

CO-ANALYTIC COUNTEREXAMPLES TO MARSTRAND'S PROJECTION THEOREM

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ABSTRACT. Assuming $V=L$, we construct a plane set E of Hausdorff dimension 1 whose every orthogonal projection onto straight lines through the origin has Hausdorff dimension 0. This is a counterexample to J. M. Marstrand's seminal projection theorem [30]. While counterexamples had already been constructed decades ago, initially by R. O. Davies [8], the novelty of our result lies in the fact that E is co-analytic. Following Marstrand's original proof [30] (and R. Kaufman's newer, and now standard, approach [18] based on capacities), a counterexample to the projection theorem cannot be analytic, hence our counterexample is optimal. We then extend the result in a strong way: we show that for each $\epsilon \in (0, 1)$ there exists a co-analytic set E_ϵ of dimension $1 + \epsilon$, each of whose orthogonal projections onto straight lines through the origin has Hausdorff dimension ϵ . The constructions of E and E_ϵ are by induction on the countable ordinals, applying a theorem by Z. Vidnyánszky [42].

1. INTRODUCTION

Hausdorff measure is one of the fundamental notions of classical fractal geometry. While Lebesgue measure limits itself to well-behaved structures such as planes or lines, or remains uninformative otherwise, Hausdorff measure is much finer (in particular, it extends Lebesgue measure). Hausdorff measure is defined similarly to Lebesgue measure: one considers covers of a set. However, for Hausdorff measure, there are two important differences: (1) the covers need not be open; and (2), the diameter of the cover sets is raised to a chosen power $\alpha > 0$. It is easily seen that for every set X there exists $s \geq 0$ such that the α -dimensional Hausdorff measure of X is 0 if $\alpha > s$; and that it is infinite if $\alpha < s$. What value the α -dimensional Hausdorff measure of X takes with $\alpha = s$ is anyone's guess. It could be 0, infinite, or take on some other finite value. For this reason, s is called the critical value—it is also called the *Hausdorff dimension* of X .

An illuminating quote explaining the intuition behind the critical value is the following [10, p. 168]:

This idea of dimension is an abstraction of what we already know from elementary geometry. If A is a [...] curve, then its length is a useful way to measure its size; but its “area” and “volume” are 0. The dimensions 2 and 3 are too large to help in measuring the size of A . If B is the surface of a sphere, then its area is positive and finite. We can say its “length” is infinite (for example, since it contains curves that are as long as we like which spiral around); its “volume” is 0, since it is contained in a solid

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spherical shell whose thickness is as small as we like. So for the set B , the dimension 1 is too small, the dimension 3 is too large, and the dimension 2 is just right. The α -dimensional Hausdorff measure give us a way of measuring the size of a set for dimensions α other than the integers 1, 2, 3, ...

Since the early development of fractal geometry, a few theorems have turned out to be foundational. Marstrand’s projection theorem [30] dating back to 1954 is one of them: it forms a cornerstone of classical fractal geometry today. While ignored for decades (with the term “fractal geometry” only arriving in the 1970s), fractal geometry, and projection theorems like Marstrand’s, are researched intensively nowadays [12].

In essence, the theorem states that orthogonal projections of analytic sets cannot drop too far in dimension—but there are exceptions. The first to notice this was R. O. Davies [8], who constructed a counterexample non-constructively, assuming the Continuum Hypothesis CH. Simplifications [18] and generalisations [32] of Marstrand’s theorem followed over the subsequent decades. Nowadays, the standard argument to prove Marstrand’s projection theorem is Kaufman’s proof [18] which is based on *energy potential characterisations* of Hausdorff dimension. Refinements of the theorem are sought after today [2, 13].

While fractal geometry traditionally depended on measure arguments, other approaches have advanced over the last decades. With the first such being R. Kaufman’s aforementioned proof [18] of Marstrand’s Theorem, more recently, advances in bridging fractal geometry to information theoretical tools have succeeded [22, 33, 15, 24]: standing out is the recent *point-to-set principle* of Lutz and Lutz [25], which relates the Hausdorff dimension of a *set* to the dimensions of its constituent *points*. The dimension of points, in this context, is given using tools of algorithmic information theory: one considers computability theory, and in particular, Kolmogorov randomness. We present the required background in subsequent sections.

The point-to-set principle has already proven to be a very useful tool in the theory of orthogonal projections. This applies to both Hausdorff dimension \dim_H as well as packing dimension \dim_P : Lutz and Stull [28] have shown using algorithmic arguments that if $X \subset \mathbb{R}^2$ satisfies $\dim_H(X) = \dim_P(X)$ then Marstrand’s theorem applies, and so the requirement of being analytic can be dropped. They also give a new bound on the packing dimension of orthogonal projections under packing dimension, and provide a new proof of Marstrand’s theorem in the same paper. It is noted that it was already known that packing dimension does not admit a Marstrand-like result [17].

In the present paper we show that being analytic is in fact sharp for Marstrand’s theorem, a fact previously unknown: we construct an optimal counterexample to Marstrand’s projection theorem using non-classical tools. In particular, we use Lutz’ and Lutz’ point-to-set principle [25] to construct a set of Hausdorff dimension 1, all of whose projections vanish. Using a result by Vidnyánszky [42] we show that the constructed set is co-analytic. We then extend our result to produce a co-analytic Marstrand-failing set of dimension $1 + \epsilon$ for each $\epsilon \in (0, 1)$ in a strong way: for each ϵ we produce a co-analytic set X such that $\dim_H(X) = 1 + \epsilon$ while its projection onto every line through the origin has

dimension ϵ , the minimal allowable value.

Throughout our work we make the set-theoretic assumption that $V=L$. This is for combinatorial reasons, as it allows us to argue by transfinite induction in our construction, following Vidnyánszky's theorem.

We would like to note that the results in this paper have independently been obtained by T. Slaman and D. Stull.

1.1. Marstrand's Projection Theorem. The original statement of John Marstrand's Projection Theorem is as follows (we have flipped the order from how it appears in the source).

Theorem 1.1 (MPT, [30, Thm I & II]). *Let $E \subset \mathbb{R}^2$ be analytic. Let $\text{proj}_\theta(E)$ denote the projection of E onto the unique line passing through the origin at angle θ with the first axis. Let μ denote the one-dimensional Lebesgue measure.*

(1) *If $\dim_H(E) \leq 1$ then for almost all $\theta \in [0, \pi)$ we have $\dim_H(\text{proj}_\theta(E)) = \dim_H(E)$.*

(2) *If $\dim_H(E) > 1$ then for almost all $\theta \in [0, \pi)$ we have $\mu(\text{proj}_\theta(E)) > 0$.*

It is noted that item 2 is strictly stronger than an assertion about Hausdorff dimension [3]. This follows from the standard fact that, in \mathbb{R}^m , Hausdorff measure generalises m -dimensional Lebesgue measure (the proof is a straightforward measure argument [10, 6.1]). In particular, if $A \subset \mathbb{R}$ has positive Lebesgue measure, then $\dim_H(A) = 1$. An easy Corollary of item 2 is hence:

Corollary 1.2. *If $E \subset \mathbb{R}^2$ is analytic and $\dim_H(E) > 1$ then for almost all $\theta \in [0, \pi)$ we have $\dim_H(\text{proj}_\theta(E)) = 1$.*

1.2. Our Theorems. We provide co-analytic counterexamples to both items 1 and 2 of theorem 1.1. Since analytic sets satisfy theorem 1.1, our results are sharp. The first theorem is proven in section 5, the second in section 6. Recall that we assume $V=L$.

Theorem 5.1. *There exists a co-analytic set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1$ while, for every $\theta \in [0, 2\pi)$ we have $\dim_H(\text{proj}_\theta(E)) = 0$.*

The construction is carried out via a theorem due to Z. Vidnyánszky [42], which allows us to build co-analytic sets using co-analytic conditions by recursion. Let \mathbb{D} denote the first quadrant of the unit disc, which we shall consider in more detail in section 3.

Given $F \subset \mathbb{D}^{\leq \omega} \times [0, \pi/2] \times \mathbb{D}$, a set $X = \{x_\alpha \mid \alpha < \omega_1\}$ is *compatible with F* if the following exist:

- an enumeration $\{p_\alpha \mid \alpha < \omega_1\}$ of B ; and
- an enumeration $\{A_\alpha \mid \alpha < \omega_1\} \subset \mathbb{D}^{\leq \omega}$ such that if $\alpha < \omega_1$ then $A_\alpha = X \upharpoonright \alpha$

such that for each $\alpha < \omega_1$ we have $(A_\alpha, p_\alpha, x_\alpha) \in F$.

Theorem 4.3 ([42]). *Let $F \subset \mathbb{D}^{\leq \omega} \times [0, \pi/2] \times \mathbb{D}$. If F is co-analytic and if for all $(A, p) \in \mathbb{D}^{\leq \omega} \times [0, \pi/2]$ the section*

$$F(A, p) = \{x \in \mathbb{D} \mid F(A, p, x)\}$$

is cofinal in the Turing degrees, then there exists a co-analytic set $X \subset \mathbb{D}$ that is compatible with F .

We revisit Vidnyánszky's theorem in more detail in section 4.

Our second contribution is the following stronger result, which we prove in section 6.

Theorem 6.1. *For every $\epsilon \in (0, 1)$, there exists a co-analytic set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1 + \epsilon$ while, for every $\theta \in [0, 2\pi)$ we have $\dim_H(\text{proj}_\theta(E)) = \epsilon$.*

The arguments in the proof of this more general theorem are similar to those prior; however, the coding procedure is more involved.

