Doctoral Proposal

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1 Introduction

Considering topological properties of set theoretical constructions is an active area of research in combinatorial set theory. Various tools allowing us to compute properties of ordinals (such as the set of countable ordinals, or countable structures on ordinals in general) have been developed over the last decades. Among others, we have Jensen's fine structure theory [Jen72b] and Todorčević's walks on ordinals [Tod07]. Both of these give rich structure to the space in question. Todorčević, for example, remarks that ordinal walks "can be used to derive virtually all known other structures that have been defined so far on ω_1" [Tod07, p. 19].

Of interest to us is Bergfalk's and Lambie-Hanson's approach to consider cohomological questions. Fixing an ordinal δ under the order topology, they consider the Čech cohomology groups under presheaves of functions into a fixed abelian group A. These considerations employ both set-theoretical and algebraic tools. In dimension 1, essentially, they boil down to the following: is it consistent with ZFC that there are coherent families of functions φ_α from α into A (i.e. when φ_α is restricted to $\beta < \alpha$, it equals φ_β modulo finite difference) that are not trivial (i.e. the family of φ_α's cannot be approximated uniformly)? Every trivial family is coherent, but the converse fails in general. Similarly, a family is coherent if and only if each of its initial segments is trivial. In that light, non-trivial coherence can be considered an example of *incompactness*. This can be naturally extended to dimension $n > 1$.

Introducing Čech cohomology into the equation, this provides a strong characterisation of the non-trivial cohomology groups which is the backbone of the theory: as it turns out, the coherent non-trivial families are precisely the non-trivial elements of the Čech cohomology groups.

The algebraic and cohomological background is given in [Jen72a] and [Mar00].

The link with mathematical logic and set theory in particular allows us to reason as follows: we start with a question about topological spaces, and interpret their properties using algebra (this is the classical approach, the basis of algebraic topology). However, the axiomatic nature of ordinals allows us to use set-theoretical tools to answer algebraic questions. Hence the (non-)vanishing of cohomology groups of ordinals reduces to the existence of specific combinatorial objects whose constructions depend on (limitations of) ZFC.

An example of this connection between set theory and and cohomology is exhibited below: we define

$$A_f = \bigoplus_{i<\omega} \bigoplus_{j=1}^{f(i)} \mathbb{Z}$$

where $f \in \omega^\omega$, Baire space. Similarly, we define

$$B_f = \prod_{i<\omega} \prod_{j=1}^{f(i)} \mathbb{Z}$$

and hence the inverse systems $\mathcal{A} = (A_f, p_{fg})_{f\in\omega^\omega}$ and $\mathcal{B} = (B_f, p_{fg})_{f\in\omega^\omega}$. Note that A_f is a subset (and even a subgroup) of B_f. An element of A_f is a function φ_f defined on

$$\varphi_f(i)\colon f(i) \to \mathbb{Z}$$

and is finitely supported; this latter restriction does not apply to elements of B_f. Under pointwise addition, B_f/A_f is a group, and its equivalence classes are sets of functions whose differences $\varphi_f - \psi_f$ are finitely supported. For a trivial example, suppose $f = \langle 1, 2, 3, \ldots\rangle$ and $\varphi_f(i,j) = j$ while $\psi_f(i,j) = j+1$; this determines elements of B_f whose pointwise differences are of the form

$$(\varphi_f - \psi_f)(i,j) = (\varphi_f(i) - \psi_f(i))(j) = \varphi_f(i,j) - \psi_f(i,j) = j - (j+1) = -1$$

for any $i \in \omega$ and $j \leq f(i)$, and hence infinitely supported.

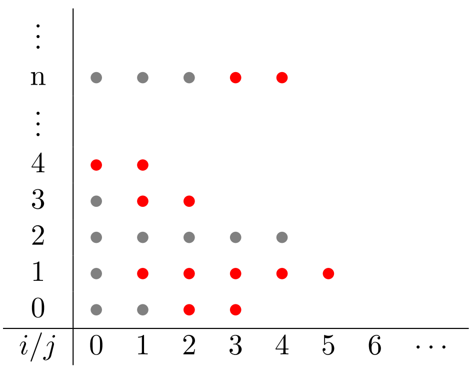
Let $f \in \omega^\omega$ and its associated function $\varphi_f \in B_f$. We can characterise its domain as

$$D_f := \{\langle i,j\rangle \in \omega \times (\omega \setminus \{0\}) \mid j \leq f(i)\}$$

For $f, g \in \omega^\omega$, suppose $f(i) \leq g(i)$ for all $i \in \omega$ (i.e. f precedes g in the product ordering). Then

$$\mathrm{dom}(\varphi_f(i)) = f(i) < g(i) = \mathrm{dom}(\varphi_f(i))$$

So the intersection $D_g(i) \cap D_f(i) = D_g(i) \restriction f(i) = D_f(i)$ in this case. If f and g are incomparable, this reduces to truncating $\varphi_f(i)$ to $\mathrm{dom}(\varphi_g(i))$ whenever $f(i) \geq g(i)$, and vice versa.



