# DUAL GRAMIAN ANALYSIS: DUALITY PRINCIPLE AND UNITARY EXTENSION PRINCIPLE

ZHITAO FAN, HUI JI, AND ZUOWEI SHEN

ABSTRACT. Dual Gramian analysis is one of the fundamental tools developed in a series of papers [37, 40, 38, 39, 42] for studying frames. Using dual Gramian analysis, the frame operator can be represented as a family of matrices composed of the Fourier transforms of the generators of (generalized) shift-invariant systems, which allows us to characterize most properties of frames and tight frames in terms of their generators. Such a characterization is applied in the above-mentioned papers to two widely used frame systems, namely Gabor and wavelet frame systems. Among many results, we mention here the discovery of the duality principle for Gabor frames [40] and the unitary extension principle for wavelet frames [38]. This paper aims at establishing the dual Gramian analysis for frames in a general Hilbert space and subsequently characterizing the frame properties of a given system using the dual Gramian matrix generated by its elements. Consequently, many interesting results can be obtained for frames in Hilbert spaces, e.g., estimates of the frame bounds in terms of the frame elements and the duality principle. Moreover, this new characterization provides new insights into the unitary extension principle in [38], e.g., the connection between the unitary extension principle and the duality principle in a weak sense. One application of such a connection is a simplification of the construction of multivariate tight wavelet frames from a given refinable mask. In contrast to the existing methods that require completing a unitary matrix with trigonometric polynomial entries from a given row, our method greatly simplifies the tight wavelet frame construction by converting it to a constant matrix completion problem. To illustrate its simplicity, the proposed construction scheme is used to construct a few examples of multivariate tight wavelet frames from box splines with certain desired properties, e.g., compact support, symmetry or anti-symmetry.

## 1. INTRODUCTION

This paper is to build up the dual Gramian analysis for studying frames in separable Hilbert spaces. The dual Gramian analysis allows us to study various properties of frames in general Hilbert spaces, including the generator based characterization of frames, estimates of the frame bounds, and the canonical dual frame construction in terms of the dual Gramian matrix. In particular, a duality principle in Hilbert spaces is derived in this paper. The dual Gramian analysis in such a general setting also provides new insights into the unitary extension principle, which results in a simple construction scheme of multivariate tight wavelet frames from box splines.

The basic blocks of the dual Gramian analysis consists of the pre-Gramian matrix, the Gramian matrix and the dual Gramian matrix. The pre-Gramian matrix

2010 *Mathematics Subject Classification.* 42C15, Secondary 42C40, 42C30, 65T60.
The work of the authors was partially supported by Singapore MOE Research Grant R-146-000-165-112 and R-146-000-178-112.

and its adjoint matrix are the matrix representations of the synthesis and analysis operators, while the Gramian matrix and the dual Gramian matrix are the matrix representations of the compositions of the analysis and synthesis operators in different orders. These matrices provide basic tools for studying Bessel, Riesz and frame properties of a system in terms of its elements. The Gramian matrix has been widely used in the study of Riesz property and orthonormality property of a given system. However, the dual Gramian matrix is not as popular as its counterpart. The dual Gramian matrix and its associated analysis were introduced and used as the main tool in the papers [37, 40, 38, 39, 42] for studying frames and tight frames for $L_2(\mathbb{R}^d)$. As already shown in these papers, it will be shown again in this paper that the dual Gramian matrix and its associated analysis are also the right tool for studying frames in general Hilbert spaces.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. A system $X$ is a countable sequence in $\mathcal{H}$. For a given system $X$, let $\ell_2(X)$ denote the space of square summable sequences (here the sequence $X$ is used as the index set) and let $\ell_0(X)$ denote the space of sequences with finite support. The **synthesis operator** $T$ is defined as

$$T : \ell_0(X) \to \mathcal{H} : c \mapsto \sum_{x\in X} c[x]x. \tag{1.1}$$

Since $\ell_0(X)$ is dense in $\ell_2(X)$, $T$ is densely defined on $\ell_2(X)$. For $f \in \mathcal{H}$, the **analysis operator** is defined as

$$T^* : f \quad \mapsto \quad \{\langle f, x\rangle\}_{x\in X}. \tag{1.2}$$

The operator $T^*$ is defined only formally as it may not map $\mathcal{H}$ into $\ell_2(X)$. The operator $T$ is a bounded operator from $\ell_2(X)$ to $\mathcal{H}$ if and only if $T^*$ is a bounded operator from $\mathcal{H}$ to $\ell_2(X)$. The operators $T$ and $T^*$ are an adjoint pair.

A system $X$ is a **Bessel system** if and only if $T$ (hence $T^*$) is a bounded operator from $\ell_2(X)$ into $\mathcal{H}$ (from $\mathcal{H}$ to $\ell_2(X)$). The norm of the operator $\|T\|$, as well as $\|T^*\|$, is the Bessel bound of the system $X$. A Bessel system $X$ is **fundamental** if and only if $T^*$ is injective in $\mathcal{H}$, or equivalently the space spanned by $X$ is dense in $\mathcal{H}$ since the injectivity of $T^*$ is equivalent to that the range of $T$ is dense in $\mathcal{H}$. A Bessel system $X$ is **$\ell_2$-independent** if and only if $T$ is injective in $\ell_2(X)$, i.e. for $c \in \ell_2(X)$, $\sum_{x\in X} c[x]x = 0$ implies that $c = 0$, or equivalently it is invertible on its range.

A Bessel system $X$ is a **Riesz sequence** if and only if $T$ has a bounded inverse, or equivalently $T$ is bounded below. Recall that a bounded operator $T$ is bounded below on $\ell_2(X)$ if there exists a constant $A > 0$ such that $\|Tc\| \geq A\|c\|$ for all $c \in \ell_2(X)$. In short, a system $X$ forms a Riesz sequence if there exist two positive constants $A, B$ such that

$$A^2\|c\|^2 \leq \|\sum_{x\in X} c[x]x\|^2 \leq B^2\|c\|^2, \quad \text{for all } c \in \ell_2(X). \tag{1.3}$$

The largest possible constant $A$ is called the lower Riesz bound and the smallest possible constant $B$ is called the upper Riesz bound. The lower Riesz bound equals $\|T^{-1}\|^{-1}$ and the upper Riesz bound equals $\|T\|$. When $A = B = 1$, the system $X$ forms an **orthonormal sequence**. A Riesz (orthonormal resp.) sequence is called a **Riesz** (**orthonormal** resp.) **basis** for $\mathcal{H}$ if it is fundamental in $\mathcal{H}$.

A system $X$ forms a **frame** if and only if $T^*$ is bounded and has a bounded inverse, or equivalently $T^*$ is both bounded up and bounded below. In other words, a system $X$ forms a frame if there exist two positive constants $A, B$ such that

$$A^2\|f\|^2 \le \sum_{x\in X} |\langle f, x\rangle|^2 \le B^2\|f\|^2, \quad \text{for all } f \in \mathcal{H}. \tag{1.4}$$

The largest possible constant $A$ is called the lower frame bound and the smallest possible constant $B$ is called the upper frame bound. The lower frame bound equals $\|(T^*)^{-1}\|^{-1}$ and the upper frame bound equals $\|T^*\|$. When $A = B > 0$, the system $X$ is called a **tight frame** and $A$ is called the tight frame bound with default value 1 through out this paper. When a frame $X$ is $\ell_2$-independent, it becomes a **Riesz basis**. It is implied by the definition that a frame $X$ is fundamental. A system $X$ is called a **frame sequence** in $\mathcal{H}$ if it is a frame of a closed subspace of $\mathcal{H}$.

We use the Gramian matrix and the dual Gramian matrix to refer to the representations of the two self-adjoint operators $T^*T$ and $TT^*$ in matrix form. It is well known (see e.g. [37]) that the operator $T^*T$ can be used to characterize various Riesz properties of a system. For example, a system $X$ is a Bessel system if and only if $T^*T$ (and $TT^*$) are bounded operators. Furthermore, a Bessel system $X$ is $\ell_2$-independent if and only if $T^*T$ is injective; it forms a Riesz sequence if and only if $T^*T$ has a bounded inverse; and it is an orthonormal sequence if and only if $T^*T = I$. The operator $TT^*$ is the so-called frame operator which is very suitable for characterizing frames. For example, a Bessel system $X$ is fundamental if and only if $TT^*$ is injective; it is a frame if and only if $TT^*$ has a bounded inverse; and it is a tight frame if and only if $TT^* = I$.

The rest of paper is organized as follows. In Section 2, we first introduce the pre-Gramian matrix, the dual Gramian matrix and their analysis. Then we use the dual Gramian matrix to characterize frame properties, to estimate the frame bounds, and to construct the canonical dual frames. The adjoint system is introduced in Section 3 via the pre-Gramian matrix, which leads to the duality principle. In Section 4, we first briefly review the dual Gramian analysis introduced in [38] for wavelet systems. Then, using the results established in the previous sections, we present a new interpretation of the unitary extension principle, which shows the connection between the duality principle and the unitary extension principle. Such a connection leads to a new construction scheme of tight wavelet frames from a given refinement mask. In contrast to the existing construction schemes which require completing a trigonometric polynomial matrix, the proposed one only need to complete a constant matrix. This dramatically simplifies the construction of tight wavelet frames, especially for the multivariate case. In the end, a few examples of multivariate tight wavelet frames are constructed from some refinable box splines.

## 2. Dual Gramian analysis

This section starts with the introduction of the dual Gramian matrix. Then the dual Gramian matrix is applied to the analysis of frames, the construction of the canonical dual frame and the estimation of the frame bounds.

### 2.1. Dual Gramian analysis for shift-invariant systems.

The dual Gramian analysis was first established in [37] for shift-invariant systems, which is built on the matrix representation of the frame operator under a unitary transform, the Fourier transform. It is then applied in [40] for studying Gabor frames and applied in [38] for studying wavelet frames. The results demonstrate the power and convenience of the dual Gramian analysis for studying frames.

Take $t \in \mathbb{R}^d$ and $f \in L_2(\mathbb{R}^d)$, let $E^t$ be the translation operator $E^t f(x) = f(x-t)$ and let $M^t$ be the modulation operator $M^t f(x) = e^{it\cdot x} f(x)$. A system $X$ is **shift-invariant** if $X$ is a collection of integer translations of a countable set of $L_2$-functions, i.e.

$$X := \{E^k \phi \mid \phi \in \Phi, k \in \mathbb{Z}^d\},$$

where $\Phi$, called the generators, is a countable subset of $L_2(\mathbb{R}^d)$. In [37], the pre-Gramian matrix is defined in terms of the Fourier transform of the generators. The Fourier transform of $f \in L_2(\mathbb{R}^d)$ is defined as

$$\hat{f}(w) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot w} dx.$$

For each $w \in \mathbb{T}^d := [-\pi, \pi]^d$, the pre-Gramian matrix is defined as the $(2\pi\mathbb{Z}^d \times \Phi)$-matrix:

$$\mathrm{J}_\Phi(w) := (\, \hat{\phi}(w+\alpha)\, )_{\alpha,\phi}. \tag{2.1}$$

The dual Gramian analysis allows us to decompose the synthesis operator, the analysis operator and the frame operator to a collection of simple operators ("fibers"). The synthesis operator $T$ is represented in Fourier domain by

$$(\widehat{Tc}(w+\alpha))_{\alpha\in 2\pi\mathbb{Z}^d} = \mathrm{J}_\Phi(w)\hat{c}(w), \quad \text{for } c \in \ell_0(X) \text{ and a.e. } w \in \mathbb{T}^d,$$

where $\hat{c} := (\hat{c}_\phi)_{\phi\in\Phi}$ and $\hat{c}_\phi$ is the Fourier series of $c$ indexed by $E^\alpha\phi$, $\alpha \in \mathbb{Z}^d$. The collection $(\mathrm{J}^*_\Phi(w))_{w\in\mathbb{T}^d}$ is the representation of the analysis operator $T^*$, i.e.

$$\widehat{T^* f}(w) = \mathrm{J}^*_\Phi(w)(\hat{f}(w+\alpha))_{\alpha\in 2\pi\mathbb{Z}^d},$$

for $f \in L_2(\mathbb{R}^d)$ and a.e. $w \in \mathbb{T}^d$. Moreover, the frame and Riesz properties of a system $X$ can now be characterized by the properties of the columns of the fibers $\mathrm{J}_\Phi(w)$ for $w \in \mathbb{T}^d$. Roughly speaking, a shift-invariant system $X$ forms a frame (Riesz sequence resp.) if and only if the columns of the pre-Gramian matrix $\mathrm{J}_\Phi(w)$ of $X$ are frames (Riesz sequences resp.) at a.e. $w \in \mathbb{T}^d$ and the collection of fibers has uniform upper and lower bounds. The decomposition of the operator $T$ into fibers $\mathrm{J}_\Phi(\cdot)$ simplifies the analysis in many aspects.

The pre-Gramian matrix can be used to create the Gramian matrix

$$\mathrm{G}_\Phi(w) = \mathrm{J}^*_\Phi(w)\mathrm{J}_\Phi(w) = \left(\sum_{\alpha\in 2\pi\mathbb{Z}^d} \hat{\phi}'(w+\alpha)\overline{\hat{\phi}(w+\alpha)}\right)_{\phi,\phi'\in\Phi},$$

and the dual Gramian matrix

$$\tilde{\mathrm{G}}_\Phi(w) = \mathrm{J}_\Phi(w)\mathrm{J}^*_\Phi(w) = \left(\sum_{\phi\in\Phi} \hat{\phi}(w+\alpha)\overline{\hat{\phi}(w+\alpha')}\right)_{\alpha,\alpha'\in 2\pi\mathbb{Z}^d}.$$

The entries of the Gramian matrix $\mathrm{G}_\Phi(w)$ are well-defined almost everywhere as long as the generators are in $L_2(\mathbb{R}^d)$. In order to make the entries of the dual Gramian matrix well-defined almost everywhere, we need to impose the condition $\sum_{\phi\in\Phi} |\hat{\phi}(w)|^2 < \infty$ for a.e. $w \in \mathbb{R}^d$. The collection $(\mathrm{G}_\Phi(w))_{w\in\mathbb{T}^d}$ and the collection $(\tilde{\mathrm{G}}_\Phi(w))_{w\in\mathbb{T}^d}$ are used to decompose the operators $T^*T$ and $TT^*$ in the Fourier transform domain into simple fibers which are then used to characterize various properties of shift-invariant systems. For example, let the unitary operator

$$U_1 : L_2(\mathbb{R}^d) \to L_2(\mathbb{T}^d, 2\pi\mathbb{Z}^d)$$

be defined by $(U_1 f)(w, \alpha) := (2\pi)^{-d/2}\hat{f}(w + \alpha)$. Then, as proved by [37], we have

$$\|T^* f\|^2 = \int_{\mathbb{T}^d} (U_1 f)(w, \cdot)^* \tilde{\mathrm{G}}_\Phi(w)(U_1 f)(w, \cdot) dw, \tag{2.2}$$

for band-limited $f \in L_2(\mathbb{R}^d)$ (i.e. $\hat{f}$ has compact support). This leads to the characterization of Bessel systems and frames in terms of the dual Gramian matrix fibers, which are much simpler operators defined by only the generators of the system. More specifically, define

$$\Lambda(w) := \|\tilde{\mathrm{G}}_\Phi(w)\|, \qquad \lambda(w) := \|\tilde{\mathrm{G}}_\Phi(w)^{-1}\|$$

as the operator norms of $\tilde{\mathrm{G}}_\Phi(w)$ and $\tilde{\mathrm{G}}_\Phi(w)^{-1}$ at each $w \in \mathbb{T}^d$, and $\lambda(w)$ is $\infty$ if $\tilde{\mathrm{G}}_\Phi(w)$ is not invertible. The shift-invariant system $X$ is Bessel if and only if the function $\Lambda$ is essentially bounded on $\mathbb{T}^d$. Moreover, the Bessel bound is $\|\Lambda\|_{L_\infty}^{1/2}$. When $X$ is Bessel, this system is a frame if and only if the function $\lambda$ is essentially bounded on $\mathbb{T}^d$. The lower frame bound is $\|\lambda\|_{L_\infty}^{-1/2}$. The system is a tight frame if and only if $\Lambda(w) = \lambda(w) = 1$, or equivalently, $\tilde{\mathrm{G}}(w) = I$ for a.e. $w \in \mathbb{T}^d$.

