Multivariate Compactly Supported Fundamental Refinable Functions, Duals and Biorthogonal Wavelets †

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Abstract: In areas of geometric modeling and wavelets, one often needs to construct a compactly supported refinable function ϕ with sufficient regularity which is fundamental for interpolation (that means, $\phi(0) = 1$ and $\phi(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \backslash \{0\}$).

Low regularity examples of such functions have been obtained numerically by several authors and a more general numerical scheme was given in [RiS1]. This paper presents several schemes to construct compactly supported fundamental refinable functions with higher regularity directly from a given continuous compactly supported refinable fundamental function ϕ. Asymptotic regularity analyses of the functions generated by the constructions are given. The constructions provide the basis for multivariate interpolatory subdivision algorithms that generate highly smooth surfaces.

A very important consequence of the constructions is a natural formation of pairs of dual refinable functions, a necessary element in the construction of biorthogonal wavelets. Combined with the biorthogonal wavelet construction algorithm for a pair of dual refinable functions given in [RiS2], we are thus able to obtain symmetric compactly supported multivariate biorthogonal wavelets with arbitrarily high regularity. Several examples are computed.

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1. Introduction

Let M be an $s \times s$ **dilation** matrix, that is, an $s \times s$ integer matrix whose eigenvalues lie outside the closed unit disk. The compactly supported distribution ϕ is M-**refinable** with finite **mask** sequence a, when ϕ satisfies the refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} |\det M| a(\alpha) \phi(M \cdot - \alpha) \tag{1.1}$$

for a finitely supported sequence a. When $M = 2I_{s\times s}$, we simply say ϕ is refinable. In most of the applications, ϕ should be a function with some regularity (smoothness). If ϕ is continuous and equal to 1 at the origin and 0 at the other integers:

$$\phi(\alpha) = \delta_\alpha := \begin{cases} 1, & \text{if } \alpha = 0; \\ 0, & \text{if } \alpha \in \mathbb{Z}^s \backslash \{0\}, \end{cases} \tag{1.2}$$

then ϕ is called **fundamental** (because any sequence of data can be interpolated at the lattice by an infinite linear combination of the shifts of ϕ with the data as coefficients). All the essential information about the M-refinable function ϕ is carried in the refinement mask a.

The construction of compactly supported fundamental M-refinable functions with desirable regularity properties is motivated from needs arising in geometric modeling and the application of wavelet theory to signal processing (as well as in many other applications of wavelets and biorthogonal wavelets). It is also the logical starting point for the construction of biorthogonal wavelets. The generator for the wavelets and its dual function is in some sense a "splitting" of a fundamental M-refinable function with the regularity of the resulting pair at least partially determined by the original function.

Examples in the univariate case were first given by [Du]. Daubechies in [D] obtained a general construction and also provided a good asymptotic regularity analysis. For higher dimensions this analysis has been limited to a few cases that can be reduced to a univariate setting.

The goal of the present paper is to carry out the following program: **(i)** "Iterate" a fundamental M-refinable function of known smoothness to obtain a smoother fundamental M-refinable function. An asymptotic analysis of the regularity of the new function can be given in terms of the regularity of the original function and the number of steps taken. The latter quantity can be related in turn to other quantities, such as the length of the refinement mask. **(ii)** "Split" the new function into a pair of dual functions, for purposes of generating biorthogonal wavelets in such a way that the original function is one of the pair. This provides compactly supported M-refinable functions of arbitrary regularity which are dual to a given compactly supported fundamental M-refinable function. Combined with the biorthogonal wavelet construction for a pair of dual refinable functions given in [RiS2], we are able to obtain symmetric compactly supported multivariate biorthogonal wavelets with arbitrarily high regularity.

We next motivate the construction of such functions by a brief description of how they arise in geometric modeling and signal processing. After that we provide the basic notation and Fourier analysis formulations that will be used in the sequel. The remainder of the introductory section will discuss more fully earlier work and the connections between fundamental M-refinable functions and biorthogonal wavelets. In §2, we will give the implementable constructions alluded to above, while the asymptotic analysis of regularity for these constructions is derived in §3. How the constructions can be used for the formation of dual pairs of functions is the topic of §4. Finally, in §5 we will illustrate our methods by examples.

In geometric modeling, subdivision algorithms are defined by refinement masks. For simplicity, assume that the dilation matrix is $M = 2I$. Subdivision algorithms begin with some initial set of discrete data, $c := \{c_\alpha : \alpha \in \mathbb{Z}^s\}$, called control points, which can be visualized as the vertices of a given polyhedral surface. The subdivision operator S is defined by the mask a as

$$S(c)_\alpha := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - 2\beta) c_\beta,\ \alpha \in \mathbb{Z}^s. \tag{1.3}$$

This gives 2^s sets of new control points

$$c^1_{\nu+2\alpha} = \sum_{\beta \in \mathbb{Z}^s} a(\nu + 2\alpha - 2\beta) c_\beta,\ \nu \in \{0,1\}^s$$

Thus, the new control point sequence c^1 is determined linearly from c by 2^s different convolution rules, and the sequence c^1 consists of 2^s different copies of the original control point sequence c which are averaged by different dyadic cosets of the mask. With the scaling factor 2, the new control polygon is parameterized so that the points c^1_α correspond to the finer grid $2^{-1}\mathbb{Z}^s$. Continuing this process, we get control point sequences $c^n = S^n c$ corresponding to the grids $2^{-n}\mathbb{Z}^s$. If the mask a is well chosen, these data sets will approach some continuous limiting surface in a computationally stable manner. The subdivision operator can be visualized as an operator which smoothes the corners of a given polyhedral surface.

If the mask a comes from a fundamental refinable function, then from (1.1) and (1.2) we find $a(2\beta) = \delta_\beta$. Thus, $c^1_{2\beta} = c(\beta)$, and the new control point sequence interpolates the previous one on the grid $\mathbb{Z}^s$. More generally, the sequences $c^n = S^n c$ corresponding to the grids $2^{-n}\mathbb{Z}^s$ interpolate the previous control points c^{n-1} on the grids $2^{n-1}\mathbb{Z}^s$. This iterative process is called an interpolatory subdivision algorithm. The limiting surface can be written as

$$\sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\phi(\cdot - \alpha),$$

and thus has the same regularity as the function ϕ. That is, smoother surfaces require smoother fundamental refinable functions. We refer the reader to [CDM] for the details.

As mentioned earlier, the construction of interpolatory subdivision algorithms is directly connected to construction of orthogonal and biorthogonal wavelets which provides the second motivation described next. We refer to [St], [M] and [RiS2] for the more details.

Biorthogonal wavelets start with the construction of a **dual pair** of compactly supported refinable functions, ϕ and ϕ^d in $L_2(\mathbb{R}^s)$ such that

$$\langle \phi, \phi^d(\cdot - \alpha)\rangle = \delta_\alpha; \quad \alpha \in \mathbb{Z}^s. \tag{1.4}$$

To construct orthogonal wavelets, ϕ should be equal to ϕ^d.

The functions ϕ and ϕ^d are then used to construct biorthogonal wavelets (cf. §4). If ϕ and ϕ^d are compactly supported, then the wavelet families they generate will also be compactly supported and will have the same regularity as their respective generator. When the wavelet transform is used to analyze signals, the compactness gives localization in the time domain, while the regularity provides localization in the frequency domain. Ideal wavelets would possess high regularity and small support. However, the Heisenberg uncertainty principle asserts the contrary; higher regularity leads to larger support. A balance between the time-frequency localization requirements is often needed. By constructing wavelets with larger support to increase the regularity, we lose accuracy in time domain to gain accuracy in frequency domain. Given one of the dual refinable functions ϕ or ϕ^d, we may wish to choose the approximate regularity of the other, which is precisely what our method allows.

The construction of dual refinable functions, or what is the same, the choice of two masks a and a^d, can be used to design a pair of (biorthogonal) low pass filters. The corresponding construction of the wavelet masks from a and a^d given in §4 can be used to construct the biorthogonal high pass filters. The filters constructed by the methods in this paper are linear phase (symmetric) filters. Wavelet construction and filter bank design in signal processing are interrelated subjects; see [SN] and [VH].

We now turn to the Fourier analysis formulations needed in the paper. A compactly supported continuous function ϕ is fundamental if and only if

$$\sum_{\alpha\in\mathbb{Z}^s} \widehat{\phi}(\omega + 2\pi\alpha) = 1. \tag{1.5}$$

Applying the Fourier transform to the refinement equation (1.1) give the relation

$$\widehat{\phi}(\omega) = \hat{a}({M^t}^{-1}\omega)\widehat{\phi}({M^t}^{-1}\omega), \tag{1.6}$$

where $\hat{a}(\omega) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\exp(-i\alpha\omega)$ and ${M^t}^{-1}$ is the inverse of the transpose M^t of the matrix M. We also call $\hat{a}$ the mask (sometimes it is called the **symbol** of the mask) and often find it convenient to write it as a Laurent polynomial

$$A(z) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)z^\alpha.$$

When $\hat{a}(0) = 1$, then there is a unique compactly supported distributional solution of refinement equation (1.1) with $\widehat{\phi}(0) = 1$. The Fourier transform of the solution ϕ of the M-refinement equation (1.1) can be obtained as the infinite product

$$\widehat{\phi}(\omega) := \prod_{j=1}^{\infty} \hat{a}({M^t}^{-j}\omega). \tag{1.7}$$

Let ϕ be a compactly supported fundamental M-refinable function with the mask $\hat{a}$. Condition (1.5) has consequences for the mask $\hat{a}$ which result from applying (1.6). Before describing this result, we need some notation. The integers $\mathbb{Z}^s$ can be decomposed into the disjoint sets (cosets) $\{\nu + M^t\mathbb{Z}^s\}$, $\nu \in \mathbb{Z}^s_{M^t}$, where $\mathbb{Z}^s_{M^t} = \mathbb{Z}^s/\big(M^t\mathbb{Z}^s\big)$. Combining (1.6) and (1.5) using this decomposition of the sum, we find that if the M-refinable compactly supported continuous function ϕ is fundamental, then its mask $\hat{a}$ satisfies

$$\tag{1.8} \begin{gathered} \sum_{\nu\in\mathbb{Z}^s_{M^t}} \hat{a}\Big(\omega + 2\pi {M^t}^{-1}\nu\Big) = 1,\ \omega \in \mathbb{R}^s, \quad \text{or, equivalently,} \\ \sum_{\nu\in\mathbb{Z}^s_{M^t}} A(\zeta_\nu z) = 1, \quad |z| = 1, \end{gathered}$$

where

$$\zeta_\nu := \exp\big(-2\pi i {M^t}^{-1}\nu\big).$$

Condition (1.8) is called the **interpolatory condition** for the mask on $\mathbb{Z}^s_{M^t}$.