Assume, for example, that $f = \langle 2, 1, 5, 1, 0, \ldots\rangle$ and $g = \langle 4, 6, 5, 3, 2, \ldots\rangle$. Consider row n in the table to the left. We see that $f(n) = 3$ while $g(n) = 5$. The black bullets indicate an element of $\mathrm{dom}(\varphi_f(n)) \cap \mathrm{dom}(\varphi_g(n))$. The red bullets indicate points in $\mathrm{dom}(\varphi_g(n)) \setminus \mathrm{dom}(\varphi_f(n))$. Of course, each row of consecutive dots must be an initial segment of ω. For each row in which $f(i) \leq g(i)$, the restriction of $\varphi_g(i)$ to f is just $\varphi_f(i)$.

Suppose $\Phi \in \prod_{f\in\omega^\omega} B_f$ is a set of functions $\varphi_f \in B_f$. We define $\varphi_f =^* \varphi_g$ if and only if the set of $n \in D_f \cap D_g$ where $\varphi_f(n) \neq \varphi_g(n)$ is finite. The family Φ is *coherent* if $\varphi_f =^* \varphi_g$ for all $f, g \in \omega^\omega$. There is a natural connection between coherent and trivial families of functions in this space, which is captured by the *inverse limit* of the poset of sets B_f.

The *inverse limit* is an algebraic object whose definition is, intuitively, based on coherence. Given an upwards directed set that indexes a sequence of groups, the elements of the associated inverse limit are infinite strings picking an element from each group so that their restrictions agree. Clearly, Baire space is upwards directed; for each $f, g \in \omega^\omega$, define $h(i) = f(i) + g(i) + 1$ for each $i \in \omega$. In this context, the coherence property reads as follows: for a restriction map $p_{fg}\colon B_g \to B_f$ we have

$$p_{fg}(x_g) = x_f$$

where $x = \langle x_f\rangle_{f\in\omega^\omega}$ is an element of the inverse limit of $\mathcal{B}$ (written as $\varprojlim \mathcal{B}$). We will make this precise below, but we give an important characterisation straight away.

Supposing Φ is coherent, we know that $\varphi_f =^* \varphi_g$ for all $f, g \in \omega^\omega$. Consider the equivalence classes of said maps, i.e. classes $[\varphi_f]$ for $f \in \omega^\omega$ where $\varphi_f \sim \psi_f$ if and only if their difference is an element of A_f (i.e. finitely supported). Then $p_{fg}([\varphi_g]) = [p_{fg}(\varphi_g)] = [\varphi_f]$ since $\varphi_g =^* \varphi_f$ implies $\varphi_g - \varphi_f \in A_f$. Therefore the set $\{[\varphi_f] \mid f \in \omega^\omega\} \in \varprojlim \mathcal{B}/\mathcal{A}$. Similarly, every element of $\varprojlim \mathcal{B}/\mathcal{A}$ gives rise to a coherent family of functions indexed by Baire space.

Observation 1. *There is a natural isomorphism between the set of coherent $\Phi = \{\varphi_f \in B_f \mid f \in \omega^\omega\}$ and the inverse limit $\varprojlim \mathcal{B}/\mathcal{A}$.*

The reasoning we have done above can be extended to the n-dimensional case. Then, families are indexed by n-tuples of elements of Baire space; the notions of coherence and triviality are naturally adapted. The crucial connection is the following:

Theorem 1 ([Ber18, Thm. 3.3])**.** *For $n > 0$, $\varprojlim^n \mathcal{A} = 0$ if and only if every n-coherent family defined on Baire space is n-trivial.*

So showing the (non-)vanishing of derived limits is now a combinatorial problem: can a given family be trivialised?