Similarly, one can use the Gramian matrix to investigate the Bessel and Riesz properties of a shift-invariant system. Interested readers are referred to [37] for more details on the fiberization technique and dual Gramian analysis for shift-invariant systems.

2.2. **Definitions.** In order to define the dual Gramian matrix for a general system $X$, we need to first introduce the concept of pre-Gramian matrix. The pre-Gramian matrix of a given system $X$ represents the synthesis operator in a matrix form composed of only the elements of $X$. With this representation, one hopefully can characterize various properties of the system $X$ in terms of its elements. For a given system $X$, the key to the dual Gramian analysis is to find a pre-Gramian matrix $J_X$ that represents the synthesis operator $T$ by its elements and the corresponding adjoint $J_X^*$ satisfies the following identity with a unitary operator $U_2 : \mathcal{H} \to \ell_2$:

$$\|T^* f\|^2 = \|J_X^* U_2 f\|^2 = (U_2 f)^* \tilde{G}_X (U_2 f), \quad \text{for } f \in \mathcal{H},$$

where $\tilde{G}_X := J_X J_X^*$ denotes the dual Gramian matrix.

Depending on the properties of the given system and the associated underlying Hilbert space, there are many ways to define a pre-Gramian matrix. For example, one may define the pre-Gramian matrix via the Fourier transform of the generators for a shift-invariant system in $L_2(\mathbb{R}^d)$ as discussed in the previous section. In general, for a given system $X$ of a Hilbert space $\mathcal{H}$, the **pre-Gramian matrix** $J_X$ of $X$ associated with an orthonormal basis $\mathcal{O}$ in $\mathcal{H}$ is defined as:

$$J_X := (\langle x, e\rangle)_{e \in \mathcal{O}, x \in X}, \tag{2.3}$$

where the rows are indexed by $\mathcal{O}$ and the columns are indexed by $X$, and the $(e, x)$-entry is the inner product of $x$ with $e$.

The dual Gramian matrix is defined as $J_X J_X^*$. In order to ensure that each entry of the dual Gramian matrix is well defined, we assume in this paper that

$$\sum_{x\in X} |\langle x, e\rangle|^2 < \infty \quad \text{for all } e \in \mathcal{O}. \tag{2.4}$$

Such an assumption holds true for any orthonormal basis if $X$ is a Bessel system. The condition (2.4) also ensures that the analysis operator $T^*$ is densely defined on $\mathcal{H}$, since it is well-defined on the span of $\mathcal{O}$ into $\ell_2(X)$. Therefore, the operator $T$ is a bounded operator on its domain if and only if $T^*$ is a bounded operator on its domain. Thus, by (2.4), the system $X$ is a Bessel system if and only if $T$ (or $T^*$) is a bounded operator on its domain.

Under the assumption (2.4), the **dual Gramian matrix** of $X$ associated with $\mathcal{O}$ is defined as

$$\tilde{G}_X := J_X J_X^* = \left(\sum_{x\in X}\langle e', x\rangle\langle x, e\rangle\right)_{e,e'}. \tag{2.5}$$

The **Gramian matrix** of $X$ is defined as:

$$G_X := J_X^* J_X = \left(\sum_{e\in\mathcal{O}}\langle x', e\rangle\langle e, x\rangle\right)_{x,x'} = (\langle x', x\rangle)_{x,x'}. \tag{2.6}$$

The entries of the Gramian matrix are well defined, since $\mathcal{O}$ is an orthonormal basis of $\mathcal{H}$. The last equality in (2.6), which follows again from the fact that $\mathcal{O}$ is an orthonormal basis of $\mathcal{H}$, shows that this definition coincides with the traditional definition of Gramian matrix of a given system $X$. Hence, the definition of the Gramian matrix is independent of the choice of the orthonormal basis $\mathcal{O}$.

For a shift-invariant system $X = \{E^k\phi : \phi \in \Phi \subset L_2(\mathbb{R}^d), k \in \mathbb{Z}^d\}$, the two definitions of pre-Gramian matrices, (2.1) and (2.3), are closely related. It is shown in [21] that if the orthonormal basis $\mathcal{O}$ used in (2.3) is

$$\{(2\pi)^{-d/2}E^k M^\alpha \hat{\chi}_{\mathbb{T}^d}(-\cdot) : k \in \mathbb{Z}^d, \alpha \in 2\pi\mathbb{Z}^d\},$$

the pre-Gramian matrix (2.3) of $X$ is then

$$J_X = (2\pi)^{-d}(\langle E^{k'}\phi, E^k M^\alpha \hat{\chi}_{\mathbb{T}^d}(-\cdot)\rangle)_{(k\in\mathbb{Z}^d,\alpha\in 2\pi\mathbb{Z}^d),(k'\in\mathbb{Z}^d,\phi\in\Phi)},$$

where $\chi_{\mathbb{T}^d}$ is the characteristic function of $\mathbb{T}^d$. Then, for any sequence $c \in \ell_0(X)$, the Fourier series of $J_X c$ will be the same as the pre-Gramian (2.1) evaluated at $\hat{c}$, i.e.

$$(J_X c)^\wedge(w) = \mathrm{J}_\Phi(w)\hat{c}(w), \text{ for a.e. } w \in \mathbb{T}^d.$$

Interested readers are referred to [21] for more details.

2.3. **Analysis.** In order to link the dual Gramian matrix $\tilde{G}_X$ to the frame operator, we need the synthesis operator corresponding to the orthonormal basis $\mathcal{O}$ of $\mathcal{H}$, which is the unitary operator

$$U : \ell_2(\mathcal{O}) \mapsto \mathcal{H} : c \mapsto \sum_{e\in\mathcal{O}} c[e]e.$$

The corresponding adjoint operator is the analysis operator

$$U^* : f \mapsto \{\langle f, e\rangle\}_{e\in\mathcal{O}}.$$

The unitary operator $U$ maps the sequence space $\ell_2(\mathcal{O})$ to $\mathcal{H}$ and the adjoint operator $U^*$ maps $\mathcal{H}$ to the sequence space $\ell_2(\mathcal{O})$. Using this unitary operator

$U$, the link between the pre-Gramian matrix of $X$ and the synthesis operator of $X$ is stated as follows.

*Proposition* 2.1. Let $X \subset \mathcal{H}$ be a given system and let $\mathcal{O}$ be an orthonormal basis of $\mathcal{H}$. Assume that $X$ and $\mathcal{O}$ satisfy (2.4). Then we have

$$Tc = UJ_X c, \qquad \text{for any } c \in \ell_0(X); \tag{2.7}$$

and

$$T^* Ud = J_X^* d, \qquad \text{for any } d \in \ell_0(\mathcal{O}). \tag{2.8}$$

Consequently, $X$ is a Bessel system if and only if $J_X$ (or $J_X^*$) is bounded. The Bessel bound equals $\|J_X\| = \|J_X^*\|$. A Bessel system $X$ is a Riesz sequence (frame resp.) if and only if $J_X$ ($J_X^*$ resp.) is bounded below. The lower Riesz bound (lower frame bound resp.) equals $\|J_X^{-1}\|^{-1}$ ($\|J_X^{*-1}\|^{-1}$ resp.).

*Proof.* For any $c \in \ell_0(X)$, we have

$$UJ_X c = \sum_{e \in \mathcal{O}} \sum_{x \in X} c[x]\langle x, e\rangle e = \sum_{x \in X} c[x] \sum_{e \in \mathcal{O}} \langle x, e\rangle e = \sum_{x \in X} c[x] x = Tc.$$

In the above derivation, the sequence $\left(\sum_{x\in X} c[x]\langle x, e\rangle\right)_{e\in\mathcal{O}}$ is in $\ell_2(\mathcal{O})$ since $c \in \ell_0(X)$ and $\mathcal{O}$ is an orthonormal basis. The summation order can be changed because the summation indexed by $X$ is finite as $c \in \ell_0(X)$. To prove (2.8), for any $d \in \ell_0(\mathcal{O})$, we have

$$T^* Ud = (\langle Ud, x\rangle)_{x\in X} = \left(\langle \sum_{e\in\mathcal{O}} d[e]e, x\rangle\right)_{x\in X} = \left(\sum_{e\in\mathcal{O}} d[e]\langle e, x\rangle\right)_{x\in X} = J_X^* d.$$

Notice that $J_X^* d \in \ell_2(X)$, because $X$ and $\mathcal{O}$ satisfy (2.4). With the two relationships (2.7) and (2.8), the characterizations of various properties of the system $X$ can be transferred from the synthesis operator $T$ and the analysis operator $T^*$ to the corresponding pre-Gramian matrix $J_X$ and its adjoint $J_X^*$. Hence, the rest of the results follow from the definitions of Bessel systems, Riesz sequences or frames that are given in terms of the operator $T$ or $T^*$. $\square$

It is noted that the proof of (2.7) does not require the assumption (2.4). However, the assumption (2.4) makes the matrix-vector product $J_X c$ well-defined for any vector $c$ in $\ell_2(X)$. As a result, the matrix $J_X$ can be formally used to define an operator on $\ell_2(X)$, but it may not map to $\ell_2(\mathcal{O})$. Now we are ready to build the bridge between the dual Gramian matrix of $X$ and the frame operator of $X$.

*Proposition* 2.2. Let $X \subset \mathcal{H}$ be a system and let $\mathcal{O}$ be an orthonormal basis of $\mathcal{H}$. Assume that $X$ and $\mathcal{O}$ satisfy (2.4). Then we have

$$\langle Tc, Td\rangle = d^* G_X c, \qquad \text{for any } c, d \in \ell_0(X), \tag{2.9}$$

and

$$\langle T^* Uc, T^* Ud\rangle = d^* \tilde{G}_X c, \quad \text{for any } c, d \in \ell_0(\mathcal{O}). \tag{2.10}$$

Furthermore, $X$ is a Bessel system if and only if the Gramian matrix $G_X$ (the dual Gramian matrix $\tilde{G}_X$ resp.) defines a bounded operator of $\ell_2(X)$ ($\ell_2(\mathcal{O})$ resp.). The Bessel bound equals $\|G_X\|^{1/2} = \|\tilde{G}_X\|^{1/2}$. If the system $X$ is Bessel, we have

$$\begin{aligned} T^* Tc &= G_X c, && \text{for any } c \in \ell_2(X), \\ U^* TT^* Ud &= \tilde{G}_X d, && \text{for any } d \in \ell_2(\mathcal{O}). \end{aligned}$$

*Proof.* For any $c, d \in \ell_0(X)$, we have

$$\langle Tc, Td\rangle = \langle \sum_{x'\in X} c[x']x', \sum_{x\in X} d[x]x\rangle = \sum_{x\in X} \overline{d[x]} \sum_{x'\in X} c[x']\langle x', x\rangle = d^* G_X c.$$

Next we prove (2.10). Let $c, d \in \ell_0(\mathcal{O})$. Then $T^*Uc, T^*Ud \in \ell_2(X)$ by (2.4) and

$$\begin{aligned}\langle T^*Uc, T^*Ud\rangle &= \sum_{x\in X}\langle Uc, x\rangle\langle x, Ud\rangle = \sum_{x\in X}\langle \sum_{e'\in\mathcal{O}} c[e']e', x\rangle\langle x, \sum_{e\in\mathcal{O}} d[e]e\rangle \\ &= \sum_{e\in\mathcal{O}} \overline{d[e]} \sum_{e'\in\mathcal{O}} c[e'] \sum_{x\in X}\langle e', x\rangle\langle x, e\rangle = d^*\tilde{G}_X c.\end{aligned}$$

It follows from (2.9) and (2.10) that $T$ ($T^*$ resp.) is bounded if and only if $G_X$ ($\tilde{G}_X$ resp.) is bounded. Therefore, $X$ is Bessel if and only if $G_X$ or $\tilde{G}_X$ defines a bounded operator, and its Bessel bound equals $\|G_X\|^{1/2} = \|\tilde{G}_X\|^{1/2}$.

When a system $X$ is Bessel, both operators $T$ and $T^*$ are bounded. By (2.9), we have that $G_X$ is bounded and satisfies

$$\langle T^*Tc, d\rangle = d^*G_Xc, \quad \text{for all } c, d \in \ell_2(X).$$

Hence $G_X = T^*T$. Similarly, $U^*TT^*U = \tilde{G}_X$. □

From (2.10), by taking limit, we conclude that

$$\|T^*Uc\|^2 = c^*\tilde{G}_Xc, \quad \text{for arbitrary } c \in \ell_2(\mathcal{O}),$$

although both sides may equal to infinity for some cases. As already shown in the study of the frame properties of shift-invariant systems, the equality above plays an important role in the dual Gramian analysis. In fact, it shows that both the upper bound and the lower bound of the operator $T^*$, which is equivalent to the frame property of $X$, can be characterized by the bounds of the nonnegative Hermitian matrix $\tilde{G}_X$. Since the Bessel property has already been characterized by the upper bounds of $\tilde{G}_X$ or $G_X$, the following proposition characterizes the lower bound of frames and Riesz systems in terms of the dual Gramian and Gramian matrices.

