varphi}(2r^\top \cdot \xi) = \prod_{r \in \mathrm{col}(\Xi)} \tau(r^\top \cdot \xi) \widehat{\varphi}(r^\top \cdot \xi) = \tau_\Xi(\xi) \widehat{\phi_\Xi}(\xi), \quad \xi \in \mathbb{R}^d,$$

and $\tau_\Xi(\xi) = \prod_{r \in \mathrm{col}(\Xi)} \tau(r^\top \cdot \xi)$ is a 2π-periodic function of $\mathbb{R}^d$, since all elements of r are integers and τ is a 2π-periodic function.

To construct MRA-based wavelet tight frames using two extension principles, we assume that the constructed refinable function ϕ satisfies the following condition, named as *Condition A*:

(i) $\widehat{\phi}$ *is real-valued, continuous at the origin with* $\widehat{\phi}(0) = 1$*;*
(ii) $[\widehat{\phi}, \widehat{\phi}]$ *is essentially bounded.*

It is noted that if ϕ satisfies Condition A, then it also satisfies (3). Now, suppose that we have a band-limited refinable function ϕ with mask τ satisfying Condition A, and suppose that there also exists a set of real-valued measurable 2π-periodic functions, denoted by $\{p(\xi), a_s(\xi), s = 1, \ldots, N\}$, satisfying the following condition:

$$\sum_{\kappa \in \mathbb{Z}_2^d} |\tau(\xi + \kappa\pi)|^2 + \sum_{s=1}^{N} \sum_{\mu \in \mathbb{Z}_2^d} |a_s(\xi + \mu\pi)|^2 + |p(\xi)|^2 = 1, \tag{12}$$

then we may define a set of 2π-periodic functions

$$\{\tau_\kappa,\, \kappa \in \mathbb{Z}_2^d \backslash \{0\}^d;\; a_{s,\mu},\, \mu \in \mathbb{Z}_2^d, s = 1, \dots, N\}$$

as follows:

$$\begin{cases} \tau_\kappa(\xi) = \exp(i\eta(\kappa)\cdot\xi)\tau(\xi+\kappa\pi), & \kappa \in \mathbb{Z}_2^d \backslash \{0\}^d; \\ a_{s,\mu}(\xi) = \exp(i\eta(\mu)\cdot\xi)a_s(\xi+\mu\pi), & \mu \in \mathbb{Z}_2^d,\; s = 1, \dots, N. \end{cases} \tag{13}$$

The following theorem shows that the above set of 2π-periodic functions defines the generators of a wavelet tight frame of $L_2(\mathbb{R}^d), d = 2, 3$.

Theorem 3. *Suppose $\phi \in L_2(\mathbb{R}^d)$ is a band-limited refinable function with mask τ and it satisfies Condition A. Suppose that $\{p(\xi), a_s(\xi), s = 1, \dots, N\}$ is a set of real-valued measurable 2π-periodic functions satisfying (12) and let $\{\tau_\kappa,\, \kappa \in \mathbb{Z}_2^d \backslash \{0\}^d;\; a_{s,\mu},\, \mu \in \mathbb{Z}_2^d, s = 1, \dots, N\}$ be given as (13). Define a set of functions Ψ as follows:*

$$\Psi = \{\psi_\kappa,\, \kappa \in \mathbb{Z}_2^d \backslash \{0\}^d\} \cup \{\psi_{s,\mu},\, \mu \in \mathbb{Z}_2^d, s = 1, \dots, N\} \cup \{\psi_p\} \tag{14}$$

where

$$\begin{cases} \widehat{\psi_\kappa}(\xi) = \tau_\kappa(\frac{\xi}{2})\widehat{\phi}(\frac{\xi}{2}), & \kappa \in \mathbb{Z}_2^d \backslash \{0\}^d; \\ \widehat{\psi_{s,\mu}}(\xi) = a_{s,\mu}(\frac{\xi}{2})\widehat{\phi}(\frac{\xi}{2}), & \mu \in \mathbb{Z}_2^d,\; s = 1, \dots, N; \\ \widehat{\psi_p}(\xi) = p(\xi)\tau(\frac{\xi}{2})\widehat{\phi}(\frac{\xi}{2}) = p(\xi)\widehat{\phi}(\xi). & \end{cases} \tag{15}$$

Then the dyadic affine system $X(\Psi)$ generated by Ψ forms a tight frame of $L_2(\mathbb{R}^d)$.

Proof. See Section 6.1.

In the case that the function p vanishes in (12), the corresponding wavelet ψ_p defined in (15) also vanishes and we have the following corollary.

Corollary 4. *Suppose that $\phi \in L_2(\mathbb{R}^d)$ is a band-limited refinable function satisfying Condition A. Let its mask τ and $\{a_s, 1 \leqslant s \leqslant N\}$ be a set of real-valued measurable 2π-periodic functions satisfying*

$$\sum_{s=1}^{N} \sum_{\mu \in \mathbb{Z}_2^d} |a_s(\xi+\mu\pi)|^2 + \sum_{\kappa \in \mathbb{Z}_2^d} |\tau(\xi+\kappa\pi)|^2 = 1. \tag{16}$$

Define

$$\Psi' = \{\psi_\kappa,\, \kappa \in \mathbb{Z}_2^d \backslash \{0\}^d\} \cup \{\psi_{s,\mu},\, \mu \in \mathbb{Z}_2^d, s = 1, \dots, N\}, \tag{17}$$

where ψ_κ and $\psi_{s,\mu}$ are given as (15). Then the dyadic affine system $X(\Psi')$ generated by Ψ' forms a tight frame of $L_2(\mathbb{R}^d)$.

Theorem 3 and Corollary 4 provide explicit construction schemes to construct wavelet tight frames of $L_2(\mathbb{R}^d), d = 2, 3$. The requirement for applying Theorem 3 and Corollary 4 is that there exists a refinable function ϕ that satisfies Condition A and a set of 2π-periodic functions $\{p(\xi),\, a_s, s = 1, \ldots, N\}$ that satisfies (12) or (16). It is noted that Condition A imposed on refinable functions is very mild as it is based on (3) required by the UEP or OEP. In contrast, some other existing methods for constructing wavelet tight frames (e.g., [1, 2]) require that the refinable function ϕ not only generates an MRA but also satisfies

$$\{\phi(\cdot - k), k \in \mathbb{Z}^d\} \text{ is a tight frame of } V_0 \text{ defined as in (2)}, \tag{18}$$

which is equivalent to (see [30, 1])

$$[\widehat{\phi}, \widehat{\phi}](\xi) = 1 \text{ a.e. on } \{\gamma : [\widehat{\phi}, \widehat{\phi}](\gamma) > 0\}.$$

Thus, if $\{\phi(\cdot - k), k \in \mathbb{Z}^d\}$ forms a tight frame but not an orthonormal basis of V_0, then the set $\{\gamma : [\widehat{\phi}, \widehat{\phi}](\gamma) > 0\}$ differs from $\mathbb{R}^d$ by a set of non-zero measure; see [30, 1] for more details. Therefore, the term $[\widehat{\phi}, \widehat{\phi}]$ cannot be a continuous function. In summary, if ϕ is a band-limited refinable function satisfying (18), then $\widehat{\phi}$ will not be continuous unless it is degenerated to the orthonormal case. Such a refinable function with discontinuities in frequency domain decays very slowly in spatial domain. On the contrary, the refinable function ϕ needed in Theorem 3 or Corollary 4 can be arbitrarily smooth in frequency domain, as long as it satisfies Condition A. Such refinable functions can have very rapid decay in spatial domain. In summary, the UEP/OEP based Theorem 3 or Corollary 4 provides a very convenient way to construct wavelet tight frames with fast decay in spatial domain.

To apply Theorem 3 or Corollary 4,we need to construct refinable functions that satisfy Condition A and sets of 2π-periodic functions $\{p(\xi),\, a_s, s = 1, \ldots, N\}$ that satisfy (12) or (16). Based on the univariate Meyer's refinable function Q_Ω, we first construct a class of non-separable band-limited refinable functions that satisfies Condition A using the representative box-spline direction matrices Ξ. For instance, in $\mathbb{R}^2$,

$$\Xi = \left[\begin{array}{ccccccccc} \overbrace{1 & \cdots & 1}^{m_1} & \overbrace{0 & \cdots & 0}^{m_2} & \overbrace{1 & \cdots & 1}^{m_3} \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{array} \right], \tag{19}$$

and in $\mathbb{R}^3$,

$$\Xi = \left[\begin{array}{cccccccccccc} \overbrace{1 & \cdots & 1}^{m_1} & \overbrace{0 & \cdots & 0}^{m_2} & \overbrace{0 & \cdots & 0}^{m_3} & \overbrace{1 & \cdots & 1}^{m_4} \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{array} \right]. \tag{20}$$

Plugging any above direction matrix into (10), we have a refinable function $\phi^{\Omega}_{d,m} \in L_2(\mathbb{R}^d)$ defined by

$$\widehat{\phi^{\Omega}_{d,m}}(\xi) = \prod_{r\in \mathrm{col}(\Xi)} \widehat{Q_\Omega}(r^\top \cdot \xi) = \widehat{Q_\Omega}^{\,m_{d+1}}(\sum_{j=1}^{d} \xi_j) \prod_{j=1}^{d} \widehat{Q_\Omega}^{\,m_j}(\xi_j) \tag{21}$$

for any $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, where Q_Ω is the Meyer's refinable function as in (9), $\frac{\pi}{2} < \Omega \leqslant \frac{2\pi}{3}$, and $m = (m_1, \ldots, m_{d+1}) \in (\mathbb{Z}^+)^{d+1}$ for $d = 2$ or 3. The associated refinement mask of $\phi^{\Omega}_{d,m}$ is

$$\tau_{d,m}(\xi) = \tau_\Omega^{m_{d+1}}(\sum_{j=1}^{d} \xi_j) \prod_{j=1}^{d} \tau_\Omega^{m_j}(\xi_j),\ \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \tag{22}$$

where τ_Ω is the refinement mask of the univariate function Q_Ω defined by (8). The refinable function defined by (21) indeed satisfies Condition A, as we will show later in the proof of Theorem 5. The remaining is then about constructing a set of measurable 2π-periodic functions $\{p(\xi), a_s(\xi), s = 1, \ldots, N\}$ such that (12) or (16) holds. The solution to (12) is certainly not unique. One trivial solution is setting $a_s = 0, s = 1, \ldots, N$ and $p(\xi) = (1 - \sum_{\kappa \in \mathbb{Z}_2^d} |\tau(\xi + \kappa\pi)|^2)^{1/2}$. However, the resulting wavelet $\widehat{\psi_p}$ may not be continuously differentiable, even though $\widehat{\phi}$ is sufficiently smooth. In the next theorem, we propose a solution to (12) whose resulting wavelets have sufficient smoothness in frequency domain. More specifically, suppose that the univariate Meyer's refinable function Q_Ω satisfies $\widehat{Q_\Omega} \in C^n(\mathbb{R})$ for some nonnegative integer n. We are seeking for the wavelet set Ψ defined in (14) (or Ψ' defined in (17)) such that $\widehat{\psi} \in C^n(\mathbb{R}^d)$, for any $\psi \in \Psi$ (or Ψ').