The question of non-trivial coherence and, in turn, the existence of witnesses, permeates several tightly connected areas of set theory and (co-)homology theory. The observation that non-trivial coherence provides witnesses to non-trivial Čech cohomology groups is the thread connecting the subjects. In [Ber17], Bergfalk explores the setup we have outlined above. In this context, classical set theoretical axioms influence the (non-)vanishing of homology groups (in aforementioned source, Bergfalk relates his findings to results by Mardesic and Prasolov [MP88] on the strong homology of the topological sum of countably many k-dimensional Hawaiian earrings – this is one motivation, and its connection to set theory is outlined in Question 5, Thm. 1 and Thm. 2 in [MP88]). The Proper Forcing Axiom, for example, decides the non-triviality of the second derived limit in this new context, but also forces the vanishing of all other derived limits [Ber17, Thm. 4.1]. In particular, deciding characteristics of the continuum entails the existence of non-trivial coherent families. From a set-theoretical perspective, this is a combinatorial property: $\mathfrak{b} = \mathfrak{d} = \aleph_2$ (a consequence of PFA) yields an *ω_2-scale*, which in turn provides a non-trivial coherent family.

Remark. *In the ordinal case, there is a tight relationship between the cofinality of the underlying ordinal and its non-trivial coherent families defined. Interpreting the dominating number $\mathfrak{d}$ as the cofinality of ω^ω renders this an instance of the same phenomenon.*

Going beyond Baire space and considering ω^κ (with associated inverse system $\mathcal{A}_\kappa$) for any $\kappa > \omega$ opens up the question of the globality of these properties: Bergfalk has shown that the first derived limit of $\mathcal{A}$ vanishes if and only if the first derived limit of $\mathcal{A}_\kappa$ vanishes [Ber17, Thm. 5.1]. One half of this result follows easily:

$$\text{If } \varprojlim^1 \mathcal{A}_\kappa \text{ vanishes then so does } \varprojlim^1 \mathcal{A}. \tag{1}$$

Indeed, if every 1-coherent family in ω^κ can be 1-trivialised, then so can every family in Baire space. One can show that a family is 1-coherent if and only if each initial segment is 1-trivial. Equivalently, a family is 1-trivial if and only if it can be properly extended to a 1-coherent family. If there were a non-trivial 1-coherent family Φ on Baire space then it could be extended to a 1-coherent family Φ^* in ω^κ. By assumption, Φ^* is trivialised by some function $\varphi\colon \kappa \to \mathbb{Z}$. But $\varphi \restriction \omega$ now trivialises Φ.

There are various open questions. Two stand out as they are set-theoretical first and foremost. These are problems 1 and 6 in [Ber17].

Firstly, what about the converse to statement (1): does local behaviour at $\varprojlim^1 \mathcal{A}$ determine global properties on all $\kappa > \omega$? Formally, does the vanishing of $\varprojlim^1 \mathcal{A}$ imply triviality of $\varprojlim^1 \mathcal{A}_\kappa$ for $\kappa > \omega$?

Another question concerns the witnesses of non-vanishing in a descriptive sense. Todorčević showed that a witness of $\varprojlim^1 \mathcal{A} \neq 0$ cannot be analytic [Tod98] (we show below that the non-vanishing of the first derived limit equates to showing that the quotient map between the raised inverse limits is not surjective; Todorčević's proof directly uses a family F witnessing the non-surjectivity). The remainder of the argument is based on a combinatorial axiom (cf. same source), which is implied by PFA:

Axiom 1 (Open Coloring Axiom, OCA)**.** *Let X be a topological space. Assume $R \subset X^2$ is an open symmetric relation. Then exactly one of the following two holds:*

(i) there is an uncountable $Z \subseteq X$ such that $[Z]^2 \subseteq R$; or

(ii) X can be covered by countably many sets X_n such that $[X_n]^2 \cap R = \emptyset$.

In other words, either we can find an uncountable set in which every pair satisfies R, or we can cover X by countably many sets, all of whose pairs of elements defy R. With a suitable choice of relation we can now trivialise the witnessing family in question, a contradiction.

Bergfalk suggests that "showing witnesses for $\varprojlim^2 \mathcal{A} \neq 0$ are non-analytic (which they surely are) could entail isolating some higher-dimensional OCA-like principle, and is in fact a reasonable framework for pursuing such (and for avoiding the more naive/false generalizations of OCA)" [1].

1.1 Considering Inverse Limits

We provide more background on the algebraic basis of the theory. This will allow us to formalise further open questions.