*Proposition* 2.3. Let $X \subset \mathcal{H}$ be a Bessel system and let $\mathcal{O}$ be an orthonormal basis of $\mathcal{H}$. Then

(a) $X$ is $\ell_2$-independent if and only if $G_X$ is injective. $X$ forms a Riesz sequence if and only if $G_X$ has a bounded inverse and the lower Riesz bound is $\|G_X^{-1}\|^{-1/2}$. $X$ is an orthonormal sequence if and only if $G_X = I$.

(b) $X$ is fundamental if and only if $\tilde{G}_X$ is injective. $X$ is a frame if and only if $\tilde{G}_X$ has a bounded inverse and the lower frame bound is $\|\tilde{G}_X^{-1}\|^{-1/2}$. $X$ is a tight frame if and only if $\tilde{G}_X = I$.

*Proof.* If the system $X$ is Bessel, then by Proposition 2.2, we have

$$T^*T = G_X, \quad U^*TT^*U = \tilde{G}_X.$$

Hence (a) and (b) follow immediately from the characterization by $T^*T$ and $TT^*$. □

It can be seen that when $X$ is a Bessel system with upper bound $B$, the summation $\sum_{x\in X} |\langle x, e\rangle|^2$ is uniformly bounded by $B^2$ for all $e \in \mathcal{O}$. Hence the condition (2.4) holds. Furthermore, the elements $\{\sum_{x\in X} |\langle x, e\rangle|^2, e \in \mathcal{O}\}$ form the diagonal entries of the dual Gramian matrix $\tilde{G}_X$. Hence, the necessary condition for $X$ being a tight frame is $\sum_{x\in X}|\langle x,e\rangle|^2=1$ for all $e\in\mathcal{O}$ and it becomes sufficient when $X$ is a Bessel system with bound 1.

*Corollary* 2.4. Let $X$ be a given system in $\mathcal{H}$ and let $\mathcal{O}$ be an orthonormal basis of $\mathcal{H}$. Assume $X$ is a Bessel system of $\mathcal{H}$ with bound 1. Then the system $X$ is a tight frame if and only if

$$\sum_{x\in X}|\langle x,e\rangle|^2=1,\quad \text{for all } e\in\mathcal{O}. \tag{2.11}$$

*Proof.* The necessity part is easy to see, as each element $\sum_{x\in X}|\langle x,e\rangle|^2$ is one of the diagonal entries of $\tilde{G}_X$. For the sufficiency part, consider the sequence $c\in\ell_2(\mathcal{O})$ whose $e'$-th element has value 1 and others have value 0. Then $\tilde{G}_X c$ gives the $e'$-th column of matrix $\tilde{G}_X$. By Proposition 2.2 and the fact that $X$ is a Bessel system with bound 1, we have $\|\tilde{G}_X c\|\le 1$. Moreover,

$$\begin{aligned}\|\tilde{G}_X c\|^2 &= \|\{\sum_{x\in X}\langle e',x\rangle\langle x,e\rangle\}_{e\in\mathcal{O}}\|^2=\left(\sum_{x\in X}|\langle x,e'\rangle|^2\right)^2+\sum_{e\in\mathcal{O}\backslash\{e'\}}|\sum_{x\in X}\langle e',x\rangle\langle x,e\rangle|^2\\ &= 1+\sum_{e\in\mathcal{O}\backslash e'}|\sum_{x\in X}\langle e',x\rangle\langle x,e\rangle|^2,\end{aligned}$$

which implies that

$$\sum_{x\in X}\langle e',x\rangle\langle x,e\rangle=0,\quad \text{for } e\in\mathcal{O}\backslash\{e'\}.$$

Hence the dual Gramian matrix $\tilde{G}_X=I$, and therefore $X$ is a tight frame by Proposition 2.3. □

As a direct application of Corollary 2.4, an orthonormal sequence $X$ is clearly a Bessel system with bound 1 and it becomes an orthonormal basis if it satisfies the additional condition (2.11) for some orthonormal basis $\mathcal{O}$, i.e. $X$ is also fundamental. For a general Bessel system $X$ with bound $B$, following the same argument as Corollary 2.4, the condition $\sum_{x\in X}|\langle x,e\rangle|^2=B^2$ for all $e\in\mathcal{O}$ implies that the system is a tight frame with bound $B$, i.e. $\sum_{x\in X}|\langle f,x\rangle|^2=B^2\|f\|^2$ for all $f\in\mathcal{H}$.

The properties of a frame can also be characterized by the Gramian matrix. In general, as an operator, the Gramian matrix has a non-trivial null set for a frame system. Thus, the analysis of frame properties via the Gramian matrix needs to involve the partial inverse and its boundedness. The interested reader is referred to [37] for the details of the characterization of a frame via Gramian matrix. While the Gramian matrix is very handy for studying the Riesz and orthonormal properties of a system, the dual Gramian matrix is more convenient for studying the frame and tight frame properties of a system. We illustrate this by the following simple example in $\mathbb{C}^n$.

*Example* 2.5. Let $\mathcal{H}$ be the finite dimensional Hilbert space $\mathbb{C}^n$ and let $\{e_i\}_{i=1}^n$ denote its canonical orthonormal basis. Let $X := \{f_k\}_{k=1}^m \subset \mathbb{C}^n$. Then, the pre-Gramian matrix of $X$ is

$$J_X := \begin{pmatrix} f_1(1) & \cdots & f_k(1) & \cdots & f_m(1) \\ \vdots & & \vdots & & \vdots \\ f_1(j) & \cdots & f_k(j) & \cdots & f_m(j) \\ \vdots & & \vdots & & \vdots \\ f_1(n) & \cdots & f_k(n) & \cdots & f_m(n) \end{pmatrix}$$

which is the matrix representation of the synthesis operator associated with $X$:

$$T : \ell_2(X) \to \mathbb{C}^n : c \mapsto \sum_{k=1}^m c_k f_k.$$

Similarly, the adjoint matrix

$$J_X^* := \begin{pmatrix} \overline{f_1(1)} & \cdots & \overline{f_1(j)} & \cdots & \overline{f_1(n)} \\ \vdots & & \vdots & & \vdots \\ \overline{f_k(1)} & \cdots & \overline{f_k(j)} & \cdots & \overline{f_k(n)} \\ \vdots & & \vdots & & \vdots \\ \overline{f_m(1)} & \cdots & \overline{f_m(j)} & \cdots & \overline{f_m(n)} \end{pmatrix}$$

is the matrix representation of the analysis operator

$$T^* : \mathbb{C}^n \to \ell_2(X) : f \mapsto \{\langle f, f_k\rangle\}_{k=1}^m.$$

Its corresponding Gramian matrix and the dual Gramian matrix are

$$G_X := J_X^* J_X = (\langle f_{k'}, f_k\rangle)_{k,k'}, \quad \tilde{G}_X := J_X J_X^* = \left(\sum_{k=1}^m f_k(j)\overline{f_k(j')}\right)_{j,j'},$$

which are the matrix representations of the linear operators $T^*T$ and $TT^*$.

2.4. **Canonical dual frame.** In this section, we demonstrate the convenience brought by the dual Gramian analysis in the construction of the canonical dual frame from a given frame or in the construction of tight frames. If a system $X$ is a frame, the frame operator $S := TT^*$ is self-adjoint, positive definite and invertible. It is well-known that the system $S^{-1}X$ is also a frame and is often called the **canonical dual frame** of $X$ (see e.g. [16]). Recall that the **dual frame** of a frame $X$ is a frame $RX$ that satisfies

$$\sum_{x\in X} \langle f, x\rangle Rx = f = \sum_{x\in X} \langle f, Rx\rangle x, \text{ for all } f \in \mathcal{H}$$

where $R$ is a map from $X$ to $\mathcal{H}$. In general, for a given frame $X$, there exist many dual frames. The canonical dual frame $S^{-1}X$ is distinguished from the others by having the following property:

$$\sum_{x\in X} |\langle f, S^{-1}x\rangle|^2 \le \sum_{x\in X} |\langle f, Rx\rangle|^2.$$

Proposition 2.2 implies that the dual Gramian matrix makes the computation of the canonical dual frame feasible while the Gramian matrix makes the computation of dual Riesz basis feasible, as shown in the following proposition.

*Proposition* 2.6. Let $X$ be a frame in $\mathcal{H}$ with frame bounds $A, B$. The system $U\tilde{G}_X^{-1}U^*X$ is a frame with bounds $B^{-1}, A^{-1}$, and is the canonical dual frame of $X$.

As one can use the Gramian matrix to construct an orthonormal basis from a Riesz basis, we can use the dual Gramian matrix to construct a tight frame from a frame. Let $S^{-1/2}$ denote the inverse of the positive square root of $S$. Notice that

$$f = S^{-1/2}SS^{-1/2}f = S^{-1/2}\sum_{x\in X}\langle S^{-1/2}f, x\rangle x = \sum_{x\in X}\langle f, S^{-1/2}x\rangle S^{-1/2}x.$$

Thus, $S^{-1/2}X$ forms a tight frame.

*Proposition* 2.7. Let $X$ be a frame in $\mathcal{H}$. Let $\tilde{G}_X^{-1/2}$ denote the inverse of the positive square root of $\tilde{G}_X$. Then, the system $U\tilde{G}_X^{-1/2}U^*X$ forms a tight frame.

The following example illustrates that the computation of the dual frame becomes straightforward for the finite dimensional case.

*Example* 2.8. Let $\mathcal{H}$ be the finite dimensional Hilbert space $\mathbb{C}^n$ and let $\{e_i\}_{i=1}^n$ denote its canonical orthonormal basis. Let $X := \{f_k\}_{k=1}^m$ be a frame in $\mathbb{C}^n$. The dual Gramian matrix $\tilde{G}_X$ is Hermitian and positive definite hence invertible. Thus $\{\tilde{G}_X^{-1}f_k\}_{k=1}^m$ forms the canonical dual frame of system $\{f_k\}_{k=1}^m$.

Example 2.8 can be extended to the construction of tight frames. Let $\tilde{G}_X^{-1/2}$ denote the inverse of the positive square root of $\tilde{G}_X$, which can be found, for example, by a unitary diagonalization of the positive definite matrix $\tilde{G}_X$. Then $\{\tilde{G}_X^{-1/2}f_k\}_{k=1}^m$ is a tight frame.

2.5. **Frame bound estimation.** Dual Gramian analysis can be used to estimate the frame bounds. Let $\mathcal{I}$ be a countable index set, and let $M$ be a complex valued non-negative Hermitian matrix with its rows and columns indexed by $\mathcal{I}$. The matrix $M$ can be viewed as an operator from $\ell_2(\mathcal{I})$ to $\ell_2(\mathcal{I})$. We use the following estimates of $\|M\|$:

$$\sup_{i\in\mathcal{I}}\left(\sum_{j\in\mathcal{I}}|M(i,j)|^2\right)^{1/2} \le \|M\| \le \sup_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}}|M(i,j)|.$$

Together with Proposition 2.2, we give an estimate of the Bessel bound of a given system $X$.

*Proposition* 2.9. Let $X$ be a system in a Hilbert space $\mathcal{H}$ and $\mathcal{O}$ be an orthonormal basis of $\mathcal{H}$ such that (2.4) is satisfied.

(a) Let

$$\tilde{B}_1 : e \mapsto \sum_{e'\in\mathcal{O}}|\sum_{x\in X}\langle e', x\rangle\langle x, e\rangle|.$$

Then $X$ is a Bessel system whenever $\sup_{e\in\mathcal{O}}\tilde{B}_1(e) < \infty$ and its Bessel bound is not larger than $(\sup_{e\in\mathcal{O}}\tilde{B}_1(e))^{1/2}$.

(b) Assume that $X$ is a Bessel system, then

$$\tilde{B}_2 : e \mapsto (\sum_{e'\in\mathcal{O}}|\sum_{x\in X}\langle e', x\rangle\langle x, e\rangle|^2)^{1/2}$$

is bounded and the Bessel bound is not smaller than $(\sup_{e\in\mathcal{O}} \tilde{B}_2(e))^{1/2}$.

The lower frame bound can be obtained when the dual Gramian matrix is diagonally dominant. Recall that for a Hermitian diagonally dominant matrix $M$,

$$\|M^{-1}\| \leq \sup_{i\in\mathcal{I}} \left( |M(i,i)| - \sum_{j\in\mathcal{I}\setminus i} |M(i,j)| \right)^{-1}.$$

This leads to the following proposition.

*Proposition* 2.10. Let $X$ be a system in Hilbert space $\mathcal{H}$ and $\mathcal{O}$ be an orthonormal basis of $\mathcal{H}$ such that (2.4) is satisfied. Let

$$\tilde{b}_1 : e \mapsto \left( \sum_{x\in X} |\langle e,x\rangle|^2 - \sum_{e'\neq e} |\sum_{x\in X} \langle e',x\rangle\langle x,e\rangle| \right)^{-1}.$$

Then $X$ is a frame whenever $\sup_{e\in\mathcal{O}} \tilde{b}_1(e) < \infty$ and the lower frame bound is not smaller than $(\sup_{e\in\mathcal{O}} \tilde{b}_1(e))^{-1/2}$.

Similarly, Riesz bounds can be estimated by using the Gramian matrix $G_X$ and we omit the details here.

## 3. Duality principle

As the Gabor systems are shift-invariant systems, the dual Gramian analysis for shift-invariant systems established in [37] was first applied to study Gabor systems in [36]. One important result presented in [36] is the duality principle. The essential ingredient of the duality principle is that the dual Gramian matrix of a given Gabor system is the Gramian matrix of another Gabor system, called the adjoint system. Based on this essential observation, we introduce the adjoint system for a given system in general Hilbert spaces, which leads to a duality principle between a given system and its adjoint system.