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2. PRELIMINARIES

In this section, we give some necessary preliminaries on Hausdorff measure and dimension, as well as computability theory and the theory of complexity. These are required in order to use the point-to-set principle theorem 2.13 in our construction.

2.1. A review of Hausdorff measure and dimension. While classically geometric measure theory formed the backbone of fractal geometry, this is not necessarily so today. When working with the point-to-set principle, measure theory is mostly unnecessary. However, we occasionally switch between one and the other; some proofs are expressed and proven more simply in the language of measures than in that of computability theory. Hence we provide a very brief introduction to Hausdorff measure, dimension, and in particular Lipschitz maps below.

Consider a subset $E \subset \mathbb{R}^2$. In order to define Hausdorff measure, we first consider

$$\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_{i < \omega} |U_i|^s \mid E \subset \bigcup_{i < \omega} U_i \wedge (\forall i < \omega)(|U_i| < \delta) \right\}$$

where d is the usual Euclidean distance and $|U| = \sup\{d(x, y) \mid x, y \in U\}$, the *diameter of U* .

Remark 2.1. Observe that as δ increases, we get to include more covers in our infimum, and hence

$$\text{if } 0 < \delta < \delta' \text{ then } \mathcal{H}_\delta^s < \mathcal{H}_{\delta'}^s.$$

So, as δ decreases, the term $\mathcal{H}_\delta^s(E)$ increases. In particular, the limit $\lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^s(E)$ always exists (it might be infinite!).

Definition 2.2. Let $E \subset \mathbb{R}^2$. We define the *s-dimensional Hausdorff measure of E* as follows:

$$\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^s(E).$$

It is easily seen that there must exist a *critical value* for s at which the s -dimensional Hausdorff measure changes from ∞ to 0—this is the *Hausdorff dimension*.

Definition 2.3. Let $E \subset \mathbb{R}^2$. Then the *Hausdorff dimension* $\dim_H(E)$ is defined as

$$\dim_H(E) = \sup\{s \geq 0 \mid \mathcal{H}^s(E) = \infty\} = \inf\{s \geq 0 \mid \mathcal{H}^s(E) = 0\}.$$

Finally, and quite importantly, Hausdorff dimension is well-behaved under Lipschitz maps. Recall that a map $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is *Lipschitz with constant* $M > 0$ if for all $x, y \in \mathbb{R}^m$ we have $|f(x) - f(y)| \leq M|x - y|$. (Here, $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^m .) Fixing $m = 2$ it is noted that Lipschitz maps cannot increase dimension:

Lemma 2.4. *Let $E \subset \mathbb{R}^2$. If $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies a Lipschitz condition then $\dim_H(f(E)) \leq \dim_H(E)$.*

We use the following simple lemma.

Lemma 2.5. *Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is Lipschitz with constant $M > 0$. Then*

$$\mathcal{H}^s(f(E)) \leq M^s \mathcal{H}^s(E)$$

for all $s \geq 0$.

Proof. Suppose f has Lipschitz constant M , and that (U_i) is a δ -cover for E . Then $(f(U_i))$ is an $M\delta$ -cover for $f(E)$. Thus

$$\mathcal{H}_{M\delta}^s(f(E)) \leq \sum_{i < \omega} |f(U_i)|^s \leq \sum_{i < \omega} M^s |U_i|^s$$

since, for any $f(x), f(y) \in f(U_i)$ we have $|f(x) - f(y)| \leq M|x - y|$, whence $|f(U_i)| \leq M|U_i|$ follows after taking suprema. Finally, taking $\lim_{\delta \rightarrow 0^+}$ above yields the result. \square

Proof of Lemma 2.4. This follows straight from the lemma above: suppose $\dim_H(E) = s$, and assume f is Lipschitz with constant $M > 0$. Then

$$\mathcal{H}^s(f(E)) \leq M^s \mathcal{H}^s(E) < \infty.$$

Since $\dim_H(E) = \sup\{s \mid \mathcal{H}^s(E) = \infty\}$ we have $\mathcal{H}^s(f(E)) < \infty$. Therefore $\dim_H(f(E)) \leq s$, as needed. \square

As a special (but extremely useful) case we note that if f is an isometry then Hausdorff measure is in fact fixed. This yields that:

Corollary 2.6. *Hausdorff dimension is preserved under isometries. In particular, it is preserved under rotation and translation.*

2.2. Kolmogorov complexity. Our main arguments rely on the notion of information density, which we describe in terms of *prefix-free Kolmogorov complexity*. Standard, and very comprehensive, references for Kolmogorov complexity are Downey and Hirschfeldt [9] as well as Li and Vitány [21]. A very short but very readable introduction is Fortnow [14]. We give a brief account of the most relevant notions and some history below.

We choose to express the basic notions of algorithmic information theory using Turing machines. In particular, we pick *universal prefix-free machine* as our *reference machine* U . Such a machine exists by construction: the meat of the argument lies in the standard result that every prefix-free p.c. (*partial computable*) function has a prefix-free machine that computes it (e.g. [9, 3.5]). Our universal machine U satisfies the following: every prefix-free p.c. function $f: 2^{<\omega} \rightarrow 2^{<\omega}$ has a program code p_f for which $U(p_f, x) = f(x)$.

The machine U has two tapes: an *input tape* and a *condition tape* (it also has a work tape, but this will be of no consequence to our arguments, so we shall ignore this fact). Define a p.c. function h such that whenever p_f is a prefix-free program for a prefix-free p.c. function f then

$$h(0^{|p_f|}1p_fx) = U(p_f, x) = f(x).$$

Observe that h is prefix-free. If $A \in 2^\omega$ is an oracle, let U^A be the universal prefix-free machine that has access to the oracle A (this means, the machine can perform a step of the type “does k belong to A ?” for any $k < \omega$, and branch accordingly), and define h_A analogously. Further, if $\tau \in 2^{<\omega}$ then let U^τ be the machine with τ written on the condition tape.

We introduce the following notation: if $\sigma \in 2^{<\omega}$ let $\ell(\sigma)$ denote the length of σ . To avoid confusion with the absolute value on \mathbb{R} which we use heavily in later section, we reserve the symbol $|\cdot|$ for the space \mathbb{R} .

Definition 2.7. Let $\sigma \in 2^{<\omega}$. The *Kolmogorov complexity* of σ is

$$K(\sigma) = \min\{\ell(\rho) \mid h(\rho) = \sigma\}.$$

If $\tau \in 2^{<\omega}$ then the *conditional Kolmogorov complexity* of σ given τ is

$$K(\sigma \mid \tau) = \min\{\ell(\rho) \mid h_\tau(\rho) = \sigma\}.$$

If A is an oracle, $K^A(\sigma)$ is defined analogously, with h_A in place of h_τ . In that case, the condition tape is empty (accessing the oracle does not require any tapes), and hence we can define $K^A(\sigma \mid \tau)$ as above.

From now on, we need to take the prefix-freeness of all machines into account. Note that all log in this paper are \log_2 . As a result of prefix-freeness we immediately obtain that K is *subadditive up to a constant*: for all strings $\sigma, \tau \in 2^{<\omega}$ we have $K(\sigma\tau) \leq K(\sigma) + K(\tau) + c$. Similarly, the following fact is basic yet useful.

Lemma 2.8. *Let $\sigma \in 2^{<\omega}$. Then $K(\sigma) \leq \ell(\sigma) + 2\log(\ell(\sigma)) + c$ for some constant c .*

2.3. Randomness. We give a brief introduction to the basic notions of randomness. For in-depth reviews see Downey and Hirschfeldt [9, 6.2] and Li and Vitányi [21, 3.5]. We give one important result here: fortunately, the two definitions given above are equivalent. (This is not the only equivalence: Martin-Löf [31], Schnorr [38], Levin [20], and Chaitin [7] all yield equivalent notions of randomness. This is strong evidence that the notion is, in fact, “correct”.)

The central objects of discussion in Kolmogorov complexity on 2^ω comprise *random sequences*: elements of 2^ω that defy short descriptions up to a constant for all of its initial segments. This definition goes back to Kolmogorov [19] and Solomonoff [40] who simultaneously and independently developed the early theory of algorithmic information. Another definition of randomness for sequences uses statistical (measure-theoretic) tests, the Martin-Löf-tests (or ML-tests), and goes back to Martin-Löf [31].

Recall that an open subset of 2^ω is the countable union of basic clopen sets, and that every basic clopen set is determined by some $\sigma \in 2^{<\omega}$. A set $U \subset 2^\omega$ is (*lightface*) Σ_1^0 , or

effectively open, if the set of finite strings that determine U is computably enumerable. A sequence $\mathcal{U} = \{U_n \mid n < \omega\}$ of sets $U_n \subset 2^\omega$ is *uniformly* Σ_1^0 if every U_n is Σ_1^0 witnessed by some c.e. set $V_n \subset 2^{<\omega}$, and the sequence $\mathcal{V} = \{V_n \mid n < \omega\}$ is uniformly c.e. itself. Martin-Löf randomness uses the natural Lebesgue measure generated by the clopen sets: if $\sigma \in 2^{<\omega}$ then $\lambda(\{\sigma f \mid f \in 2^\omega\}) = 2^{-\ell(\sigma)}$. An ML-test is then a computably enumerable sequence of effectively open sets U_n of decreasing diameter. A *Martin-Löf-random sequences* is not captured by any ML-test.

The following proposition connects the two notions.