It was shown in [LLS1] that a continuous refinable function ϕ is fundamental if and only if ϕ is stable and its mask, $\hat{a}$, satisfies the interpolatory condition (1.8). Here, we recall that ϕ is **stable**, when $\phi \in L_2(\mathbb{R}^s)$ and its shifts form a Riesz basis of $S(\phi)$, where $S(\phi)$ is the shift invariant subspace in $L_2(\mathbb{R}^s)$ generated by ϕ. A function $\phi \in L_2(\mathbb{R}^s)$ is stable if and only if the inequality

$$\tag{1.9} c \le \sum_{\alpha\in\mathbb{Z}^s} |\widehat{\phi}(\omega + 2\pi\alpha)|^2 \le C; \quad \text{a.e.} \quad \omega \in \mathbb{R}^s$$

holds for some constants $0 < c \le C \le \infty$. A compactly supported distribution ϕ is **pre-stable**, if the inequality

$$c \le \sum_{\alpha\in\mathbb{Z}} |\widehat{\phi}(\omega + 2\pi\alpha)|^2$$

holds for some constant $0 < c$, or equivalently, if $\widehat{\phi}$ has no 2π-periodic zeros. If a compactly supported pre-stable function is in $L_2(\mathbb{R}^s)$, then it is stable.

The construction of compactly supported fundamental functions starts with the construction of masks that satisfy the interpolatory condition (1.8). Since the masks considered here are Laurent polynomials, the problem of constructing masks to satisfy the interpolatory condition (1.8) can be reduced to a problem of solving a system of equations. However, this is not enough for the corresponding refinable function to be fundamental as the following examples will show that the interpolatory condition (1.8) is only a necessary condition. The first simple example shows that the corresponding refinable function can be stable and have a mask that satisfies the interpolatory condition (1.8), but may not be continuous.

Example 1.10. *Let $\hat{a}(\omega) = \frac{(1+\exp(-i\omega))}{2}$. Then $\hat{a}$ satisfies the interpolatory condition (1.8) and the corresponding refinable function is the characteristic function on $[0, 1]$, which is not continuous.*

The next example shows that the mask can satisfy the interpolatory condition and the corresponding refinable function can be very smooth, however, it may not be fundamental.

Example 1.11. *Let $\hat{a} := \cos^2(3\omega/2)$. Then $\hat{a}$ satisfies the interpolatory condition (1.8). Since the corresponding refinable function ϕ is the autocorrelation function of the characteristic function of $[0,3]$, it is not stable, hence not fundamental. Further, let*

$$\hat{a}_k := \hat{a}_{k-1}^2(3 - 2\hat{a}_{k-1}), \ k \geq 0$$

where $\hat{a}_0 = \hat{a}$. Then, by Theorem 2.15 of [JiS], $\hat{a}_k$ satisfies the interpolatory condition (1.8) and the regularity of the corresponding refinable function ϕ_k of the mask $\hat{a}_k$ can be made as high as we wish by taking k sufficiently large. However, the function ϕ_k cannot be fundamental, since $\widehat{\phi_k}$ has $\widehat{\phi_0} = \widehat{\phi}$ as a factor which implies that ϕ_k is not stable.

The procedure commonly used in the literature to construct examples of compactly supported fundamental refinable functions is:

(i) Solve a system of equations derived from the interpolatory condition (1.8) to obtain a mask satisfying (1.8).

(ii) Check the stability and calculate the regularity of the corresponding refinable function via the transition operator.

This procedure faces the following difficulties: First, except in very few cases, the masks obtained from the system of equations are numerical in nature and have no explicit closed form. Secondly, after the numerical solutions are obtained, one still needs to check whether the corresponding refinable functions are fundamental and to determine their regularity. A general method of construction should not only build the masks to satisfy the interpolatory condition (1.8), but also should provide the analysis of (a) whether the resulting functions are fundamental and (b) the asymptotic regularity.

In this regard, there are few such constructions available. Daubechies in [D] obtained such a general construction for the univariate case where each mask is explicitly given as the convolution of the mask of a B-spline with a mask of a refinable distribution. It is this construction that leads to the general construction of compactly supported orthogonal wavelets, and later to the construction of compactly supported biorthogonal wavelets (see [CDF]). This construction also is the basis of the bivariate constructions in [CD] and [HL], where the bivariate problem is reduced to a univariate problem to which the Daubechies' construction can be applied directly. Strictly bivariate examples of continuous and continuously differentiable compactly supported bivariate fundamental refinable functions were obtained by [DDD] and [DGL] respectively. A set of examples of compactly supported bivariate fundamental refinable functions with increasing regularity was provided in [RiS1] from masks formed by the convolution of box spline masks with masks of refinable distributions. A construction was given in [HJ2] for examples of continuous fundamental refinable functions with various optimal properties.

Several iterative methods for the construction of compactly supported fundamental refinable functions were obtained in [JiS] in their construction of univariate biorthogonal wavelets from a multiresolution generated by fundamental refinable function. In §2, the ideas of [JiS] are more fully developed to obtain methods for the construction of compactly supported fundamental M-refinable functions for any dilation matrix and in any number of variables starting from a given compactly supported M-refinable fundamental function. The examples mentioned in the previous paragraph provide a sufficient number of starting points to justify our methods.

For dual pairs of M-refinable functions, ϕ and ϕ^d in $L_2(\mathbb{R}^s)$, we require that the functions be stable and satisfy the biorthgonality relation (1.4). Then the function ϕ^d in $L_2(\mathbb{R}^s)$ is called a **dual function** to ϕ. Often, one of these two functions is given. For a given stable function $\phi \in L_2(\mathbb{R}^s)$, it is possible to find noncompactly supported dual refinable functions ϕ^d, but the construction of compactly supported dual refinable functions requires more as proved in [BR]: A compactly supported function has a compactly supported (not necessary refinable) dual function if and only if its shifts are **linearly independent**, that is, if and only if for an arbitrary sequence a, the equality

$$\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(\cdot-\alpha)=0$$

implies that $a=0$. The linear independence of the function $\phi \in L_2(\mathbb{R}^s)$ and its shifts is a stronger requirement than the stability of ϕ.

One method for the construction of a dual pair of refinable functions is as follows: First, find a finitely supported mask $\hat{a}$ whose corresponding compactly supported refinable function φ is fundamental. To obtain a dual pair, one factors the mask $\hat{a}$ in an appropriate way and then separates the factors into two masks. In the univariate case, when φ is fundamental, the mask $\hat{a}$ always has the mask of a B-spline as a factor and this method leads to the construction of the biorthogonal wavelets in [CDF]. Further, in the univariate case, when $\hat{a}=\hat{a}(-\cdot)$ and is non-negative, one can separate the mask into identical pairs of factors to obtain a refinable function with orthonormal shifts. That is the case in Daubechies' construction of compactly supported orthonormal wavelets.

For the multivariate case, we do not have factorization of the mask any more. In [CS], an example of biorthogonal bivariate wavelets with dilation matrix $M=2I$ was constructed where one basis is continuous piecewise linear polynomials while its dual basis is a compactly supported L_2 function found by solving linear equations. In [RiS2], pairs of dual refinable functions are obtained by separating the masks from [RiS1] where $\hat{a}$ are the products of box splines masks with trigonometric polynomials. Thus, one set of wavelets consisted of finite linear combinations of box splines while various dual functions of differing smoothness were obtained. In some cases, the construction of compactly supported M-refinable dual functions can be reduced to the problem of the construction of compactly supported fundamental M-refinable functions as follows: If the refinable M-function ϕ has linearly independent shifts, then one finds a compactly supported M-refinable function $\phi^d \in L_2(\mathbb{R}^s)$, so that the function $\varphi=\phi*\phi(-\cdot)^d$ is fundamental. Then ϕ^d is a dual function of ϕ. In this regard, our particular methods will lead naturally to the construction of dual functions, as we will show in §4.

2. Methods for Constructions of Masks

Suppose we have a fundamental M-refinable function in hand, there are two questions we wish to address here. (1) If the smoothness of the given refinable function is not high enough, can we "iterate" in some fashion to obtain a smoother one from it while still preserving the interpolatory property for the new mask? One likely way to obtain higher smoothness is to obtain higher powers of the given mask as a factor of the mask obtained through the iterative procedure. This would require controlling the remaining factor so as to not detract too much from the advantage gained. If such is the case, then there is hope that after removing one power of the original mask from the new mask, what remains will be a mask which answers the second question: (2) Can we find a dual function with smoothness as high as we wish?

The constructions of new compactly supported M-refinable fundamental functions from a given one and the construction of a dual refinable function given below are based on the following simple idea: Let P_0 be the mask symbol $\hat{a}$ of the given fundamental M-refinable function ϕ. Define

$$P_\nu(z) := P_0(\zeta_\nu z), \qquad \nu \in \mathbb{Z}^s_{M^t}, \quad |z| = 1. \tag{2.1}$$

Then the interpolatory condition (1.8) becomes

$$\begin{gathered} \sum_{\nu \in \mathbb{Z}^s_{M^t}} P_\nu(z) = 1, \quad |z| = 1, \text{ which implies} \\ \Big(\sum_{\nu \in \mathbb{Z}^s_{M^t}} P_\nu(z) \Big)^{mN} = \sum_{|\gamma| = mN} \Big(C^\gamma_{mN} \prod_{\mu \in \mathbb{Z}^s_{M^t}} P_\mu^{\gamma_\mu}(z) \Big) = 1, \quad |z| = 1 \end{gathered} \tag{2.2}$$

for any integer N and m. In particular, we take $m = |\det M|$. Since we mainly consider this polynomial on the torus, we shall abuse notation and write $P_\nu(\omega)$ instead of $P_\nu(\exp(-i\omega))$.