Theorem 5. *Let $\phi^{\Omega}_{d,m}$ be a refinable function defined as* (21) *with its mask $\tau_{d,m}$ given by* (22) *and $\frac{\pi}{2} < \Omega \leqslant \frac{2\pi}{3}$, $m \in (\mathbb{Z}^+)^{d+1}$ for $d = 2$ or 3. Define a set of 2π-periodic functions $\{h_j, j = 1, \ldots, d+1\}$ as follows:*

$$\begin{cases} h_j(\xi) = \sqrt{1 - \tau_\Omega^{2(m_j-1)}(\xi_j)} \prod_{s=1}^{j} \tau_\Omega(\xi_s) \prod_{t=j+1}^{d} \tau_\Omega^{m_t}(\xi_t),\ j = 1, \ldots, d; \\ h_{d+1}(\xi) = \tau_\Omega(\sum_{j=1}^{d} \xi_j + \pi)\sqrt{1 + \sum_{s=1}^{m_{d+1}-1} \tau_\Omega^{2s}(\sum_{j=1}^{d} \xi_j)} \prod_{j=1}^{d} \tau_\Omega^{m_j}(\xi_j), \end{cases} \tag{23}$$

where τ_Ω is the refinement mask of Q_Ω given by (8). *Then we have $\phi_{d,m}$ satisfies Condition A, and*

$$\sum_{j=1}^{d+1} \sum_{\nu \in \mathbb{Z}_2^d} |h_j(\xi + \nu\pi)|^2 + \sum_{\nu \in \mathbb{Z}_2^d} |\tau_{d,m}(\xi + \nu\pi)|^2 = 1. \tag{24}$$

Proof. See Section 6.2.1.

Theorem 5 provides a solution to (16). Together with Theorem 3, we have

Corollary 6. *Let $\phi^{\Omega}_{d,m}$ be a refinable function defined as* (21) *with its mask $\tau_{d,m}$ given by* (22) *and $\frac{\pi}{2} < \Omega \leqslant \frac{2\pi}{3}$, $m \in (\mathbb{Z}^+)^{d+1}$ for $d = 2$ or 3. Define*

$$\begin{cases} \tau_\kappa(\xi) = \exp(i\eta(\kappa)\cdot\xi)\tau_{d,m}(\xi+\kappa\pi), & \kappa \in \mathbb{Z}_2^d\backslash\{0\}^d; \\ h_j^\mu(\xi) = \exp\big(i\eta(\mu)\cdot\xi\big)h_j(\xi+\mu\pi), & \mu \in \mathbb{Z}_2^d,\ j = 1,\dots,d+1, \end{cases} \tag{25}$$

where $\{h_j,\ j = 1,\dots,d+1\}$ is given by (23)*. Let*

$$\Psi = \{\psi_\kappa,\ \kappa \in \mathbb{Z}_2^d\backslash\{0\}^d\} \cup \{\psi_j^\mu,\ \mu \in \mathbb{Z}_2^d, j = 1,\dots,d+1\},$$

where

$$\begin{cases} \widehat{\psi_\kappa}(\xi) = \tau_\kappa(\frac{\xi}{2})\widehat{\phi}(\frac{\xi}{2}), & \kappa \in \mathbb{Z}_2^d\backslash\{0\}^d; \\ \widehat{\psi_j^\mu}(\xi) = h_j^\mu(\frac{\xi}{2})\widehat{\phi}(\frac{\xi}{2}), & \mu \in \mathbb{Z}_2^d,\ j = 1,\dots,d+1. \end{cases} \tag{26}$$

Then the dyadic affine system $X(\Psi)$ generated by Ψ as (1) *forms a tight frame of $L_2(\mathbb{R}^d)$. Moreover, if $\widehat{Q_\Omega} \in C^n(\mathbb{R})$ for some $n \in \mathbb{Z}^*$, then $\widehat{\psi} \in C^n(\mathbb{R}^d)$ for any $\psi \in \Psi$.*

Proof. See Section 6.2.2.

4. Band-limited Stable Refinable Functions and Wavelets

In last section, for constructing framelets, a class of non-separable band-limited refinable functions is first constructed and then using two extension principles, i.e. UEP and OEP. However, these two extension principles are not applicable to the construction of Riesz wavelets or orthonormal wavelets. One existing approach to construct Riesz wavelets and orthonormal wavelets in low dimensions requires that the constructed refinable function is stable ([29]). Unfortunately, the refinable function ϕ constructed using (21) in the last section is not stable.

Proposition 7. *Let $\phi^{\Omega}_{d,m}$, $d = 2$ or 3, be a refinable function defined as* (21)*, where Q_Ω is any univariate Meyer's refinable function with $\frac{\pi}{2} < \Omega \leqslant \frac{2\pi}{3}$, then $\phi^{\Omega}_{d,m}$ is not stable.*

Proof. A corollary of Theorem 8 with $\frac{\pi}{2} < \Omega \leqslant \frac{2\pi}{3}$.

Thus, in order to construct Riesz wavelets and orthonormal wavelets in low dimensions using the approach proposed in [29], we need to construct a new class of stable band-limited refinable functions satisfying (7). The basic idea of our construction is using a specially designed class of direction matrices Ξ such that the resulting refinable function is stable. Motivated by the direction matrices of the form (19) and (20). We proposed the following direction matrices for the construction of stable refinable functions in $\mathbb{R}^2$ and $\mathbb{R}^3$: in $\mathbb{R}^2$,

$$\Xi = \Bigg[\ \overbrace{\begin{matrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{matrix}}^{m_1}\quad \overbrace{\begin{matrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{matrix}}^{m_2}\quad \overbrace{\begin{matrix} \rho^{-1} & \cdots & \rho^{-1} \\ \rho^{-1} & \cdots & \rho^{-1} \end{matrix}}^{m_3}\ \Bigg], \tag{27}$$

and in $\mathbb{R}^3$,

$$\Xi = \left[\begin{array}{ccccccccccccc} \overbrace{\begin{array}{ccc} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{array}}^{m_1} & \overbrace{\begin{array}{ccc} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{array}}^{m_2} & \overbrace{\begin{array}{ccc} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{array}}^{m_3} & \overbrace{\begin{array}{ccc} \rho^{-1} & \cdots & \rho^{-1} \\ \rho^{-1} & \cdots & \rho^{-1} \\ \rho^{-1} & \cdots & \rho^{-1} \end{array}}^{m_4} \end{array} \right],$$

where ρ is a positive scalar and $m_i, i = 1, \ldots, 4$ are positive integers. By using the above direction matrices, we define the following band-limited functions in $L_2(\mathbb{R}^d)$ for $d = 2, 3$:

$$\widehat{\phi_{d,m}^{\rho,\Omega}}(\xi) = \widehat{Q_\Omega}^{m_{d+1}}\big(\rho^{-1}(\sum_{j=1}^{d}\xi_j)\big)\prod_{j=1}^{d}\widehat{Q_\Omega}^{m_j}(\xi_j) \tag{28}$$

for any $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, where Q_Ω is the Meyer's refinable function of the form (9), $\frac{\pi}{2} < \Omega \leqslant \frac{2}{3}\pi$, and $m = (m_1, \ldots, m_{d+1}) \in (\mathbb{Z}^+)^{d+1}$ for $d = 2$ or 3.

Theorem 8. *The function $\phi_{d,m}^{\rho,\Omega}$ defined in (28), with $d = 2$ or 3 and $\frac{\pi}{2} < \Omega \leqslant \frac{2}{3}\pi$, is refinable whenever $\rho > 0$. Moreover, the function $\phi_{d,m}^{\rho,\Omega}$ is stable if and only if*

$$\rho > \frac{d(\pi - \Omega)}{\Omega}. \tag{29}$$

Proof. See Section 6.4 and 6.5.

By Theorem 8, the refinable function defined by (28) with $\Omega = \frac{2}{3}\pi$ is stable if and only if $\rho > 1$ for $\mathbb{R}^2$ and $\rho > \frac{3}{2}$ for $\mathbb{R}^3$. Moreover, a direct calculation shows that the area of the support of the stable refinable function in $\mathbb{R}^2$ in frequency domain satisfies

$$\inf_{\rho > 1} \text{area of supp}(\widehat{\phi_{2,m}^{\rho,\frac{2}{3}\pi}}) = \frac{48}{9}\pi^2.$$

In the next, we show that, by using some other type of directional matrices, it is possible to construct stable refinable functions in $\mathbb{R}^2$ with smaller support in frequency domain. The direction matrix for $\mathbb{R}^2$ we discussed is defined as follows:

$$\Xi = \left[\begin{array}{ccc} \overbrace{\begin{array}{ccc} 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{array}}^{m_1} & \overbrace{\begin{array}{ccc} 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{array}}^{m_2} & \overbrace{\begin{array}{ccc} \sigma^{-1}\cos\theta & \cdots & \sigma^{-1}\cos\theta \\ \sigma^{-1}\sin\theta & \cdots & \sigma^{-1}\sin\theta \end{array}}^{m_3} \end{array} \right]. \tag{30}$$

The above direction matrix leads to the following band-limited function in $L_2(\mathbb{R}^2)$:

$$\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\xi) = \widehat{Q_\Omega}^{m_1}(\xi_1)\widehat{Q_\Omega}^{m_2}(\xi_2)\widehat{Q_\Omega}^{m_3}\big(\sigma^{-1}(\xi_1\cos\theta + \xi_2\sin\theta)\big) \tag{31}$$

for any $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, where Q_Ω is the Meyer's refinable function of the form (9), $\frac{\pi}{2} < \Omega \leqslant \frac{2}{3}\pi$, $m = (m_1, m_2, m_3) \in (\mathbb{Z}^+)^3$, $\sigma > 0$, and $0 \leqslant \theta \leqslant \frac{\pi}{2}$. It is easy to see that the direction matrix defined by (27) is indeed a special case of the matrix defined by (30) with $\theta = \frac{\pi}{4}$ and $\sigma = \frac{\sqrt{2}}{2}\rho$.

Theorem 9. *The function $\phi_{m,\Omega}^{\sigma,\theta}$ defined in* (31)*, with $\frac{\pi}{2} < \Omega \leqslant \frac{2\pi}{3}$ and $\sigma > 0$, is refinable. For $0 \leqslant \theta \leqslant \frac{\pi}{2}$, $\phi_{m,\Omega}^{\sigma,\theta}$ is stable if and only if $\sigma > \sigma_0(\theta, \Omega)$, where $\sigma_0(\theta, \Omega)$ is defined by*

$$\sigma_0(\theta,\Omega) = \begin{cases} \frac{\pi\cos\theta}{2\Omega}, & 0 \leqslant \theta \leqslant \arctan\frac{2\Omega-\pi}{2(\pi-\Omega)}; \\ \frac{(\pi-\Omega)(\cos\theta+\sin\theta)}{\Omega}, & \arctan\frac{2\Omega-\pi}{2(\pi-\Omega)} < \theta < \frac{\pi}{2} - \arctan\frac{2\Omega-\pi}{2(\pi-\Omega)}; \\ \frac{\pi\sin\theta}{2\Omega}, & \frac{\pi}{2} - \arctan\frac{2\Omega-\pi}{2(\pi-\Omega)} \leqslant \theta \leqslant \frac{\pi}{2}. \end{cases} \tag{32}$$

Proof. See Section 6.5.