Proposition 2.9. *Let $f \in 2^\omega$. The following notions are equivalent:*

- (1) $f \in 2^\omega$ is Kolmogorov random: there exists a constant c such that for all $n < \omega$ we have $K(f \upharpoonright n) \geq n - c$;
- (2) f is Martin-Löf-random: for any uniformly Σ_1^0 sequence $\mathcal{U} = \{U_n \mid n < \omega\}$ of sets $U_n \subset 2^\omega$ for which $\lambda(U_n) \leq 2^{-n}$ we have $f \notin \bigcap \mathcal{U}$ (so f passes the test).

These results relativise (see [9, 6.4]).

Definition 2.10. Let $A \in 2^\omega$ be an oracle. A string $f \in 2^\omega$ is *Kolmogorov random relative to A* if there exists a constant c such that for all $n < \omega$ we have $K^A(f \upharpoonright n) \geq n - c$.

It is a standard result that a universal ML-test \mathcal{V} exists, from which it follows immediately that the Kolmogorov random sequences have measure 1 in 2^ω : every ML-test is Lebesgue null, and hence $2^\omega \setminus \mathcal{V}$ has measure 1. If f passes the universal test \mathcal{V} , it passes all ML-tests, and hence $2^\omega \setminus \mathcal{V}$ is the set of ML-randoms. Thus every $\sigma \in 2^{<\omega}$ has a Kolmogorov random extension. Further, if $A \in 2^\omega$ is an oracle, then we can construct a universal ML-test relative to A . Hence every $\sigma \in 2^{<\omega}$ has a Kolmogorov random extension relative to A .

2.4. Coding objects in $2^{<\omega}$. While, formally, our arguments take place in $2^{<\omega}$, we naturally identify certain finite strings with objects in the domain of discourse; these are usually rational numbers (elements of \mathbb{Q}) and natural numbers (elements of ω). As is common, this identification takes place in the meta-theory; however, determining whether a certain string is to be identified as a rational or natural number is computable. We normally denote the string representation of such objects using an overline: if x is an object in the domain of discourse, then \bar{x} denotes the string which we shall identify as x .

Fixing a particular coding is illustrative—we code objects in the domain of discourse as follows. The implied operation on finite strings below is always concatenation. We remark that these codings are not necessarily optimal.

- If $k < \omega$ then let \bar{k} be the string whose digits are given by the binary expansion of k .
- If $n \in \mathbb{Z}$ then let w be the binary expansion of n with each digit doubled ($n = 101$ becomes $w = 110011$). Then let $\bar{n} = w01$ if $n \geq 0$, and $\bar{n} = w10$ otherwise.
- If $q \in \mathbb{Q}$ then suppose $q = a/b$. Then let $\bar{q} = \bar{a}\bar{b}$.
- If $q = (q_1, \dots, q_m) \in \mathbb{Q}^m$ then let $\bar{q} = \bar{q}_1 \cdots \bar{q}_m$.
- If $x \in \mathbb{R}$, suppose $k < \omega$, and express x in binary. Take the integer part of x and double each digit; denote this string by w . Take the first k bits of x after the binary point, denoted by z . If $x \geq 0$, let $\bar{x}[k] = w01z$; otherwise define $\bar{x}[k] = w10z$.
- If $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, suppose $k < \omega$. Then let $\bar{x}[k] = \bar{x}_1[k] \cdots \bar{x}_m[k]$.

- If $x \in \mathbb{R}$ then let $\bar{x} \in 2^\omega$ be the limit of $\bar{x}[k]$ in the obvious fashion. If $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ then interweave $\bar{x}_1, \dots, \bar{x}_m$ bit by bit.

Observe that, using this coding, if $k < \omega$ then $\ell(\bar{k}) \leq \log(k) + 1$.

The distinction between strings and objects is particularly important when we discuss real numbers (i.e. objects in \mathbb{R}), and their truncated approximations. In other cases we are more casual; for instance, we normally write $K(k)$ and $K(q)$ instead of the formally correct $K(\bar{k})$ and $K(\bar{q})$.

2.5. Dimension of points and the point-to-set principle. Using effective tools in order to answer geometrical and measure-theoretical questions has been an avenue for research for a couple of decades. This development goes back at least to the beginning of this century [22, 33, 24]. Among others, further discoveries were made by Hitchcock [15, 16] and Mayordomo [33], who related the notion of effective dimension of reals (in 2^ω) away from the notion of gales towards Kolmogorov complexity (the connection between martingales and Hausdorff dimension had previously been investigated by Ryabko [36, 37], Staiger [41], and Cai and Hartmanis [5]; gales, a generalisation of martingales, are due to Lutz [23]). As a result of their work, we may now determine the Hausdorff dimension of a set solely from its elements, using randomness notions.

Definition 2.11. Let $f \in 2^\omega$. We define the *dimension of f* by

$$\dim(f) = \liminf_{r \rightarrow \infty} \frac{K(f \upharpoonright r)}{r}$$

This relativises: if $A \in 2^\omega$ is an oracle then define

$$\dim^A(f) = \liminf_{r \rightarrow \infty} \frac{K^A(f \upharpoonright r)}{r}.$$

This notion can be naturally extended to Euclidean space: first, consider the complexity of a point.

Definition 2.12. Let $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. Then we define the *Kolmogorov complexity of x at precision $t < \omega$* by

$$K_t(x) = \min\{K(q) \mid q \in \mathbb{Q}^m \cap B_{2^{-t}}(x)\}$$

where $B_s(y)$ is the open ball with respect to the Euclidean metric, with radius s and centre y . The *effective Hausdorff dimension* of x is then given by

$$\dim(x) = \liminf_{t \rightarrow \infty} \frac{K_t(x)}{t}.$$

Both of these notions relativise.

The ideas behind the definitions stem from effectivising Hausdorff dimension, an idea that is due to Lutz [23] and their notion of gales. An investigation of effective packing dimension followed suit [1], also in terms of gales. The characterisation of effective Hausdorff dimension of reals given in definition 2.12 is due to Mayordomo [33]. We also remark that replacing \liminf by \limsup in definition 2.12 yields a characterisation of effective packing dimension [27], which is denoted by $\text{Dim}(x)$.

We can now state Lutz' and Lutz' point-to-set principle, which forms a cornerstone of our subsequent arguments.

Theorem 2.13 (Point-to-set Principle, [25, Thm. 1]). *Let $n < \omega$ and $E \subset \mathbb{R}^n$. Then*

$$\dim_H(E) = \min_{A \in 2^\omega} \sup_{x \in E} \dim^A(x).$$

Using this theorem, we are now in a position to control the dimension of a set by focussing on individual points. Lutz and Lutz [25] and Lutz and Stull [28] provide outlines and applications of the point-to-set principle. Recently, the point-to-set principle has been extended to arbitrary separable metric spaces [26].

Crucial to our arguments in this paper is the following technical lemma, which allows us to work with finite strings instead of approximating rationals. As we will be working with sequences in Cantor space 2^ω while talking about reals in \mathbb{R} , a convenient identification is useful. Instead of moving back and forth between 2^ω and \mathbb{R} , we use an observation by Lutz and Stull [28, p. 6].

Lemma 2.14 ([29, Corollary 2.4]). *For every $m < \omega$ there exists a constant c such that for all $t < \omega$ and $x \in \mathbb{R}^m$ we have*

$$|K_t(x) - K(\bar{x}[t])| \leq K(t) + c.$$

The proof uses the fact that $x[t]$ provides a reasonable approximation to x , in the sense that its distance to x is bounded by a function that depends on m . In section 3, we provide a similar identification argument for polar coordinates; see proposition 3.1. We note an important corollary right here.

Corollary 2.15. *If $m \geq 1$ and $x \in \mathbb{R}^m$ then $\dim(x) = \liminf_{r \rightarrow \infty} \frac{K(\bar{x}[r])}{r}$.*

3. ARGUING IN POLAR COORDINATES

In the course of our constructions in both theorems 5.1 and 6.1, it will be easier to work in polar coordinates than in Euclidean coordinates. A point (x, y) in Euclidean space *has polar coordinates* (r, θ) if and only if $x = r \cos \theta$ and $y = r \sin \theta$. We will restrict our attention to the first quadrant of the unit disc, which we denote by

$$\mathbb{D} = \left\{ (x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \wedge \sqrt{x^2 + y^2} \leq 1 \right\}.$$

Thus $r \in [0, 1]$ and $\theta \in [0, \pi/2]$. Importantly, all points expressed in the proofs below are given in Euclidean coordinates. When we write (r, θ) we do *not* mean $(r \cos \theta, r \sin \theta)$.

The following lemma can be considered an analogue to lemma 2.14; its proof follows the proof of lemma 2.14 from [29, Corollary 2.4].

Proposition 3.1. *Suppose $(x, y) \in \mathbb{D}$ has polar coordinates (r, θ) . Then*

$$\dim(x, y) = \dim(r, \theta).$$

This argument relativises.

We provide proofs to both directions of proposition 3.1 individually below. We require the following result, which is due to Casey and J. Lutz [6].

Lemma 3.2. *There exists a constant c such that for all $m, s, \Delta s < \omega$ and all $x \in \mathbb{R}^m$ we have*

$$K_s(x) \leq K_{s+\Delta s}(x) \leq K_s(x) + K(s) + c_m(\Delta s) + c$$

where $c_m(\Delta s) = K(\Delta s) + m\Delta s + 2 \log(\lceil \frac{1}{2} \log(m) \rceil + \Delta s + 3) + (\lceil \frac{1}{2} \log(m) \rceil + 3)m + K(m) + 2 \log(m)$.