Theorem 2.3. *Let $P = P_0$ be a Laurent polynomial satisfying (2.2) (i.e. (1.8)) for a dilation matrix M with $m = |\det M|$. Define*

$$\begin{aligned} G_0 &:= \Big\{ \gamma \in \mathbb{N}_0^m : |\gamma| = mN,\ \gamma_0 > N \text{ and } \gamma_0 > \gamma_\nu, \nu \in \mathbb{Z}^s_{M^t} \backslash \{0\} \Big\} \\ G_j &:= \Big\{ \gamma \in \mathbb{N}_0^m : |\gamma| = mN,\ \gamma_0 > N \text{ and } \gamma_0 \geq \gamma_\nu, \nu \in \mathbb{Z}^s_{M^t} \backslash \{0\}, \\ &\qquad\qquad \text{with exactly } j \text{ equalities} \Big\}, \quad j = 1, \ldots, m-2. \end{aligned}$$

and define

$$H := \sum_{j=0}^{m-2} \frac{1}{j+1} \Big(\sum_{\gamma \in G_j} C^\gamma_{mN} P_0^{\gamma_0 - 1} \prod_{\nu \in \mathbb{Z}^s_{M^t} \backslash \{0\}} P_\nu^{\gamma_\nu} \Big) + C^{(N,\ldots,N)}_{mN} \prod_{\nu \in \mathbb{Z}^s_{M^t}} P_\nu^N,$$

where C^γ_{mN} are the multinomial coefficients. Then, as a symbol, the Laurent polynomial PH also satisfies (1.8).

Remark. It should be noted that some of the sets G_j, $j = 1, \ldots, m-2$, may be empty for particular choices of m and N. In this case, the corresponding terms in the definition of H are zero.

Proof. Let $Q_\nu(z) := (PH)(\zeta_\nu z)$, $\nu \in \mathbb{Z}^s_{M^t}$. Observe that the mapping $\nu \mapsto \nu + \mu$ is one-to-one and onto $\mathbb{Z}^s_{M^t}$ for each fixed $\mu \in \mathbb{Z}^s_{M^t}$. Hence, when z is replaced by $\zeta_\nu z$ in the sequence $\left(P_\mu\right)_{\mu\in\mathbb{Z}^s_{M^t}}$ only a permutation of the sequence is obtained. For the last term of $\sum_{\mu\in\mathbb{Z}^s_{M^t}}(PH)_\mu$, we find

$$\begin{aligned} C^{(N,\ldots,N)}_{mN} \sum_{\mu\in\mathbb{Z}^s_{M^t}} P^{N+1}_\mu \prod_{\nu\neq\mu} P^N_\nu &= C^{(N,\ldots,N)}_{mN}\left(\prod_{\mu\in\mathbb{Z}^s_{M^t}} P^N_\mu\right)\sum_{\mu\in\mathbb{Z}^s_{M^t}} P_\mu \\ &= C^{(N,\ldots,N)}_{mN} \prod_{\mu\in\mathbb{Z}^s_{M^t}} P^N_\mu \end{aligned}$$

by (1.8) for P.

The term in PH corresponding to j, $0 \le j \le m-2$ with $G_j \neq \emptyset$, is

$$\frac{1}{j+1}\left(\sum_{\gamma\in G_j} C^\gamma_{mN} P_0^{\gamma_0} \prod_{\nu\in\mathbb{Z}^s_{M^t}\backslash\{0\}} P_\nu^{\gamma_\nu}\right). \tag{2.4}$$

When z is replaced by $\zeta_\mu z$, we obtain a similar sum with G_j replaced by

$$\begin{aligned} G_j(\mu) := \Big\{\gamma \in \mathbb{N}_0^m : &|\gamma| = mN,\ \gamma_\mu > N \text{ and } \gamma_\mu \ge \gamma_\nu, \nu \in \mathbb{Z}^s_{M^t}\backslash\{\mu\}, \\ &\text{with exactly } j \text{ equalities}\Big\}. \end{aligned}$$

Summing over $\mu \in \mathbb{Z}^s_{M^t}$, we obtain

$$\frac{1}{j+1}\left(\sum_{\mu\in\mathbb{Z}^s_{M^t}}\sum_{\gamma\in G_j(\mu)} C^\gamma_{mN}\prod_{\nu\in\mathbb{Z}^s_{M^t}} P^{\gamma_\nu}_\nu\right) = \sum_{\gamma\in G_j^*}\left(C^\gamma_{mN}\prod_{\nu\in\mathbb{Z}^s_{M^t}} P_\nu^{\gamma_\nu}\right),$$

where

$$G_j^* := \Big\{\gamma\in\mathbb{N}_0^m : |\gamma| = mN, \ \text{ with exactly } j+1 \text{ of the } \gamma_\mu \text{ equal to } |\gamma|_\infty\Big\},$$

(because $G_j(\mu)$, $\mu\in\mathbb{Z}^s_{M^t}$ covers G_j^* exactly $j+1$ times). Thus for $|z| = 1$,

$$\begin{aligned} \sum_{\mu\in\mathbb{Z}^s_{M^t}} Q_\nu &= \sum_{j=0}^{m-1}\sum_{\gamma\in G_j^*}\left(C^\gamma_{mN}\prod_{\nu\in\mathbb{Z}^s_{M^t}} P^{\gamma_\nu}_\nu\right) + C^{(N,\ldots,N)}_{mN}\prod_{\mu\in\mathbb{Z}^s_{M^t}} P^N_\mu \\ &= \sum_{|\gamma|=mN}\left(C^\gamma_{mN}\prod_{\mu\in\mathbb{Z}^s_{M^t}} P^{\gamma_\mu}_\mu\right) = \left(\sum_{\nu\in\mathbb{Z}^s_{M^t}} P_\nu\right)^{mN} = 1. \end{aligned}$$

$\square$

Here are a few cases that will be used in later examples:

$$
\begin{aligned}
&m = 2,\ N = 1: \\
&\qquad H = P_0 + 2P_0P_1 = P_0\big(1 + 2P_1\big). \\
&m = 2,\ N = 2; \\
&\qquad H = P_0^2(P_0 + 4P_1 + 6P_1^2). \\
&m = 2,\ N = 3; \\
&\qquad H = P_0^3(P_0^2 + 6P_0P_1 + 15P_1^2 + 20P_1^3). \\
&m = 2,\ N = 4; \\
&\qquad H = P_0^4(P_0^7 + 8P_0^6P_1 + 28P_0^5P_1^2 + 56P_0^4P_1^3 + 70P_0^4P_1^4). \\
&m = 3,\ N = 1; \\
&\qquad H = P_0\big(P_0 + 3(P_1 + P_2) + 6P_1P_2\big). \\
&m = 3,\ N = 2; \\
&\qquad H = P_0^2\big(P_0^3 + 6P_0^2(P_1 + P_2) + 15P_0(P_1 + P_2)^2 \\
&\qquad\qquad + 60(P_1^2P_2 + P_1P_2^2) + 10(P_1^3 + P_2^3) + 90P_1^2P_2^2\big). \\
&m = 4,\ N = 1; \\
&\qquad H = P_0\big(P_0^2 + 4P_0(P_1 + P_2 + P_3) + 12(P_1P_2 + P_2P_3 + P_1P_3) \\
&\qquad\qquad + 3(P_1^2 + P_2^2 + P_3^2) + 24P_1P_2P_3\big).
\end{aligned}
\tag{2.5}
$$

Of course, to make use of these equations, we must assign some ordering to the coset representers ν, $\nu \in \mathbb{Z}^s_{M^t}\backslash\{0\}$, say $\nu_1, \ldots, \nu_{m-1}$, and set $P_j = P_{\nu_j}$, $j = 1, \ldots, m-1$.

The Theorem will be used in the following ways: Since PH satisfies the interpolatory condition for a mask, PH is a candidate as a mask for a new fundamental function with hopefully higher smoothness, while H would then be the mask for a dual function. The trick of placing the extra power of P_0 in the last term in H means that H has the factorization $P_0^N T$. With good initial choice of $P = P_0$, the power of P_0 will add to the smoothness of the functions generated from the masks PH and H, provided that T can be bounded suitably. The next result addresses the bounds on T.

Lemma 2.6. *On $|z| = 1$, if $P = P_0$ is non-negative, then the function H of Theorem 2.3 has the form $H = P^N T$ where*

$$0 \le T(\omega) \le C_{mN}^{N+1} + C_{mN}^{(N,\ldots,N)}(m-1)^{-(m-1)N},$$

with the zero set of T being a subset of the zero set of P_0. Therefore,

$$\tau_{N,m} := \max_{\omega} T(\omega) \le C(N,m)\frac{m^{Nm+1/2}}{\sqrt{N}(m-1)^{N(m-1)-1/2}} \tag{2.7}$$

where $C(N,m) \le 1.9542$ if $N = 1,\ m \ge 2$ and $C(N,m) \le 5.1$ if $N \ge 2,\ m \ge 2$.

Proof. Since $P_0 + \ldots + P_{m-1} = 1$ and P_0 is non-negative, we conclude that $0 \le P_\nu \le 1$. The conclusion ensures that T is the sum of non-negative terms hence, the zero set of T is a subset of P_0, since the term with highest power of P_0 has only P_0 as a factor.

Note further that PH contains only "monomials" in the expansion $\left(\sum_{\nu \in \mathbb{Z}^s_{M^t}} P_\nu\right)^{mN}$ with $\gamma_0 \ge N$ but perhaps with smaller coefficients (even zero in some cases). Since $1 - P_0 = \sum_{\nu \in \mathbb{Z}^s_{M^t}\setminus\{0\}} P_\mu$, we may use instead the expansion of $(P_0 + (1-P_0))^{mN}$ to obtain

$$
\begin{aligned}
T &\le \sum_{j=N+1}^{mN} C^j_{mN} P_0^{j-1-N}(1-P_0)^{mN-j} + C^{(N,\ldots,N)}_{mN} \prod_{\nu \in \mathbb{Z}^s_{M^t}\setminus\{0\}} P_\nu \\
&\le \sum_{j=N+1}^{mN} C^j_{mN} P_0^{j-1-N}(1-P_0)^{mN-j} + C^{(N,\ldots,N)}_{mN} \left(\frac{1-P_0}{m-1}\right)^{(m-1)N},
\end{aligned}
\tag{2.8}
$$

where the last inequality comes from the fact that the maximum of $\prod_{\nu \in \mathbb{Z}^s_{M^t}\setminus\{0\}} P_\nu$ subject to the constraints

$$
P_\mu \ge 0 \quad \text{and} \quad \sum_{\nu \in \mathbb{Z}^s_{M^t}\setminus\{0\}} P_\mu = \text{const}, \ \text{(namely, the constant } 1 - P_0 \ge 0,)
$$

occurs when all the P_ν are equal.