For a given Ω, the area of the support of $\widehat{\phi_{m,\Omega}^{\sigma,\theta}}$ defined by (31) with $\sigma = \sigma_0(\theta, \Omega)$ will achieve its minimum at $\theta = \arctan\frac{2\Omega-\pi}{2\pi-2\Omega}$. For the Meyer's refinable function Q_Ω with $\Omega = \frac{2}{3}\pi$, the value of θ should be set as $\theta = \arctan\frac{1}{2}$ to minimize the support of the resulting stable refinable function in frequency domain. A direction calculation leads to

$$\inf_{\sigma > \sigma_0(\arctan\frac{1}{2}, \frac{2}{3}\pi)} \text{area of supp}(\widehat{\phi_{m,\frac{2}{3}\pi}^{\sigma,\arctan\frac{1}{2}}}) = \frac{46}{9}\pi^2.$$

Once a stable refinable function ϕ is constructed via (28) or (31), a set of Riesz wavelets Ψ^d of $L_2(\mathbb{R}^d)$ can be immediately constructed using (6) of [29]:

$$\Psi^d = \left\{ \psi_\mu : \widehat{\psi_\mu}(\xi) = \exp\left(i\eta(\mu)\cdot\frac{\xi}{2}\right)[\widehat{\phi},\widehat{\phi}](\frac{\xi}{2}+\mu\pi)\tau(\frac{\xi}{2}+\mu\pi)\widehat{\phi}(\frac{\xi}{2}), \mu \in \mathbb{Z}_2^d\backslash\{0\}^d \right\},$$

where τ is the refinement mask of ϕ. Suppose that $\widehat{Q_\Omega} \in C^n(\mathbb{R})$ for some $n \in \mathbb{Z}^*$, then $\widehat{\phi} \in C^n(\mathbb{R}^d)$ by its definition.

Moreover, any band-limited stable refinable function ϕ given by (28) or (31) can also be used to construct orthonormal band-limited wavelets using the classical orthonormalization technique. Define a new band-limited function $\widetilde{\phi}$ as

$$\widehat{\widetilde{\phi}} := \frac{\widehat{\phi}}{\sqrt{[\widehat{\phi},\widehat{\phi}]}}. \tag{33}$$

Then we have $[\widehat{\widetilde{\phi}}, \widehat{\widetilde{\phi}}] \equiv 1$, i.e., $\widetilde{\phi}$ is orthonormal. Moreover,

$$\widehat{\widetilde{\phi}}(2\xi) = \frac{\widehat{\phi}(2\xi)}{\sqrt{[\widehat{\phi},\widehat{\phi}](2\xi)}} = \frac{\sqrt{[\widehat{\phi},\widehat{\phi}](\xi)}}{\sqrt{[\widehat{\phi},\widehat{\phi}](2\xi)}}\tau(\xi)\widehat{\widetilde{\phi}}(\xi),$$

where τ is the mask of ϕ. Thus, the refinement mask $\widetilde{\tau}$ of $\widetilde{\phi}$ is given by

$$\widetilde{\tau}(\xi) = \frac{\sqrt{[\widehat{\phi}, \widehat{\phi}](\xi)}}{\sqrt{[\widehat{\phi}, \widehat{\phi}](2\xi)}}\tau(\xi). \tag{34}$$

Again, using (6) of [29], we can immediately obtain non-separable band-limited orthonormal wavelets $\widetilde{\Psi}^d$ of $L_2(\mathbb{R}^d)$:

$$\widetilde{\Psi}^d = \left\{ \widetilde{\psi}_\mu : \widehat{\widetilde{\psi}_\mu}(\xi) = \exp\left(i\eta(\mu) \cdot \frac{\xi}{2}\right)\widetilde{\tau}(\frac{\xi}{2} + \mu\pi)\widehat{\widetilde{\phi}}(\frac{\xi}{2}), \mu \in \mathbb{Z}_2^d \backslash \{0\}^d \right\}, \tag{35}$$

where $\widetilde{\tau}$ is the mask of $\widetilde{\phi}$.

It is seen that once we have a stable band-limited refinable function in hand, the construction of orthonormal band-limited wavelets is very easy by the simple normalization technique in (33). In contrast, the construction of orthonormal compactly supported wavelets is much more difficult. One reason is that a compactly supported function in spatial domain will extend the support of the resulting function to infinity after applying (33), while a band-limited function is still band-limited after applying (33). Thus, the normalization (33) can be used for constructing orthonormal wavelets with compact support in frequency domain, but cannot be used for constructing orthonormal wavelets with compact support in spatial domain. In the end, we summarize the spatial decay property of the orthonormal wavelets defined by (35) in terms of their smoothness in frequency domain in the following proposition.

Proposition 10. *Let ϕ be a band-limited stable refinable function defined by* (28) *or* (31)*, and let $\widetilde{\phi}$ be defined by* (33)*. Suppose that $\widehat{Q_\Omega} \in C^n(\mathbb{R})$ for some $n \in \mathbb{Z}^*$. Then $\widehat{\widetilde{\phi}} \in C^n(\mathbb{R}^d)$ and for the set of orthonormal wavelets $\widetilde{\Psi}^d$ defined by* (35)*, we have $\widehat{\widetilde{\psi}} \in C^n(\mathbb{R}^d), \forall \widetilde{\psi} \in \widetilde{\Psi}^d$.*

Proof. See Section 6.6.

5. Examples of Non-separable Band-limited Framelets, Riesz Wavelets and Orthonormal Wavelets in $\mathbb{R}^2$

In this section, we will apply the results developed in previous sections to construct sample non-separable band-limited framelets, Riesz wavelets and orthonormal wavelets. The construction of these framelets and wavelets starts with the univariate Meyer's refinable function. Same as [12], the term h in the mask τ_Ω of the form (8) corresponding to the Meyer's refinable function is given as follows:

$$h(\xi) = \cos\left(\frac{\pi}{2} \cdot \beta\left(\frac{1}{2}(\frac{|2\xi| - \pi}{2\Omega - \pi} + 1)\right)\right), \tag{36}$$

where $\beta(x) = (\int_0^1 t^n(1-t)^n dt)^{-1} \int_0^x t^n(1-t)^n dt$ for some nonnegative integer n. The smoothness of the associated Meyer's refinable function in frequency domain is closely related to the value of n. More specifically, let Q_Ω denote the Meyer's refinable function determined by (36) and (8). Then $\widehat{Q_\Omega}$ is differentiable up to order n, i.e., $\widehat{Q_\Omega} \in C^n(\mathbb{R})$ (see [24] for more details). Through all examples constructed in this section, we always set $n = 3$ such that

$$\beta(x) = x^4(35 - 84x + 80x^2 - 20x^3),$$

and start with the Meyer's refinable function with $\Omega = \frac{2}{3}\pi$, denoted by $Q_{\frac{2}{3}\pi}$.

Example 5.1. In this example, we use the direction matrix of the form (19) with $m = (1, 1, 1)$ to generate the non-separable refinable function $\phi \in L_2(\mathbb{R}^2)$ as follows:

$$\widehat{\phi}(\xi_1, \xi_2) = \widehat{Q_{\frac{2}{3}\pi}}(\xi_1)\widehat{Q_{\frac{2}{3}\pi}}(\xi_2)\widehat{Q_{\frac{2}{3}\pi}}(\xi_1 + \xi_2).$$

Then the refinement mask of ϕ is given by (22),

$$\tau_{2,m}(\xi_1, \xi_2) = \tau_{\frac{2}{3}\pi}(\xi_1)\tau_{\frac{2}{3}\pi}(\xi_2)\tau_{\frac{2}{3}\pi}(\xi_1 + \xi_2),$$

where $\tau_{\frac{2}{3}\pi}$ is the mask of the Meyer's refinable function given by (8). Following (23), we define

$$\begin{cases} h_1(\xi_1, \xi_2) = \sqrt{1 - \tau_{\frac{2}{3}\pi}^{2\times 0}(\xi_1)}\tau_{\frac{2}{3}\pi}(\xi_1)\tau_{\frac{2}{3}\pi}(\xi_2)\tau_{\frac{2}{3}\pi}(\xi_1 + \xi_2) = 0, \\ h_2(\xi_1, \xi_2) = \sqrt{1 - \tau_{\frac{2}{3}\pi}^{2\times 0}(\xi_2)}\tau_{\frac{2}{3}\pi}(\xi_2)\tau_{\frac{2}{3}\pi}(\xi_1)\tau_{\frac{2}{3}\pi}(\xi_1 + \xi_2) = 0, \\ h_3(\xi_1, \xi_2) = \tau_{\frac{2}{3}\pi}(\xi_1 + \xi_2 + \pi)\tau_{\frac{2}{3}\pi}(\xi_1)\tau_{\frac{2}{3}\pi}(\xi_2). \end{cases}$$

By Corollary 6, we have a framelet system for $L_2(\mathbb{R}^2)$ whose associated masks are given as follows:

$$\begin{cases} \tau_\kappa = \exp\big(i\eta(\kappa) \cdot \xi\big)\tau(\xi + \kappa\pi), & \kappa \in \mathbb{Z}_2^2 \backslash \{0\}^2; \\ h_3^\mu = \exp\big(i\eta(\mu) \cdot \xi\big)h_3(\xi + \mu\pi), & \mu \in \mathbb{Z}_2^2, \end{cases} \tag{37}$$

where $\xi = (\xi_1, \xi_2)$. See Fig. 1 for the plots of the refinable function ϕ and the seven framelets derived from (37) in spatial domain.

Example 5.2. In this example, we use the power vector $m = (2, 1, 1)$ to generate the non-separable refinable function $\phi \in L_2(\mathbb{R}^2)$ defined as follows:

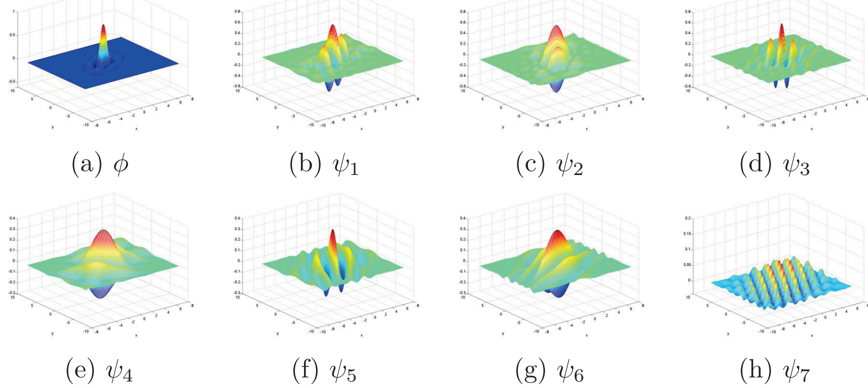
$$\widehat{\phi}(\xi_1, \xi_2) = \widehat{Q_{\frac{2}{3}\pi}}^2(\xi_1)\widehat{Q_{\frac{2}{3}\pi}}(\xi_2)\widehat{Q_{\frac{2}{3}\pi}}(\xi_1 + \xi_2).$$

Then by (22), the refinement mask of ϕ is given as

$$\tau_{2,m}(\xi) = \tau_{\frac{2}{3}\pi}^2(\xi_1)\tau_{\frac{2}{3}\pi}(\xi_2)\tau_{\frac{2}{3}\pi}(\xi_1 + \xi_2).$$

If we use the same construction scheme as Example 5.1, it will generate totally 11 framelets for the definable function ϕ defined above. However, if we use Theorem 3, we actually can construct a framelet system with fewer framelets whose smoothness in Fourier domain is the same as the associated definable function ϕ. More explicitly, we define

$$a_1(\xi) = \tau_{\frac{2}{3}\pi}^2(\xi_1)\tau_{\frac{2}{3}\pi}(\xi_2)\tau_{\frac{2}{3}\pi}(\xi_1 + \xi_2 + \pi)$$



(a) ϕ (b) ψ_1 (c) ψ_2 (d) ψ_3

(e) ψ_4 (f) ψ_5 (g) ψ_6 (h) ψ_7

Figure 1: Graphs of the refinable function and its associated framelets as introduced in Example 5.1. (a) Refinable function; and (b)–(h) the associated framelets.

and

$$p(\xi) = \sqrt{2}\tau_{\frac{2}{3}\pi}(\xi_1)\tau_{\frac{2}{3}\pi}(\xi_1 + \pi),$$

Then the set of 2π-periodic functions $\{p(\xi), \tau(\xi), a_1(\xi)\}$ satisfies (12):

$$|p(\xi)|^2 + \sum_{\kappa\in\mathbb{Z}_2^2} |\tau(\xi + \kappa\pi)|^2 + \sum_{\mu\in\mathbb{Z}_2^2} |a_1(\xi + \mu\pi)|^2 = 1.$$

By Theorem 3, we have a framelet system with *eight* tight framelets for $L_2(\mathbb{R}^2)$, whose associated masks are given as follows:

$$\begin{cases} \exp\big(i\eta(\kappa)\cdot\xi\big)\tau(\xi+\kappa\pi), & \kappa \in \mathbb{Z}_2^2\backslash\{0\}^2; \\ \exp\big(i\eta(\mu)\cdot\xi\big)a_1(\xi+\mu\pi), & \mu \in \mathbb{Z}_2^2; \\ p(2\xi)\tau(\xi)\,. \end{cases} \tag{38}$$

In our example, the mask $\tau_{\frac{2}{3}\pi}$ given by (36) satisfies $\tau_{\frac{2}{3}\pi} \in C^3(\mathbb{R})$, and thus all masks defined by (38) are also in $C^3(\mathbb{R}^2)$. As a result, the Fourier transforms of the associated framelets are in $C^3(\mathbb{R}^2)$ too. See Fig. 2 for the graphs of the refinable function and its associated eight framelets in spatial domain.