Observe that the term $c_m(\Delta s)$ does not depend on s .

Before we commence with the proofs of the directions of proposition 3.1, we require the following lemma. This is a standard result [4, p. 151]).

Lemma 3.3. *Suppose $C \subset \mathbb{R}^2$ is compact and convex. If the function $f: C \rightarrow \mathbb{R}^2$ sending (x, y) to $f(x, y)$ is continuously differentiable on C then it satisfies a Lipschitz condition on C .*

The first halves of the proofs below follow the same argument as Lutz and Stull [29, Lemma 2.3]. We briefly observe that the map $(r, \theta) \mapsto (r \cos \theta, r \sin \theta)$ is continuously differentiable everywhere, and that $[0, 1] \times [0, \pi/2]$ is of course compact and convex.

Lemma 3.4 (First half of Proposition 3.1). *There exists a constant c such that whenever $(x, y) \in \mathbb{D}$ has polar coordinates (r, θ) then for all $s < \omega$ we have*

$$K_s(x, y) \leq K(\bar{r}[s]\bar{\theta}[s]) + K(s) + c.$$

Proof. By lemma 3.3, the map translating polar into Cartesian coordinates satisfies a Lipschitz condition as $[0, 1] \times [0, \pi/2]$ is compact and convex: there exists $M > 0$ such that if $(r, \theta), (r', \theta') \in [0, 1] \times [0, \pi/2]$ then

$$(*) \quad |(r \cos \theta, r \sin \theta) - (r' \cos \theta', r' \sin \theta')| \leq M|(r, \theta) - (r', \theta')|.$$

Let $(r, \theta) \in [0, 1] \times [0, \pi/2]$, and suppose $(x, y) = (r \cos \theta, r \sin \theta)$. We define the following:

- let $r_s = r[s]$, and $\theta_s = \theta[s]$, the truncation of r and θ to s bits after the binary point;
- we will consider the approximation $r_s \cos \theta_s$ of $r \cos \theta$ (and similarly $r_s \sin \theta_s$ for $r \sin \theta$); however, this approximation will in general not yield a finite string; hence we define

$$x[s] = (r_s \cos \theta_s)[s] \quad \text{and} \quad y[s] = (r_s \sin \theta_s)[s]$$

which are the truncations to s bits after the binary point of such approximations.

These truncations allow us to approximate the point (x, y) effectively:

Claim 1. $(x[s], y[s]) \in B_{2^{-s}(1+M\sqrt{2})}(x, y)$

Proof of Claim 1. Recall that $x = r \cos \theta$ and $y = r \sin \theta$, and that $x[s] = (r_s \cos \theta_s)[s]$; hence $|(x[s], y[s]) - (r_s \cos \theta_s, r_s \sin \theta_s)| \leq 2^{-s}$, by construction. Using the nomenclature introduced and the Lipschitz condition (*) above we can compute the maximum error

when $(x[s], y[s])$ approximates (x, y) :

$$\begin{aligned}
|(x[s], y[s]) - (x, y)| &= |(x[s], y[s]) - (r_s \cos \theta_s, r_s \sin \theta_s) + (r_s \cos \theta_s, r_s \sin \theta_s) - (x, y)| \\
&\leq |(x[s], y[s]) - (r_s \cos \theta_s, r_s \sin \theta_s)| + |(r_s \cos \theta_s, r_s \sin \theta_s) - (x, y)| \\
&\leq 2^{-s} + |(r_s \cos \theta_s, r_s \sin \theta_s) - (r \cos \theta, r \sin \theta)| \\
&\leq 2^{-s} + M|(r_s, \theta_s) - (r, \theta)| \\
&\leq 2^{-s} + M\sqrt{(r - r_s)^2 + (\theta - \theta_s)^2} \\
&< 2^{-s} + M\sqrt{(2)2^{-2s}} \\
&= 2^{-s} + \sqrt{2}M2^{-s} \\
&= 2^{-s}(1 + M\sqrt{2})
\end{aligned}$$

as required. \dashv

Observe that $2^{-t} = 2^{-s}(1 + M\sqrt{2})$ if and only if $t = s - \log(1 + M\sqrt{2})$. Therefore, if we can compute $(x[s], y[s])$ then we can compute (x, y) at precision $t = s - \log(1 + M\sqrt{2})$. Letting $\Delta t = \log(1 + M\sqrt{2})$ we hence have

$$K_{s-\Delta t}(x, y) \leq K(x[s], y[s]) \leq K(\bar{x}[s]\bar{y}[s]) + c'$$

where c' is the machine constant that turns the string representations $\bar{x}[s]$ and $\bar{y}[s]$ into the real approximations, and hence rationals, $x[s]$ and $y[s]$.

Now, $t + \Delta t = s$, so the right-hand side of lemma 3.2 implies

$$\begin{aligned}
K_s(x, y) &\leq K_{s-\Delta t}(x, y) + K(t) + c_2(\Delta t) + c \\
&\leq K(\bar{x}[s]\bar{y}[s]) + c' + K(t) + c_2(\Delta t) + c
\end{aligned}$$

where $c_2(\Delta t)$ is as in lemma 3.2 and hence does not depend on t , and thus not on s .

Finally, we prove the following claim:

Claim 2. $K(\bar{x}[s]\bar{y}[s]) \leq K(\bar{r}[s]\bar{\theta}[s]) + c''$ for some constant c'' .

Proof of Claim 2. Recall that $r_s = r[s]$ and $\theta_s = \theta[s]$. Then $x[s] = (r_s \cos \theta_s)[s]$, which (as well as $y[s]$) is easily computable via approximations and Taylor's Theorem (multiplication and cos are obviously computable, with associated machine constant c''). \dashv

To recap, we established so far that

$$K_s(x, y) \leq K(\bar{x}[s]\bar{y}[s]) + c' + K(t) + c_2(\Delta t) + c.$$

By the claim we now have

$$\leq K(\bar{r}[s]\bar{\theta}[s]) + c' + c'' + K(t) + c_2(\Delta t) + c.$$

Recall that $\Delta t = \log(1 + M\sqrt{2})$ which is constant. Further, $t = s - \Delta t$, and thus there exists a constant c''' (which only depends on Δt , and hence only on M and not on s) for which $K(t) = K(s - \Delta t) \leq K(s) + c'''$. Hence we have

$$K_s(x, y) \leq K(\bar{r}[s]\bar{\theta}[s]) + K(s) + d$$

where $d = c' + c'' + c''' + c_2(\Delta t) + c$ as required. \square

For the second half, we make the following brief observation. The argument below will focus on points in \mathbb{D} that do not lie on the first axis; this is necessary for a bounding argument involving Lipschitz conditions: for each point (x, y) not on the first axis, we can find a nice neighbourhood on which the coordinate transformation map from Euclidean to polar coordinates is nicely behaved. What about the points *on* the first axis? There is nothing to do, for if $x \geq 0$, the polar coordinates and Euclidean coordinates of the point $(x, 0)$ coincide. Hence proposition 3.1 holds on the first axis trivially.

Lemma 3.5 (Second half of Proposition 3.1). *There exists a constant c such that whenever $(x, y) \in \mathbb{D}$ has polar coordinates (r, θ) then there exist $N_{(x,y)} < \omega$ and $\Delta < \omega$ such that if $s > N_{(x,y)}$ then*

$$K(\bar{r}[s - \Delta]\bar{\theta}[s - \Delta]) \leq K_s(x, y) + K(s) + c.$$

Proof. First, we make an approximating observation.

Claim 1. *For $a \in \mathbb{Q}^2 \cap B_{2^{-r}}(x, y)$ we have $(x[r], y[r]) \in B_{2^{-r}(1+\sqrt{2})}(a)$.*

Proof of Claim 1. By assumption, $|(x, y) - a| < 2^{-r}$, so by the triangle inequality we have

$$\begin{aligned} |(x[r], y[r]) - a| &\leq |(x[r], y[r]) - (x, y)| + |(x, y) - a| \\ &= \sqrt{(x[r] - x)^2 + (y[r] - y)^2} + |(x, y) - a| \\ &\leq \sqrt{2(2^{-2r})} + 2^{-r} \\ &\leq 2^{-r}\sqrt{2} + 2^{-r} \\ &= 2^{-r}(1 + \sqrt{2}) \end{aligned}$$

as required. \dashv

Further, in the notation of Lutz and Stull [29], let $\mathcal{Q}_r^2 = \{2^{-r}z \mid z \in \mathbb{Z}^2\}$ denote the set of r -dyadics. Observe that r -dyadics have at most r -many non-zero post-binary-point bits. It is easy to bound the number of r -dyadics in any open ball. In particular, we have:

Claim 2. *For any $a \in \mathbb{Q}^2$ and $r < \omega$, we have*

$$|\mathcal{Q}_r^2 \cap B_{2^{-r}(1+\sqrt{2})}(a)| \leq (4(1 + \sqrt{2}))^2.$$

Proof of Claim 2. Let C_2 be the square with side length $2(1 + \sqrt{2})2^{-r}$ that is centred at a . It is clear that $B_{2^{-r}(1+\sqrt{2})} \subset C_2$ and thus

$$|\mathcal{Q}_r^2 \cap B_{2^{-r}(1+\sqrt{2})}| \leq |\mathcal{Q}_r^2 \cap C_2|.$$