Suppose $\mathcal{A} = (A^{(n)})_{n<\omega}$ is a sequence of cochain complexes. Similarly, assume we have complexes $\mathcal{B}$ and $\mathcal{C}$ such that

$$0 \to A^{(n)} \to B^{(n)} \to C^{(n)} \to 0$$

is short exact for every $n < \omega$ (hence, in particular, $C \cong B/A$). Suppose we have maps $p^{(m,n)}$ for $m < n < \omega$.

This gives rise to an ω-tower, whose inverse limit is of interest for a variety of reasons. For example, algebraic structures such as the p-adic integers, can be identified as the inverse limit of such a tower of groups (pick $A^{(n)} = \mathbb{Z}/p^n\mathbb{Z}$). Crucially, as we shall outline below when considering ordinals, the inverse limits will coincide with the respective cohomology groups.

Formally, we define the following: let

$$\mathcal{A} = (A^{(\lambda)}, p^{(\lambda,\lambda')})_{\lambda\in\Lambda}$$

be an inverse system of cochain complexes, Λ is upwards directed. This is called a *pro-chain complex.* The cochain maps $p^{(\lambda,\lambda')}$ satisfy

- $p^{(\lambda,\lambda)}$ is the identity function on $A^{(\lambda)}$;
- if $\lambda'' > \lambda' > \lambda$ then $p^{(\lambda,\lambda'')} = p^{(\lambda,\lambda')}p^{(\lambda',\lambda'')}$.

[1] Private communication, 31/01/2020

We write $A^{(\lambda,n)}$ for the set of n-cochains in $A^{(\lambda)}$.

Hence we obtain the following diagram from $\mathcal{A}$:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A^{(\lambda+2,0)} \arrow[r, "d"] \arrow[d, "p"] & A^{(\lambda+2,1)} \arrow[r] \arrow[d] & A^{(\lambda+2,2)} \arrow[r] \arrow[d] & \dots \\
A^{(\lambda+1,0)} \arrow[r] \arrow[d] & A^{(\lambda+1,1)} \arrow[r] \arrow[d] & A^{(\lambda+1,2)} \arrow[r] \arrow[d] & \dots \\
A^{(\lambda,0)} \arrow[r] \arrow[d] & A^{(\lambda,1)} \arrow[r] \arrow[d] & A^{(\lambda,2)} \arrow[r] \arrow[d] & \dots \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A^{(0,0)} \arrow[r] & A^{(0,1)} \arrow[r] & A^{(0,2)} \arrow[r] & \dots
\end{tikzcd}$$

The vertical maps are the cochain maps p; the lateral maps are the boundary maps d. All squares commute. Both are indicated exemplary in the diagram above.

Suppose Λ is an ordinal. In particular, consider an ω-tower. Then the inverse limit of the inverse system $\mathcal{A}$ is defined by

$$\left(\varprojlim \mathcal{A}\right)^0 = \varprojlim \mathcal{A} = \left\{ \langle x_n \rangle \in \prod_{n<\omega} A^{(n,0)} \mid p^{(m,n)}(x_n) = x_m \text{ for all } m < n < \omega \right\}$$

So each coordinate, when projected (or *restricted*) agrees with its initial segment in the sequence. Essentially, the inverse limit is generated by the groups and, equally, by the restriction maps p. Similarly, a boundary map d on the inverse limit is induced by the boundary maps at the cochain levels. We define

$$d(\langle x_n \rangle_{n<\omega}) = \langle d(x_n) \rangle_{n<\omega}$$

Hence we have the following sequence at the top of the diagram above:

$$\varprojlim \mathcal{A} \xrightarrow{d} \left(\varprojlim \mathcal{A}\right)^1 \longrightarrow \left(\varprojlim \mathcal{A}\right)^2 \longrightarrow \dots$$

Note that the inverse limit (as obtained above) is a cochain complex itself.