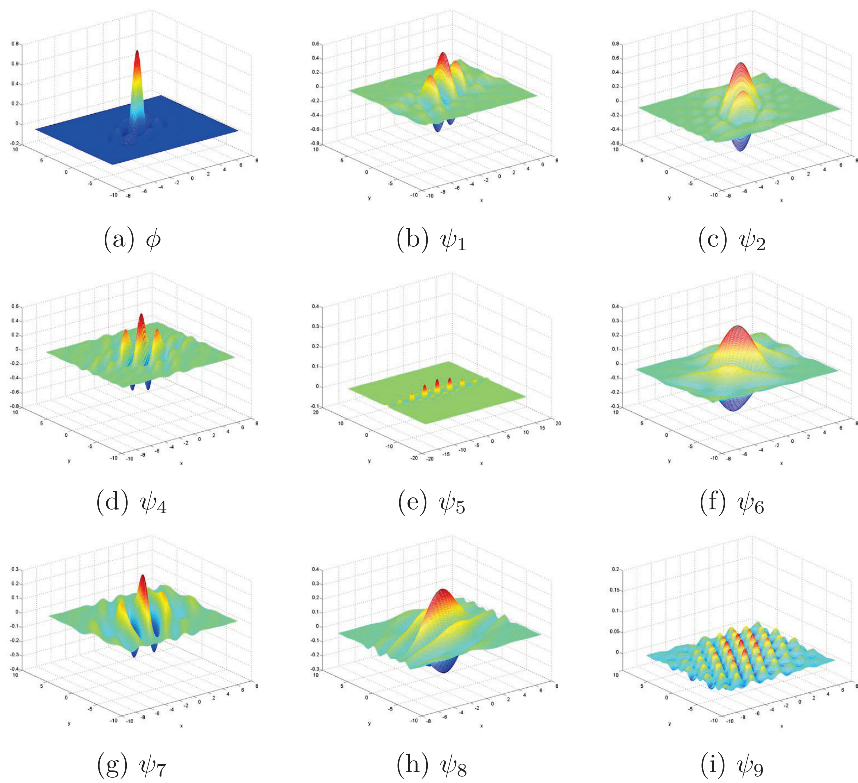
Example 5.3. In this example, we use the direction matrix of the form (27), with $m = (1, 1, 1)$ and $\rho = \frac{5}{4}$, to generate the non-separable band-limited stable function $\phi \in L_2(\mathbb{R}^2)$ defined as follows:

$$\widehat{\phi}(\xi_1, \xi_2) = \widehat{Q_{\frac{2}{3}\pi}}(\xi_1)\widehat{Q_{\frac{2}{3}\pi}}(\xi_2)\widehat{Q_{\frac{2}{3}\pi}}\Big(\frac{4}{5}(\xi_1 + \xi_2)\Big).$$

By Theorem 8, ϕ is refinable and stable.

After obtaining a stable refinable function, we can then use the construction scheme proposed in (6) to construct a Riesz wavelet system of three wavelets, whose associated masks are given as follows:

$$\exp(i\eta(\kappa)\cdot\xi)[\widehat{\phi}, \widehat{\phi}](\xi + \kappa\pi)\tau(\xi + \kappa\pi),\ \kappa \in \mathbb{Z}_2^2\backslash\{0\}^2,$$



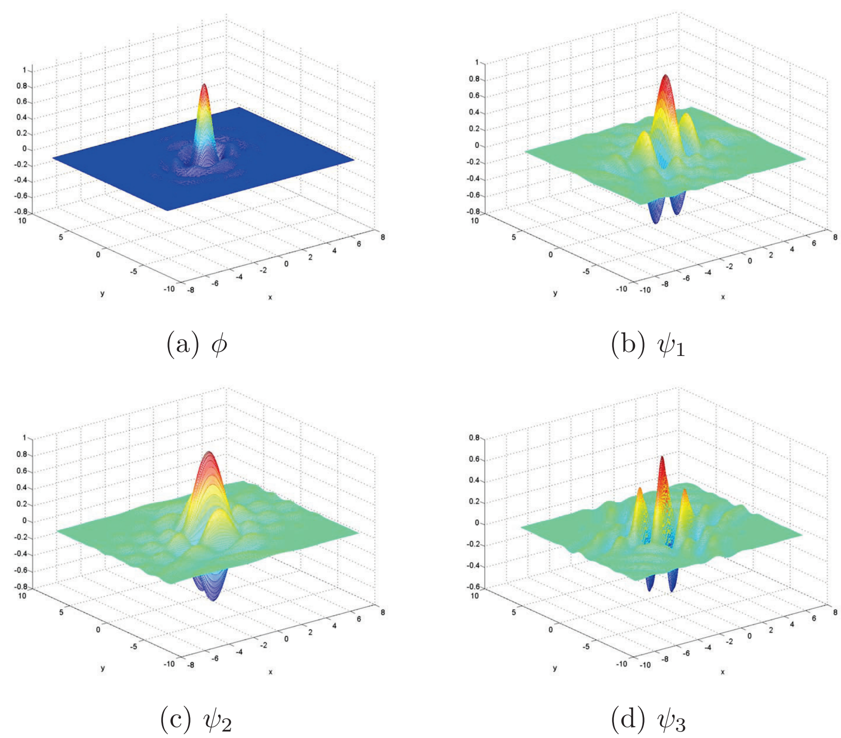
(a) ϕ (b) ψ_1 (c) ψ_2

(d) ψ_4 (e) ψ_5 (f) ψ_6

(g) ψ_7 (h) ψ_8 (i) ψ_9

Figure 2: Graphs of the refinable function and its associated framelets as introduced in Example 5.2. (a) Refinable function, and (b)–(i) the associated framelets.

where τ is the refinement mask of ϕ. See Fig. 3 for the graphs of the refinable function and its associated three Riesz wavelets in spatial domain.



(a) ϕ (b) ψ_1

(c) ψ_2 (d) ψ_3

Figure 3: Graphs of the refinable function and its associated three Riesz wavelets as introduced in Example 5.3. (a) Refinable function, and (b)–(d) the associated Riesz wavelets.

Example 5.4. In this example, we use the direction matrix of the form (30), with $m = (1, 1, 1)$, $\sigma = \frac{\sqrt{5}}{2}$ and $\theta = \arctan \frac{1}{2}$, to generate the non-separable band-limited function $\phi \in L_2(\mathbb{R}^2)$ defined as the follows:

$$\widehat{\phi}(\xi_1, \xi_2) = \widehat{Q_{\frac{2\pi}{3}}}(\xi_1)\widehat{Q_{\frac{2\pi}{3}}}(\xi_2)\widehat{Q_{\frac{2\pi}{3}}}(\xi_1 + \frac{1}{2}\xi_2). \tag{39}$$

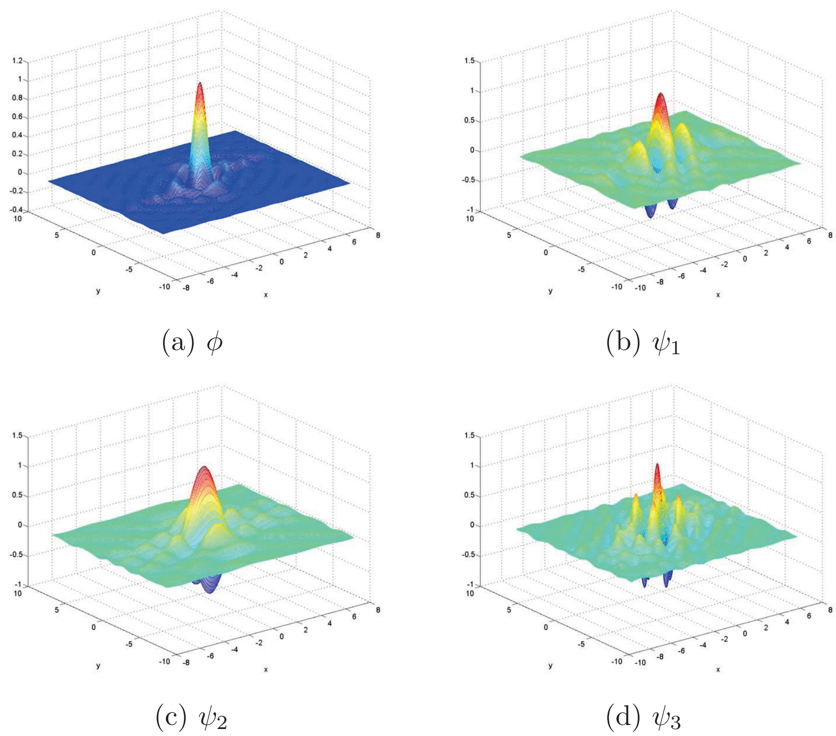
By Theorem 9, ϕ is refinable and stable. Then, by using the standard orthonormalization technique, we have an orthonormal refinable function $\widetilde{\phi}$ defined by

$$\widehat{\widetilde{\phi}} = \frac{\widehat{\phi}}{\sqrt{[\widehat{\phi}, \widehat{\phi}]}},$$

and the refinement mask $\tilde{\tau}$ of $\tilde{\phi}$ is $\widetilde{\tau}(\xi) = \frac{\sqrt{[\widehat{\phi},\widehat{\phi}](\xi)}}{\sqrt{[\widehat{\phi},\widehat{\phi}](2\xi)}}\tau(\xi)$, where τ is the refinement mask of ϕ in (39). Again, using the scheme of (6), we can construct a wavelet system of three orthonormal wavelets from $\tilde{\phi}$, whose masks are given as follows:

$$\exp(i\eta(\kappa)\cdot\xi)\widetilde{\tau}(\xi+\kappa\pi),\ \kappa\in\mathbb{Z}_2^2\backslash\{0\}^2.$$

See Fig. 4 for the graphs of the refinable function and its associated three orthonormal wavelets in spatial domain.



(a) ϕ (b) ψ_1

(c) ψ_2 (d) ψ_3

Figure 4: Graphs of the refinable function and its associated orthonormal wavelets as introduced in Example 5.4. (a) Refinable function; and (b) – (d) the associated three orthonormal wavelets.