Observe that C_2 has area $(2(1 + \sqrt{2}))^2 2^{-2r}$. Now, if $x, y \in \mathcal{Q}_r^2$ and $x \neq y$ then $|x - y| \geq 2^{-r}$ (since the elements in \mathcal{Q}_r^2 have at most r -many non-zero post-binary-point bits). Hence consider a *small square*: a square of side length 2^{-r} . Such a small square has area 2^{-2r} and cannot contain more than 4 r -dyadics: one on each of its vertices. Hence, dividing the area of C_2 by the area of a small square and multiplying by 4 for each vertex gives an upper bound for the number of r -dyadics:

$$|\mathcal{Q}_r^2 \cap B_{2^{-r}(1+\sqrt{2})}| \leq \frac{(2(1 + \sqrt{2}))^2 2^{-2r}}{2^{-2r}} 2^2 = (2(1 + \sqrt{2}))^2 2^2 = (4(1 + \sqrt{2}))^2$$

as required. \dashv

Let M with program P be a machine that does the following: on input $\pi = \pi_1\pi_2\pi_3$ if $h(\pi_1) = \bar{k}$ with $k < \omega$, and $h(\pi_2) = \bar{t}$ with $t < \omega$, and $h(\pi_3) = \bar{a}$ with $a = (p, q) \in \mathbb{Q}^2$, then M outputs the k -th dyadic rational in $B_{2^{-t}(1+\sqrt{2})}(a)$. Suppose $a \in \mathbb{Q}^2$ witnesses the complexity of $K_s(x, y)$; then the claims together imply that $(x[s], y[s])$ is the k -th element in $\mathcal{Q}_r^2 \cap B_{2^{-r}(1+\sqrt{2})}(a)$ for some $k < (4(1+\sqrt{2}))^2$. Let the programs π_1, π_2, π_3 be witnesses for $K(k), K(s)$ and $K(a) = K_s(x, y)$, respectively. Then

$$h(0^{\ell(P)}1P\pi_1\pi_2\pi_3) = \bar{x}[s]\bar{y}[s]$$

and thus

$$\begin{aligned} K(\bar{x}[s]\bar{y}[s]) &\leq \ell(\pi_1) + \ell(\pi_2) + \ell(\pi_3) + c \\ &= K(k) + K(s) + K_s(x, y) + c' \\ &\leq K_s(x, y) + K(s) + c \end{aligned}$$

where $K(k)$ can be bounded above and hence only contributes a constant term.

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the computable function mapping a point in Euclidean coordinates to its polar coordinates. On \mathbb{D} (excluding the first axis), this map is given by $(x, y) \mapsto (\sqrt{x^2 + y^2}, \tan^{-1}(y/x))$, and is continuously differentiable. Hence take some $\epsilon > 0$ such that the closed ball B of radius ϵ centred at (x, y) does not intersect the first axis. By lemma 3.3, the map f satisfies a Lipschitz condition on B . Now suppose $s < \omega$ is such that $2^{-s} < \epsilon$; thus $B_{2^{-s}}(x, y) \subset B$. Suppose $(p, q) \in \mathbb{Q}^2 \cap B_{2^{-s}}(x, y)$. Recalling that $f(x, y) = (r, \theta)$ we have

$$\begin{aligned} |(r, \theta) - f(p, q)| &= |f(x, y) - f(p, q)| \\ &\leq M|(x, y) - (p, q)| \\ &\leq M2^{-s} \\ &= 2^{-(s-\log M)}. \end{aligned}$$

Thus, computing $(x[s], y[s])$ yields, after applying the machine that computes f with machine constant c'' (compare with claim 2 of the proof of lemma 3.4), the polar coordinates (r, θ) up to precision $s - \log M$. In other words,

$$\begin{aligned} K(\bar{r}[s - \log M]\bar{\theta}[s - \log M]) &\leq K(\bar{x}[s]\bar{y}[s]) + c'' \\ &\leq K_s(x, y) + K(s) + c'' + c \end{aligned}$$

as needed. \square

Proof of Proposition 3.1. The proof is an easy consequence of the previous two lemmas and the following claim, which is easily seen to be true.

Claim 1. *If $\Delta < \omega$ then $|K(\bar{r}[s]\bar{\theta}[s]) - K(\bar{r}[s - \Delta]\bar{\theta}[s - \Delta])| \leq c$ for some constant c .*

Proof of Claim 1. It is easy to compute $\bar{r}[s - \Delta]\bar{\theta}[s - \Delta]$ from $\bar{r}[s]\bar{\theta}[s]$. For the other direction, let $\bar{r}(\Delta)$ be such that $\bar{r}[s] = \bar{r}[s - \Delta]\bar{r}(\Delta)$, and equally for $\bar{\theta}$. Suppose $h(\pi_1) = \bar{r}[s - \Delta]\bar{\theta}[s - \Delta]$, and $h(\pi_2) = \bar{r}(\Delta)$ and $h(\pi_3) = \bar{\theta}(\Delta)$, and all such programs are optimal. Let p be a program that on input $\pi = \pi_1\pi_2\pi_3$, merges the two strings obtained by π_2 and π_3 with the string from π_1 in the obvious way (recall the coding from section 2.4). Then

$$h(0^{\ell(p)}1p\pi_1\pi_2\pi_3) = \bar{r}[s]\bar{\theta}[s]$$

and thus

$$\begin{aligned} K(\bar{r}[s]\bar{\theta}[s]) &\leq \ell(\pi_1) + \ell(\pi_2) + \ell(\pi_3) + c \\ &= K(\bar{r}[s - \Delta]\bar{\theta}[s - \Delta]) + K(\bar{r}(\Delta)) + K(\bar{\theta}(\Delta)) + c \end{aligned}$$

by optimality. Observe that $\ell(\bar{r}(\Delta)) = \Delta$, and recall that $K(\sigma) \leq \ell(\sigma) + 2\log(\ell(\sigma)) + c'$ if $\sigma \in 2^{<\omega}$; thus

$$K(\bar{r}[s]\bar{\theta}[s]) \leq K(\bar{r}[s - \Delta]\bar{\theta}[s - \Delta]) + 2\ell(\bar{\Delta}) + 4\log(\ell(\bar{\Delta})) + c$$

which, since $\Delta < \omega$ is fixed, is as required. \dashv

The claim now yields the result from the previous two lemmas: let $(x, y) \in \mathbb{D}$ with polar coordinates (r, θ) . Suppose Δ is as in lemma 3.5. Then

$$\begin{aligned} \dim(r, \theta) &= \liminf_{s \rightarrow \infty} \frac{K_s(r, \theta)}{s} \\ &= \liminf_{s \rightarrow \infty} \frac{K(\bar{r}[s]\bar{\theta}[s])}{s} \\ &= \liminf_{s \rightarrow \infty} \frac{K(\bar{r}[s - \Delta]\bar{\theta}[s - \Delta])}{s} \\ &= \liminf_{s \rightarrow \infty} \frac{K_s(x, y)}{s} \\ &= \dim(x, y) \end{aligned}$$

using the fact that $K(s) \leq \log(s) + 2\log(\log(s) + 1) + c$ for some constant. \square

We may now pass to polar coordinates as required. In particular, the points of the co-analytic sets we build in theorem 5.1 will be determined by their radius, which we will construct explicitly.

From now on, if we write (r, θ) below, we usually mean the point that has Euclidean coordinates $(r \cos \theta, r \sin \theta)$; in such cases, $(r, \theta) \in \mathbb{D}$. We will occasionally return to Euclidean coordinates, however, and we will explicitly mention when we do so.

3.1. Projections in polar coordinates. The focus of this paper is on projections of points onto straight lines. We make some simple geometric observations below that will simplify arguments later on. Consider $\theta \in [0, \pi]$, and let L_θ be the straight line that passes through the origin at angle θ with the first coordinate axis. It is clear that $[0, \pi]$ exhausts all straight lines through the origin. Let $(s, \rho) \in \mathbb{D}$ and denote by $\text{proj}_\theta(s, \rho)$ the *projection of (s, ρ) onto L_θ* : the unique point of intersection of L_θ with the unique perpendicular-to- L_θ line containing (s, ρ) . Recall that if $(s, \rho) \in \mathbb{D}$ then $0 \leq s \leq 1$. See fig. 1.

There are two cases: either $|\theta - \rho| \leq \pi/2$ or $|\theta - \rho| > \pi/2$. If $|\theta - \rho| \leq \pi/2$ then the length of the projection is given by $|\text{proj}_\theta(s, \rho)| = s \cos(\theta - \rho)$; otherwise $|\text{proj}_\theta(s, \rho)| = s \cos((\theta + \pi) - \rho)$. Since $\cos(x + \pi) = -\cos(x)$ and $0 \leq s \leq 1$, we conclude:

Lemma 3.6. *For every $(s, \rho) \in \mathbb{D}$ and every $\theta \in [0, \pi]$ we have*

$$|\text{proj}_\theta(s, \rho)| = s|\cos(\theta - \rho)|.$$

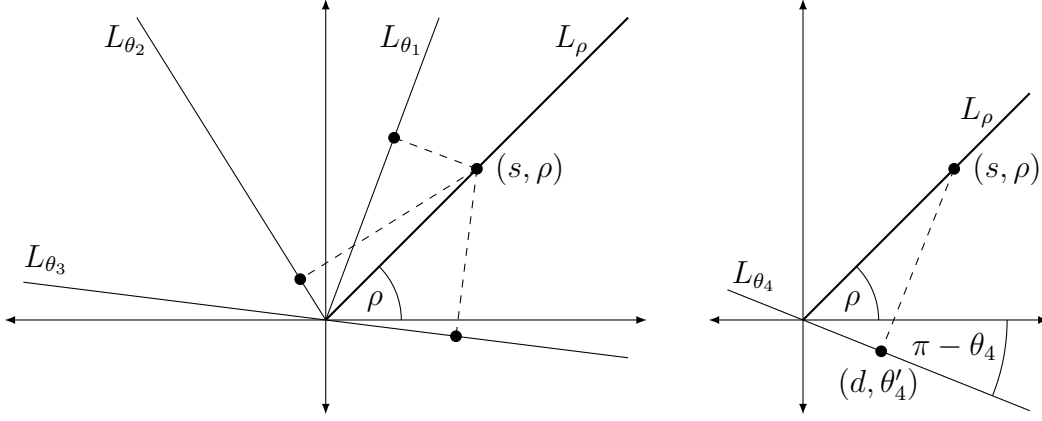


FIGURE 1. With $(s, \rho) \in L_\rho$, the projections onto lines with angles θ_1, θ_2 are straightforward. For θ_3, θ_4 , the projections meet in the fourth quadrant. There, $\theta'_4 = \pi - \theta_4$, and the definition of \cos yields $d = s \cos(\rho + \pi - \theta_4) = s|\cos(\theta_4 - \rho)|$.