Since the derivative with respect to P_0 of the first term on the right hand side of (2.8) is (after a regrouping)

$$
\begin{aligned}
&\sum_{N+1}^{mN-1} \left[C^{k+1}_{mN}(k-N)P_0^{k-1-N}(1-P_0)^{mN-k-1} - C^k_{mN}(mN-k)P_0^{k-1-N}(1-P_0)^{mN-k-1}\right] \\
&\qquad = (-N-1)\sum_{N+1}^{mN-1} C^{k+1}_{mN} P_0^{k-1-N}(1-P_0)^{(mN-k-1)} < 0,
\end{aligned}
$$

the right hand side is decreasing on $0 \le P_0 \le 1$. Therefore, the right hand side of (2.8) has its maximum when $P_0 = 0$. That gives the desired bound.

The bound (2.7) follows from the strong form of Stirling's formula

$$
\left|\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}} - 1\right| < \frac{2}{\sqrt{2\pi x}}, \quad x \ge 1.
$$

Indeed, applying this formula, we find that

$$
C(N,m) \le \begin{cases}
\dfrac{\left(1+\frac{2}{\sqrt{2\pi m}}\right)}{2e\left(1-\frac{2}{\sqrt{2\pi(m-1)}}\right)} + \dfrac{\sqrt{2\pi}\left(1+\frac{2}{\sqrt{2\pi m}}\right)}{e^m\sqrt{m-1}}, & \text{when } N = 1 \text{ and } m \ge 2; \\[2ex]
\dfrac{N\left(1+\frac{2}{\sqrt{2\pi Nm}}\right)}{(N+1)\sqrt{2\pi}\left(1-\frac{2}{\sqrt{2\pi N(m-1)}}\right)\left(1-\frac{2}{\sqrt{2\pi Nm}}\right)} & \\[2ex]
\quad + \dfrac{\left(1+\frac{2}{\sqrt{2\pi Nm}}\right)}{\sqrt{2\pi(m-1)}\left(1-\frac{2}{\sqrt{2\pi N}}\right)^2\left(\sqrt{2\pi N}-2\right)^{m-2}}, & \text{if } N \ge 2 \text{ and } m \ge 2.
\end{cases}
$$

□

Another consideration is whether good properties of P are passed on to the newly defined PH and H. The next lemma will be useful in establishing the stability of the functions generated by masks PH and H

Lemma 2.9. *Suppose the M-refinable functions ϕ_1 and ϕ_2 have Fourier transforms which are continuous and non-vanishing at 0. Suppose further that the zero set of the mask $\hat{a}_{\phi_1}$ contains the zero set of the mask $\hat{a}_{\phi_2}$. If ϕ_1 is pre-stable, then ϕ_2 is pre-stable.*

Proof. The pre-stability of ϕ_2 will follow from the fact that any zero for $\widehat{\phi}_2$ is a zero for $\widehat{\phi}_1$. Suppose $\widehat{\phi}_2(\omega_0) = 0$. Then for k large enough,

$$\widehat{\phi}_2(\omega_0) = \left(\prod_{j=1}^{k} \hat{a}_{\phi_2}\left(M^{t^{-j}}\omega_0\right) \right) \widehat{\phi}_2\left(M^{t^{-k}}\omega_0\right)$$

with $\widehat{\phi}_2\left(M^{t^{-k}}\omega_0\right) \neq 0$ by the nonvanishing and continuity the origin and the fact that $M^{t^{-k}}\omega_0 \to 0$ as $k \to \infty$. Hence, $M^{t^{-j}}\omega_0$ is a zero of $\hat{a}_{\phi_2}$ for some j. But then the same $M^{t^{-j}}\omega_0$ is a zero of $\hat{a}_{\phi_1}$ as well, which implies ω_0 is a zero of $\widehat{\phi}_1$. $\square$

Corollary 2.10. *If $P = \hat{a}$ is the non-negative mask of a continuous compactly supported fundamental M-refinable function ϕ, then the M-refinable functions generated by the masks PH and H in Theorem 2.3 are pre-stable.*

Proof. We have noted already that such a ϕ is stable. Clearly $\widehat{\phi}$ is continuous since ϕ is compactly supported. Since P is non-negative, Lemma 2.6 implies that the zero sets of P, PH and H coincide. Hence, the result follows from Lemma 2.9. $\square$

We have clearly established that Theorem 2.3 provides a family (indexed by N) of functions H so that PH and H preserve many of the good properties inherent in P; positivity, interpolatory condition (for PH) and pre-stability. Further, we have the following result:

Corollary 2.11. *Let $P = \hat{a}$ be the non-negative mask of a continuous compactly supported fundamental M-refinable function ϕ. If the M-refinable function corresponding to the mask PH is continuous, then it is fundamental. If the M-refinable function corresponding to the mask H is in $L_2(\mathbb{R}^s)$, then it is stable and dual to ϕ.*

Further, since both PH and H are real, we have the following corollary:

Corollary 2.12. *Let $P = \hat{a}$ be the non-negative mask of a continuous compactly supported fundamental M-refinable function ϕ. Then the M-refinable functions corresponding to the mask PH and to the mask H are symmetric to the origin.*

In the next section, we will discuss the question of gain in smoothness by this procedure. Since larger N gives higher powers of P in PH, this may already be a method of building smoothness and indeed, it is! But, a quick glance at the examples (2.5) shows that with larger m or N, the complexity of H increases rapidly. Hence, it may be better to obtain the higher smoothness through iteration.

The Iteration Algorithm 2.13. *Given a mask a for an M-refinable continuous function ϕ with a mask $\hat{a} \geq 0$ which satisfies the interpolatory condition (1.5). Fix an $N \geq 1$ and set $P = \hat{a}$, $k = 0$. DO:*

Step 1. Set $k = k+1$, $P_0 = P$, and compute P_ν, $\nu \in \mathbb{Z}^s_{M^t} \backslash \{0\}$ as in (2.1).

Step 2. Form H according to Theorem 2.3 (as in (2.5)).

Step 3. Set $P = P_0 H$ and/or when $k > 1$, set $H^d = P_0 H / \hat{a}$.

Step 4. Define

$$\widehat{\phi}_{N,k}(\omega) := \prod_{j=0}^{\infty} P(M^{t-j}\omega) \quad \text{and/or} \quad \widehat{\phi}^d_{N,k-1}(\omega) := \prod_{j=0}^{\infty} \overline{H(M^{t-j}\omega)}, \text{and/or}$$

$$\widehat{\phi}^{d,N,k}(\omega) := \prod_{j=0}^{\infty} \overline{H^d(M^{t-j}\omega)}.$$

Step 5. STOP if the smoothness of the desired function(s) ($\phi_{N,k}$ and/or $\phi^d_{N,k-1}$ and/or $\phi^{d,N,k}$) is reached, ELSE repeat Steps 1–5.

A word about the choices of output. The function $\phi_{N,k}$ will be a new, smoother function with a non-negative mask which satisfies the interpolatory condition (1.5). The function $\phi^d_{N,k-1}$ is a candidate for a dual function to $\phi_{N,k-1}$ since the product of the mask of $\phi_{N,k-1}$ (P_0 in Step 3) and the mask of $\overline{\phi^d_{N,k-1}}$ (H in Step 3) satisfies the interpolatory condition (1.5) (we elaborate on this in Section 4). For a similar reason, the functions $\phi^{d,N,k}$ are candidates (with smoothness increasing in k) for a dual function to the original ϕ.

The first three steps in Algorithm 2.13 are easy to implement in symbolic software such as MAPLE. Step 5 can also be carried out with only the information provided by the new masks through the procedure described in the next section.

Remark 2.14. The bivariate fundamental refinable functions constructed by [RiS1] provide us many initial functions for the construction here. In general, one can use box spline as suggested by [RiS1] to generate examples of the fundamental refinable function with low regularity numerically by solving a system of equations, or one can always obtain examples by using tensor product of fundamental refinable functions with lower regularity. Then apply the constructions in this section on those fundamental refinable functions to build nonseparable fundamental refinable functions with higher regularity.

3. Regularity

We first show that provided the initial function ϕ with mask $\hat{a} = P$ is suitably smooth, then the regularity of PH will increase with N or with each iteration in Algorithm 2.13.

A function ϕ belongs to C^α for $n < \alpha < n+1$, provided that $\phi \in C^n$ and

$$|D^\gamma \phi(x+t) - D^\gamma \phi(x)| \leq \text{const}\, |t|^{\alpha-n}, \quad \text{for all } |\gamma| = n \text{ and } |t| \leq 1 \tag{3.1}$$

for some constant independent of x. The number α is related to weighted L_1 exponents κ defined as

$$\kappa_{sup} := \sup\{\kappa : \int_{\mathbb{R}^s} (1+|\omega|)^{\kappa} |\widehat{\phi}(\omega)|\, d\omega < \infty\}. \tag{3.2}$$

The relation is given by the inequality $\sup \alpha \geq \kappa_{sup}$.

Therefore, an increase in the decay rate of the Fourier transform at infinity will mean that the corresponding function will have increased smoothness. We say $\widehat{\phi}$ has decay rate γ if

$$|\widehat{\phi}(\omega)| \leq C(1+|\omega|)^{-\gamma}.$$

Our object is to show that the constructions in the last section lead to smoother functions provided the original functions have sufficient smoothness. We will show that the Fourier transforms of the constructed functions have increasing decay rate provided the Fourier transform of the initial function has sufficient rate of decay.

The analysis depends on the factorization of H as $P^N T$, the estimates on T as given in Lemma 2.6, and on the characteristics of the dilation matrix M^t. For simplicity, we require that M has a complete orthonormal set of eigenvectors. Then there is an equivalent norm on $\mathbb{R}^s$ for which

$$|\lambda_{min}|\, ||\omega|| \leq ||M^t \omega|| \leq |\lambda_{max}|\, ||\omega||,$$

where $|\lambda_{min}|$ (respectively, $|\lambda_{max}|$) is the minimum (respectively, maximum) modulus of the eigenvalues of M^t. Hence, if $B := B(\mathbb{R}^s)$ is the closed unit ball in $\mathbb{R}^s$, then

$$\omega \in {M^t}^{K+1} B \backslash \left({M^t}^K B\right) \Longrightarrow |\lambda_{min}|^K \leq ||\omega|| \leq |\lambda_{max}|^{K+1}. \tag{3.3}$$

The matrix M will remain fixed, but we must label the various elements of the construction in the last section to identify how they arise. We do this with a subscript N denoting the choice of N and a subscript k to denote the iteration step k. Thus, we use $H_{N,k}$ with a factorization $\left(P_{N,k-1}\right)^N T_{N,k}$ to obtain the function $\widehat{\phi}_{N,k}$, while $\widehat{\phi}^d_{N,k-1}$ is the function obtained from the mask $P_{N,k} = P_{N,k-1} H_{N,k}$. We make use of the bound on $T_{N,k}$ found in Lemma 2.6.