6. Proof of the Main Results

6.1. Proof of Theorem 3

Define

$$P(\xi)=1-\sum_{s=1}^{N}\sum_{\mu\in\mathbb{Z}_2^d}|a_s(\xi+\mu\pi)|^2-\sum_{\kappa\in\mathbb{Z}_2^d}|\tau(\xi+\kappa\pi)|^2,$$

and let $B(\xi) = p(2\xi)\tau(\xi), \Theta(\xi) = 1 - P(\xi)$. Then we have $|p(\xi)|^2 = P(\xi)$ and

$$\begin{aligned}
&\Theta(2\xi)\tau(\xi)\overline{\tau(\xi)} + \sum_{\kappa\in\mathbb{Z}_2^d\backslash\{0\}^d} \tau_\kappa(\xi)\overline{\tau_\kappa(\xi)} + \sum_{s=1}^{N}\sum_{\mu\in\mathbb{Z}_2^d} a_{s,\mu}(\xi)\overline{a_{s,\mu}(\xi)} + B(\xi)\overline{B(\xi)} \\
&= -P(2\xi)\tau(\xi)\overline{\tau(\xi)} + 1 - P(\xi) + |p(2\xi)|^2\tau(\xi)\overline{\tau(\xi)} \\
&= -P(2\xi)\tau(\xi)\overline{\tau(\xi)} + 1 - P(\xi) + P(2\xi)\tau(\xi)\overline{\tau(\xi)} \\
&= 1 - P(\xi) = \Theta(\xi).
\end{aligned} \tag{40}$$

Let $\tau_{\{0\}^d} = \tau$. By Lemma 2.12 of [29], we have for all $\nu \in \mathbb{Z}_2^d\backslash\{0\}^d$

$$\sum_{\kappa\in\mathbb{Z}_2^d} \tau_\kappa(\xi)\overline{\tau_\kappa(\xi+\nu\pi)} = 0, \text{ and } \sum_{\mu\in\mathbb{Z}_2^d} a_{s,\mu}(\xi)\overline{a_{s,\mu}(\xi+\nu\pi)} = 0, \;\; s = 1,\ldots,N.$$

Thus for any $\nu \in \mathbb{Z}_2^d\backslash\{0\}^d$,

$$\begin{aligned}
&\Theta(2\xi)\tau(\xi)\overline{\tau(\xi+\nu\pi)} + \sum_{\kappa\in\mathbb{Z}_2^d\backslash\{0\}^d} \tau_\kappa(\xi)\overline{\tau_\kappa(\xi+\nu\pi)} + \sum_{s=1}^{N}\sum_{\mu\in\mathbb{Z}_2^d} a_{s,\mu}(\xi)\overline{a_{s,\mu}(\xi+\nu\pi)} \\
&\quad + B(\xi)\overline{B(\xi+\nu\pi)} \\
&= -P(2\xi)\tau(\xi)\overline{\tau(\xi+\nu\pi)} + |p(2\xi)|^2\tau(\xi)\overline{\tau(\xi+\nu\pi)} \\
&= -P(2\xi)\tau(\xi)\overline{\tau(\xi+\nu\pi)} + P(2\xi)\tau(\xi)\overline{\tau(\xi+\nu\pi)} \\
&= 0.
\end{aligned} \tag{41}$$

The equalities in (40) and (41) imply that the combined masks

$$\{\tau_\kappa(\xi), \kappa \in \mathbb{Z}_2^d;\, a_{s,\mu}(\xi), \mu \in \mathbb{Z}_2^d, \quad s = 1,\ldots,N;\, B(\xi)\}$$

satisfy (5) in OEP. Together with Condition A on ϕ, one can conclude via OEP that the system $X(\Psi)$ as in (1), with Ψ given as (14), forms a tight frame of $L_2(\mathbb{R}^d)$.

6.2. Proof of Theorem 5 and Corollary 6

6.2.1. Proof of Theorem 5

PROOF. We first verify that the function $\phi_{d,m}^{\Omega}$ in (21) satisfies *Condition A*. Since the function $\widehat{\phi_{d,m}^{\Omega}}$ is finitely supported, $[\widehat{\phi_{d,m}^{\Omega}}, \widehat{\phi_{d,m}^{\Omega}}]$ is a finite summation of the sequence $\{|\widehat{\phi_{d,m}^{\Omega}}|^2(\cdot + 2\pi k), k \in \mathbb{Z}^d\}$ in any finite interval. Thus, $[\widehat{\phi_{d,m}^{\Omega}}, \widehat{\phi_{d,m}^{\Omega}}]$ is continuous as long as $|\widehat{\phi_{d,m}^{\Omega}}|^2$ is continuous. The facts that $\widehat{\phi_{d,m}^{\Omega}}(0) = 1$ and $\widehat{\phi_{d,m}^{\Omega}}(2\pi k) = 0$ for all $k \in \mathbb{Z}^d\backslash\{0\}^d$ lead to $[\widehat{\phi_{d,m}^{\Omega}}, \widehat{\phi_{d,m}^{\Omega}}](0) = 1$. The verification of Condition A on $\phi_{d,m}^{\Omega}$ is done.

Secondly, it is seen that h_j given by (23) is real for $j = 1, \dots, d, d+1$. Moreover,

$$\begin{cases} |h_j(\xi)|^2 = \left(1 - \tau_\Omega^{2(m_j-1)}(\xi_j)\right) \prod_{s=1}^{j} \tau_\Omega^2(\xi_s) \prod_{t=j+1}^{d} \tau_\Omega^{2m_t}(\xi_t),\ j = 1, \dots, d; \\ |h_{d+1}(\xi)|^2 = \left(1 - \tau_\Omega^{2m_{d+1}}(\sum_{j=1}^{d} \xi_j)\right) \prod_{j=1}^{d} \tau_\Omega^{2m_j}(\xi_j). \end{cases}$$

Thus, we have

$$\begin{aligned} |\tau_{d,m}(\xi)|^2 + |h_{d+1}(\xi)|^2 &= \tau_\Omega^{2m_{d+1}}(\sum_{j=1}^{d} \xi_j) \prod_{j=1}^{d} \tau_\Omega^{2m_j}(\xi_j) + \left(1 - \tau_\Omega^{2m_{d+1}}(\sum_{j=1}^{d} \xi_j)\right) \prod_{j=1}^{d} \tau_\Omega^{2m_j}(\xi_j) \\ &= \prod_{j=1}^{d} \tau_\Omega^{2m_j}(\xi_j). \end{aligned}$$

By induction, we have

$$|\tau_{d,m}(\xi)|^2 + |h_{d+1}(\xi)|^2 + \sum_{j=1}^{s} |h_j(\xi)|^2 = \prod_{j=1}^{s} \tau_\Omega^2(\xi_j) \prod_{j=s+1}^{d} \tau_\Omega^{2m_j}(\xi_j), \tag{42}$$

holds for any s, $s = 1, \dots, d$. Plugging $s = d$ into (42), we have

$$|\tau_{d,m}(\xi)|^2 + \sum_{j=1}^{d+1} |h_j(\xi)|^2 = |\tau(\xi)|^2 + |h_{d+1}(\xi)|^2 + \sum_{j=1}^{d} |h_j(\xi)|^2 = \prod_{j=1}^{d} \tau_\Omega^2(\xi_j).$$

By the fact that $\tau_\Omega^2(\cdot) + \tau_\Omega^2(\cdot + \pi) = 1$,

$$\begin{aligned} &\sum_{\nu \in \mathbb{Z}_2^d} \left(|\tau_{d,m}(\xi + \nu\pi)|^2 + \sum_{j=1}^{d+1} |h_j(\xi + \nu\pi)|^2\right) = \sum_{\nu \in \mathbb{Z}_2^d} \prod_{j=1}^{d} \tau_\Omega^2(\xi_j + \nu_j \pi) \\ &= \prod_{j=1}^{d} \left(\tau_\Omega^2(\xi_j) + \tau_\Omega^2(\xi_j + \pi)\right) = 1, \end{aligned}$$

Therefore

$$\sum_{\nu \in \mathbb{Z}_2^d} \left(|\tau_{d,m}(\xi + \nu\pi)|^2 + \sum_{j=1}^{d+1} |h_j(\xi + \nu\pi)|^2\right) = 1,$$

which is (24). The proof is done.

6.2.2. Proof of Corollary 6

Proof. Firstly, by Theorem 5 and Corollary 4, the dyadic system $X(\Psi)$ of the form (6) is a tight frame of $L_2(\mathbb{R}^d)$ for the set Ψ defined by (26). Secondly, suppose that $\widehat{Q_\Omega} \in C^n(\mathbb{R})$ for some $n \in \mathbb{Z}^*$, then for the function $\phi_{d,m}^\Omega$ defined by (21), we have $\widehat{\phi_{d,m}^\Omega} \in C^n(\mathbb{R}^d)$ and $\tau_\Omega \in C^n(\mathbb{R})$. Then, for h_{d+1} in (23), one has $\sqrt{1+\sum_{s=1}^{m_d-1}\tau_\Omega^{2s}(\sum_{j=1}^{d}\xi_j)} \in C^n(\mathbb{R}^d)$ since $\sum_{s=1}^{m_d-1}\tau_\Omega^{2s}(\sum_{j=1}^{d}\xi_j) \in C^n(\mathbb{R}^d)$ is nonnegative. Then, we observe that

$$\sqrt{1-\tau_\Omega^{2(m_j-1)}(\xi_j)} = \begin{cases} 0 \in C^n(\mathbb{R}^d), & \text{if } m_j = 1; \\ \tau_\Omega(\xi_j+\pi)\sqrt{1+\sum_{\ell=1}^{m_j-2}\tau_\Omega^{2\ell}(\xi_j)} \in C^n(\mathbb{R}), & \text{if } m_j \geqslant 2. \end{cases}$$

Hence $\sqrt{1-\tau_\Omega^{2(m_j-1)}(\xi_j)} \in C^n(\mathbb{R})$, which leads to $h_j \in C^n(\mathbb{R}^d)$, $j = 1, 2, \ldots, d$. Since the exponential function $\exp\big(i\eta(\nu)\cdot\xi\big)$ is C^∞ for all $\nu \in \mathbb{Z}_2^d$, the set of wavelet masks defined in (25) satisfies

$$\{\tau_\kappa,\, \kappa \in \mathbb{Z}_2^d\backslash\{0\}^d;\, h_j^\mu,\, \mu \in \mathbb{Z}_2^d, j = 1, \ldots, d+1\} \subset C^n(\mathbb{R}^d).$$

By the definition (26) of the framelets Ψ and the fact that $\widehat{\phi_{d,m}^\Omega} \in C^n(\mathbb{R}^d)$, we have $\widehat{\psi} \in C^n(\mathbb{R}^d)$ for any $\psi \in \Psi$. The proof is done.

6.3. Auxiliary Results for the Proof of Theorem 8 and Theorem 9

Before proving Theorem 8 and Theorem 9, we first prove some lemmas and propositions which will be used in the proof of these two theorems.

Lemma 11. *For any $\phi \in L_2(\mathbb{R}^d)$, suppose that the set $S_1 = \operatorname{supp}\big(\widehat{\phi}\big)$ is a subset of $[-\frac{4}{3}\pi, \frac{4}{3}\pi]^d$ and the set $S_2 = \{\xi : \widehat{\phi}(2\xi) \neq 0\} \cap \{\xi : \widehat{\phi}(\xi) = 0\}$ is of measure zero. Then there exists a 2π-periodic measurable function τ (not necessarily unique) such that*

$$\widehat{\phi}(2\xi) = \tau(\xi)\widehat{\phi}(\xi) \ \text{a.e. on } \mathbb{R}^d,$$

i.e. ϕ is refinable with mask τ.