In particular, the polar coordinates of the projection of (s, ρ) onto L_θ are

$$\text{proj}_\theta(s, \rho) = \begin{cases} (s|\cos(\theta - \rho)|, \theta) & \text{if } |\theta - \rho| \leq \pi/2 \\ (s|\cos(\theta - \rho)|, \theta + \pi) & \text{otherwise.} \end{cases}$$

Now suppose $E \subset \mathbb{D}$ and fix some $\theta \in [0, \pi]$. Define

$$E(\theta) = \{s|\cos(\theta - \rho)| \mid (s, \rho) \in E\} \subset \mathbb{R}.$$

We show below that, in fact, $\dim_H(E(\theta)) = \dim_H(\text{proj}_\theta(E))$.

We need the following notions: a real number $x \in \mathbb{R}$ is *computable* if there exists a machine that uniformly on input $k < \omega$ (or rather \bar{k}) outputs a rational $q \in \mathbb{Q}$ (or rather \bar{q}) such that $q \in B_{2^{-k}}(x)$; this naturally extends to \mathbb{R}^m for $m \geq 1$.

Lemma 3.7. *Let $m \geq 1$. Every computable real $x \in \mathbb{R}^m$ has dimension 0.*

Proof. Suppose M with program p is a machine that on input \bar{s} for $s < \omega$ computes \bar{q}_s for some $q_s \in \mathbb{Q}^m \cap B_{2^{-s}}(x)$. Then $h(0^{\ell(p)}1p\bar{s}) = \bar{q}_s \in \mathbb{Q}^m \cap B_{2^{-s}}(x)$ and so $K_s(x) \leq \ell(\bar{s}) + c$. Recall that $\ell(\bar{s}) \leq \log(s) + 1$ and thus

$$\dim(x) \leq \liminf_{s \rightarrow \infty} \frac{\log(s) + 1 + c}{s} = 0$$

as needed. □

Lemma 3.8. *Every countable set $E \subset \mathbb{R}^2$ has Hausdorff dimension 0.*

Proof. Suppose $E = \{x_i \mid i < \omega\}$, and let $X = \bigoplus \bar{x}_i$, the infinite join. Let M with program p be a machine with oracle to access to X that on input (\bar{i}, \bar{s}) computes $\bar{x}_i[s]$. Then it is clear that M computes all x_i , and hence by lemma 3.7 and the point-to-set principle theorem 2.13 we have

$$\dim_H(E) \leq \sup_{x \in E} \dim^X(x) = 0$$

as needed. □

Lemma 3.9. *Let $r \in \mathbb{R}$. Then for every oracle $A \in 2^\omega$ the following hold.*

- (1) $\dim^A(r) = \dim^A(r, 0)$
- (2) $\dim^A(r) = \dim^A(-r)$

Proof. It is easily seen, modulo machine constants, that

$$K(\bar{r}[s]) \leq K(\bar{r}[s]\bar{0}[s]) \leq K(\bar{r}[s]) + K(\bar{0}[s]).$$

Since 0 is computable, lemma 3.7 implies $\lim_{s \rightarrow \infty} \frac{K(\bar{0}[s])}{s} = 0$. Applying \liminf yields item 1. For item 2, observe that it is easy to compute $-\bar{r}[s]$ from $\bar{r}[s]$, from which the result follows immediately. Both arguments relativise. \square

Lemma 3.10. *Let $\theta \in [0, \pi]$. If $E \subset \mathbb{D}$ then*

$$\dim_H(\text{proj}_\theta(E)) = \dim_H(E(\theta)).$$

Proof. Fix $\theta \in [0, \pi]$ and suppose $(s, \rho) \in \mathbb{D}$. For brevity, define $p(s, \rho)$ so that

$$p(s, \rho) = |\text{proj}_\theta(s, \rho)| = s|\cos(\theta - \rho)|$$

by lemma 3.6. Now item 1 of lemma 3.9 implies

$$\dim^A(p(s, \rho)) = \dim^A(p(s, \rho), 0)$$

for every oracle $A \in 2^\omega$. Hence let

$$P_\theta(E) = \{(p(s, \rho), 0) \mid (s, \rho) \in E\} \subset \mathbb{R}^2.$$

It is now easy to see that $\dim_H(E(\theta)) = \dim_H(P_\theta(E))$ by the point-to-set principle theorem 2.13.

We now aim to appeal to corollary 2.6: Hausdorff dimension is invariant under rotations. However, rotating $P_\theta(E)$ by θ anti-clockwise is not necessarily equal to $\text{proj}_\theta(E)$: if there exists $(s, \rho) \in E$ for which $|\theta - \rho| > \pi/2$ then $\text{proj}_\theta(s, \rho) = (p(s, \rho), \theta + \pi)$, not $(p(s, \rho), \theta)$. This is easily accounted for: whenever $(s, \rho) \in E$ and $|\theta - \rho| > \pi/2$ then, passing to Euclidean coordinates, consider $(-p(s, \rho), 0)$ instead. To this end, let

$$p^*(s, \rho) = \begin{cases} p(s, \rho) & \text{if } |\theta - \rho| \leq \pi/2 \\ -p(s, \rho) & \text{otherwise.} \end{cases}$$

and hence define, in Euclidean coordinates, the set

$$P_\theta^*(E) = \{(p^*(s, \rho), 0) \mid (s, \rho) \in E\};$$

see fig. 2. By items 1 and 2 of lemma 3.9, it is immediate that $\dim^A(p(s, \rho), 0) = \dim^A(p^*(s, \rho), 0)$ for all oracles $A \in 2^\omega$. Hence the point-to-set principle implies that $\dim_H(P_\theta(E)) = \dim_H(P_\theta^*(E))$. Further, rotating $P_\theta^*(E)$ by θ yields $\text{proj}_\theta(E)$. Hence applying corollary 2.6 shows

$$\dim_H(E(\theta)) = \dim_H(P_\theta(E)) = \dim_H(P_\theta^*(E)) = \dim_H(\text{proj}_\theta(E))$$

as required. \square

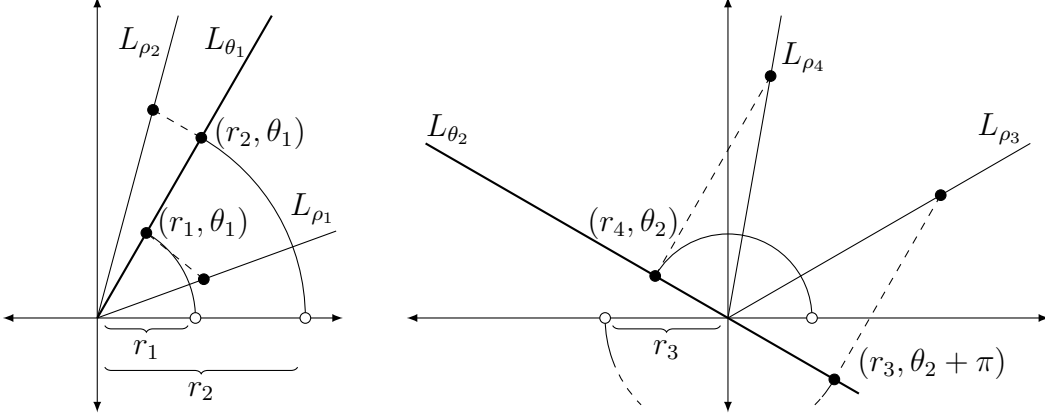


FIGURE 2. If $|\rho - \theta| \leq \pi/2$ then it suffices to consider the length of the projections on the first axis, and rotate (see ρ_1, ρ_2 and θ_1). Otherwise, we need to mirror along the second axis and then rotate by θ ; see θ_2 and ρ_3 .

4. CONSTRUCTING CO-ANALYTIC SETS BY RECURSION

Z. Vidnyánszky [42] has developed a general recursive construction of co-analytic sets using co-analytic conditions, assuming $V=L$. While their result is very general (it uses descriptive set theoretical tools and applies to all Polish spaces), we only focus on a special case, which we describe below. The result is a generalisation of a method used by A. Miller [35]. The earliest version of the method is due to Erdős, Kunen, and Mauldin [11].

For notational simplicity, if $X = \{x_\alpha \mid \alpha < \omega_1\}$ we define $X \upharpoonright \alpha = \{x_\beta \mid \beta < \alpha\}$.