3.1. **Duality principle for Gabor systems.** A Gabor system is defined via applying the translation and modulation operators on a window function. Given a window function $\phi \in L_2(\mathbb{R}^d)$, the **Gabor system** is defined as

$$X = (K,L)_\phi := \{E^k M^l \phi \mid k \in K, l \in L\},$$

where $K$ is a lattice defined as $A_K\mathbb{Z}^d$ with a linear invertible map $A_K : \mathbb{R}^d \to \mathbb{R}^d$, and so is $L$. The dual lattice of $K$ is defined as $\tilde{K} := \{\tilde{k} \in \mathbb{R}^d \mid \tilde{k}\cdot k \in 2\pi\mathbb{Z},\ \text{for all } k \in K\}$. These two lattices always satisfy

$$|K|\cdot|\tilde{K}| = (2\pi)^d,$$

where $|K| = |\det(A_K)|$. The number $\mathrm{den}(X) := \frac{(2\pi)^d}{|K|\cdot|L|}$ is called the density parameter of the Gabor system $(K,L)_\phi$. The adjoint system $Y$ of a Gabor system is defined as

$$Y := \mathrm{den}(X)^{1/2}(\tilde{L},\tilde{K})_\phi = \{\mathrm{den}(X)^{1/2} E^{\tilde{l}} M^{\tilde{k}} \phi \mid \tilde{l} \in \tilde{L}, \tilde{k} \in \tilde{K}\}.$$

Notice that $X$ becomes a $K$-shift-invariant system if $L_\phi := \{M^l\phi \mid l \in L\}$ is used as the generators. The fiberization technique for shift-invariant systems can be easily extended to $K$-shift-invariant systems by replacing the lattice $\mathbb{Z}^d$ by the lattice $K$. The pre-Gramian matrix of a Gabor system $X$ is a $(\tilde{K}\times L)$-matrix whose

$(k,l)$-entry is given by $|K|^{-1/2}\hat{\phi}(\cdot+k+l)$. The pre-Gramian matrix of the adjoint system $Y$ is an $(L\times\tilde{K})$-matrix whose $(l,k)$-entry is given by $|K|^{-1/2}\hat{\phi}(\cdot+l+k)$. Hence we have

$$\mathrm{J}_Y(\cdot)=\overline{\mathrm{J}_X^*}(\cdot).$$

The identity above is the essential observation made in [40] regarding the relationship between a Gabor system $X$ and its adjoint system $Y$. It states that the dual Gramian matrix of a Gabor system $X$ is (unitarily-equivalent to) the Gramian matrix of its adjoint system $Y$:

$$\mathrm{G}_Y(\cdot)=\overline{\widetilde{\mathrm{G}}_X(\cdot)}.$$

Such a connection leads to the so-called duality principle in [40]: a system $X$ is Bessel if and only if its adjoint system $Y$ is Bessel with the same Bessel bound; a Bessel system $X$ is fundamental if and only if its adjoint system $Y$ is Bessel and $\ell_2$-independent; a system $X$ is a frame if and only if its adjoint system $Y$ is a Riesz sequence and the frame bounds of $X$ coincide with the Riesz bounds of $Y$. Moreover, a system $X$ is a tight frame if and only if its adjoint system $Y$ is an orthonormal sequence. The duality principle presented in [40] was first announced in [36]. It was also obtained independently by [18] and [31] without using dual Gramian analysis, but these results lack the estimation of the frame bounds.

Here we remark that instead of using the Fourier transform of the window function, one can also build the dual Gramian matrix by using the window function. The reason is that the Fourier transform of a Gabor system is still a Gabor system with only different lattices and the Fourier transform does not change the frame property of a given system. The same technique is still applicable for estimating the frame bound via the dual Gramian matrix. In fact, most existing frame bound estimators implicitly use the dual Gramian matrix built either by the window function or by the Fourier transform of the window function. Interested readers are referred to [40] for more details.

3.2. **Duality principle for general systems.** For the adjoint system of a given Gabor system, the key observation is that its pre-Gramian matrix is the adjoint matrix of the pre-Gramian matrix of the original Gabor system. Thus the Gramian matrix of the adjoint system is the dual Gramian matrix of the original Gabor system, which in turn leads to the duality principle. This observation inspires us with the definition of the adjoint system of a general system in a Hilbert space.

*Definition* 3.1. Let $X$ be a given system in a Hilbert space $\mathcal{H}$ and let $\mathcal{O}$ be an orthonormal basis of $\mathcal{H}$ such that (2.4) holds. Let $J_X$ be the pre-Gramian matrix of $X$ defined in (2.3) associated with $\mathcal{O}$. A system $Y$ in a Hilbert space $\mathcal{H}'$ is called the **adjoint system** of $X$ , if there exists an orthonormal basis $\mathcal{O}'$ of $\mathcal{H}'$ such that $Y$ and $\mathcal{O}'$ satisfy (2.4) and the corresponding pre-Gramian matrix $J_Y$ of $Y$ associated with $\mathcal{O}'$ is the adjoint matrix of the pre-Gramian matrix $J_X$(up to unitary equivalence), i.e.

$$J_Y=V_1J_X^*V_2, \tag{3.1}$$

where $V_1$ and $V_2$ are two unitary operators.

The matrix relation (3.1) of the system $X$ and its adjoint system $Y$ can also be defined up to a complex conjugation since the characterization of Bessel, Riesz and frame property stays the same. For a given system $X$, there are many ways to construct a pre-Gramian matrix that is the same as the synthesis operator up to unitary equivalence. The definition of the adjoint system can be adapted to any given pre-Gramian matrix of the synthesis operator of the given system, which leads to different ways to define an adjoint system. The following example shows that the R-dual sequence defined in [8] is indeed an adjoint system of a given system.

*Example* 3.2 ([8]). Let $X := \{f_k\}_{k\in\mathbb{N}}$ be a system in a Hilbert space $\mathcal{H}$ and let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ such that (2.4) holds. Suppose $\{h_k\}_{k\in\mathbb{N}}$ is another orthonormal basis of $\mathcal{H}$ and define $Y := \{g_i = \sum_{k\in\mathbb{N}}\langle f_k, e_i\rangle h_k\}_{i\in\mathbb{N}}$. Then $Y$ is indeed an adjoint system of $X$. Firstly, the system $Y$ satisfies (2.4) since

$$\sum_{i\in\mathbb{N}} |\langle g_i, h_k\rangle|^2 = \sum_{i\in\mathbb{N}} |\langle f_k, e_i\rangle|^2 < \infty, \quad \text{for all } k \in \mathbb{N}.$$

Secondly, it is easy to see that

$$J_Y = (\langle g_i, h_k\rangle)_{k,i} = (\langle f_k, e_i\rangle)_{k,i} = \overline{J_X^*}.$$

Thus, the conclusion that $Y$ is an adjoint system of $X$.

The papers [8, 9] discussed whether the adjoint system of a Gabor system defined in [40] is an R-dual sequence. By our definition of the adjoint system via the pre-Gramian matrix, the link between the two systems goes back to a relationship between the synthesis operator and the analysis operator. In short, the synthesis operator of the adjoint system is the analysis operator of the given system up to unitary equivalence. This is the essence of the duality principle. For Gabor systems, the relationship between the original system and its adjoint on the operators is reduced to the relationship between a matrix and its transpose by the fiberization technique for shift-invariant systems. From this viewpoint, the duality principle presented in this paper is more general.

By Definition 3.1 and Proposition 2.1, we can conclude that the original system is Bessel if and only if its adjoint system is Bessel; the original system is a frame if and only if its adjoint system is a Riesz sequence. Moreover, by Definition 3.1, we have the following two identities:

$$\begin{cases} G_X = J_X^* J_X = V_1^* J_Y V_2^* V_2 J_Y^* V_1 = V_1^* \tilde{G}_Y V_1, \\ \tilde{G}_X = J_X J_X^* = V_2 J_Y^* V_1 V_1^* J_Y V_2^* = V_2 G_Y V_2^*. \end{cases}$$

In other words, the dual Gramian matrix of a system is the Gramian matrix of its adjoint system counterpart up to unitary equivalence. Applying Proposition 2.3 to a given system $X$ and its adjoint system $Y$, we immediately have the duality principle between these two systems.

*Theorem* 3.3. Let $X$ be a given system in $\mathcal{H}$, and suppose that $Y$ is its adjoint system in $\mathcal{H}'$ as defined in Definition 3.1. Then

(a) A system $X$ is Bessel in $\mathcal{H}$ if and only if its adjoint system $Y$ is Bessel in $\mathcal{H}'$ with the same Bessel bound.
(b) A Bessel system $X$ is $\ell_2$-independent if and only if its adjoint system $Y$ is Bessel and fundamental.
(c) A system $X$ forms a frame in $\mathcal{H}$ if and only if its adjoint system $Y$ forms a Riesz sequence in $\mathcal{H}'$. The frame bounds of $X$ coincide with the Riesz bounds of $Y$.
(d) A system $X$ forms a tight frame in $\mathcal{H}$ if and only if its adjoint system $Y$ forms an orthonormal sequence in $\mathcal{H}'$.

Since the adjoint system of the adjoint system is the original system itself, the role of $X$ and $Y$ in the above theorem is inter-changable. The duality principle for the sequence pair in Example 3.2, i.e. the sequence $\{f_k\}_{k\in\mathbb{N}}$ and its R-dual sequence $\{g_i\}_{i\in\mathbb{N}}$ in [8], follows immediately from Theorem 3.3. Indeed, understanding the results in [8] from the viewpoint of dual Gramian analysis is one of the motivations of this paper. In Example 3.2, the Hilbert spaces of the original system and its adjoint system are the same. But the Hilbert space $\mathcal{H}'$ might be different from $\mathcal{H}$ as we will see in the next example in a finite dimensional Hilbert space.

*Example* 3.4. Proceeding with Example 2.5, let $X := \{f_k\}_{k=1}^m \subset \mathbb{C}^n$. The pre-Gramian matrix is

$$J_X = \begin{pmatrix} f_1(1) & \cdots & f_k(1) & \cdots & f_m(1) \\ \vdots & & \vdots & & \vdots \\ f_1(j) & \cdots & f_k(j) & \cdots & f_m(j) \\ \vdots & & \vdots & & \vdots \\ f_1(n) & \cdots & f_k(n) & \cdots & f_m(n) \end{pmatrix}.$$

By Definition 3.1, the rows of $J_X$ form the adjoint system of $X$. Notice that the rows are the elements in $\mathbb{C}^m$, a space different from $\mathbb{C}^n$. The duality principle for the finite case can be understood in terms of matrix terminology. The columns are fundamental (equivalent to be a frame) if and only if the rows are linearly independent (equivalent to be a Riesz sequence). The columns form a tight frame if and only if the rows form an orthonormal sequence.

The observation in Example 3.4 can be used to introduce the duality principle for a shift-invariant system at each fiber. In such a case, the adjoint system of a shift-invariant system at each fiber is formed by the rows of the pre-Gramian matrix $\mathrm{J}(w)$ of the given column system on each $w \in \mathbb{T}^d$. In general, the collection of rows of the fibers might not be generated as the pre-Gamian of another shift-invariant system. In other words, the adjoint system may not have a simple form. However, the dual Gramian analysis converts the analysis of frame properties by the whole system to the analysis on fibers. Such a weak form of duality principle is still of some interest. However, it is possible to find an adjoint system with explicit definition in certain cases, e.g. the adjoint system of a Gabor system is still a Gabor system as discussed in Section 3.1.

## 4. Unitary Extension Principle

The dual Gramian analysis, established for shift-invariant systems in [37], is used in [38] for studying wavelet frames. The main idea is to define a shift-invariant system from the given wavelet system, the so-called quasi-affine system. It is shown in [38] that the (tight) frame property of a wavelet system is equivalent to the (tight) frame property of its quasi-affine system counterpart. Thus, the dual Gramian analysis can be carried out for the quasi-affine system to obtain a complete characterization of (tight) frame property of its corresponding wavelet system in terms of its generators.

In [38], such a characterization of wavelet frames in terms of their generators via the dual Gramian analysis is applied to a special class of wavelet frames generated from a multiresolution analysis (MRA). Then, in some special cases, the huge dual Gramian matrix can be factorized through the MRA such that the dual Gramian matrix is reduced to a finite order matrix in terms of the masks associated with the refinable function and wavelets. This leads to the so-called unitary extension principle (UEP) first presented in [38]. The UEP greatly simplifies the construction of tight wavelet frames, particularly for the univariate case. For example, using univariate B-splines of any order as the refinable functions, a family of B-spline tight wavelet frames is constructed in [38].

The UEP also reduces the study of the tight frame property of an entire wavelet system to the study of one-level perfect reconstruction property of discrete decomposition and reconstruction. Such a discrete property, in a sense, can be viewed as the tight frame property of a system formulated by the masks associated with the MRA wavelet system in $\ell_2(\mathbb{Z}^d)$. The dual Gramian analysis of such system in $\ell_2$ provided in this paper brings new insights into the UEP, and one of them is the connection between the UEP and the duality principle. This connection leads to a new construction scheme of multivariate tight wavelet frames from box splines with many desired properties. For example, the supports of the constructed wavelets are small, which are not larger than that of the associated refinable function in MRA. All wavelets are symmetric or anti-symmetric. The number of wavelets is relatively small compared to the number of wavelets obtained from the tensor product of univariate B-spline framelets in [38].

The 2D tensor product of univariate B-spline wavelet frames has been widely used in many image restoration tasks, e.g., image inpainting [2, 19], image denoising [7, 44], image enhancement [30], and image deblurring [4, 5, 6]. The 3D tensor product of B-spline wavelet frames also has been used for 3D reconstruction task in electronic microscopy [33].

Interested readers are referred to [20, 43] for a detailed review of MRA-based tight wavelet frames and their applications. Moreover, as pointed out in [3], the widely used total variation based approach for image restorations can be approximated by a special case of the tight wavelet frame based approach. Using tensor product tight frames is convenient for the computation of frame decomposition/reconstruction, but it may be limited for certain applications in image processing since many types of images are non-separable multi-dimensional data. So far, the existing non-separable tight wavelet frames are not as widely used as the tensor product B-spline tight wavelet frames. One possible reason is that they lack certain desired properties including small support, symmetry/anti-symmetry, and relatively small number of wavelets. We hope that the examples of the multivariate box spline tight wavelet frames constructed in this paper will inspire some new applications that benefit from the nice properties of multivariate box spline tight wavelet frames.

4.1. **Wavelet frames.** A **wavelet system** $X \subset L_2(\mathbb{R}^d)$ is a collection of functions of the form

$$X = X(\Psi) := \bigcup_{k\in\mathbb{Z}} D^k E(\Psi) \tag{4.1}$$

where $\Psi$ is a finite subset of $L_2(\mathbb{R}^d)$, $E(\Psi)$ is the set of the integer translations of the functions in $\Psi$, and $D^k$ is the dilation operator $D^k : f \mapsto 2^{kd/2}f(2^k\cdot)$. Notice that wavelet systems are not shift-invariant, since the shift lattice becomes coarser for decreasing and negative dilation parameter $k$. In order to apply the dual Gramian analysis for shift-invariant systems established in [37], a quasi-affine system is introduced in [38]. For a given wavelet system $X$, the quasi-affine system $X^q$ is a shift-invariant system generated by adding in $2^{-dk}-1$ functions

$$E^{\gamma}D^k\psi(\cdot - j)$$

at each dilation level $k<0$, and for each $\psi\in\Psi$, $j\in\mathbb{Z}^d$, where each entry of the non-zero $\gamma\subset\mathbb{Z}^d$ takes values in $\{0,1,2,\ldots,2^{-k}-1\}$. The dual Gramian matrix of this shift-invariant system at $w\in\mathbb{T}^d$ is given by

$$\tilde{G}(w)=\left(\sum_{\psi\in\Psi}\sum_{k=\kappa(\alpha-\beta)}^{\infty}\hat{\psi}(2^k(w+\alpha))\overline{\hat{\psi}(2^k(w+\beta))}\right)_{\alpha,\beta\in2\pi\mathbb{Z}^d}, \tag{4.2}$$

where $\kappa$ denote the dyadic valuation

$$\kappa:\mathbb{R}^d\to\mathbb{Z}^d : w\mapsto\inf\{k\in\mathbb{Z}:2^kw\in2\pi\mathbb{Z}^d\}.$$

It is proven in [38] that the wavelet system $X$ is a frame if and only if the quasi-affine system $X^q$ is a frame and these two systems have the same frame bounds. Therefore, the frame property of the wavelet system $X$ is completely characterized by the dual Gramian matrix (4.2). Particularly, the wavelet system $X$ forms a tight frame if and only if the quasi-affine system $X^q$ forms a tight frame, i.e. the wavelet system $X$ is a tight frame if and only if $\tilde{G}(w)(\alpha,\beta)=\delta_{\alpha,\beta}$ for $\alpha,\beta\in2\pi\mathbb{Z}^d$. In fact, using the same method as Section 2.5, we can obtain many wavelet frame bounds estimates via the dual Gramian matrix. Furthermore, the oversampling theory for the wavelet frame can also be obtained by the observation that the sub-matrix of the dual Gramian matrix of the wavelet system still preserves the same operator bounds as the dual Gramian matrix. Interested readers are referred to [38, 42] for more details.