Proof. Firstly, we define a compactly supported function $b(\xi)$, such that

$$b(\xi) = \begin{cases} \frac{\widehat{\phi}(2\xi)}{\widehat{\phi}(\xi)}, & \text{if } \widehat{\phi}(\xi) \neq 0; \\ 0, & \text{otherwise }. \end{cases}$$

By the definition of $b(\xi)$, we have

$$\widehat{\phi}(2\xi) = b(\xi)\widehat{\phi}(\xi),\, \forall \xi \in \{\gamma : \widehat{\phi}(\gamma) \neq 0\} \cup \{\gamma : \widehat{\phi}(\gamma) = 0, \widehat{\phi}(2\gamma) = 0\}.$$

Since the complement of $\{\gamma : \widehat{\phi}(\gamma) \neq 0\} \cup \{\gamma : \widehat{\phi}(\gamma) = 0, \widehat{\phi}(2\gamma) = 0\}$ is the set $\{\gamma : \widehat{\phi}(2\gamma) \neq 0\} \cap \{\gamma : \widehat{\phi}(\gamma) = 0\}$, which is of Lebesgue measure zero, we have then

$$\widehat{\phi}(2\xi) = b(\xi)\widehat{\phi}(\xi) \text{ a.e. on } \mathbb{R}^d.$$

Secondly, since $\operatorname{supp}(\widehat{\phi}) \subseteq [-\frac{4\pi}{3}, \frac{4\pi}{3}]^d$, then

$$\operatorname{supp}\big(b(\xi)\big) \subseteq [-\frac{2\pi}{3}, \frac{2\pi}{3}]^d \subset (-\pi, \pi)^d.$$

by its definition. Let τ be a 2π-periodic extension of b such that

$$\tau(\xi) = b(\xi),\, \forall\, \xi \in [-\pi,\pi)^d,$$

or equivalently

$$\tau(\xi) = \sum_{k\in\mathbb{Z}^d} b(\xi + 2\pi k),\, \xi \in \mathbb{R}^d.$$

Then we have

$$\tau(\xi)\widehat{\phi}(\xi) = b(\xi)\widehat{\phi}(\xi) = \widehat{\phi}(2\xi) \text{ a.e. on } [-\pi,\pi)^d. \tag{43}$$

For those $\xi \in \mathbb{R}^d\backslash[-\pi,\pi)^d$, $\tau(\xi)\widehat{\phi}(\xi) \neq 0$ only if $\xi \in \operatorname{supp}(\tau) \cap \operatorname{supp}(\widehat{\phi})$. Note that $\operatorname{supp}(\tau)$ is a subset of $\bigcup_{k\in\mathbb{Z}^d} \Big([-\frac{2\pi}{3}, \frac{2\pi}{3}]^d + 2\pi k\Big)$, and $\operatorname{supp}(\widehat{\phi})$ is a subset of $[-\frac{4}{3}\pi, \frac{4}{3}\pi]^d$, thus $\big(\mathbb{R}^d\backslash[-\pi,\pi)^d\big) \cap \operatorname{supp}(\tau) \cap \operatorname{supp}(\widehat{\phi})$ is a subset of

$$\big(\mathbb{R}^d\backslash[-\pi,\pi)^d\big) \cap \Big(\bigcup_{k\in\mathbb{Z}^d} \big([-\frac{2\pi}{3}, \frac{2\pi}{3}]^d + 2\pi k\big)\Big) \cap [-\frac{4}{3}\pi, \frac{4}{3}\pi]^d, \tag{44}$$

It is seen that the set as in (44) is contained within $\partial([-\frac{4}{3}\pi, \frac{4}{3}\pi]^d)$ (the set of all boundary points of the cube $[-\frac{4}{3}\pi, \frac{4}{3}\pi]^d$), which is of Lebesgue measure zero in $\mathbb{R}^d$. Thus, the set $\big(\mathbb{R}^d\backslash[-\pi,\pi)^d\big) \cap \operatorname{supp}(\tau) \cap \operatorname{supp}(\widehat{\phi})$ is of Lebesgue measure zero in $\mathbb{R}^d$. In other words,

$$\tau(\xi)\widehat{\phi}(\xi) = 0 = \widehat{\phi}(2\xi), \text{ a.e. on } \mathbb{R}^d\backslash[-\pi,\pi)^d. \tag{45}$$

By both (43) and (45), we have

$$\widehat{\phi}(2\xi) = \tau(\xi)\widehat{\phi}(\xi) \text{ a.e. on } \mathbb{R}^d.$$

The proof is complete.

Lemma 12. *Let ϕ be a band-limited refinable function on $\mathbb{R}^d$. If $\widehat{\phi}$ is continuous, then ϕ is stable if and only if $[\widehat{\phi}, \widehat{\phi}](\xi) > 0$ for all $\xi \in [-\pi,\pi]^d$.*

Proof. Since $\widehat{\phi}$ is continuous and compactly supported, $[\widehat{\phi}, \widehat{\phi}]$ is also continuous by its definition. If ϕ is stable, then by definition there exists some constant $c > 0$ such that $[\widehat{\phi}, \widehat{\phi}] \geqslant c$ a.e. on $\mathbb{R}^d$. Since $[\widehat{\phi}, \widehat{\phi}]$ is continuous, we have $[\widehat{\phi}, \widehat{\phi}](\xi) \geqslant c$ for all $\xi \in \mathbb{R}^d$, which implies that $[\widehat{\phi}, \widehat{\phi}](\xi) > 0$ for all $\xi \in \mathbb{R}^d$, and the same inequality holds true for all $\xi \in [-\pi,\pi]^d$.

If $[\widehat{\phi}, \widehat{\phi}] > 0$ for all $\xi \in [-\pi,\pi]^d$, then since $[\widehat{\phi}, \widehat{\phi}]$ is continuous and $[-\pi,\pi]^d$ is compact, the infimum of $[\widehat{\phi}, \widehat{\phi}]$ can be achieved and it is greater than 0, i.e., there exists a positive constant c_1 such that $[\widehat{\phi}, \widehat{\phi}] \geqslant c_1$ on $[-\pi,\pi]^d$. Also since $[\widehat{\phi}, \widehat{\phi}]$ is continuous, the supremum of $[\widehat{\phi}, \widehat{\phi}]$ on the compact set $[-\pi,\pi]^d$ is less than infinity, i.e., there exists a positive constant c_2 such that $[\widehat{\phi}, \widehat{\phi}] \leqslant c_2$ on $[-\pi,\pi]^d$. Then since $[\widehat{\phi}, \widehat{\phi}]$ is 2π-periodic, we conclude $0 < c_1 \leqslant [\widehat{\phi}, \widehat{\phi}] \leqslant c_2 < \infty$ on $\mathbb{R}^d$, which implies that ϕ is stable. The proof is complete.

Proposition 13. *Let $\phi_{d,m}^{\rho,\Omega}$ be a band-limited refinable function defined by* (28)*, then $\widehat{\phi_{d,m}^{\rho,\Omega}}(\xi) > 0$ if and only if $\xi = (\xi_1, \ldots, \xi_d)$ satisfies the following two conditions:*

(i) $\xi \in (-2\Omega, 2\Omega)^d$;

(ii) $-2\rho\Omega < \sum_{j=1}^{d} \xi_j < 2\rho\Omega$.

Proof. The necessity and sufficiency of the two conditions can be easily justified by the definition (28) and the fact that $\widehat{Q}_\Omega(\gamma) > 0$ if and only if $\gamma \in (-2\Omega, 2\Omega)$.

Proposition 14. *Let $\phi_{m,\Omega}^{\sigma,\theta}$ be a band-limited refinable function defined by* (31)*, then $\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\xi) > 0$ if and only if $\xi = (\xi_1, \xi_2)$ satisfies the following two conditions:*

(i) $\xi \in (-2\Omega, 2\Omega)^2$;

(ii) $-2\sigma\Omega < \xi_1 \cos\theta + \xi_2 \sin\theta < 2\sigma\Omega$.

Proof. The necessity and sufficiency of two conditions can be easily justified by the definition (31) and the fact that $\widehat{Q}_\Omega(\gamma) > 0$ if and only if $\gamma \in (-2\Omega, 2\Omega)$.

6.4. Proof of Theorem 8 for $d = 3$

In this section, we only give the proof of Theorem 8 for the case of $d = 3$. The proof of Theorem 8 for $d = 2$ is essentially the same as that of Theorem 9 with $\theta = \frac{\pi}{4}$ and $\sigma = \frac{\sqrt{2}}{2}\rho$, which is given in the next section.

Proof. Given a band-limited function $\phi_{3,m}^{\rho,\Omega}$ defined in (28), we have $\phi_{3,m}^{\rho,\Omega} \in L_2(\mathbb{R}^3)$. Moreover, one can verify that for any $\rho > 0$,

(i) $\mathrm{supp}\big(\widehat{\phi_{3,\mathrm{m}}^{\rho,\Omega}}\big) \subseteq [-2\Omega, 2\Omega]^3 \subseteq [-\frac{4\pi}{3}, \frac{4\pi}{3}]^3$, since $\Omega \leqslant \frac{2}{3}\pi$,

(ii) $\{\xi : \widehat{\phi_{3,m}^{\rho,\Omega}}(2\xi) \neq 0\}$ is a subset of $\{\xi : \widehat{\phi_{3,m}^{\rho,\Omega}}(\xi) \neq 0\}$, which implies the set $\{\xi : \widehat{\phi_{3,m}^{\rho,\Omega}}(2\xi) \neq 0\} \cap \{\xi : \widehat{\phi_{3,m}^{\rho,\Omega}}(\xi) = 0\}$ is empty.

Therefore by Lemma 11, any band-limited function $\phi_{3,m}^{\rho,\Omega}$ as in (28) is refinable.