Definition 4.1. Given $F \subset \mathbb{D}^{\leq \omega} \times [0, \pi/2] \times \mathbb{D}$, a set $X = \{x_\alpha \mid \alpha < \omega_1\}$ is *compatible with F* if the following exist:

- an enumeration $\{p_\alpha \mid \alpha < \omega_1\}$ of $[0, \pi/2]$; and
- an enumeration $\{A_\alpha \mid \alpha < \omega_1\} \subset \mathbb{D}^{\leq \omega}$ such that if $\alpha < \omega_1$ then $A_\alpha = X \upharpoonright \alpha$

such that for each $\alpha < \omega_1$ we have $(A_\alpha, p_\alpha, x_\alpha) \in F$.

Observe that since $A_\alpha \in \mathbb{D}^{\leq \omega}$ each A_α has order type $\leq \omega$. This is well-defined since ω_1 is the least uncountable ordinal, hence for every $\omega \leq \beta < \omega_1$ there is a bijection between β and ω , providing the bounded ordering of order type ω .

Definition 4.2. A set $X \subset 2^\omega$ is *cofinal in the Turing degrees* if it is cofinal in the partial ordering of Turing degrees. If $m \geq 1$ and $X \subset \mathbb{R}^m$ then X is cofinal in the Turing degrees if the set $\{\bar{x} \mid x \in X\}$ is.

Theorem 4.3 ([42, Thm. 1.3], $V=L$). *Let $F \subset \mathbb{D}^{\leq \omega} \times [0, \pi/2] \times \mathbb{D}$. If F is co-analytic and if for all $(A, p) \in \mathbb{D}^{\leq \omega} \times [0, \pi/2]$ the section*

$$F(A, p) = \{x \in \mathbb{D} \mid F(A, p, x)\}$$

is cofinal in the Turing degrees, then there exists a co-analytic set $X \subset \mathbb{D}$ that is compatible with F .

It should be noted that the theorem above has been proven to hold for all Polish spaces and all uncountable Borel subsets of an arbitrary Polish space [42].

Theorem 4.3 proves a type of recursion principle: if $[0, \pi/2] = \{p_\alpha \mid \alpha < \omega_1\}$ is considered the set of *conditions*, then \mathbb{D} is the set of *candidates*. The theorem guarantees the existence of a set $X = \{x_\alpha \mid \alpha < \omega_1\}$ which, a posteriori, can be considered as being constructed in stages: at each stage a particular condition is satisfied (by picking a suitable candidate) without violating already satisfied conditions.

We outline the intuition: at stage α we have access to A_α (the set of elements we have already enumerated into X) and to the current condition to be satisfied, p_α . The section $F(A_\alpha, p_\alpha)$ now gives the set of *suitable candidates* which both satisfy condition p_α and respect A_α . Since $A_\alpha = X \upharpoonright \alpha$, we see that X satisfies all conditions and is coherent (in a sense similar to that of coherence in inverse limits in category theory).

In our particular case, we think of \mathbb{D} as the set of candidates, and of $[0, \pi/2]$ as the set of conditions: these are the angles we project on. We satisfy conditions differently in theorems 5.1 and 6.1: let $\theta \in [0, \pi/2]$ be the current condition to be satisfied. In theorem 5.1, we pick a suitable candidate on the straight line through the origin at angle θ ; in theorem 6.1, we use θ as an oracle to find a point (lying on a different line) that is sufficiently complicated relative to it.

We then show that the sections containing suitable candidates are cofinal in the Turing degrees. We prove that for every $X \in 2^\omega$ there exists $x \in F(A, p)$ such that x codes some $Y \geq_T X$ via some m -reduction; we incorporate those reductions via what we call *folding maps* in section 5.2. (For an introduction to computability theoretic notions such as m -reductions see Soare [39].)

Vidnyánszky and Medini have provided applications of theorem 4.3 [42, 34]: among others, they construct a co-analytic two-point set as well as a co-analytic Hamel basis and MAD (maximally almost disjoint) family of sets. All of these results had been obtained previously by A. Miller [35].

5. THE PROOF OF THEOREM 5.1

In this section, we construct the following counterexample: a plane set of Hausdorff dimension 1, all of whose projections have dimension 0. Using the results from section 3, we will argue in polar coordinates.

Theorem 5.1 ($V=L$). *There exists a co-analytic set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1$ while, for every $\theta \in [0, \pi]$ we have $\dim_H(\text{proj}_\theta(E)) = 0$.*

We make use of the following classical theorem. We give a proof using effective dimension, and the point-to-set principle theorem 2.13.

Lemma 5.2. *If $E \subset \mathbb{R}^2 \setminus \{0\}$ intersects every line through the origin in \mathbb{D} , then $\dim_H(E) \geq 1$.*

Proof. Let $A \in 2^\omega$ be an oracle. There exists $B \in 2^\omega$ random relative to A . Thus $\bar{\theta} = 0001B \in 2^\omega$ codes a real $\theta \in (0, 1)$. Since B is random relative to A , we know $K^A(B \upharpoonright s) \geq s - c$ for some constant c . As $B \upharpoonright s$ is easily computable from $\bar{\theta}[s]$ we have

$$s - c \leq K^A(B \upharpoonright s) \leq K^A(\bar{\theta}[s]) + c'$$

for some machine constant c' . Thus

$$\dim^A(\theta) = \liminf_{s \rightarrow \infty} \frac{K^A(\bar{\theta}[s])}{s} \geq \liminf_{s \rightarrow \infty} \frac{K^A(B \upharpoonright s)}{s} \geq \liminf_{s \rightarrow \infty} \frac{s - c}{s} = 1.$$

Since E intersects the line with angle θ , there exists $r > 0$ such that $(r, \theta) \in E$. Therefore

$$\dim^A(r, \theta) = \liminf_{s \rightarrow \infty} \frac{K^A(\bar{r}[s]\bar{\theta}[s])}{s}.$$

We can easily compute $\bar{\theta}[s]$ from $\bar{r}[s]\bar{\theta}[s]$, so $K^A(\bar{\theta}[s]) \leq K^A(\bar{r}[s]\bar{\theta}[s]) + c''$ for some machine constant c'' . Hence

$$\dim^A(r, \theta) = \liminf_{s \rightarrow \infty} \frac{K^A(\bar{r}[s]\bar{\theta}[s])}{s} \geq \liminf_{s \rightarrow \infty} \frac{K^A(\bar{\theta}[s])}{s} = \dim^A(\theta) = 1.$$

Since A was arbitrary, the result follows. \square

5.1. The roadmap towards a proof. We assume $V=L$, and hence let $B = \{\theta_\alpha \mid \alpha < \omega_1\}$ be an enumeration of $[0, \pi/2]$. We want to argue by induction on ω_1 and hence build $E \subset \mathbb{D}$ satisfying theorem 5.1 in stages; we think of the angles in B as the *conditions* (or *requirements*) which need to be satisfied. During our construction, when considering condition φ , we also handle $\varphi + \pi/2$ at the same time. By theorem 4.3, at stage α we have access to all points (r_i, θ_i) already enumerated into E . We aim to satisfy condition θ_α . As a shorthand, denote $\theta = \theta_\alpha$. We argue as follows:

- (1) Let $A_\alpha = \{(r_i, \theta_i) \mid i < \omega\}$, the set of points already enumerated into E . For each $i < \omega$ the angular coordinate θ_i tells us which condition we have already satisfied.
- (2) Construct $r \in (0, 1)$ such that $\dim(r|\cos(\theta - \theta_i)|) = 0$ and $\dim(r|\cos(\theta + \pi/2 - \theta_i)|) = 0$ for all $i < \omega$. This suffices by lemma 3.10.
- (3) Enumerate the pair (r, θ) into E .

Observe that the set of reals in item (2) must be cofinal in the Turing degrees for theorem 4.3 to apply. The following proposition is essential.

Proposition 5.3. *Suppose $a_i \in (0, 1)$ for all $i < \omega$. There exists $r \in (0, 1)$ such that $\dim(a_i r) = 0$ for all $i < \omega$. The set of such r is cofinal in the Turing degrees.*

We will postpone the proof of proposition 5.3 to section 5.2. However, having it in hand we may already give a proof of theorem 5.1. One additional lemma is needed before we do so.

Lemma 5.4. *For every $A \in 2^\omega$ and $a \in \mathbb{R}$ the set $\{x \in \mathbb{R} \mid \dim^A(x) = a\}$ is Borel.*

Proof. We need to show that \dim as a function of x is Borel measurable. Recall its definition

$$\dim(x) = \liminf_{n \rightarrow \infty} \frac{K_n(x)}{n} = \liminf_{n \rightarrow \infty} \frac{\min\{K(q) \mid q \in \mathbb{Q} \cap B_{2^{-n}}(x)\}}{n}.$$

Since the \liminf of a sequence of Borel measurable functions is itself Borel measurable, it suffices to show that $K_n(x) = \min\{K(q) \mid q \in \mathbb{Q} \cap B_{2^{-n}}(x)\}$ is Borel measurable. This is

easily seen: observe that

$$\begin{aligned} K_n(x) < c &\iff \exists q \in \mathbb{Q} \cap B_{2^{-n}}(x) (K(q) < c) \\ &\iff x \in \bigcup_{q \in \mathcal{K}(c)} B_{2^{-n}}(q) \end{aligned}$$

where $\mathcal{K}(c) = \{p \in \mathbb{Q} \mid K(p) < c\}$. Hence K_n is Borel measurable, and thus so is \dim . This argument relativises. \square

Proof of Theorem 5.1. We use theorem 4.3 and define $F \subset \mathbb{D}^{\leq \omega} \times [0, \pi/2] \times \mathbb{D}$ such that

$$\begin{aligned} (A, \varphi, (r, \theta)) &\in F \text{ if and only if} \\ \varphi = \theta \text{ and for all } (r', \theta') &\in \text{ran}(A) \text{ we have} \\ \dim(r|\cos(\varphi - \theta')) &= \dim(r|\cos(\varphi + \pi/2 - \theta')) = 0. \end{aligned}$$

In particular, observe that every point witnessing that condition φ is satisfied lies on the line L_φ .