We follow the well-known analysis of [D2] for the univariate case. We begin with the observation that $P_{N,k-1}(0) = P_{N,k}(0) = 1$ implies that $T_{N,k}(0) = 1$. This in turn implies that $T_{N,k}(\omega) \leq 1 + C|\omega|$, and consequently allows the estimate,

$$\sup_{|\omega|\leq 1} \prod_{j=1}^{\infty} T_{N,k}({M^t}^{-j}\omega) \leq \sup_{|\omega|\leq 1} \prod_{j=1}^{\infty} \exp(|C{M^t}^{-j}\omega|) \leq C. \tag{3.4}$$

Therefore, when $\omega \in {M^t}^{K+1} B \backslash \left({M^t}^K B\right)$, we have that

$$\begin{aligned}\prod_{j=1}^{\infty} T_{N,k}({M^t}^{-j}\omega) &= \prod_{j=1}^{K} T_{N,k}({M^t}^{-j}\omega) \prod_{j=1}^{\infty} T_{N,k}({M^t}^{-j-K}\omega) \leq C\tau_{N,m}^K \\ &= C|\lambda_{min}|^{K \log\left(\tau_{N,m}\right)/\log\left(|\lambda_{min}|\right)} \leq C(1+|\omega|)^{\log\left(\tau_{N,m}\right)/\log\left(|\lambda_{min}|\right)},\end{aligned}$$

by (3.4) and (3.3). For brevity in the ensuing expressions, we set

$$\eta_N := log\big(\tau_{N,m}\big)/\log(|\lambda_{min}|). \tag{3.5}$$

If we assume that $\widehat{\phi}_{N,k-1}$ has rate of decay $\gamma_{N,k-1}$ (with $\gamma_{N,0} := \gamma_0$, the decay rate of the original ϕ), then since

$$\begin{aligned}\widehat{\phi}_{N,k}(\omega) &= \prod_{j=1}^{\infty}\Big(P_{N,k-1}\big(M^{t^{-j}}\omega\big)\Big)^{N+1} T_{N,k}\big(M^{t^{-j}}\omega\big)\\ &= \Big(\widehat{\phi}_{N,k-1}(\omega)\Big)^{N+1}\prod_{j=1}^{\infty} T_{N,k}\big(M^{t^{-j}}\omega\big), \qquad \text{and, similarly}\\ \widehat{\phi}^{d}_{N,k-1}(\omega) &= \Big(\widehat{\phi}_{N,k-1}(\omega)\Big)^{N}\prod_{j=1}^{\infty} T_{N,k}\big(M^{t^{-j}}\omega\big)\end{aligned}$$

we find that

$$\begin{aligned}\widehat{\phi}_{N,k}(\omega) &\leq C(1+|\omega|)^{-(N+1)\gamma_{N,k-1}+\eta_N}, \quad \text{and}\\ \widehat{\phi}^{d}_{N,k-1}(\omega) &\leq C(1+|\omega|)^{-N\gamma_{N,k-1}+\eta_N}.\end{aligned} \tag{3.6}$$

It is easy to show inductively from the estimate (3.6) that the following formulas for decay rates hold for $\widehat{\phi}_{N,k}$ and $\widehat{\phi}^{d}_{N,k-1}$ respectively:

$$\begin{aligned}\gamma_{N,k} &= (N+1)^k\big(\gamma_0 - \eta_N/N\big) + \eta_N/N \quad \text{and}\\ \gamma^{d}_{N,k} &= N(N+1)^{k-1}\big(\gamma_0 - \eta_N/N\big).\end{aligned} \tag{3.7}$$

For the decay rate of $\widehat{\phi}^{d,N,k}$, we take $\widehat{\phi}^{d,N,1} = \widehat{\phi}^{d}_{N,1}$ and observe via induction that

$$\widehat{\phi}^{d,N,k}(\omega) = \widehat{\phi}^{d,N,k-1}(\omega)\widehat{\phi}^{d}_{N,k-1}(\omega) = \prod_{j=1}^{k}\widehat{\phi}^{d}_{N,j}(\omega).$$

Hence, from (3.7), we obtain the decay rate of $\widehat{\phi}^{d,N,k}$ as

$$\gamma^{d,N,k} = \big((N+1)^k - 1\big)\big(\gamma_0 - \eta_N/N\big) \quad \text{for } k > 1.$$

It follows from (2.7) that η_N/N is bounded by a constant dependent only on the matrix M. Hence, we have established

Theorem 3.8. *Let ϕ be an M-refinable function with non-negative mask $P = \hat{a}$. If the decay rate γ_0 of $\widehat{\phi}$ is sufficiently large, then the decay rate of $\widehat{\phi}_{N,k}$, $\widehat{\phi}^{d}_{N,k-1}$ and $\widehat{\phi}^{d,N,k}$ can be made arbitrarily large by increasing N or k. In particular, a decay rate of*

$$\gamma_0 > \log\big(\tau_{N,m}\big)/N\log\big(|\lambda_{min}|\big),$$

will suffice, where $\tau_{N,m}$ can be bounded as in (2.7).

Remark. We note that with $k = 1$ and N increasing, both the diameter of the support of the mask and the rate of decay of the Fourier transform are increasing linearly in N. For a fixed N and increasing k, both the diameter of the support and the rate of decay of the Fourier transform are increasing geometrically with k. Therefore, to increase regularity efficiently while maintaining control on the size of the support, it is better to increase N. On the other hand iteration with low values of N is easier to implement by (e.g.) MAPLE, because of the complexity of H for large N.

Example.3.9 Let ϕ be the the simplest fundamental function constructed in [RiS1] which is obtained by convolving the mask of a three direction box spline of equal multiplicity 2 in each direction with a distribution (see §5). The mask $\hat{a}$ of ϕ is non-negative. By choosing a sufficiently large N, or a sufficiently large iteration k, we get fundamental refinable functions with high order box spline factor. Further, the regularity of the refinable function grows linearly with N (geometrically with k). Hence, the constructions given in §2 together with the constructions given in [RiS1] provide several methods for the construction of fundamental refinable functions with high order box spline factor. The regularity of the refinable function increases as the order of the box spline factor does. $\square$

The crude bounds on T_N served their purpose to establish Theorem 3.8, but they may mislead one to believe that the starting point needs to be unrealistically high. However, in practice things are better since the smoothness can be assessed much more accurately. Here we summarize the approach to be used in our examples at the end. By now it has received treatment at many levels, by several people [CGV], [CD], [E], [H], [J], [RiS1], [RS4] and [V].

The criterion to be used to bound κ_{sup} from below is contained in the following statement: *For an integer r, let*

$$V_r := \big\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha \in \mathbb{Z}^s} p(\alpha) v(\alpha) = 0, \quad \forall\, p \in \Pi_r \big\},$$

where Π_r denotes the polynomials of total degree r. Assume M is a dilation matrix with a complete set of orthonormal eigenvectors. If the mask $\hat{a}$ for the stable fundamental M-refinable function ϕ satisfies

$$\text{(3.10)} \qquad \hat{a} \ge 0, \quad \hat{a}(0) = 1, \textit{ and } \quad D^\beta \hat{a}\big(2\pi {M^t}^{-1} \nu\big) = 0 \quad \textit{for } |\beta| \le r \textit{ and } \nu \in \mathbb{Z}^s_{M^t} \backslash \{0\},$$

then for a suitable choice of Ω with $\operatorname{supp} a \subseteq \Omega$, *$V_r$ is invariant under the matrix*

$$\mathbb{H} := \Big[a(M\alpha - \beta) \Big]_{\alpha \in \Omega, \beta \in \Omega}.$$

Let $\rho_{\mathbb{H}|_{V_r}}$ be the spectral radius of $\mathbb{H}|_{V_r}$. Then the exponent κ_{sup} satisfies

$$\text{(3.11)} \qquad \kappa_{sup} \ge -\frac{\log\big(\rho_{\mathbb{H}|_{V_r}}\big)}{\log\big(|\lambda_{max}|\big)}.$$

The proof of this statement can be obtained by modifying the proof in [RiS1] or from [CGV], [J] and [RS4]. The invariant set Ω was defined in [LLS1] and [LLS2] and [HJ1]. An explicit formula for the invariant set was given in [HJ1, Theorem 4.2]:

$$\Omega := \sum_{j=1}^{\infty} M^{-j} \operatorname{supp} a. \tag{3.12}$$

4. Dual Functions and Biorthogonal Wavelets

In this section we combine the construction of biorthogonal wavelets in our previous paper [RiS2] with the constructions of fundamental refinable functions in section 2 to construct dual functions of arbitrary smoothness.

Suppose that ϕ is a continuous, compactly supported, M-refinable function on $\mathbb{R}^s$, and the set of functions $\{\phi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^s}$ are linearly independent. We want to construct a stable compactly supported M-refinable function ϕ^d in $L_2(\mathbb{R}^s)$ so that the set of functions $\{\phi^d(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^s}$ forms a Riesz basis of $S(\phi^d)$ and

$$\langle \phi, \phi^d(\cdot - \alpha)\rangle = \delta_\alpha, \quad \alpha \in \mathbb{Z}^s. \tag{4.1}$$

The latter equations hold (see e.g. [RS1]) if and only if the Fourier transform for ϕ and ϕ^d respectively satisfy

$$\sum_{\alpha\in 2\pi\mathbb{Z}^s} \widehat{\phi}(\omega+\alpha)\overline{\widehat{\phi^d}(\omega+\alpha)} = 1, \quad \omega \in \mathbb{T}^s. \tag{4.2}$$

Therefore, when (4.1) holds, the masks $\hat{a}$ and $\hat{a}^d$ must satisfy the following necessary condition:

$$\sum_{\nu\in\mathbb{Z}^s_{M^t}} \hat{a}(\cdot + 2\pi {M^t}^{-1}\nu)\overline{\hat{a}^d(\cdot + 2\pi {M^t}^{-1}\nu)} = 1. \tag{4.3}$$

This can be used to advantage when compared to the interpolatory condition (1.8), because if the mask $\hat{h}$ for a fundamental M-refinable function factors into a product of two masks $\hat{h} = \hat{a}\overline{\hat{a}}^d$, then $\hat{a}$ and $\hat{a}^d$ can be considered as candidates for the masks of an M-refinable function and its dual function. In our case, we will deal only with real masks and therefore can dispense with the conjugation.