Sufficiency of (29). By Lemma 12, to verify $\phi_{3,m}^{\rho,\Omega}$ is stable, we only need to verify that

$$[\widehat{\phi_{3,m}^{\rho,\Omega}}, \widehat{\phi_{3,m}^{\rho,\Omega}}](\xi) > 0, \forall\, \xi \in [-\pi, \pi]^3. \tag{46}$$

We prove (46) on two disjoint subsets of $[-\pi, \pi]^3 = S_1 \cup S_2$, where

$$S_1 = \{\xi : \xi = (\xi_1, \xi_2, \xi_3) \in [-\pi, \pi]^3, |\sum_{j=1}^{3} \xi_j| < 2\rho\Omega\},$$

and

$$S_2 = \{\xi : \xi = (\xi_1, \xi_2, \xi_3) \in [-\pi, \pi]^3, |\sum_{j=1}^{3} \xi_j| \geqslant 2\rho\Omega\}.$$

If $\xi \in S_1$, then by the definition of $\phi_{3,m}^{\rho,\Omega}$, one has $\widehat{\phi_{3,m}^{\rho,\Omega}}(\xi) > 0$, which implies $[\widehat{\phi_{3,m}^{\rho,\Omega}}, \widehat{\phi_{3,m}^{\rho,\Omega}}](\xi) > 0$. The next step is to verify the inequality (46) holds for all $\xi \in S_2$. Suppose $\xi = (\xi_1, \xi_2, \xi_3) \in S_2$. Without loss of generality, we may assume $\sum_{j=1}^{3} \xi_j \geqslant 2\rho\Omega$ and $\xi_1 = \max\{\xi_1, \xi_2, \xi_3\}$, then

$$3\xi_1 \geqslant \sum_{j=1}^{3} \xi_j \geqslant 2\rho\Omega. \tag{47}$$

By substituting (29), i.e. $\rho > \frac{3(\pi-\Omega)}{\Omega}$ into (47), we get

$$3\xi_1 > 2\frac{3(\pi - \Omega)}{\Omega}\Omega = 6(\pi - \Omega),$$

or equivalently

$$\xi_1 - 2\pi > -2\Omega.$$

Meanwhile, since $\xi_1 \leqslant \pi$, we have

$$\xi_1 - 2\pi \leqslant \pi - 2\pi = -\pi < 0 < 2\Omega.$$

Together with the fact that $\xi_j \in [-\pi, \pi] \subset (-2\Omega, 2\Omega)$, $j = 1, 2, 3$, we can conclude that the point $(\xi_1 - 2\pi, \xi_2, \xi_3)$ satisfies Condition (i) of Proposition 13. Moreover, we have

$$(\xi_1 - 2\pi) + \xi_2 + \xi_3 \leqslant 3\pi - 2\pi = \pi < 6(\pi - \Omega) < 2\rho\Omega$$

and

$$\xi_1 - 2\pi + \xi_2 + \xi_3 \geqslant 2\rho\Omega - 2\pi > 6(\pi - \Omega) - 2\pi = 4\pi - 6\Omega \geqslant 0 > -2\rho\Omega.$$

Thus, the point $(\xi_1 - 2\pi, \xi_2, \xi_3)$ satisfies Condition (ii) of Proposition 13. By Proposition 13, we have $\widehat{\phi_{3,m}^{\rho,\Omega}}(\xi_1 - 2\pi, \xi_2, \xi_3) > 0$. This implies, by the definition of $[\cdot, \cdot]$ in (3), that

$$[\widehat{\phi_{3,m}^{\rho,\Omega}}, \widehat{\phi_{3,m}^{\rho,\Omega}}](\xi) > 0, \ \xi = (\xi_1, \xi_2, \xi_3).$$

The proof for the sufficiency of (29) is complete.

Necessity of (29). By contrapositive, assume that (29) does not hold true, i.e. $\rho \leqslant \frac{3(\pi-\Omega)}{\Omega}$. Consider the point $\tilde{\xi} = (\xi_1, \xi_2, \xi_3) \in [-\pi, \pi]^3$, with $\xi_1 = \xi_2 = \xi_3 = 2(\pi - \Omega)$. Since $\sum_{j=1}^{3} \xi_j = 6(\pi - \Omega) \geqslant 2\rho\Omega$, it is seen that Condition (ii) of Proposition 13 does not hold true for $\tilde{\xi}$, which leads to $\widehat{\phi_{3,m}^{\rho,\Omega}}(\tilde{\xi}) = 0$.

Also, we have $\tilde{\xi}-2\pi k \notin (-2\Omega,2\Omega)^3$ for any $k\in\mathbb{Z}_2^3\backslash\{0\}^3$ as $2(\pi-\Omega)-2\pi=-2\Omega$. Consequently, Condition (i) of Proposition 13 does not hold true for the point $\tilde{\xi}-2\pi k$ with any $k\in\mathbb{Z}_2^3\backslash\{0\}^3$ either. As for any $k\in\mathbb{Z}^3\backslash\mathbb{Z}_2^3$, clearly we have $\tilde{\xi}-2\pi k$ lies outside the cube $(-2\Omega,2\Omega)^3$, thus $\widehat{\phi}(\tilde{\xi}-2\pi k)=0$ for all $k\in\mathbb{Z}^3\backslash\mathbb{Z}_2^3$. In summary, we have shown that if $\rho\leqslant\frac{3(\pi-\Omega)}{\Omega}$, then for $\tilde{\xi}=\big(2(\pi-\Omega),2(\pi-\Omega),2(\pi-\Omega)\big)\in[-\pi,\pi]^3$, one has $\widehat{\phi_{3,m}^{\rho,\Omega}}(\tilde{\xi}-2\pi k)=0$ for all $k\in\mathbb{Z}^3$, which implies

$$[\widehat{\phi_{3,m}^{\rho,\Omega}},\widehat{\phi_{3,m}^{\rho,\Omega}}](\tilde{\xi})=0,\ \text{for } \tilde{\xi}=\big(2(\pi-\Omega),2(\pi-\Omega),2(\pi-\Omega)\big).$$

By Lemma 12, $\phi_{3,m}^{\rho,\Omega}$ is not stable. The proof is complete.

6.5. Proof of Theorem 9

Proof. Firstly, for any band-limited function $\phi_{m,\Omega}^{\sigma,\theta}$ in (31), one can verify that

(i) $\text{supp}(\widehat{\phi_{m,\Omega}^{\sigma,\theta}})\in[-2\Omega,2\Omega]^2\subseteq[-\frac{4}{3}\pi,\frac{4}{3}\pi]^2$, since $\Omega\leqslant\frac{2\pi}{3}$,

(ii) the set $\{\xi:\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(2\xi)\neq 0\}$ is a subset of $\{\xi:\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\xi)\neq 0\}$, which implies that the set $\{\xi:\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(2\xi)\neq 0\}\cap\{\xi:\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\xi)=0\}$ is empty.

Then by Lemma 14, any band-limited function $\phi_{m,\Omega}^{\sigma,\theta}$ as in (31) is refinable. By the definition of $\phi_{m,\Omega}^{\sigma,\theta}$ in (31), we have $\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\xi_1,\xi_2)=\widehat{\phi_{m,\Omega}^{\sigma,\frac{\pi}{2}-\theta}}(\xi_2,\xi_1)$ for any $\theta\in[0,\frac{\pi}{2}]$. Thus, we only need to prove the result for $\theta\in[0,\frac{\pi}{4}]$.
Sufficiency. We first prove that if $\sigma>\sigma_0(\theta,\Omega)$, then $\phi_{m,\Omega}^{\sigma,\theta}$ is stable. By Lemma 12, the stability of $\phi_{m,\Omega}^{\sigma,\theta}$ is guaranteed as long as

$$[\widehat{\phi_{m,\Omega}^{\sigma,\theta}},\widehat{\phi_{m,\Omega}^{\sigma,\theta}}](\xi)>0,\ \forall\,\xi\in[-\pi,\pi]^2 \tag{48}$$

for any $\sigma>\sigma_0(\theta,\Omega)$. We prove the above inequality (48) on two separate subsets of $[-\pi,\pi]^2=S_1\cup S_2$, where

$$S_1=\{(\xi_1,\xi_2):(\xi_1,\xi_2)\in[-\pi,\pi]^2,|\xi_1\cos\theta+\xi_2\sin\theta|<2\sigma\Omega\},$$

and

$$S_2=\{(\xi_1,\xi_2):(\xi_1,\xi_2)\in[-\pi,\pi]^2,|\xi_1\cos\theta+\xi_2\sin\theta|\geqslant 2\sigma\Omega\}.$$

The inequality (48) on S_1 is obvious. The fact that $\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\xi)>0$ for any $\xi\in S_1$ implies

$$[\widehat{\phi_{m,\Omega}^{\sigma,\theta}},\widehat{\phi_{m,\Omega}^{\sigma,\theta}}](\xi)>0,\ \forall\xi\in S_1.$$

The next is to prove (48) for any $\xi\in S_2$. By the symmetry of S_2, it is sufficient to prove (48) holds true on the subset $S_3\subset S_2$ given by

$$S_3=\{(\xi_1,\xi_2):(\xi_1,\xi_2)\in[-\pi,\pi]^2,\xi_1\cos\theta+\xi_2\sin\theta\geqslant 2\sigma\Omega\}.$$

Furthermore, by the definition of $[\cdot,\cdot]$, the inequality (48) holds for all $\xi \in S_3$ as long as the following statement holds true: $\forall\, \xi = (\xi_1, \xi_2) \in S_3$,

$$\begin{cases} \widehat{\phi^{\sigma,\theta}_{m,\Omega}}(\xi_1 - 2\pi, \xi_2) > 0, & \text{if } \xi_1 \geqslant \xi_2; \\ \widehat{\phi^{\sigma,\theta}_{m,\Omega}}(\xi_1, \xi_2 - 2\pi) > 0, & \text{otherwise.} \end{cases} \tag{49}$$

The proof of the two inequalities in (49) is based on Proposition 14. We will only give a detailed proof of the first inequality and the proof of the second inequality in (49) is essentially the same as that of the first one. Assume that $\xi_1 \geqslant \xi_2$.

The case of $\mathbf{0} \leqslant \theta \leqslant \arctan \frac{\mathbf{2\Omega}-\pi}{\mathbf{2}(\pi-\mathbf{\Omega})}$. Since $\xi_1 \geqslant \xi_2$, $(\xi_1, \xi_2) \in S_3$, we have

$$\xi_1(\cos\theta + \sin\theta) \geqslant \xi_1 \cos\theta + \xi_2 \sin\theta \geqslant 2\sigma\Omega.$$

Since $\sigma > \sigma_0(\theta, \Omega) = \frac{\pi\cos\theta}{2\Omega}$ and $0 \leqslant \tan\theta \leqslant \frac{2\Omega-\pi}{2(\pi-\Omega)}$, we have then

$$\xi_1 \geqslant \frac{2\sigma\Omega}{\cos\theta + \sin\theta} > \frac{2\Omega \frac{\pi\cos\theta}{2\Omega}}{\cos\theta + \sin\theta} = \frac{\pi}{1 + \tan\theta} \geqslant \frac{\pi}{1 + \frac{2\Omega-\pi}{2(\pi-\Omega)}} = 2(\pi - \Omega),$$

which implies

$$\xi_1 - 2\pi > -2\Omega.$$

Together with the fact that $\xi_1 - 2\pi \leqslant -\pi < 0$ as $\xi_1 \leqslant \pi$, we have

$$(\xi_1 - 2\pi, \xi_2) \in (-2\Omega, 2\Omega)^2.$$

Hence $(\xi_1 - 2\pi, \xi_2)$ satisfies Condition (i) of Proposition 14. Next, the inequality $\xi_1 \cos\theta + \xi_2 \sin\theta \geqslant 2\sigma\Omega$ implies that for $\sigma > \frac{\pi\cos\theta}{2\Omega}$,

$$(\xi_1 - 2\pi)\cos\theta + \xi_2 \sin\theta \geqslant 2\sigma\Omega - 2\pi\cos\theta > \pi\cos\theta - 2\pi\cos\theta = -\pi\cos\theta > -2\sigma\Omega.$$

Meanwhile since $\xi_1 \leqslant \pi$, $\xi_2 \leqslant \pi$, and $0 \leqslant \theta \leqslant \frac{\pi}{4}$,

$$(\xi_1 - 2\pi)\cos\theta + \xi_2 \sin\theta \leqslant (\pi - 2\pi)\cos\theta + \pi\sin\theta = \pi(\sin\theta - \cos\theta) \leqslant 0.$$

The above two inequalities imply that Condition (ii) of Proposition 14 also holds true for $\sigma > \frac{\pi\cos\theta}{2\Omega}$ since

$$-2\sigma\Omega < (\xi_1 - 2\pi)\cos\theta + \xi_2 \sin\theta \leqslant 0 < 2\sigma\Omega.$$

The first inequality in (49) is then proved by Proposition 14.