When the wavelet system is generated by an MRA, the dual Gramian matrix defined in (4.2) can be factorized further through the MRA to a finite order matrix under some mild assumptions; see e.g. [38, Theorem 6.5] for a complete characterization of MRA-based tight wavelet frames. One special case of the characterization is the UEP. An MRA starts with a refinable function. Recall that a function $\phi\in L_2(\mathbb{R}^d)$ is called a **refinable function** if

$$\hat{\phi}(2\cdot)=\hat{a}_0\hat{\phi} \tag{4.3}$$

for some $a_0\in\ell_2(\mathbb{Z}^d)$ where $\hat{a}_0$ is the Fourier series of $a_0$. The sequence $a_0$ or its Fourier series $\hat{a}_0$ is called the **refinement mask** of $\phi$.

Let $V_0$ be the closed linear span of $E(\phi)$. Recall that the sequence of spaces $\{V_k=D^k(V_0),\ k\in\mathbb{Z}\}$ forms an MRA if (i) $V_k\subset V_{k+1}$; (ii) $\cup_kV_k$ is dense in $L_2(\mathbb{R}^d)$ and (iii) $\cap_kV_k=\{0\}$. If $\phi\in L_2(\mathbb{R}^d)$ is refinable and $\hat{\phi}$ is continuous at 0 with $\hat{\phi}(0)\neq0$, then $\{V_k,\ k\in\mathbb{Z}\}$ forms an MRA (see e.g. [20]). With such an MRA in hand, the wavelets $\Psi:=\{\psi_l\}_{l=1}^r\subset L_2(\mathbb{R}^d)$ are then defined as

$$\hat{\psi}_l(2\cdot)=\hat{a}_l\hat{\phi} \tag{4.4}$$

for some $a_l\in\ell_2(\mathbb{Z}^d)$. The sequence $a_l$ or its Fourier series $\hat{a}_l$ is called the **wavelet mask** and the function $\psi_l\in\Psi$ is called a **wavelet**. For simplicity, we assume that the refinable function $\phi$ is compactly supported with $\hat{\phi}(0)=1$ and its masks are finitely supported. Interested readers are referred to [38] for the UEP under a weaker condition on the refinable function and the masks. The UEP of [38] for MRA tight wavelet frames is stated as follows.

*Corollary* 4.1 ([38]). Let $\phi$ be a compactly supported refinable function with $\hat{\phi}(0) = 1$ and the refinement mask $a_0$ is finitely supported. Let $\Psi = \{\psi_l\}_{l=1}^r$ be the wavelets with finitely supported wavelet masks $\{a_l\}_{l=1}^r$. Denote $\hat{a} := (\hat{a}_l)_{l=0}^r$. If a.e. $w \in \mathbb{R}^d$, and $\nu \in 2^{-1}\mathbb{Z}^d/\mathbb{Z}^d$,

$$\sum_{l=0}^{r} \hat{a}_l(w)\overline{\hat{a}_l(w + 2\pi\nu)} = \delta_\nu,$$

then the wavelet system $X(\Psi)$ is a tight frame.

The UEP in Corollary 4.1 can be expressed by the unitary property of the matrix $H$ defined by the masks $\{a_l\}_{l=0}^r$ as

$$(4.5) H(w) := \begin{pmatrix} \hat{a}_0(w + 2\pi\nu_1) & \hat{a}_1(w + 2\pi\nu_1) & \dots & \hat{a}_r(w + 2\pi\nu_1) \\ \hat{a}_0(w + 2\pi\nu_2) & \hat{a}_1(w + 2\pi\nu_2) & \dots & \hat{a}_r(w + 2\pi\nu_2) \\ \vdots & \vdots & & \vdots \\ \hat{a}_0(w + 2\pi\nu_{N_0}) & \hat{a}_1(w + 2\pi\nu_{N_0}) & \dots & \hat{a}_r(w + 2\pi\nu_{N_0}) \end{pmatrix},$$

where $w \in \mathbb{T}^d$ and $\nu_j \in 2^{-1}\mathbb{Z}^d/\mathbb{Z}^d$ for $j = 1, 2, \dots, N_0 = 2^d$. The UEP essentially says that a wavelet system $X(\Psi)$ is a tight frame system if $H(w)H^*(w) = I$ for a.e. $w \in \mathbb{T}^d$. In other words, for MRA tight wavelet frames, the huge dual Gramian matrix (4.2) is reduced to a finite order matrix $H$. In general, the dual Gramian matrix defined in (4.2) can be factorized through the MRA to a finite order matrix under some assumptions. In some special cases, it becomes $H(\cdot)H^*(\cdot)$ with some additional conditions. Interested readers are referred to [38] for more details.

In addition, the UEP condition also reduces the tight frame property of a wavelet system of infinitely many levels to the tight frame property of its masks in the space of $\ell_2$ sequences. In fact, the UEP condition for the tight wavelet frame is equivalent to the single-level perfect reconstruction property of the decomposition and reconstruction for sequences. We will elaborate it more in the next section.

4.2. **UEP condition and dual Gramian analysis.** The UEP condition given in Corollary 4.1 is expressed in terms of the wavelet masks in the Fourier domain, which actually can be viewed as the tight frame property of the masks in $\ell_2(\mathbb{Z}^d)$. For simplicity, we restrict to the case that all masks are finitely supported as we are only interested in constructing compactly supported tight wavelet frames. It is shown in [13, 24, 26] that the condition of the UEP on the masks can be re-written as:

$$(4.6) \qquad 2^d \sum_{l=0}^{r} \sum_{k\in\mathbb{Z}^d} \overline{a_l(n + 2k + \ell)} a_l(2k + \ell) = \delta_n, \text{ for any } n, \ell \in \mathbb{Z}^d.$$

Next, we show that the condition (4.6) is also equivalent to the statement that the dual Gramian matrix of the system generated by the masks in $\ell_2(\mathbb{Z}^d)$ is the identity matrix. Such a view of the UEP condition provides a simple construction scheme of MRA tight wavelet frames.

Let $\mathcal{H} := \ell_2(\mathbb{Z}^d)$. For a given set of finitely supported masks $a_l \in \mathcal{H}$, $l \in \mathbb{Z}_{r+1} := \{0, 1, \dots, r\}$, define the system

$$(4.7) \qquad X := \{f_{l,k} := (2^{d/2} a_l(n - 2k))_{n\in\mathbb{Z}^d} \mid l \in \mathbb{Z}_{r+1}, k \in \mathbb{Z}^d\} \subset \mathcal{H}.$$

Note that $X$ is generated by the 2-shifts of the given masks in $\mathcal{H}$ and $\mathcal{H}' := \ell_2(X) = \ell_2(\mathbb{Z}_{r+1} \times \mathbb{Z}^d)$. The associated synthesis operator $T : \mathcal{H}' \to \mathcal{H}$ is

$$Tc = \sum_{l \in \mathbb{Z}_{r+1}} \sum_{k \in \mathbb{Z}^d} c(l,k) f_{l,k} = 2^{d/2} \sum_{l \in \mathbb{Z}_{r+1}} \sum_{k \in \mathbb{Z}^d} a_l(\cdot - 2k) c(l,k), \tag{4.8}$$

for any $c \in \mathcal{H}'$ and the analysis operator is

$$T^* v = \{\langle v, f_{l,k} \rangle\}_{(l,k) \in \mathbb{Z}_{r+1} \times \mathbb{Z}^d}, \text{ for } v \in \mathcal{H}, \tag{4.9}$$

where

$$(T^* v)(l,k) = 2^{d/2} \sum_{n \in \mathbb{Z}^d} \overline{a_l(n-2k)} v(n), \text{ for } l \in \mathbb{Z}_{r+1}, k \in \mathbb{Z}^d.$$

Let $\mathcal{O}$ be the canonical orthonormal basis of $\ell_2(\mathbb{Z}^d)$. Then the pre-Gramian matrix of the system $X$ is the $\mathbb{Z}^d \times (\mathbb{Z}_{r+1} \times \mathbb{Z}^d)$ matrix:

$$J_X := (\ 2^{d/2} a_l(n-2k)\ )_{n,(l,k)}. \tag{4.10}$$

Hence the associated dual Gramian matrix is

$$\tilde{G}_X = J_X J_X^* = 2^d \left( \sum_{l \in \mathbb{Z}_{r+1}} \sum_{k \in \mathbb{Z}^d} a_l(n-2k) \overline{a_l(n'-2k)} \right)_{n,n'}. \tag{4.11}$$

The UEP condition (4.6) is equivalent to that the dual Gramian matrix $\tilde{G}_X$ (4.11) is the identity matrix, which in turn is equivalent to that the system $X$ defined in (4.7) forms a tight frame for $\ell_2(\mathbb{Z}^d)$. In other words, if the system $X$ defined in (4.7) is a tight frame in $\ell_2(\mathbb{Z}^d)$, the underlying wavelet system $X(\Psi)$ generated by the wavelet masks $\{a_l\}_{l=1}^r$ forms a tight frame in $L_2(\mathbb{R}^d)$. In summary, for MRA tight wavelet frames, the UEP reduces the tight frame property of the system $X(\Psi)$ to the tight frame property of a much simpler system $X$ defined by (4.7). Such a connection between the UEP and the tight frame property of the system $X$ in $\ell_2(\mathbb{Z}^d)$ is also discussed in [24].

Notice that the analysis operator (4.9) represents the discrete wavelet decomposition algorithm through $J_X^*$ and the synthesis operator (4.8) represents the discrete wavelet reconstruction algorithm through $J_X$. The tight frame property of $X$ defined in (4.7) is equivalent to the perfect reconstruction property of the one-level discrete wavelet decomposition and reconstruction algorithms. The one-level perfect reconstruction property guarantees the tight frame property of the wavelet system $X(\Psi)$. This is the beauty of the UEP. Here we remark that although the UEP condition (4.6) guarantees that the system $X$ defined in (4.7) forms a tight frame for $\ell_2(\mathbb{Z}^d)$ and the wavelet system $X(\Psi)$ is a tight frame for $L_2(\mathbb{R}^d)$, it does not guarantee that the shifts of all elements in $\Psi$ form a tight frame in the space spanned by them.

Here is a simple observation on the connection between the UEP and the duality principle. A re-examination on the pre-Gramian (4.10) shows that its columns are the elements of $X$ defined in (4.7). If we define the rows of the pre-Gramian (4.10) as the adjoint system $Y \subset \mathcal{H}'$ of $X$, then $X$ is a tight frame in $\mathcal{H}$ if and only if its adjoint system $Y$ is an orthonormal sequence in $\mathcal{H}'$. This is the duality principle. On the other hand, when using the UEP to construct tight wavelet frames, the columns of $J_X$ (4.10) are unknown except for the ones determined by the refinement mask $a_0$. Thus, one construction scheme is to complete the pre-Gramian matrix $J_X$ by filling the missing columns generated by wavelet masks, so that all the columns of

$J_X$, i.e. the system $X$, form a tight frame in $\mathcal{H}$. However, as we will show next, sometimes it is easier to fill the missing entries which are related to the unknown wavelet masks in the rows so that all rows form an orthonormal sequence. In other words, we propose to construct a system satisfying the UEP condition (4.6) via constructing an adjoint system that forms an orthonormal sequence in $\mathcal{H}'$. In the next section, we show how to use such an idea to construct multivariate tight wavelet frames in $L_2(\mathbb{R}^d)$.

4.3. **Tight wavelet frame constructions via constant matrix completion.** As discussed above, the UEP-based construction of a tight wavelet frame $X$ defined as in (4.7) is about completing the pre-Gramian matrix (4.10) with only $(0,n)$ columns available so that the columns of the completed pre-Gramian matrix form a tight frame in $\ell_2(\mathbb{Z}^d)$. From the definition of the pre-Gramian matrix (4.10), the adjoint system $Y \subset \mathcal{H}'$ of the system $X \subset \mathcal{H}$ given in (4.7) is

$$Y := \{(2^{d/2}a_l(n-2k))_{(l,k)\in\mathbb{Z}_{r+1}\times\mathbb{Z}^d} |\ n \in \mathbb{Z}^d\}.$$

The structure of the adjoint system $Y$ will be clearer by re-ordering the columns of the pre-Gramian matrix $J_X$ as follows. Based on the masks $\{a_l\}_{l=0}^r$, the columns of $J_X$ are re-ordered by grouping different $a_l$'s (omitting the shift) together so that the pre-Gramian matrix $J_X$ is formed by shifts of a small block matrix given by

$$\mathcal{A} = \begin{pmatrix} a_0(n_1) & a_0(n_2) & \cdots & a_0(n_N) \\ a_1(n_1) & a_1(n_2) & \cdots & a_1(n_N) \\ \vdots & \vdots & & \vdots \\ a_r(n_1) & a_r(n_2) & \cdots & a_r(n_N) \end{pmatrix} \tag{4.12}$$

where $n_i \in \mathbb{Z}^d,\ i = 1, 2, \ldots, N$, is the coordinate that contains the support of masks $\{a_l\}_{l=0}^r$. Here we assume that all masks $\{a_l\}_{l=0}^r$ are finitely supported.