It was shown in [S] that ϕ and ϕ^d are a dual pair if and only if ϕ and ϕ^d are stable and their masks satisfy (4.3). Therefore, if masks $\hat{a}$ and $\hat{a}^d$ satisfy (4.3), then one needs show that the refinement equations of both masks $\hat{a}$ and $\hat{a}^d$ have compactly supported solutions (ϕ, ϕ^d, respectively) in $L_2(\mathbb{R}^s)$, and that the functions ϕ and ϕ^d are stable.

These verifications are much easier through the constructions and results of § 2. We begin with a continuous, compactly supported fundamental M-refinable function ϕ with mask $\hat{a} = P$. An application of the constructions provide new mask PH which also satisfies the interpolatory condition (1.5), and therefore, we have (4.3) with $\hat{a} := P$, our original mask, and $\hat{a}^d := \overline{H}$. The existence and stability for ϕ are given while the existence in $L_2(\mathbb{R}^s)$ of the solution ϕ^d is assured if ϕ is smooth enough, and the stability of ϕ^d is assured if the mask $\hat{a}$ in non-negative (by Corollary 2.10). Thus, if we begin with a continuous, compactly supported fundamental M-refinable function ϕ with mask $\hat{a} \geq 0$, then the existence of $L_2(\mathbb{R}^s)$ solutions and the stability of ϕ^d are essentially automatic (one may have to iterate a few times or choose a larger N to ensure that ϕ^d is in $L_2(\mathbb{R}^s)$, but as Theorem 3.8 shows, we may then achieve any regularity we wish for the dual).

Here is a summary of two constructions for dual functions and biorthogonal wavelets using the results of § 2. The two constructions differ only in how the $\hat{a}$ and $\hat{a}^d$ are chosen in Step 1 of Algorithm 4.5 to commence the actual construction.

The first assumes that we want to find a dual function for the given fundamental function ϕ (which may have been arrived at after an iterative procedure to gain smoothness) with the smoothness of the dual function to be determined solely by the choice of N, namely, the case $k = 1$ and N fixed. In this case, we take $\hat{a} = P$ and $\hat{a}^d = \overline{H}$.

In the second construction, we want to find a dual function for the given fundamental function ϕ with the smoothness of the dual enhanced by repeated iterations. In this case, after a suitable number of iterations to obtain the smoothness of the candidate for the dual function, we take $\hat{a}$ and set the dual mask $\hat{a}^d = \overline{PH^d/\hat{a}}$, since $\hat{a}$ is a factor of the iterations.

Let ϕ and ϕ^d be a dual pair. Then, it was shown in [BDR] (also see [JS]) that the sequence of subspaces defined by

$$S^k(\phi) := \{f(M^k\cdot) : f \in S(\phi)\}; \quad k \in \mathbb{Z},$$

and the sequence of subspaces defined by

$$S^k(\phi^d) := \{f(M^k\cdot) : f \in S(\phi^d)\}; \quad k \in \mathbb{Z},$$

form a 'dual' pair of multiresolutions of $L_2(\mathbb{R}^s)$. Here we recall that a sequence $S^k(\phi)$ forms a multiresolution, when the following conditions are satisfied: (i) $S^k(\phi) \subset S^{k+1}(\phi)$; (ii) $\cup_{k\in\mathbb{Z}} S^k(\phi) = L_2(\mathbb{R}^s)$ and $\cap_{k\in\mathbb{Z}} S^k(\phi) = \{0\}$; (iii) ϕ and its shifts form a Riesz basis of $S(\phi)$.

Once a dual pair of the multiresolutions are available, the construction of the biorthogonal wavelets from the pair of multiresolutions is equivalent to a problem of the matrix extension. The algorithm presented here for biorthogonal wavelet construction makes use of an algorithm for design of matrix pairs provided in [RiS2]:

Algorithm for Matrix Pairs 4.4. *For given $1 \times m$ vectors of Laurent polynomials $P = [P_1, P_2, \ldots, P_m]$ and Q with $P(z)Q^T(z) = 1$ for all $z \in \{\mathbb{C}\backslash\{0\}\}^s$, DO:*

Step 1. Extend the row P to an $m \times m$ matrix K in $\mathcal{P}$, where $\mathcal{P}$ is the set of all finite order matrices with entries being Laurent polynomials in $(\mathbb{C}\backslash\{0\})^s$.

Step 2. *Alter the last $m-1$ rows of K to be orthogonal to Q: Let K_j, $j=2,\ldots,m$, be the last $m-1$ rows of the matrix K and define*

$$G_j := K_j - (K_j Q^T)P, \quad j = 2, \ldots m.$$

Step 3. *Define $X := \left[P^T, G_2^T, \ldots, G_m^T\right]^T$. Then $X \in \mathcal{P}$ and is nonsingular on $(\mathbb{C}\backslash\{0\})^s$.*

Step 4. *Find $X^{-1} := \left[F_1^T, F_2^T, \ldots, F_m^T\right]$ and set $Y^T := \left[Q^T, F_2^T, \ldots, F_m^T\right]$ Then $Y \in \mathcal{P}$ and*

$$X(z)Y^T(z) = I_m, \quad z \in (\mathbb{C}\backslash\{0\})^s.$$

Step 1 is definitely trivial here, since ϕ is fundamental, for then substituting $\phi(\alpha) = \delta(\alpha)$ into the refinement equation (1.1) shows that $a(M\alpha) = \delta(\alpha)/|det(M)|$. Hence,

$$\sum_{\alpha \in \mathbb{Z}^s} a(M\alpha) z^{M\alpha} = 1/|det(M)|.$$

Thus, if $\nu = 0$ is the first element in our ordering of $\mathbb{Z}^s_{M^t}$, then the matrix can be extended simply by placing $1/|\det(M)|$'s in the diagonal. Once the step one is ready, the other steps can be implemented easily.

Remark. If each entry of the polynomial vectors P and Q^T are real on $|z| = 1$, and if P_1 is a real constant (as in the case just described), then each entry of X is real on $|z| = 1$ and each entry of Y is real on $|z| = 1$ except that the rows $2, \ldots, m$ may be multiplied by $|\det X|$ which is a monomial, that is, a real constant times an exponential $\exp(-i\alpha\omega)$ on $|z| = 1$. Therefore, if we start with a fundamental M-refinable function and a dual function both symmetric to the origin, then the wavelets and dual wavelets constructed in the next algorithm will also be symmetric.

Algorithm for Biorthogonal Wavelets from Interpolatory Subdivision 4.5. *For a given continuous, compactly supported fundamental M-refinable function ϕ with mask $\hat{a} \geq 0$. DO:*

Step 1. *Apply Algorithm 2.13 to a desired level k to obtain H or H^d as required. We take $\hat{a}$ and $\hat{a}^d = \overline{H}$ for $k = 1$ or $\hat{a}^d = \overline{H^d}$ for $k > 1$. Let*

$$\widehat{\phi}^d(\omega) := \prod_{j=1}^{\infty} \hat{a}^d\big(M^{t-j}\omega\big).$$

Step 2. *Define polynomials corresponding to the masks restricted to the cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$ by*

$$A_{0,\nu}(z) := \sum_{\alpha \in \mathbb{Z}^s} a(\nu + M\alpha) z^{M\alpha}, \nu \in \mathbb{Z}^s/M\mathbb{Z}^s,$$

and

$$A^d_{0,\nu}(z) := \sum_{\alpha \in \mathbb{Z}^s} \overline{a^d(\nu + M\alpha)} z^{-M\alpha}, \nu \in \mathbb{Z}^s/M\mathbb{Z}^s.$$

Step 3. *Apply Algorithm 4.4 to $P(z) := A_0(z) := (A_{0,\nu}(z))_{\nu \in \mathbb{Z}^s/M\mathbb{Z}^s}$, and $Q(z) := A^d_0 := (A^d_{0,\mu}(z))_{\nu \in \mathbb{Z}^s/M\mathbb{Z}^s}$ to obtain matrices X and Y.*

Step 4. Label the rows and columns of X and Y by $\mathbb{Z}^s/M\mathbb{Z}^s$ with the first row labeled by 0 and the remaining rows in some fixed order (e.g. by the lexicographic order). For the μ-th rows $A_\mu = (A_{\mu\nu})_{\nu\in\mathbb{Z}^s/M\mathbb{Z}^s}$ and $A^d_\mu := (A^d_{\mu\nu})_{\nu\in\mathbb{Z}^s/M\mathbb{Z}^s}$ respectively of X and Y, define

$$A_\mu(\omega) := \sum_{\nu\in\mathbb{Z}^s/M\mathbb{Z}^s} \exp(-i\nu\omega)A_{\mu\nu}(\exp(-iM^t\omega)),$$

and

$$A^d_\mu(\omega) := \overline{\sum_{\nu\in\mathbb{Z}^s/M\mathbb{Z}^s} \exp(-i\nu\omega)A^d_{\mu\nu}(\exp(-iM^t\omega))}.$$

Step 5. Define two sets of functions

$$\widehat{\psi}_\mu(M^t\omega) := A_\mu(\omega)\widehat{\phi}(\omega), \text{ and } \widehat{\psi}^d_\mu(M^t\omega) := A^d_\mu(\omega)\widehat{\phi}^d(\omega),\ \mu\in\mathbb{Z}^s/M\mathbb{Z}^s. \tag{4.6}$$

Then

(i) $\psi_0 = \phi$ and $\psi^d_0 = \phi^d$.

(ii) The functions ψ_μ, $\mu \in \mathbb{Z}^s/M\mathbb{Z}^s\backslash\{0\}$, are called the wavelets for the refinable function ϕ.

(iii) The functions ψ^d_μ, $\mu \in \mathbb{Z}^s/M\mathbb{Z}^s\backslash\{0\}$, are called the dual wavelets, that is, the wavelets of the dual function ϕ^d.