Given a sequence of groups $\langle G^{(n)} \rangle_{n\in\omega}$ (we fix ω for simplicity, this notion can be defined on any upwards directed ordered set, just like pro-chain complexes above), we can construct cochain complexes from tuples from $G^{(n)}$ – the inverse system obtained from these complexes and the associated restriction maps (which are group homomorphism) is called a *pro-group*. Indeed, suppose each $p^{(m,n)}$ is a group homomorphism (still satisfying the composition- and identity-property). We may then define cochain complexes on the groups: let

$$G^{(n,m)} = \prod_{g_0<g_1<\ldots<g_m<\omega} G^{(n)}$$

Assuming the towers are *nice* (the restriction functions are surjective, and the tower is continuous, i.e. each $G^{(\lambda)} = \varprojlim_{\delta<\lambda} G^{(\delta)}$, which immediately holds if we restrain ourselves to ω) we can prove that the n-th cohomology group of the associated tower of chain complexes equals the n-th derived limit induced by the towers. This provides the link between derived limits and cohomology groups. It also provides an extremely suitable framework for Čech cohomology which, by definition, associates open sets with groups.

In a different direction, we can consider more algebraic questions. Given a short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of directed inverse systems of abelian groups, we can consider the inverse limits $\varprojlim \mathcal{A}$, $\varprojlim \mathcal{B}$, and $\varprojlim \mathcal{C}$. Even more so, the short-exactness at the cochain levels induces a long exact sequence

$$0 \longrightarrow \varprojlim \mathcal{A} \longrightarrow \varprojlim \mathcal{B} \longrightarrow \varprojlim \mathcal{C} \longrightarrow \varprojlim{}^1 \mathcal{A} \longrightarrow \dots$$

which is in general not short exact. Indeed, if this sequence is short exact then the first derived limit of $\mathcal{A}$ vanishes. (Equally, if the quotient map $q\colon \varprojlim \mathcal{B} \to \varprojlim \mathcal{B}/\mathcal{A}$ is surjective, the result follows.) In that sense, the (non)-vanishing of the first derived limit gives insight about the exactness of the induced sequence of inverse limits, just like, for example, the (non)-vanishing of the kernel of a homomorphism informs how injective said map is. If the systems are defined on a countable index set (or, as in our cases, on some countable ordinal δ), the *Mittag-Leffler condition* gives a sufficient condition for the vanishing of $\varprojlim^1 \mathcal{A}$: if the inverse system $\mathcal{A}$ stabilises, i.e. for each $\lambda < \delta$ there is a $\kappa \geq \lambda$ such that for any $\kappa' \geq \kappa$ we have

$$p^{(\lambda,\kappa')}[A^{(\kappa')}] = p^{(\lambda,\kappa'')}[A^{(\kappa'')}]$$

then the first derived limit vanishes.

One can show that having surjective maps p proves that the system is Mittag-Leffler. However, the fact that ML implies a vanishing first derived limit depends on δ being countable (in fact, each such system can be collapsed to an ω-system, which forms part of the proof; this is clearly not possible in the uncountable case as, then, we cannot do without limits). Whether there is a similar condition for arbitrary ordinals (and hence in particular in the case of cohomology of ordinals as investigated by Bergfalk and Lambie-Hanson) is an open question.

Martino Lupini and I have made some progress on this question. We hope this provides novel perspectives to the purely topological approach, eventually answering the question: what is the homological dimension of the ordinals? These are problems I will be working on.

1.2 The Čech Cohomology of the Ordinals

The role taken on by the restriction maps in the inverse system above can be emulated by presheaves, the underlying space X topological. Fix an abelian group A. Then a presheaf $\mathcal{P}$ maps an open set $U \subset X$ to a set of functions from U into A. It is accompanied by a set of functions p_{UV} for open sets U and V, mapping between the associated groups. These are the familiar restriction maps, and they satisfy the identity and composition laws. This can be considered a recasting in the setting of ordinals of the cohomological questions outlined above on the Baire space. Bergfalk and Lambie-Hanson consider this in [BL19].

Fix $\delta \in \mathrm{ON}$. Alongside a presheaf, Čech cohomology requires an open covering $\mathcal{V}$ of the space. We define cochain complexes by

$$L^j(\mathcal{V},\mathcal{P}) = \prod_{\vec{\alpha} \in [\delta]^{j+1}} \mathcal{P}(V_{\vec{\alpha}})$$

which induces a sequence of cochain complexes

$$0 \longrightarrow L^0(\mathcal{V},\mathcal{P}) \longrightarrow \dots \longrightarrow L^j(\mathcal{V},\mathcal{P}) \xrightarrow{d^j} L^{j+1}(\mathcal{V},\mathcal{P}) \longrightarrow \dots$$

where d^j is the boundary homomorphism. Translating the constructions, a presheaf associating finitely supported functions yields the analogue to the Baire space case above. This is obtained by the presheaf $\mathcal{D}_A(U) = \bigoplus_U A$.