In the 1D case, by reordering the columns of $J_X$ or equivalently reordering the rows of $J_X^*$, the matrix $J_X^*$ can be expressed as a block-wise matrix generated by the even-integer shifts of the block matrix $\overline{\mathcal{A}}$, i.e.

$$J_X^*(n,k) = \begin{cases} 2^{d/2}\overline{\mathcal{A}_{k-2n}}, & 1 \le |k-2n| \le N; \\ 0, & \text{otherwise}, \end{cases}$$

where $\overline{\mathcal{A}_j}$ denotes the conjugate of the $j$-th column of the matrix $\mathcal{A}$. In other words, each block of the matrix $2^{-d/2}J_X^*$ is the same as the block matrix $\overline{\mathcal{A}}$ shifted to the right by two columns:

$$l\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \overline{\mathcal{A}_1} & \overline{\mathcal{A}_2} & \overline{\mathcal{A}_3} & \overline{\mathcal{A}_4} & \cdot & \cdot & \cdots & \overline{\mathcal{A}_N} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \overline{\mathcal{A}_1} & \overline{\mathcal{A}_2} & \overline{\mathcal{A}_3} & \overline{\mathcal{A}_4} & \cdots & \cdot & \overline{\mathcal{A}_{N-1}} & \overline{\mathcal{A}_N} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \overline{\mathcal{A}_1} & \overline{\mathcal{A}_2} & \cdots & \cdot & \cdot & \cdot & \overline{\mathcal{A}_{N-1}} & \overline{\mathcal{A}_N} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Thus, each element in $Y$, the adjoint system of $X$ defined in (4.7), is formed by concatenating different masks entries that lie in the same $2\mathbb{Z}$-coset of an index $n$. It is also the same for the higher-dimensional case, i.e., each element in $Y$ is formed by concatenating different masks entries that lie in the same $2\mathbb{Z}^d$-coset of an index $n$. The duality principle stated in Theorem 3.3, as well as the UEP, guarantees that the system $X$ is a tight frame if and only if the adjoint system $Y$ is an orthonormal sequence. Thus, we can find the remaining masks $\{a_l\}_{l=1}^r$ that satisfy the UEP from a given mask $a_0$ by imposing some simple condition on the constant matrix $\mathcal{A}$ to ensure the adjoint system $Y$ satisfies the orthonormality condition.

*Theorem* 4.2. Let $\{a_l\}_{l=0}^r$ be a set of finitely supported masks and let $\mathcal{A}$ denote the matrix formed by $\{a_l\}_{l=0}^r$ as (4.12). Suppose that the columns of $\mathcal{A}$ are pairwise orthogonal and

$$\sum_{l=0}^{r}\sum_{n\in\Omega_j}|a_l(n)|^2 = 2^{-d}, \tag{4.13}$$

where $j \in \mathbb{Z}_2^d := 2\mathbb{Z}^d/\mathbb{Z}^d$ and $\Omega_j = 2\mathbb{Z}^d + j$. Then, the masks $\{a_l\}_{l=0}^r$ satisfy the UEP condition (4.6).

*Proof.* For the case $n \neq 0$, we have

$$2^d\sum_{l=0}^{r}\sum_{k\in\mathbb{Z}^d}\overline{a_l(n+2k+\ell)}a_l(2k+\ell) = 2^d\sum_{k\in\mathbb{Z}^d}\sum_{l=0}^{r}\overline{a_l(n+2k+\ell)}a_l(2k+\ell) = 0$$

for any $\ell \in \mathbb{Z}^d$, since the columns of $\mathcal{A}$ are pairwise orthogonal. For the case $n = 0$ and $\ell \in \mathbb{Z}_2^d$, we have

$$\sum_{l=0}^{r}\sum_{k\in\mathbb{Z}^d}\overline{a_l(n+2k+\ell)}a_l(2k+\ell) = \sum_{l=0}^{r}\sum_{k\in\mathbb{Z}^d}|a_l(2k+\ell)|^2 = \sum_{l=0}^{r}\sum_{k\in\Omega_\ell}|a_l(k)|^2 = 2^{-d}.$$

□

Inspired by Theorem 4.2, we propose a constant matrix completion scheme for constructing multivariate tight wavelet frames from box splines. Recall that the refinement mask from any box spline, denoted by $a_0$, has the following two properties: (i) all entries of $a_0$ are nonnegative; (ii) it satisfies

$$\sum_{n\in\Omega_j} a_0(n) = 2^{-d}, \tag{4.14}$$

where $j \in \mathbb{Z}_2^d$ and $\Omega_j = (2\mathbb{Z}^d + j) \cap \operatorname{supp}(a_0)$. The second property is equivalent to $\hat{a}_0(0) = 1$ and $\hat{a}_0(j\pi) = 0$ for $j \in \mathbb{Z}_2^d\backslash\{0\}$, which is a necessary condition to generate a tight framelet filter bank $\{a_0; a_1, \ldots, a_r\}$ from such a refinement mask.

By Theorem 4.2, the construction of tight wavelet frames is reduced to the completion of the matrix $\mathcal{A}$ with the first row given by $a_0$. In the following, we present a matrix completion scheme for completing the matrix $\mathcal{A}$ provided that $a_0$ satisfies certain conditions.

*Construction* 4.3. Suppose we have a refinement mask $a_0$ with only nonnegative entries and satisfying (4.14).

- **Step 1** (initialization): define the first row of the matrix $\mathcal{A}$ by collecting only non-zero entries of $a_0$.
- **Step 2** (normalization): define a normalized vector $\tilde{a}_0$ with $\|\tilde{a}_0\| = 1$ by taking the square root of all entries in the first row of $\mathcal{A}$.
- **Step 3** (orthogonal matrix extension): construct an orthogonal matrix $\tilde{\mathcal{A}}$ with the first row being $\tilde{a}_0$.
- **Step 4** (restoration): define the matrix $\mathcal{A}$ by multiplying each column of $\tilde{\mathcal{A}}$ with the first entry of the corresponding column.

*Remark* 4.1. (1) Construction 4.3 is a special way to construct an adjoint system $Y$ of the original system $X$ generated by the masks $a_0$, which is inspired from the connection between the idea of the duality principle and the idea of UEP. There is a lot of freedom to construct an orthogonal matrix with only its first row provided, which allows us to construct wavelet masks with desired properties. For example, if the refinement mask $a_0$ has certain symmetry properties, one may impose extra symmetry conditions on the matrix extension to generate wavelet masks with the same symmetries, as we will see later. (2) Construction 4.3 is only one possible scheme to obtain a matrix $\mathcal{A}$ that satisfies the conditions specified in Theorem 4.2. One may consider a matrix $\mathcal{A}$ with more rows than columns such that there are more wavelets. Construction 4.3 contains the minimal number of wavelet masks among all the possible constructions using Theorem 4.2.

*Theorem* 4.4. Let $a_0$ of finite support be the refinement mask of a refinable function $\phi \in L_2(\mathbb{R}^d)$ with convex compact support that generates an MRA. Assume that the entries of $a_0$ are all non-negative and $a_0$ satisfies (4.14). Let $m$ denotes the number of positive entries of $a_0$ and let $\{a_l\}_{l=1}^{m-1}$ denote second to the last row of the matrix $\mathcal{A}$ constructed by Construction 4.3. Then the masks $\{a_l\}_{l=0}^{m-1}$ satisfy the UEP condition (4.6), and the wavelet system $X(\Psi)$ generated by the corresponding wavelets $\Psi = \{\psi_l\}_{l=0}^{m-1}$ forms a tight frame in $L_2(\mathbb{R}^d)$. Moreover, we have $\text{supp}(a_l) \subset \text{supp}(a_0)$ and $\text{supp}(\psi_l) \subset \text{supp}(\phi)$ for $i = 1, \ldots, m-1$.

*Proof.* By Construction 4.3, we complete a square matrix with the first row taken by the refinement mask. Thus, exactly $m-1$ masks are generated from the remaining rows and their supports satisfy $\text{supp}(a_l) \subset \text{supp}(a_0) \subset \text{supp}(\phi)$ for $l = 1, \ldots, m-1$. Since $\text{supp}(\phi)$ is convex, hence $\text{supp}(\phi(2\cdot +k)) \subset \frac{1}{2}\text{supp}(\phi) + \frac{1}{2}\text{supp}(\phi) \subset \text{supp}(\phi)$ for $k \in \text{supp}(a_0)$. By the equivalent form of (4.4) in real domain, i.e. $\psi_l(\cdot) = 2^d \sum_{k\in\mathbb{Z}^d} a_l[k]\phi(2\cdot +k)$, we have $\text{supp}(\psi_l) \subset \text{supp}(\phi)$.

The orthogonality of the columns of $\mathcal{A}$ is guaranteed by Step 3 and Step 4. Moreover, we have

$$\sum_{l=0}^{m-1} \sum_{n\in\Omega_j} |a_l(n)|^2 = \sum_{l=0}^{m-1} \sum_{n\in\Omega_j} a_0(n)|\tilde{a}_l(n)|^2 = \sum_{n\in\Omega_j} a_0(n) = 2^{-d},$$

where the assumption of the entries $a_0$ to be nonnegative is used. According to Theorem 4.2, the masks $\{a_l\}_{l=0}^{m-1}$ generated by Construction 4.3 satisfy the UEP (4.6). Thus, by the UEP, the wavelet system $X(\Psi)$ generated by the wavelets defined from these wavelet masks forms a tight wavelet frame in $L_2(\mathbb{R}^d)$. $\square$

In the existing construction schemes, the construction of a compactly supported tight wavelet frame from a given refinement mask is reduced to a problem of completing a unitary matrix with trigonometric polynomial entries. In contrast, Construction 4.3 is only a problem of completing a constant matrix. As a result, the construction of tight wavelet frames is greatly simplified in our scheme. Such a simplification is very helpful to the construction of multivariate tight wavelet frames from box splines with desired properties, as we will show in the next section.

4.4. **Multivariate tight wavelet frames from box splines.** The construction of univariate tight wavelet frames using B-spline functions has been extensively studied during the last decades. The Fourier transform of the (centered) **B-spline** function of order $m \in \mathbb{N}$, denoted by $B_m$, is

$$\hat{B}_m(w) = e^{-ijw/2} \left( \frac{\sin(w/2)}{w/2} \right)^m$$

where $j = 0$ when $m$ is even and $j = 1$ when $m$ is odd. The B-spline function $B_m \in C^{m-2}$ is a refinable function with the refinement mask $\hat{a}_0(w) = e^{-ijw/2} \cos^m(w/2)$.

Using the UEP of [38], in total $m$ wavelets are constructed for the B-spline function $B_m$ and their wavelet masks are

$$\hat{a}_\ell(w) = -i^\ell e^{-ijw/2} \sqrt{\begin{pmatrix} m \\ \ell \end{pmatrix}} \sin^\ell(w/2) \cos^{m-\ell}(w/2), \quad \ell = 1, 2, \ldots, m.$$

These wavelets have exactly the same support as that of the refinable splines, the number of wavelets is one less than the number of the nonzero coefficients in the refinement mask, and the wavelets are either symmetric or anti-symmetric.

There have been some other methods proposed to construct univariate tight wavelet frames from B-splines. By using the UEP and trigonometric polynomial matrix completion, the construction given by [13] can have only two wavelets for B-splines of any order, and have three if certain symmetry is imposed on the wavelets. Independent of which method or which spline function is used, the approximation order of the truncated tight wavelet frames constructed by the UEP from B-splines is never greater than two. To construct spline tight wavelet frame of better approximation order, it led to the discovery of the oblique extension principle (OEP), independently discovered in [17] and [15]. By using the OEP, spline tight wavelet frames with two or three wavelets are constructed in [17] with better approximation orders than the ones constructed from the UEP. In [29, 28, 27], interesting examples of symmetric tight wavelet frames with two or three wavelets are constructed by splitting a matrix of Laurant polynomials with symmetry. Further discussions on constructing univariate tight wavelet frames with symmetric or anti-symmetric filter banks are provided in [23, 25] with emphasis on complex-valued tight frames.

The construction of non-separable tight wavelet frames by using refinable box splines first appeared in [34, 35], where exponentially decaying orthogonal wavelets are constructed. After the UEP was introduced, compactly supported tight wavelet frames from box splines are first constructed in [41]. The methods provided in [41] are applicable in general to box splines of any order, however, the support of the constructed wavelet can be large. There are also a few other construction schemes of tight wavelet frames from box splines; see e.g. [14, 22, 32, 10]. Interested readers are referred to these papers and references therein for more details. The main challenge in these construction schemes is the completion of a trigonometric polynomial matrix from one single row such that the matrix satisfies the UEP condition. By considering the case of non-negative refinement masks, a fully local construction scheme of tight frames is proposed in [12], in which the problem of polynomial matrix completion is simplified to a problem of constant matrix factorization. In other words, the construction scheme proposed in [12] only requires factorizing a positive definite constant matrix $\mathcal{R}$ by $\mathcal{R} = QQ^\top$.

Compared to those methods using trigonometric polynomial matrix completion, Construction 4.3 only needs to complete a constant matrix. Thus, it provides more flexibilities in the construction of wavelets. Indeed, we can construct multivariate wavelet frames from box splines with the same properties as their univariate counterpart, although they do not have a unified explicit formula. The matrix factorization based method [12] is the closest one to Construction 4.3 as both only need to work with constant matrices. However, these two approaches are conceptually different. The approach proposed in [12] studied the structure of the UEP expressed by a trigonometric polynomial matrix, and converted the polynomial matrix completion problem to a matrix factorization problem. In contrast, Construction 4.3 is based on the intrinsic connection between the duality principle and the UEP, which leads to a constant matrix completion approach of constructing tight wavelet frames. Based on the same connection, we can also construct wavelet bi-frames in a scheme similar to Construction 4.3; see [21] for more details.

In summary, all multivariate wavelet frames constructed by Construction 4.3 have the following properties: the supports of wavelets are not larger than that of the box spline, wavelets and their masks are either symmetric or anti-symmetric, and the number of wavelets constructed is one less than the number of nonzero coefficients of the refinement mask. The last property implies that the number of wavelets is smaller than that of the tensor product of the univariate spline wavelets and the support of each wavelet is smaller as well. Such a property is attractive when applied on high-dimensional image data.

Given a set of directions $\{\xi_j\}_{j=1}^n \subset \mathbb{Z}^d$ with multiplicity $m_j$ for each $\xi_j$, the Fourier transform of the **box spline** $\phi$ associated with the given directions is defined by

$$\hat{\phi}(w) = \prod_{j=1}^{n} \left( \frac{1 - e^{-i\xi_j \cdot w}}{i\xi_j \cdot w} \right)^{m_j}.$$

Let $L$ be the minimal number of directions $\{\xi_{j_k}\}_{k=1}^L$ whose removal from this set cannot span $\mathbb{R}^d$ anymore. Then the corresponding box spline $\phi \in C^{L-2}(\mathbb{R}^d)$. The box spline $\phi$ is refinable and the refinement mask is given by

$$\hat{a}_0(w) = \prod_{j=1}^{n} \left( \frac{1 + e^{-i\xi_j \cdot w}}{2} \right)^{m_j}.$$

Interested readers are referred to [1] for a detailed introduction to box splines.