Finally, as was noted in [RiS2] and in a more general setting for dilation matrices M as discussed here in [RS3], the systems

$$\Big\{\psi_\mu(M^k\cdot-\alpha):\qquad \mu\in\mathbb{Z}^s_M\backslash\{0\}, k\in\mathbb{Z}, \alpha\in\mathbb{Z}^s\Big\} \tag{4.7}$$

and

$$\Big\{\psi^d_\mu(M^k\cdot-\alpha):\qquad \mu\in\mathbb{Z}^s_M\backslash\{0\}, k\in\mathbb{Z}, \alpha\in\mathbb{Z}^s\Big\}, \tag{4.8}$$

are the biorthogonal wavelet systems from the above construction, i.e.

$$\langle\psi(M^{k_1}\cdot-\alpha_1), \psi^d(M^{k_2}\cdot-\alpha_2)\rangle = \delta_{\alpha_1,\alpha_2}\delta_{k_1,k_2},\quad \alpha_1,\alpha_2\in\mathbb{Z}^s,\quad k_1,k_2\in\mathbb{Z}.$$

Further, they form a biorthogonal Riesz basis of $L_2(\mathbb{R}^s)$ if they are Bessel systems (see [RS3]. Let $\Psi := \big\{\psi_\mu : \mu\in\mathbb{Z}^s_M\backslash\{0\}\big\}$ and $\Psi^d := \big\{\psi^d_\mu : \mu\in\mathbb{Z}^s_M\backslash\{0\}\big\}$. It was shown in [RS2],[RS3] that the dilations and shifts of functions in Ψ' form a Bessel system if the functions

$$R_E := \max_{\psi\in\Psi'} \sum_{\alpha\in 2\pi\mathbb{Z}^s} |\widehat{\psi}(\cdot+\alpha)|, \tag{4.9}$$

and

$$R_D := \sum_{\psi\in\Psi', k\in\mathbb{Z}} |\widehat{\psi}(M^{t^k}\cdot)| \tag{4.10}$$

are in L_∞. This will be true provided the functions have certain decay rates at infinity and a certain order of the zeros at the origin.

Remark. Let ϕ be an arbitrary given compactly supported fundamental M-refinable function with mask $\hat{a}$ non-negative. By the methods given in this paper, we are able to construct a compactly supported dual refinable function with any desired regularity and the corresponding biorthogonal wavelet systems. Further, since we can construct the refinable fundamental function to have arbitrary regularity in any number of variables by Remark 2.14, we are able to construct multivariate biorthogonal wavelets with arbitrary regularity by the methods given here.

Remark. In the bivariate case, for an arbitrary given three direction mesh box spline one can construct dual refinable function with arbitrary regularity by Example 2.14. Therefore, biorthogonal wavelet systems with arbitrary regularity such that one of the bases is formed by piecewise polynomials can be constructed.

Remark. Let ϕ be a given M-refinable function with mask $\hat{a}$ that is not fundamental. Assume that ϕ and its shifts are linearly independent. One may find a refinable dual function ϕ^d with mask $\hat{a}^d$ that has a lower regularity numerically by solving a system of equations. If the mask $\hat{a}\hat{a}^d$ is non-negative, then we can use the constructions of §2 to find a mask P with $\hat{a}$ as a factor and for which the compactly supported M-refinable function corresponding to $\overline{P/\hat{a}}$ has high regularity. Hence, smoother biorthogonal wavelets can be obtained. When ϕ is a linearly independent box spline in $\mathbb{R}^s$, we may get a compactly supported dual refinable function with any desired regularity. Therefore, the problem of the construction of a compactly supported dual refinable function with any desired regularity for a given linearly independent refinable function ϕ is reduced to the problem of either finding numerically a distributional dual function or finding numerically a dual function with a lower regularity.

5. Examples

In this section we apply the techniques developed in the paper to illustrate the theory in concrete cases. We have chosen bivariate examples with different dilation matrices. The findings are summarized in Table 1.

Bivariate example with dilation matrix $M = 2I_{2\times 2}$. For the first example we use the simplest of the fundamental functions constructed in [RiS1]. It was found by multiplying the mask of a three direction box spline of equal multiplicity 2 in each direction by a suitable factor to give a mask satisfying (1.5) with smallest support and with the same symmetries as the box spline. The mask for the fundamental function, call

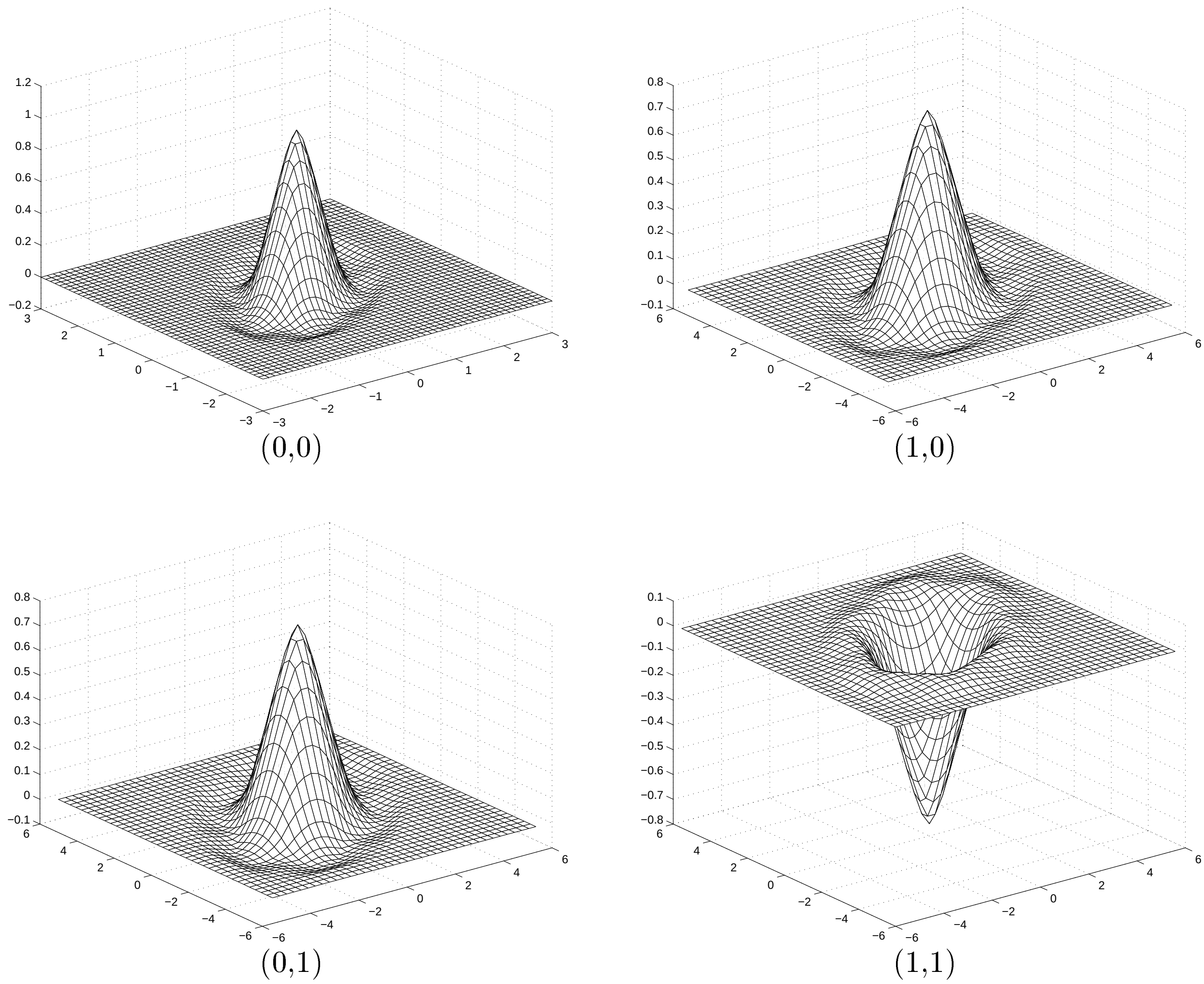


Figure 1. The interpolatory refinable function $\phi_{2,2,2}$ and the wavelets from it as derived in Algorithm (4.5).

it $\phi_{2,2,2}$, is reproduced below:

$$a_{2,2,2} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & \frac{-1}{64} & \frac{-3}{64} & \frac{-3}{64} & \frac{-1}{64} \\ 0 & 0 & \frac{-3}{64} & 0 & \frac{3}{32} & 0 & \frac{-3}{64} \\ 0 & \frac{-3}{64} & \frac{3}{32} & \frac{33}{64} & \frac{33}{64} & \frac{3}{32} & \frac{-3}{64} \\ \frac{-1}{64} & 0 & \frac{33}{64} & \mathbf{1} & \frac{33}{64} & 0 & \frac{-1}{64} \\ \frac{-3}{64} & \frac{3}{32} & \frac{33}{64} & \frac{33}{64} & \frac{3}{32} & \frac{-3}{64} & 0 \\ \frac{-3}{64} & 0 & \frac{3}{32} & 0 & \frac{-3}{64} & 0 & 0 \\ \frac{-1}{64} & \frac{-3}{64} & \frac{-3}{64} & \frac{-1}{64} & 0 & 0 & 0 \end{bmatrix}. \tag{5.1}$$

The mask $\hat{a}_{2,2,2}$ is

$$\Big(\cos\big(\frac{\omega_1}{2}\big)\cos\big(\frac{\omega_2}{2}\big)\cos\big(\frac{\omega_1+\omega_2}{2}\big)\Big)^2\Big(5-\cos(\omega_1)-\cos(\omega_2)-\cos(\omega_1+\omega_2)\Big)/2,$$

which is clearly non-negative. The support of a is in $[-3,3]^2$ and it has maximum value at the origin. In [RiS2], the corresponding function ϕ was shown to have smoothness $C^{2-\varepsilon}$ for any $\varepsilon > 0$.