Remark. *We can easily verify that, for any ordinal δ, the set $\{\alpha \mid \alpha < \delta\}$ is an open δ-cover; we shall denote it by $\mathcal{U}_\delta$. Under this cover, the notion of coherence we defined above translates faithfully: a family $\Phi = \{\varphi_\alpha \colon \alpha \to \delta \mid \alpha < \delta\}$ is coherent if and only if $\varphi_\beta \restriction \alpha =^* \varphi_\alpha$ for any $\alpha < \beta < \delta$. Note this is equivalent to saying*

$$p_{\alpha\beta}(\varphi_\beta) =^* \varphi_\alpha$$

for all $\alpha < \beta < \delta$. Even more so, if we replace $\mathcal{D}_A(U)$ with $\mathcal{F}_A(U) = (\prod_U A)/(\bigoplus_U A)$ we obtain $\varphi_\beta \restriction \alpha =^* \varphi_\alpha$ if and only if

$$p_{\alpha\beta}([\varphi_\beta]) = [\varphi_\alpha]$$

and hence recover the Baire space characterisation (which also resembles the condition defining inverse limits).

Definition 1. *The n-th Čech cohomology group of δ is the direct limit of the groups*

$$H^n(\mathcal{V}, \mathcal{P}) = \ker(d^n)/\operatorname{im}(d^{n-1})$$

where $\mathcal{V}$ is an open cover δ endowed with the order topology, and $\mathcal{P}$ is a presheaf. The groups are ordered by refinement of covers: if $\mathcal{V}$ is a cover then we say $\mathcal{W} \geq \mathcal{V}$ (i.e. $\mathcal{W}$ refines $\mathcal{V}$) if there is a map $r \colon \mathcal{W} \to \mathcal{V}$ such that $W \subset r(W)$ for each $W \in \mathcal{W}$. That is, we can embed any open set in $\mathcal{W}$ in some open set in $\mathcal{V}$.

Thus the n-th cohomology group is given by

$$\check{H}^n(\delta, \mathcal{P}) \coloneqq \varinjlim_{\mathcal{V} \text{ covers } \delta} H^n(\mathcal{V}, \mathcal{P})$$

The following theorem shows that computing the groups can be simplified by the choice of a good cover: the open cover $\mathcal{U}_\delta = \{\alpha \mid \alpha < \delta\}$ of initial segments of δ is in fact fine enough to not miss any non-trivial elements.

Theorem 2 ([BL19, Thm. 2.30])**.** *Suppose $n > 0$, $\mathcal{P}_A$ is a presheaf of functions into A, and $\mathcal{V}$ refines $\mathcal{U}_\delta$. Then*

$$H^n(\mathcal{V}, \mathcal{P}_A) \cong \check{H}^n(\delta, \mathcal{P}_A)$$

The elements of the cochain complexes are functions of functions, or equivalently, a sequence of functions indexed by increasing finite tuples of δ (whose first coordinate determines the domain of the function). The definitions of non-trivial and coherent translate intuitively from the Baire space context above.

The set-theoretical influences under ZFC are exhibited in various levels depending on the dimension n. For $n = 1$ we have the following: Todorčević's *walks on ordinals* provide us with a coherent non-trivial family defined on $\delta < \omega_1$ when $A = \mathbb{Z}$ ([BL19, Thm. 2.4], [Moo06, Fact 3]). In general, we are in a similar situation as before: the Čech cohomology groups are precisely the groups of non-trivial coherent families.

Theorem 3 ([BL19, Thm. 2.36])**.** *For any ordinal δ and abelian group A, the following char-*

acterisation holds:

$$\check{H}^n(\delta, \mathcal{D}_A) = \begin{cases} \textit{the group of } 0\textit{-coherent} & \\ \textit{functions from } \delta \textit{ into } A & \textit{if } n = 0 \\ & \\ \textit{the group of } n\textit{-coherent} & \\ \textit{families of functions } \{\varphi_{\overrightarrow{\alpha}} \colon \alpha_0 \to A \mid \overrightarrow{\alpha} \in [\delta]^n\} & \\ \textit{modulo trivial families of} & \\ \textit{functions } \{\varphi_{\overrightarrow{\alpha}} \colon \alpha_0 \to A \mid \overrightarrow{\alpha} \in [\delta]^n\} & \textit{otherwise} \end{cases}$$