*Example* 4.5. Consider the linear bivariate box spline with the following three directions:

$$\{\xi_1, \xi_2, \xi_3\} = \{(1,0)^\top,\ (0,1)^\top,\ (1,1)^\top\}.$$

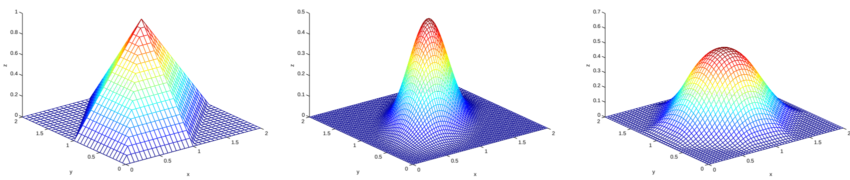
The multiplicity is 1 for each direction. The graph of the function is plotted in (a) of Figure 1. The refinement mask of this box spline is

$$a_0 = \frac{1}{8} \begin{bmatrix} 1 & 1 & \\ 1 & 2 & 1 \\ & 1 & 1 \end{bmatrix} \tag{4.15}$$

with empty entries to be 0. Construction 4.3 gives the following six wavelet masks:

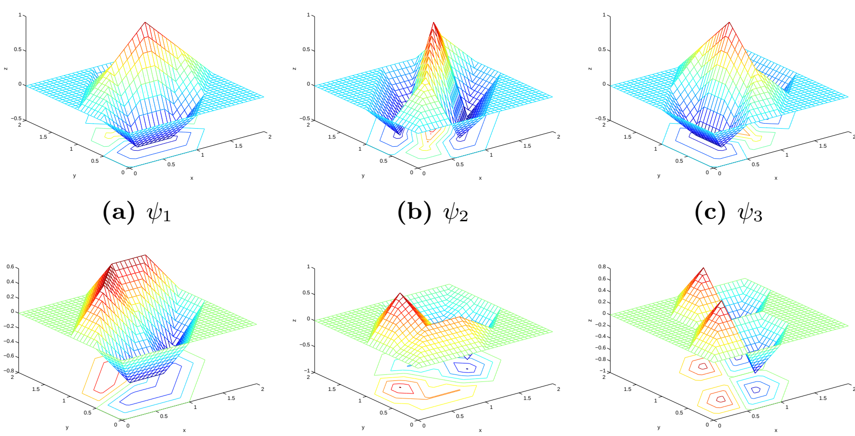
$$\tfrac{1}{8}\begin{bmatrix} -1 & -1 & \\ 1 & 2 & 1 \\ & -1 & -1 \end{bmatrix}, \quad \tfrac{1}{8}\begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix}, \quad \tfrac{1}{8}\begin{bmatrix} -1 & 1 & \\ -1 & 2 & -1 \\ & 1 & -1 \end{bmatrix},$$

$$\tfrac{\sqrt{3}}{12}\begin{bmatrix} -1 & -1 & \\ 1 & 0 & -1 \\ & 1 & 1 \end{bmatrix}, \quad \tfrac{\sqrt{6}}{24}\begin{bmatrix} 1 & 1 & \\ 2 & 0 & -2 \\ & -1 & -1 \end{bmatrix}, \quad \tfrac{\sqrt{2}}{8}\begin{bmatrix} 1 & -1 & \\ 0 & 0 & 0 \\ & 1 & -1 \end{bmatrix}.$$



**(a)** $\phi_1$ **(b)** $\phi_2$ **(c)** $\phi_3$

**Figure 1.** Graphs of refinable box splines used in the constructions: (a) the box spline in Example 4.5; (b) the box spline in Example 4.6; (c) the box spline in Example 4.7.



**(a)** $\psi_1$ **(b)** $\psi_2$ **(c)** $\psi_3$

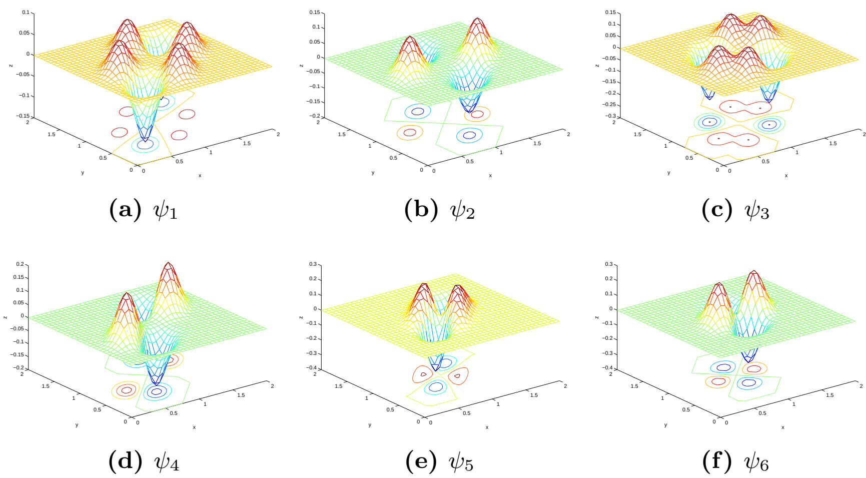
**(d)** $\psi_4$ **(e)** $\psi_5$ **(f)** $\psi_6$

**Figure 2.** Graphs of the six wavelets constructed from box spline of three directions with multiplicity one in Example 4.5

See Figure 2 for the graphs of the six corresponding wavelets. It is seen that the supports of the wavelet masks (wavelets resp.) are not larger than the support of the refinement mask (box spline resp.). All wavelets are either symmetric or anti-symmetric. In comparison, the number of bivariate wavelets obtained by the tensor product of linear B-spline wavelets in [38] is eight and they have larger support. The number of bivariate wavelets constructed in [14] is seven and their supports are the same as the support of box spline. In comparison, there are six wavelets in the construction of [32], and seven or six wavelets with the same support as the box spline in the construction of [12].

*Example* 4.6. Consider the box spline with the following three directions

$$\{\xi_1, \xi_2, \xi_3\} = \{(1,0)^\top,\ (0,1)^\top,\ (1,1)^\top\}.$$



**(a)** $\psi_1$ **(b)** $\psi_2$ **(c)** $\psi_3$

**(d)** $\psi_4$ **(e)** $\psi_5$ **(f)** $\psi_6$

**Figure 3.** Graphs of the six wavelets constructed from box spline of three directions with multiplicity two in Example 4.6

The multiplicity is 2 for each direction. The graph of the function is plotted in (b) of Figure 1. The refinement mask is

$$a_0 = \frac{1}{64}\begin{bmatrix} 1 & 2 & 1 & & \\ 2 & 6 & 6 & 2 & \\ 1 & 6 & 10 & 6 & 1 \\ & 2 & 6 & 6 & 2 \\ & & 1 & 2 & 1 \end{bmatrix}. \tag{4.16}$$

Construction 4.3 gives 18 wavelet masks; see Table 1 for their explicit expressions. The supports of all wavelets are not larger than that of the underlying refinable box spline. All wavelets are either symmetric or anti-symmetric. See Figure 3 for the plots of the first six wavelets. The same box spline is also used in [32] to generate seven wavelets, whose explicit expressions are provided in [11].

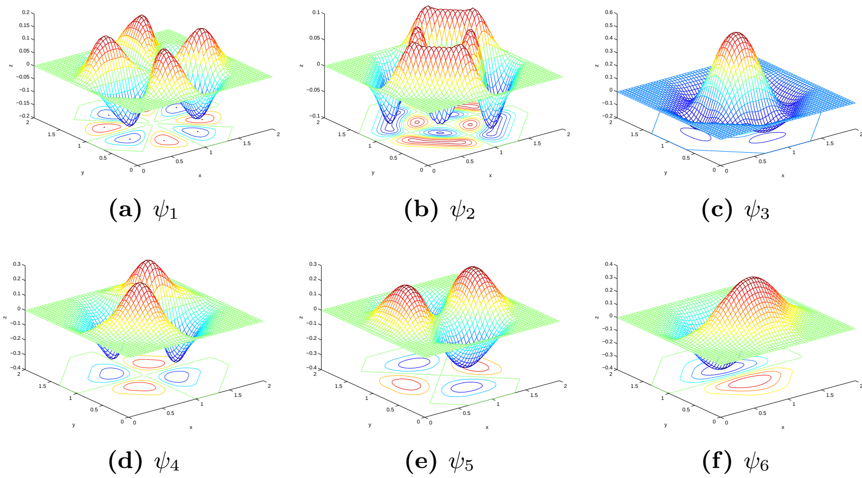
*Example* 4.7. Consider the bivariate box spline in $\mathbb{R}^2$ with the following four directions:

$$(\xi_1, \xi_2, \xi_3, \xi_4) = \{(0,1)^\top,\ (1,0)^\top,\ (1,1)^\top,\ (1,-1)^\top\}.$$

The multiplicity is 1 for each direction. The graph of the function is plotted in (c) of Figure 1. The refinement mask is

$$a_0 = \frac{1}{16}\begin{bmatrix} & 1 & 1 & \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ & 1 & 1 & \end{bmatrix}. \tag{4.17}$$

See Table 2 for the eleven wavelet masks constructed by Construction 4.3. See Figure 4 for the graphs of the first six wavelets. Again, the supports of all wavelets are not larger than the support of the underlying box spline and have various symmetries. Using the same bivariate box spline, 15 wavelets are constructed using the method proposed in [14] and six wavelets with larger support are constructed using the method in [32].



**(a)** $\psi_1$ **(b)** $\psi_2$ **(c)** $\psi_3$

**(d)** $\psi_4$ **(e)** $\psi_5$ **(f)** $\psi_6$

**Figure 4.** Graphs of five wavelets constructed from box spline of four directions with multiplicity one in Example 4.7

*Example* 4.8. Consider the refinable function the box spline in $\mathbb{R}^3$ with the following four directions:

$$(\xi_1, \xi_2, \xi_3, \xi_4) = \{(1,0,0)^\top,\ (0,1,0)^\top,\ (0,0,1)^\top,\ (1,1,1)^\top\}.$$

The multiplicity is 1 for each direction. The refinement mask is

$$a_0 = \frac{1}{16}\left[\begin{array}{ccc|ccc|ccc} 1 & 1 & & 1 & 1 & & & & \\ 1 & 1 & & 1 & 2 & 1 & & 1 & 1 \\ & & & & 1 & 1 & & 1 & 1 \end{array}\right]. \tag{4.18}$$

The above matrix is the 3D matrix aligned slice by slice by the $x$-coordinate. Construction 4.3 gives 14 wavelets; see Table 3 for their explicit expressions. When using the tensor product of univariate wavelets to construct trivariate wavelets, e.g. linear B-spline and its two wavelets, it will produce totally 26 wavelets. As a comparison, only 14 wavelets are produced with their supports no larger than the support of the box spline. The reduced number of wavelets and the relative small support of wavelet masks could benefit the applications of tight wavelet frames in high-dimensional data, in terms of both computational efficiency and memory utilization efficiency.

## References


[1] C. De Boor, K. Höllig, and S. D. Riemenschneider. *Box splines*. Springer-Verlag, New York, 1993.

[2] J. Cai, R. Chan, and Z. Shen. A framelet-based image inpainting algorithm. *Applied and Computational Harmonic Analysis*, 24:131–149, 2008.

[3] J. Cai, B. Dong, S. Osher, and Z. Shen. Image restoration: total variation, wavelet frames, and beyond. *Journal of the American Mathematical Society*, 25:1033–1089, 2012.

[4] J. Cai, H. Ji, C. Liu, and Z. Shen. Blind motion deblurring from a single image using sparse approximation. In *IEEE Conference on Computer Vision and Pattern Recognition*, Miami, 2009. IEEE.

$$
\frac{\sqrt{12}}{96}\begin{bmatrix} -2 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \\ & & 1 & 0 & -2 \end{bmatrix},\ 
\frac{1}{16}\begin{bmatrix} 0 & 0 & -1 & & \\ 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \\ & & -1 & 0 & 0 \end{bmatrix},\ 
\frac{\sqrt{6}}{48}\begin{bmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 0 & -2 & \\ 0 & 0 & 0 & 0 & 0 \\ & -2 & 0 & 0 & 1 \\ & & 0 & 1 & 0 \end{bmatrix},\ 
\frac{\sqrt{2}}{16}\begin{bmatrix} 0 & -1 & 0 & & \\ 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 0 & -1 & 0 \end{bmatrix},\ 
\frac{\sqrt{2}}{16}\begin{bmatrix} 0 & 0 & 0 & & \\ 0 & -2 & 1 & 0 & \\ 0 & 1 & 0 & 1 & 0 \\ & 0 & 1 & -2 & 0 \\ & & 0 & 0 & 0 \end{bmatrix},
$$

$$
\frac{\sqrt{6}}{16}\begin{bmatrix} 0 & 0 & 0 & & \\ 0 & 0 & -1 & 0 & \\ 0 & 1 & 0 & 1 & 0 \\ & 0 & -1 & 0 & 0 \\ & & 0 & 0 & 0 \end{bmatrix},\ 
\frac{1}{24}\begin{bmatrix} 1 & -1 & 1 & & \\ -1 & 0 & 0 & -1 & \\ 1 & 0 & 0 & 0 & 1 \\ & -1 & 0 & 0 & -1 \\ & & 1 & -1 & 1 \end{bmatrix},\ 
\frac{\sqrt{3}}{72}\begin{bmatrix} 1 & 2 & 1 & & \\ 2 & -3 & -3 & 2 & \\ 1 & -3 & 0 & -3 & 1 \\ & 2 & -3 & -3 & 2 \\ & & 1 & 2 & 1 \end{bmatrix},\ 
\frac{\sqrt{15}}{576}\begin{bmatrix} 1 & 2 & 1 & & \\ 2 & 6 & 6 & 2 & \\ 1 & 6 & -54 & 6 & 1 \\ & 2 & 6 & 6 & 2 \\ & & 1 & 2 & 1 \end{bmatrix},\ 
\frac{\sqrt{12}}{96}\begin{bmatrix} 2 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & \\ -1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \\ & & -1 & 0 & -2 \end{bmatrix},
$$

$$
\frac{1}{16}\begin{bmatrix} 0 & 0 & -1 & & \\ 0 & 0 & 0 & 0 & \\ -1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \end{bmatrix},\ 
\frac{\sqrt{6}}{48}\begin{bmatrix} 0 & 1 & 0 & & \\ -1 & 0 & 0 & -2 & \\ 0 & 0 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 1 \\ & & 0 & -1 & 0 \end{bmatrix},\ 
\frac{\sqrt{2}}{16}\begin{bmatrix} 0 & -1 & 0 & & \\ -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 0 & 1 & 0 \end{bmatrix},\ 
\frac{\sqrt{2}}{16}\begin{bmatrix} 0 & 0 & 0 & & \\ 0 & 2 & 1 & 0 & \\ 0 & -1 & 0 & 1 & 0 \\ & 0 & -1 & -2 & 0 \\ & & 0 & 0 & 0 \end{bmatrix},\ 
\frac{\sqrt{6}}{16}\begin{bmatrix} 0 & 0 & 0 & & \\ 0 & 0 & -1 & 0 & \\ 0 & -1 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 \end{bmatrix},
$$

$$
\frac{\sqrt{2}}{48}\begin{bmatrix} -1 & \sqrt{2} & 1 & & \\ -\sqrt{2} & -\sqrt{6} & \sqrt{6} & \sqrt{2} & \\ -1 & -\sqrt{6} & 0 & \sqrt{6} & 1 \\ & -\sqrt{2} & -\sqrt{6} & \sqrt{6} & \sqrt{2} \\ & & -1 & -\sqrt{2} & 1 \end{bmatrix},\ 
\frac{\sqrt{12}}{96}\begin{bmatrix} -1 & -\sqrt{2} & 1 & & \\ \sqrt{2} & 0 & 0 & -\sqrt{2} & \\ -1 & 0 & 0 & 0 & 1 \\ & \sqrt{2} & 0 & 0 & -\sqrt{2} \\ & & -1 & \sqrt{2} & 1 \end{bmatrix},\ 
\frac{1}{48}\begin{bmatrix} -1 & \sqrt{2} & 1 & & \\ -\sqrt{2} & 2\sqrt{6} & -2\sqrt{6} & \sqrt{2} & \\ -1 & 2\sqrt{6} & 0 & -2\sqrt{6} & 1 \\ & -\sqrt{2} & 2\sqrt{6} & -2\sqrt{6} & \sqrt{2} \\ & & -1 & -\sqrt{2} & 1 \end{bmatrix}.
$$