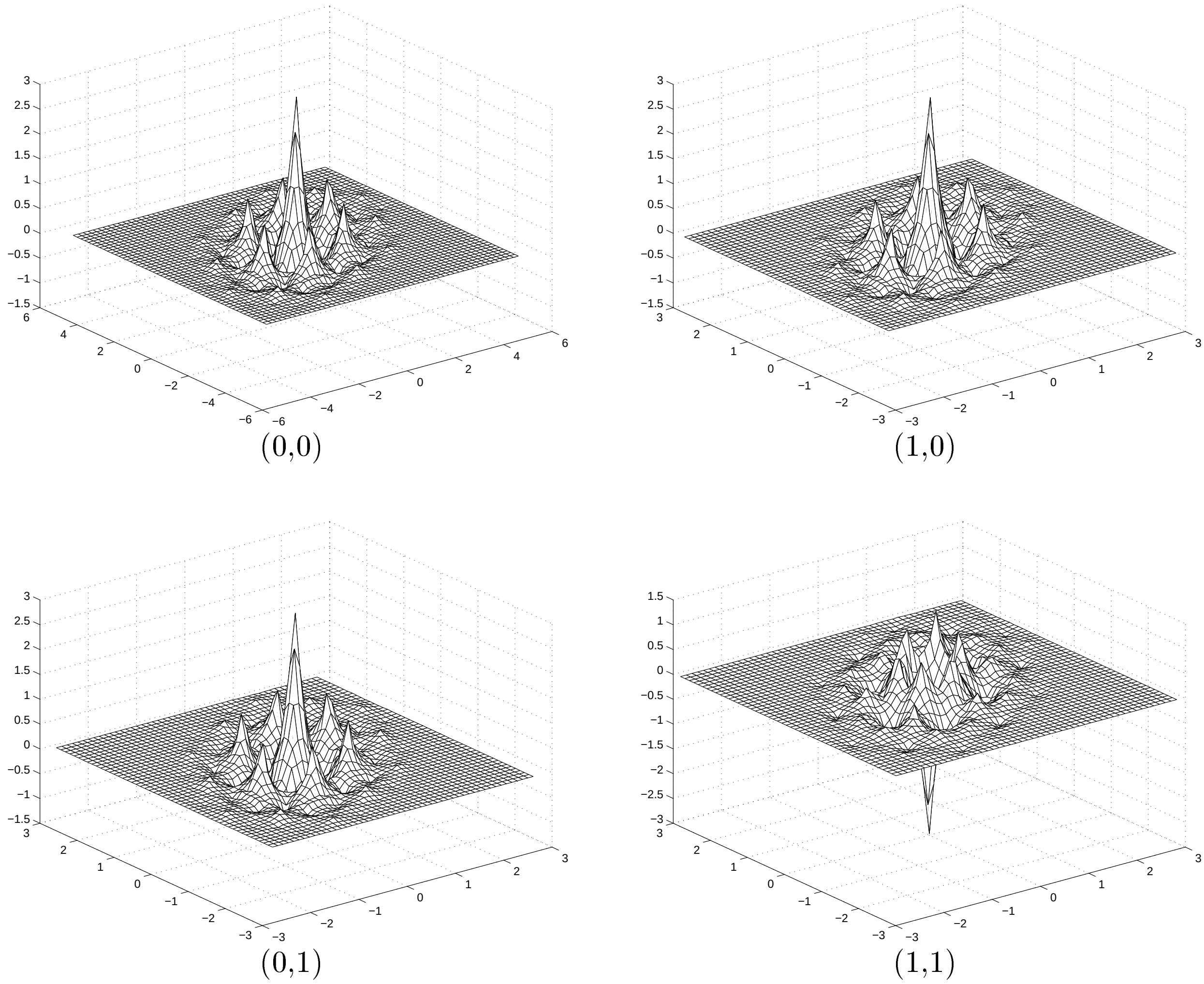
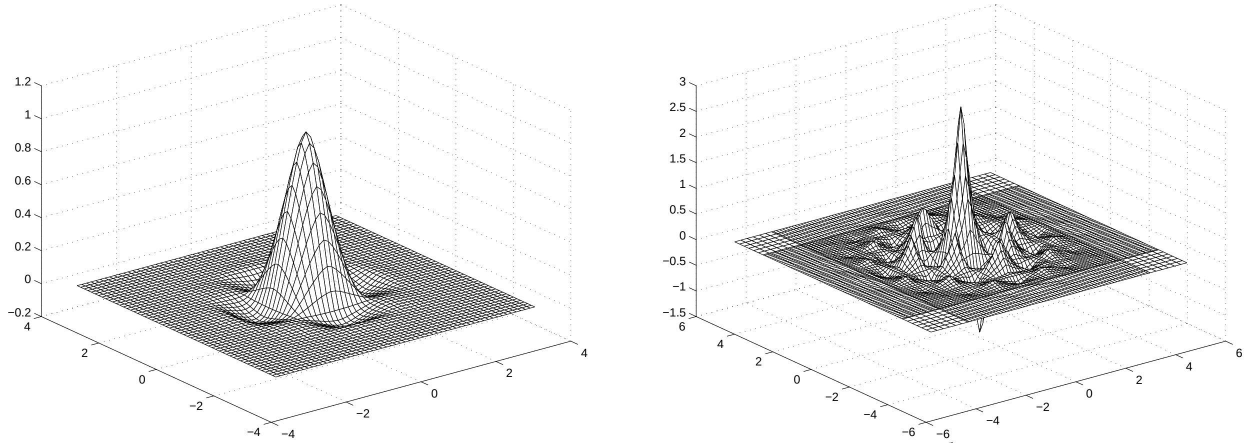


Figure 2. The dual function $\phi^d_{2,2,2}$ and the wavelets from it as derived in Algorithm (4.5).

Hence Algorithm 2.13 can be applied to $\hat{a}$ with $N = 1$ to obtain a mask H supported on $[-12,12]^2$. The mask H satisfies (3.10) for $r = 3$, hence, V_3 is invariant under the matrix $\mathbb{H}$ for the mask H. Applying the criterion at the end of §3, we find that the corresponding function $\phi^d_{2,2,2}$ is in smoothness class C^α, where $\alpha \geq .858185$. Therefore, $\phi^d_{2,2,2}$ is stable and dual to $\phi_{2,2,2}$ by Corollary 2.11.

Since we have a reasonable dual function to $\phi_{2,2,2}$, we can apply Algorithm 4.5 to construct biorthogonal wavelets for the pair $\phi_{2,2,2}$ and $\phi^d_{2,2,2}$. Figure 1 shows the refinable function $\phi_{2,2,2}$ and the corresponding wavelets, while Figure 2 shows the dual function $\phi^d_{2,2,2}$ and the dual wavelets.



ϕ_{ten1} $\qquad$ ϕ^d_{ten1}

Figure 3. The function ϕ_{ten1} and the dual function ϕ^d_{ten1} found by Algorithm 2.13 applied to a simple tensor product.

Nonseparable bivariate example from univariate hat functions with dilation matrix $M = 2I_{2\times 2}$. As mentioned earlier, the initial fundamental function may be the tensor product of simple univariate fundamental functions. The application of the constructions in § 2 will lead to nonseparable fundamental functions. Here, we take the tensor product, ϕ_{ten}, of the univariate hat function which has the mask

$$a_{ten} = \begin{bmatrix} 1/16 & 1/8 & 1/16 \\ 1/8 & 1/4 & 1/8 \\ 1/16 & 1/8 & 1/16 \end{bmatrix}.$$

Clearly, the mask $\hat{a}_{ten}$ will be non-negative since it is the product of the univariate mask in each variable. Applying Algorithm (2.13) with $N = 1$, we find a dual function in $L_2(\mathbb{R}^2)$ (but not continuous) and a new nonseparable fundamental function ϕ_{ten1} which belongs to $C^{2-\varepsilon}$, for any $\varepsilon > 0$. Iterating once more ($k = 2$) we find a dual, ϕ^d_{ten1}, for ϕ_{ten1} that has smoothness. In Table 1, we have labeled this next iteration of ϕ_{ten} as the first iteration of ϕ_{ten1} to give its dual function and to better show the comparison with $\phi_{2,2,2}$. Figure 3 shows the functions ϕ_{ten1} and ϕ^d_{ten1}. The supports of the functions derived using $\phi_{2,2,2}$ are smaller.

Examples with dilation matrix $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Here we take a quick look at an example which has already been well studied in the literature (see [DD] and [CD]). For the dilation matrix $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and the mask

$$a = \begin{bmatrix} 0 & 1/8 & 0 \\ 1/8 & 1/2 & 1/8 \\ 0 & 1/8 & 0 \end{bmatrix}, \tag{5.2}$$

we have that $\hat{a} = \frac{1}{2} + \frac{1}{4}\cos(\omega_1) + \frac{1}{4}\cos(\omega_2) \geq 0$. Therefore, the methods of §2 apply to build smoother fundamental functions and for those smoother functions we can try to find dual functions as well. Let ϕ_M be the fundamental function corresponding to the mask (5.2).

Since $\hat{a}$ is non-negative, we may apply the smoothness criterion given at the end of §3. For the matrix M, $|\lambda_{min}| = |\lambda_{max}| = \sqrt{2}$ and therefore, $\phi_{N,k}$ is in $C^{\alpha-\varepsilon}$, where $\alpha \geq \kappa := -2\log\left(\rho_{\mathbb{H}|_{V_r}}\right)/\log(2)$ and where V_r is to be determined for each choice of N and k. Table 1 summarizes our calculations for $N = 1, 2, 3, 4$ with $k = 1$ and $N = 1,\ k = 2$.

Consulting Table 1 we see quite clearly the increasing smoothness with N. An iterate, ($k = 2$ for $N = 1$), also shows the increasing smoothness with the iteration. For the dual functions however, one must progress further to obtain smoothness. With $N = 1$, we do not get an $L_2(\mathbb{R}^2)$ function for $\phi^d_{1,1}$, but with increasing N we do get existence in $L_2(\mathbb{R}^2)$ for the dual functions $\phi^d_{2,1}$, $\phi^d_{3,1}$, and a continuous dual function in $\phi^d_{4,1}$.

Function	N, k	r in V_r	$\phi_{N,k} \in C^{\alpha-\varepsilon}$ with $\alpha \geq$	$\phi^{d,N,k}$
$\phi_{2,2,2}$	$N = 0, k = 0$	$r = 3$	$\kappa = 2.0000\ldots$	—
	$N = 1, k = 1$	$r = 7$	$\kappa = 3.6594\ldots$	$\phi^{d,1,1} \in C^{0.8581}$
ϕ_{ten}	$N = 0, k = 0$	$r = 1$	$\kappa = 1.0000\ldots$	—
	$N = 1, k = 1$	$r = 3$	$\kappa = 2.0000\ldots$	$\phi^{d,1,1} \in L_2(\mathbb{R}^s)$
ϕ_{ten1}	$N = 0, k = 0$	$r = 2$	$\kappa = 2.0000\ldots$	—
	$N = 1, k = 1$	$r = 7$	$\kappa = 3.6841\ldots$	$\phi^{d,1,1} \in C^{0.8578}$
ϕ_M	$N = 0, k = 0$	$r = 1$	$\kappa = 0.6115\ldots$	—
	$N = 1, k = 1$	$r = 3$	$\kappa = 1.3581\ldots$	$\phi^{d,1,1} \notin L_2(\mathbb{R}^2)$
	$N = 2, k = 1$	$r = 5$	$\kappa = 1.9908\ldots$	$\phi^{d,2,1} \in L_2(\mathbb{R}^2)$
	$N = 3, k = 1$	$r = 7$	$\kappa = 2.5474\ldots$	$\phi^{d,3,1} \in L_2(\mathbb{R}^2)$
	$N = 4, k = 1$	$r = 9$	$\kappa = 3.0509\ldots$	$\phi^{d,4,1} \in C^{0.31326}$
	$N = 1, k = 2$	$r = 7$	$\kappa = 2.6387\ldots$	$\phi^{d,1,2} \in L_2(\mathbb{R}^2)$

Table 1. Results of applying the constructions in §2 to the Examples of this section

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