Inherent properties of the ordinals in question fundamentally determine the cohomology groups in question. For example, the cofinality essentially determines the (non)-vanishing of cohomology groups. Any non-trivial coherent family on $\mathrm{cf}(\delta)$ can be extended to a non-trivial coherent family on δ. Similarly, in the same sense in which countable cofinalities trivialise the notions of clubs and stationary sets, so do they affect Čech cohomology groups. Any infinitely supported function on a δ of countable cofinality whose initial segments are finitely supported yields a non-trivial element of the cohomology group. (This immediately proves that there are no 0-coherent functions on ω_1, for example, as one would disprove the regularity of ω_1.) The following theorem generalises this:

Theorem 4 ([BL19, Thm. 2.32])**.** *Assume* $\mathrm{cf}(\omega_n) = \omega_k$ *with* $k < n$*. Then for any* $m > k$*, any* m*-coherent family of functions defined on* ω_n *can be trivialised.*

In other words, the Čech cohomology groups $\check{H}^m(\delta, \mathcal{D}_A)$ vanish. Viewing this in terms of dimensions, this can be likened to the vanishing of all homology groups $H^m(K)$ where K is a simplicial complex of dimension $n < m$.

1.2.1 Trivialising Families

With the characterisation of Čech cohomology groups comprising non-trivial coherent families of functions, we know that any family that can be trivialised does not contribute to the Čech cohomology. The existence of non-trivial coherence does, however, affect the existence of non-trivial coherence in extensions of the space of the same cofinality; witnesses can be "stretched" via a clubset.

It is a theorem of ZFC that any ordinal ω_n possesses a non-trivial n-th Čech cohomology group [BL19, Thm. 2.35]. Similarly, the 0-th Čech cohomology group is the group of all 0-coherent functions, i.e. functions with finite support. Hence this cohomology groups is never trivial.

Beyond these limitations, the behaviour of $\check{H}^n$ is very much affected by set-theoretical assumptions. In some cases, large cardinal assumptions trivialise all possible groups. When $\delta = \kappa$ weakly compact, for example, then $\check{H}^n(\kappa, \mathcal{D}_A)$ vanishes whenever possible (i.e. for every $n > 0$); being non-n-trivial is a Π^1_1-sentence and hence reflects into some V_α, contradicting that any initial segment of an n-coherent family is n-trivial. Bergfalk and Lambie-Hanson have also shown that, assuming $V = L$, all groups that are not forced to be non-trivial by ZFC will be non-trivial; this is based on combinatorial constructions employing $\square$- and $\diamondsuit$-sequences.

In order to answer the question, one needs to determine consistency of ZFC with trivialising families of dimension $0 < n < \mathrm{cf}(\delta)$. There are several special cases that have been proven: under product forcing that preserves the uncountable cofinality of δ, no non-1-trivial 1-coherent family can be trivialised in the forcing extension [BL19, Q. 4.2]. Using strongly compact cardinal κ, one can then show that $\check{H}^1(\lambda, \mathcal{D}_A) = 0$ for any regular $\lambda \geq \kappa$ is consistent with ZFC. Using the P-Ideal Dichotomoy, Todorčević showed that $\check{H}^1(\delta, \mathcal{D}_A)$ vanishes if and only if $\mathrm{cf}(\delta) = \omega_1$

[BL19, Thm. 3.16]. But none of these results have the sweeping uniformity presented by the maximality of non-vanishing in the constructible universe. Results have been communicated piecewise.

Whether it is consistent with ZFC that, for example, $\check{H}^2(\omega_3, \mathcal{D}_A)$ can vanish, is unanswered. It is an open question whether the product-forcing lemma [BL19, Lem. 3.10] can be extended to higher dimensions. A proof of the 2-dimensional case settles "$\check{H}^2(\omega_3, \mathcal{D}_A) = 0$"-consistency – and a general affirmative result settles the cohomological possibilities for regular ordinals.

The possible cohomology of ordinals uncovers a fundamental connection to combinatorial properties. It appears as though each ordinal ω_n exhibits properties that defy general characterisation, which is witnessed by the independence of their cohomology groups. As Bergfalk and Lambie-Hanson note in [BL19]: "It appears likely, in conclusion, that what we do succeed in understanding of higher-dimensional incompactness principles will not be unconnected to what we succeed in understanding of the ordinals ω_n $(n > 1)$ themselves." These are problems of interest, whose solutions I will be working towards.

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