**Table 1.** List of 18 wavelet masks of Example 4.6

$$
\frac{\sqrt{2}}{16}\begin{bmatrix} & 1 & -1 & \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ & -1 & 1 & \end{bmatrix},\ 
\frac{1}{16}\begin{bmatrix} & 1 & -1 & \\ 1 & -2 & 2 & -1 \\ -1 & 2 & -2 & 1 \\ & -1 & 1 & \end{bmatrix},\ 
\frac{1}{16}\begin{bmatrix} & -1 & -1 & \\ -1 & 2 & 2 & -1 \\ -1 & 2 & 2 & -1 \\ & -1 & -1 & \end{bmatrix},\ 
\frac{1}{16}\begin{bmatrix} & 1 & -1 & \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ & -1 & 1 & \end{bmatrix},
$$

$$
\frac{\sqrt{2}}{16}\begin{bmatrix} & -1 & -1 & \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ & -1 & -1 & \end{bmatrix},\ 
\frac{1}{16}\begin{bmatrix} & 1 & 1 & \\ 1 & 2 & 2 & 1 \\ -1 & -2 & -2 & -1 \\ & -1 & -1 & \end{bmatrix},\ 
\frac{1}{16}\begin{bmatrix} & 1 & -1 & \\ 1 & -2 & 2 & -1 \\ 1 & -2 & 2 & -1 \\ & 1 & -1 & \end{bmatrix},\ 
\frac{1}{16}\begin{bmatrix} & 1 & 1 & \\ 1 & -2 & -2 & 1 \\ -1 & 2 & 2 & -1 \\ & -1 & -1 & \end{bmatrix},
$$

$$
\frac{1}{16}\begin{bmatrix} & 1 & -1 & \\ 1 & 2 & -2 & -1 \\ 1 & 2 & -2 & -1 \\ & 1 & -1 & \end{bmatrix},\ 
\frac{\sqrt{2}}{16}\begin{bmatrix} & 1 & -1 & \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ & 1 & -1 & \end{bmatrix},\ 
\frac{\sqrt{2}}{16}\begin{bmatrix} & -1 & -1 & \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ & 1 & 1 & \end{bmatrix}.
$$

**Table 2.** List of 11 wavelet masks of Example 4.7

$$
\frac{1}{8}\left[\begin{array}{cc|ccc|cc} 1 & -1 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & -1 & 1 \end{array}\right],\ 
\frac{1}{8}\left[\begin{array}{cc|ccc|cc} 0 & 0 & 0 & 0 & & & \\ 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ & & & 0 & 0 & 0 & 0 \end{array}\right],\ 
\frac{1}{8}\left[\begin{array}{cc|ccc|cc} 0 & 0 & 1 & -1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & -1 & 1 & 0 & 0 \end{array}\right]
$$

$$
\frac{1}{8}\left[\begin{array}{cc|ccc|cc} 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \end{array}\right],\ 
\frac{\sqrt{2}}{16}\left[\begin{array}{cc|ccc|cc} 1 & 1 & 0 & 0 & & & \\ -1 & -1 & 0 & 0 & 0 & -1 & -1 \\ & & & 0 & 0 & 1 & 1 \end{array}\right],\ 
\frac{\sqrt{2}}{16}\left[\begin{array}{cc|ccc|cc} 0 & 0 & 1 & 1 & & & \\ 0 & 0 & -1 & -2 & -1 & 0 & 0 \\ & & & 1 & 1 & 0 & 0 \end{array}\right],
$$

$$
\frac{1}{16}\left[\begin{array}{cc|ccc|cc} 1 & 1 & -1 & -1 & & & \\ 1 & 1 & -1 & -2 & -1 & 1 & 1 \\ & & & -1 & -1 & 1 & 1 \end{array}\right],\ 
\frac{\sqrt{14}}{56}\left[\begin{array}{cc|ccc|cc} 1 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 \\ & & & -1 & -1 & -1 & -1 \end{array}\right],\ 
\frac{1}{8}\left[\begin{array}{cc|ccc|cc} 1 & -1 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 1 & -1 \end{array}\right]
$$

$$
\frac{1}{8}\left[\begin{array}{cc|ccc|cc} 0 & 0 & 0 & 0 & & & \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 \\ & & & 0 & 0 & 0 & 0 \end{array}\right],\ 
\frac{1}{8}\left[\begin{array}{cc|ccc|cc} 0 & 0 & 1 & -1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & -1 & 0 & 0 \end{array}\right],\ 
\frac{\sqrt{2}}{16}\left[\begin{array}{cc|ccc|cc} 1 & 1 & 0 & 0 & & & \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ & & & 0 & 0 & -1 & -1 \end{array}\right]
$$

$$
\frac{\sqrt{3}}{24}\left[\begin{array}{cc|ccc|cc} 0 & 0 & 1 & 1 & & & \\ 0 & 0 & -2 & 0 & 2 & 0 & 0 \\ & & & -1 & -1 & 0 & 0 \end{array}\right],\ 
\frac{\sqrt{42}}{336}\left[\begin{array}{cc|ccc|cc} 3 & 3 & -4 & -4 & & & \\ 3 & 3 & -4 & 0 & 4 & -3 & -3 \\ & & & 4 & 4 & -3 & -3 \end{array}\right].
$$

**Table 3.** List of 14 wavelet masks of Example 4.8

[5] J. Cai, H. Ji, C. Liu, and Z. Shen. Blind motion deblurring using multiple images. *Journal of Computational Physics*, 228:5057–5071, 2009.

[6] J. Cai, S. Osher, and Z. Shen. Linearized Bregman iteration for frame-based image deblurring. *SIAM Journal on Imaging Sciences*, 2:226–252, 2009.

[7] J. Cai, S. Osher, and Z. Shen. Split bregman and frame based image restoration. *Multiscale Modeling and Simulation: A SIAM interdisciplinary Journal*, 8(2):337–369, 2009.

[8] P. G. Casazza, G. Kutyniok, and M. C. Lammers. Duality principles in frame theory. *Journal of Fourier Analysis and Application*, 10(4):383–408, 2004.

[9] P.G. Casazza, G. Kutyniok, and M.C. Lammers. Duality principles, localization of frames, and gabor theory. *Wavelets XI. Proceedings of the SPIE*, 5914:389–397, 2005.

[10] M. Charina, C.K. Chui, and W.J. He. Tight frames of compactly supported multivariate multi-wavelets. *Journal of Computational and Applied Mathematics*, 233:2044–2061, 2010.

[11] M. Charina, M. Putinar, C. Scheiderer, and J. Stöckler. Multivariate tight wavelet frames. *Constructive Approximation*, 38:253–276, 2013.

[12] M. Charina and J. Stöckler. Tight wavelet frames for irregular multiresolution analysis. *Applied and Computational Harmonic Analysis*, 25(1):98–113, 2008.

[13] C.K. Chui and W.J. He. Compactly supported tight frames associated with refinable functions. *Applied and Computational Harmonic Analysis*, 8(3):293–319, 2000.

[14] C.K. Chui and W.J. He. Construction of multivariate tight frames via Kronecker products. *Applied and Computational Harmonic Analysis*, 11(2):305–312, 2001.

[15] C.K. Chui, W.J. He, and J. Stöckler. Compactly supported tight and sibling frames with maximum vanishing moments. *Applied and Computational Harmonic Analysis*, 13(3):224–262, 2002.

[16] I. Daubechies. *Ten Lectures on Wavelets*, volume 61. SIAM, 1992.

[17] I. Daubechies, B. Han, A. Ron, and Z. Shen. Framelets: MRA-based constructions of wavelet frames. *Applied and Computational Harmonic Analysis*, 14(1):1–46, 2003.

[18] I. Daubechies, H.J. Landau, and Z. Landau. Gabor time-frequency lattices and the Wexler-Raz identity. *Journal of Fourier Analysis and Applications*, 1(4):437–478, 1994.

[19] B. Dong, H. Ji, J. Li, Z. Shen, and Y. Xu. Wavelet frame based blind image inpainting. *Applied and Computational Harmonic Analysis*, 32(2):268–279, 2012.

[20] B. Dong and Z. Shen. MRA-based wavelet frames and applications. *IAS Lecture Notes Series, Summer Program on "The Mathematics of Image Processing", Park City Mathematics Institute*, 19, 2010.

[21] Z. Fan, A. Heinecke, and Z. Shen. Duality for frames. preprint, National University of Singapore, 2014.

[22] B. Han. Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix. *J. Comput. Appl. Math.*, 155(1):43–67, 2003.

[23] B. Han. Matrix splitting with symmetry and symmetric tight framelet filter banks with two high-pass filters. *Applied and Computational Harmonic Analysis*, 35(2):200–227, 2013.

[24] B. Han. Properties of discrete framelet transforms. *Mathematical Modeling of Natural Phenomena*, 8(1):18–47, 2013.

[25] B. Han. Symmetric tight framelet filter banks with three high-pass filters. *Applied and Computational Harmonic Analysis*, 37(1):140–161, 2013.

[26] B. Han, G. Kutyniok, and Z. Shen. Adaptive multiresolution analysis structures and shearlet systems. *SIAM Journal on Numerical Analysis*, 49(5):1921–1946, 2011.

[27] B. Han and Q. Mo. Splitting a matrix of Laurent polynomials with symmetry and its application to symmetric framelet filter banks. *SIAM Journal on Matrix Analysis and Applications*, 26(1):97–124, 2004.

[28] B. Han and Q. Mo. Tight wavelet frames generated by three symmetric B-spline functions with high vanishing moments. *Proceedings of the American Mathematical Society*, 132(1):77–86, 2004.

[29] B. Han and Q. Mo. Symmetric MRA tight wavelet frames with three generators and high vanishing moments. *Applied and Computational Harmonic Analysis*, 18(1):67–93, 2005.

[30] L. Hou, H. Ji, and Z. Shen. Recovering over/under-exposed regions in photographs. *SIAM Journal on Imaging Sciences*, 6(4), 2013.

[31] A.J.E.M. Janssen. Duality and biorthogonality for Weyl-Heisenberg frames. *Journal of Fourier Analysis and Applications*, 1(4):403–436, 1994.

[32] M. J. Lai and J. Stöckler. Construction of multivariate compactly supported tight wavelet frames. *Applied and Computational Harmonic Analysis*, 21(3):324–348, 2006.

[33] M. Li, Z. Fan, H. Ji, and Z. Shen. Wavelet frame based algorithm for 3d reconstruction in electron microscopy. *SIAM Journal on Scientific Computing*, 36(1):B45–B69, 2014.

[34] S.D. Riemenschneider and Z. Shen. Box splines, cardinal series, and wavelets. *Approximation Theory and Functional Analysis*, pages 133–149, 1991.

[35] S.D. Riemenschneider and Z. Shen. Wavelets and pre-wavelets in low dimensions. *Journal of Approximation Theory*, 71(1):18–38, 1992.

[36] A. Ron and Z. Shen. Frames and stable bases for subspaces of $L_2(\mathbb{R}^d)$: the duality principle of Weyl-Heisenberg sets. In M. Chu, R. Plemmons, D. Browns, and D. Ellison, editors, *Proceedings of the Lanczos Centenary Conference*, pages 422–425, Raleigh, NC, 1993.

[37] A. Ron and Z. Shen. Frames and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$. *Canadian Journal of Mathematics*, 47:1051–1094, 1995.

[38] A. Ron and Z. Shen. Affine system in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator. *Journal of Functional Analysis*, 148:408–447, 1997.

[39] A. Ron and Z. Shen. Affine systems in $L_2(\mathbb{R}^d)$ II: dual systems. *Journal of Fourier Analysis and applications*, 3:617–637, 1997.

[40] A. Ron and Z. Shen. Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$. *Duke Mathematical Journal*, 89:237–282, 1997.

[41] A. Ron and Z. Shen. Compactly supported tight affine spline frames in $L_2(\mathbb{R}^d)$. *Mathematics of Computation*, 67(221):191–207, 1998.

[42] A. Ron and Z. Shen. Generalized shift-invariant systems. *Constructive Approximation*, 22(1):1–45, 2005.

[43] Z. Shen. Wavelet frames and image restorations. In *Proceedings of the International Congress of Mathematics*, volume IV, pages 2834–2863, Hyderabad, India, 2010.

[44] Z. Shen, K.C. Toh, and S. Yun. An accelerated proximal gradient algorithm for frame-based image restoration via the balanced approach. *SIAM Journal on Imaging Sciences*, 4(2):573–596, 2011.



Department of Mathematics, National University of Singapore,10, Lower Kent Ridge Road, Singapore 119076, Singapore

*E-mail address*: `a0030819@nus.edu.sg`

Department of Mathematics, National University of Singapore,10, Lower Kent Ridge Road, Singapore 119076, Singapore

*E-mail address*: `matjh@nus.edu.sg`

Department of Mathematics, National University of Singapore,10, Lower Kent Ridge Road, Singapore 119076, Singapore

*E-mail address*: `matzuows@nus.edu.sg`