The case of $\arctan \frac{\mathbf{2\Omega}-\pi}{\mathbf{2}(\pi-\mathbf{\Omega})} < \theta \leqslant \frac{\pi}{\mathbf{4}}$. Since $\xi_1 \cos\theta + \xi_2 \sin\theta \geqslant 2\sigma\Omega$ and $\xi_1 \geqslant \xi_2$, we have

$$\xi_1(\cos\theta + \sin\theta) \geqslant \xi_1 \cos\theta + \xi_2 \sin\theta \geqslant 2\sigma\Omega.$$

Since $\sigma > \sigma_0(\theta, \Omega) = \frac{(\pi-\Omega)(\cos\theta+\sin\theta)}{\Omega}$, we have then

$$\xi_1 \geqslant \frac{2\sigma\Omega}{\cos\theta + \sin\theta} > \frac{2\sigma_0(\theta, \Omega)\Omega}{\cos\theta + \sin\theta} = 2(\pi - \Omega),$$

which leads to

$$\xi_1 - 2\pi > 2(\pi - \Omega) - 2\pi = -2\Omega.$$

Together with

$$\xi_1 - 2\pi \leqslant \pi - 2\pi = -\pi < 2\Omega, \quad -2\Omega < -\pi \leqslant \xi_2 \leqslant \pi < 2\Omega,$$

we have $(\xi_1 - 2\pi, \xi_2) \in (-2\Omega, 2\Omega)^2$. Thus $(\xi_1 - 2\pi, \xi_2)$ satisfies Condition (i) of Proposition 14. Next, by the fact that $\xi_1 \cos\theta + \xi_2 \sin\theta \geqslant 2\sigma\Omega$ and $\sigma > \sigma_0(\theta, \Omega) = \frac{(\pi-\Omega)(\cos\theta+\sin\theta)}{\Omega}$, we have

$$(\xi_1 - 2\pi)\cos\theta + \xi_2 \sin\theta + 2\sigma\Omega \geqslant 4\sigma\Omega - 2\pi\cos\theta > \cos\theta\big(4(\pi-\Omega)(1+\tan\theta) - 2\pi\big).$$

Notice that $\tan\theta > \frac{2\Omega-\pi}{2(\pi-\Omega)}$, we have then

$$(\xi_1 - 2\pi)\cos\theta + \xi_2 \sin\theta + 2\sigma\Omega > \cos\theta \cdot \Big(4(\pi-\Omega)(1 + \frac{2\Omega-\pi}{2(\pi-\Omega)}) - 2\pi\Big) = 0.$$

Thus

$$(\xi_1 - 2\pi)\cos\theta + \xi_2 \sin\theta > -2\sigma\Omega. \tag{50}$$

Meanwhile, we have

$$(\xi_1 - 2\pi)\cos\theta + \xi_2 \sin\theta \leqslant (\pi - 2\pi)\cos\theta + \pi\sin\theta = \pi(\sin\theta - \cos\theta) \leqslant 0 < 2\sigma\Omega,$$

since $(\xi_1, \xi_2) \in [-\pi, \pi]^2$ and $0 \leqslant \theta \leqslant \frac{\pi}{4}$. Together with (50), we have

$$-2\sigma\Omega < (\xi_1 - 2\pi)\cos\theta + \xi_2 \sin\theta < 2\sigma\Omega.$$

Thus $(\xi_1 - 2\pi, \xi_2)$ satisfies Condition (ii) of Proposition 14 and the first inequality is justified.

Necessity. In this part, we prove the necessity by contrapositive, that is, if $0 < \sigma \leqslant \sigma_0(\theta, \Omega)$, then $\phi_{m,\Omega}^{\sigma,\theta}$ in (31) is not stable. Suppose that $0 < \sigma \leqslant \sigma_0(\theta, \Omega)$ as defined by (32), i.e.

$$\begin{cases} 0 < \sigma \leqslant \frac{\pi\cos\theta}{2\Omega}, & 0 \leqslant \theta \leqslant \arctan\frac{2\Omega-\pi}{2(\pi-\Omega)}; \\ 0 < \sigma \leqslant \frac{(\pi-\Omega)(\cos\theta+\sin\theta)}{\Omega}, & \arctan\frac{2\Omega-\pi}{2(\pi-\Omega)} < \theta \leqslant \frac{\pi}{4}. \end{cases} \tag{51}$$

The case when $\theta \in [\mathbf{0}, \arctan\frac{\mathbf{2\Omega}-\pi}{\mathbf{2}(\pi-\mathbf{\Omega})}]$. Consider the point $\tilde{\xi} = (\pi, 0) \in [-\pi, \pi]^2$. It is seen via (51) that $\pi\cos\theta + 0\sin\theta = \pi\cos\theta \geqslant 2\sigma\Omega$, which contradicts Condition (ii) of Proposition 14. Thus, $\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\tilde{\xi}) = 0$. Next, for the point $(-\pi, 0) = \tilde{\xi} - 2\pi(1, 0)$. Similarly we have $-\pi\cdot\cos\theta + 0\cdot\sin\theta \leqslant -2\sigma\Omega$, which also contradicts Condition (ii) of Proposition 14. Thus, $\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\tilde{\xi} - 2\pi(1,0)) = 0$. For the point $(\pi, -2\pi) = \tilde{\xi} - 2\pi(0, 1)$, we have $(\pi, -2\pi) \notin (-2\Omega, 2\Omega)^2$, which contradicts Condition (i) of Proposition 14, thus $\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\tilde{\xi} - 2\pi(0,1)) = 0$. So far, we have verified that for $\tilde{\xi} = (\pi, 0)$, one has $\widehat{\phi_{m,\Omega}^{\sigma,\theta}}(\tilde{\xi} - 2\pi k) = 0$, for $k = (0,0), (1,0)$

and $(0,1)$. For any other $k \in \mathbb{Z}^2$, clearly $\widehat{\phi^{\sigma,\theta}_{m,\Omega}}(\tilde{\xi} - 2\pi k) = 0$ as $\tilde{\xi} - 2\pi k$ lies outside of the support of $\widehat{\phi^{\sigma,\theta}_{m,\Omega}}$. In summary, $[\widehat{\phi^{\sigma,\theta}_{m,\Omega}}, \widehat{\phi^{\sigma,\theta}_{m,\Omega}}](\tilde{\xi}) = 0$, for $\tilde{\xi} = (\pi, 0)$. By Lemma 12, $\phi^{\sigma,\theta}_{m,\Omega}$ is not stable.

The case when $\arctan \frac{\mathbf{2\Omega}-\pi}{\mathbf{2}(\pi-\mathbf{\Omega})} < \theta \leqslant \frac{\pi}{\mathbf{4}}$. Consider the point $\tilde{\xi} = \big(2(\pi - \Omega), 2(\pi - \Omega)\big) \in [-\pi, \pi]^2$. By (51), we have

$$2(\pi - \Omega)\cos\theta + 2(\pi - \Omega)\sin\theta = 2(\pi - \Omega)(\cos\theta + \sin\theta) \geqslant 2\sigma\Omega,$$

which implies $\widehat{\phi^{\sigma,\theta}_{m,\Omega}}(\tilde{\xi}) = 0$. Also, since $2(\pi - \Omega) - 2\pi = -2\Omega$, we have $\tilde{\xi} - 2\pi k \notin (-2\Omega, 2\Omega)^2$ for $k = (1,0)$, $(0,1)$ and $(1,1)$. Thus, $\widehat{\phi^{\sigma,\theta}_{m,\Omega}}(\tilde{\xi} - 2\pi k) = 0$, for all $k \in \mathbb{Z}_2^2$. For any other $k \in \mathbb{Z}^2$, $\widehat{\phi^{\sigma,\theta}_{m,\Omega}}(\tilde{\xi} - 2\pi k) = 0$ as $\tilde{\xi} - 2\pi k$ is outside of the support of $\widehat{\phi^{\sigma,\theta}_{m,\Omega}}$. All together, we have $\widehat{\phi^{\sigma,\theta}_{m,\Omega}}(\tilde{\xi} - 2\pi k) = 0$ for all $k \in \mathbb{Z}^2$, which implies

$$[\widehat{\phi^{\sigma,\theta}_{m,\Omega}}, \widehat{\phi^{\sigma,\theta}_{m,\Omega}}](\tilde{\xi}) = 0, \text{ for } \tilde{\xi} = \big(2(\pi - \Omega), 2(\pi - \Omega)\big).$$

By Lemma 12, $\phi^{\sigma,\theta}_{m,\Omega}$ is not stable. The proof is complete.

6.6. Proof of Proposition 10

Proof. Firstly, notice that $\widehat{\phi}$ is nonnegative as $\widehat{Q_\Omega}$ is nonnegative. Together with the fact that $\widehat{\phi} \in C^n(\mathbb{R}^d)$, $|\widehat{\phi}|^2 = \widehat{\phi}^2 \in C^n(\mathbb{R}^d)$. Moreover, $\widehat{\phi}$ is compactly supported implies that $[\widehat{\phi}, \widehat{\phi}](\cdot) = \sum_{k\in\mathbb{Z}^d} |\widehat{\phi}|^2(\cdot + 2\pi k)$ is a finite summation of the sequence $\{|\widehat{\phi}|^2(\cdot + 2\pi k),\, k \in \mathbb{Z}^d\}$ in any finite interval. Thus, $[\widehat{\phi}, \widehat{\phi}] \in C^n(\mathbb{R}^d)$. Since ϕ is stable, there exists some $c > 0$ such that $[\widehat{\phi}, \widehat{\phi}] > c$. Then we have $\frac{1}{\sqrt{[\widehat{\phi}, \widehat{\phi}]}} \in C^n(\mathbb{R}^d)$ and thus $\widehat{\widehat{\phi}} \in C^n(\mathbb{R}^d)$.

Secondly, given any $\widetilde{\psi} \in \widetilde{\Psi}$, to prove that $\widetilde{\psi} \in C^n(\mathbb{R}^d)$, it is sufficient to show $\widetilde{\tau} \in C^n(\mathbb{R}^d)$. Recall that the refinement mask τ of ϕ is defined as

$$\tau(\xi) = \sum_{k\in\mathbb{Z}^d} b(\xi + 2\pi k),\; \xi \in \mathbb{R}^d,$$

where $b(\xi) = \frac{\widehat{\phi}(2\xi)}{\widehat{\phi}(\xi)}$ if $\phi(\xi) > 0$ and 0 otherwise, as shown in the proof of Lemma 11. Indeed, we have that $b(\cdot) = \widehat{\phi}(2\cdot) \in C^n(\mathbb{R}^d)$ which can be proved as follows. By the definition (28) or (31) of ϕ, it is easy to see that $\widehat{\phi}(\xi) = 1$ if $\phi(2\xi) > 0$, which implies $b(\xi) = \widehat{\phi}(2\xi)$ if $\phi(2\xi) > 0$. If $\phi(2\xi) = 0$, $b(\xi) = 0$. Thus, $b(\cdot) = \widehat{\phi}(2\cdot)$. Notice that the support of $b(\xi)$ is strictly within $(-\pi, \pi)^d$, we have $\tau(\xi) \in C^n(\mathbb{R}^d)$. Since ϕ is stable, i.e. there exists a constant $c > 0$ such that $[\widehat{\phi}, \widehat{\phi}] > c$, together with $[\widehat{\phi}, \widehat{\phi}] \in C^n(\mathbb{R}^d)$, we have $\frac{\sqrt{[\widehat{\phi}, \widehat{\phi}](\xi)}}{\sqrt{[\widehat{\phi}, \widehat{\phi}](2\xi)}} \in C^n(\mathbb{R}^d)$. Putting all together, we have then $\widetilde{\tau} \in C^n(\mathbb{R}^d)$ by its definition (34). The proof is complete.

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