In order to apply theorem 4.3, we must show that F is co-analytic; but this follows immediately from lemma 5.4. Hence let $\varphi \in [0, \pi/2]$. We now focus on the sections of F : by definition, given $\alpha < \omega_1$ we have

$$F(A, \varphi) = \{(r, \theta) \mid (A, \varphi, (r, \theta)) \in F\}.$$

Suppose $A = \{(r_i, \theta_i) \mid i < \omega\} \in \mathbb{D}^{\leq \omega}$, and hence countable. Let

$$a_i = |\cos(\varphi - \theta_i)| \text{ and } b_i = |\cos(\varphi + \pi/2 - \theta_i)|.$$

Observe that, by construction, we have $(r, \theta) \in F(A, \varphi)$ if and only if $\theta = \varphi$ and $\dim(ra_i) = \dim(rb_i) = 0$ for all $i < \omega$. Now proposition 5.3 implies that this section is cofinal in the Turing degrees. Therefore, using lemma 5.4, we see that theorem 4.3 is applicable: there exists a co-analytic set

$$E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\} \subset \mathbb{R}^2$$

which is compatible with F . In particular, there exist enumerations $\{\varphi_\alpha \mid \alpha < \omega_1\} = [0, \pi/2]$ and $\{A_\alpha \mid \alpha < \omega_1\}$ of $A_\alpha = \{(r_i, \theta_i) \mid i < \omega\} = E \upharpoonright \alpha$ such that for each $\alpha < \omega_1$,

$$(r_\alpha, \theta_\alpha) \in F(A_\alpha, \varphi_\alpha).$$

In particular, $\theta_\alpha = \varphi_\alpha$.

For the verification, let $\varphi \in [0, \pi]$. We show that $\dim_H(\text{proj}_\varphi(E)) = 0$. By lemma 3.10, it suffices to show that $\dim_H(E(\varphi)) = 0$, where $E(\varphi) = \{r|\cos(\varphi - \theta) \mid (r, \theta) \in E\}$. This is what we show below.

Firstly, observe that either $\varphi = \varphi_\delta \in [0, \pi/2]$ for some $\delta < \omega_1$; or $\varphi = \varphi_\delta + \pi/2 \in (\pi/2, \pi]$ for some $\varphi_\delta \in [0, \pi/2]$. Let δ be such, and recall that $E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\}$. We consider the points that were enumerated before condition φ_δ and those enumerated after φ_δ separately.

$\leq \delta$: At condition φ_δ , define (analogous to lemma 3.8) the oracle

$$X = \bigoplus \left\{ \overline{r_\beta|\cos(\varphi_\delta - \theta_\beta)}, \overline{r_\beta|\cos(\varphi_\delta + \pi/2 - \theta_\beta)} \mid \beta \leq \delta \right\}.$$

Then X computes $r_\beta|\cos(\varphi_\delta - \theta_\beta)|$ and $r_\beta|\cos(\varphi_\delta + \pi/2 - \theta_\beta)|$ for all $\beta \leq \delta$. Since either $\varphi = \varphi_\delta$ or $\varphi = \varphi_\delta + \pi/2$, lemma 3.7 implies in particular that

$$\dim^X(r_\beta|\cos(\varphi - \theta_\beta)|) = 0$$

for all $\beta \leq \delta$.

$> \delta$: We show that for every $\beta > \delta$ we have $\dim(r_\beta|\cos(\varphi - \theta_\beta)|) = 0$. Let $\delta < \beta < \omega_1$. Then $(r_\beta, \theta_\beta) \in F(A_\beta, \varphi_\beta) = F(E \upharpoonright \beta, \varphi_\beta)$. But the conditions we have already attended to at stage β are exactly the angular coordinates of the points enumerated into $E \upharpoonright \beta$; in particular, $E \upharpoonright \beta = \{(r_\alpha, \varphi_\alpha) \mid \alpha < \beta\}$. So for all $\gamma < \beta$, again by definition of F , we have

$$\dim(r_\beta|\cos(\varphi_\gamma - \theta_\beta)|) = \dim(r_\beta|\cos(\varphi_\gamma + \pi/2 - \theta_\beta)|) = 0.$$

Since $\delta < \beta$ and either $\varphi = \varphi_\delta$ or $\varphi = \varphi_\delta + \pi/2$ we have in particular

$$\dim(r_\beta|\cos(\varphi - \theta_\beta)|) = 0.$$

We picked $\delta < \beta < \omega_1$ arbitrarily, hence this holds for all such β , as required.

Thus, by the point-to set principle theorem 2.13 and lemma 3.10, we have

$$\begin{aligned} \dim_H(\text{proj}_\varphi(E)) &= \dim(E(\varphi)) \\ &= \min_{A \in 2^\omega} \sup_{\alpha < \omega_1} \dim^A(r_\alpha|\cos(\varphi - \theta_\alpha)|) \\ &\leq \sup_{\alpha < \omega_1} \dim^X(r_\alpha|\cos(\varphi - \theta_\alpha)|) \\ &= 0. \end{aligned}$$

Now $\dim_H(E) \geq 1$ by lemma 5.2. □

Finally, the fact that $\dim_H(E) = 1$ is a consequence of the following corollary.

Corollary 5.5. *Suppose $E \subset \mathbb{D}$. Then $\dim_H(\text{proj}_\theta(E)) \geq \dim_H(E) - 1$.*

Proof. Suppose $(r, \theta) \in \text{proj}_\theta(E)$. By lemma 3.6, we know that $r = s|\cos(\theta - \rho)|$ for some $(s, \rho) \in E$. But this means there is only one piece of information missing: from (r, θ) , we can compute s from ρ , and vice versa. Hence suppose $\dim(r, \theta) = \epsilon$. Since $\dim(s), \dim(\rho) \leq 1$ we see that $\dim(s, \rho) \leq \dim(r, \theta) + 1$, which is as desired. □

5.2. Proving Proposition 5.3. An interval is (*open*) *dyadic* if it is of the form $(j/2^k, (j+1)/2^k)$. Intervals of the form $[j/2^k, (j+1)/2^k]$ are *closed dyadic*. Observe that if $x \in (j/2^k, (j+1)/2^k)$ then $|x - j/2^k| \leq 2^{-k}$, and hence x and $j/2^k$ agree on the first k bits in their binary expansion: both start with the binary expansion of j .

In the results below, we work with open intervals in $(0, 1)$. All reals are expressed in binary. Instead of manipulating intervals directly, we will argue in terms of dyadic reals, which we will express by their finite binary expansion. For this, we introduce the following notation. If $\sigma \in 2^{\leq \omega}$, let $\tilde{\sigma} = 0.\sigma \in \mathbb{R}$. If $\sigma \in 2^{< \omega}$, let $\tilde{\sigma}_+ = 0.\sigma 1^\infty \in \mathbb{R}$; let $[\tilde{\sigma}]$ denote the open interval $(\tilde{\sigma}, \tilde{\sigma}_+)$. If $a \in \mathbb{R}$ then $a[\tilde{\sigma}] = (a\tilde{\sigma}, a\tilde{\sigma}_+)$.

Some basic facts that follow directly from our definitions are:

- If $\sigma \in 2^{< \omega}$ then $[\tilde{\sigma}]$ is a dyadic interval; so if $x \in [\tilde{\sigma}]$ then x and $\tilde{\sigma}$ agree on the initial segment of length $\ell(\sigma)$. We can think of x *extending* $\tilde{\sigma}$.

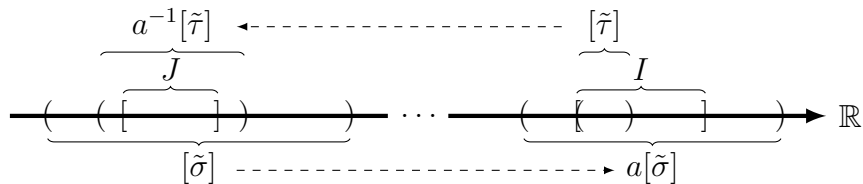


FIGURE 3. We start on the left and argue anti-clockwise: considering $a[\tilde{\sigma}]$ yields an open interval; the largest closed dyadic interval inside is I . Picking a suitable $\tilde{\tau} \in I$ yields $a^{-1}[\tilde{\tau}]$. The largest dyadic interval contained in it is J with left end-point $d = \tilde{\rho}$. Hence $[\tilde{\rho}] \subset J$, where in fact the interior of J equals $[\tilde{\rho}]$.

- Conversely, if I is dyadic and $\sigma \in 2^{<\omega}$ is such that $\tilde{\sigma}$ is the left-end point of I , then $I = [\tilde{\sigma}]$.
- If I is dyadic and $\sigma \in 2^{<\omega}$ is such that $\tilde{\sigma} \in I$ then $[\tilde{\sigma}] \subset I$.
- In particular, if $\sigma, \rho \in 2^{<\omega}$ then $\sigma \prec \rho$ if and only if $\tilde{\rho} \in [\tilde{\sigma}]$.

Lemma 5.6. *Let $\sigma \in 2^{<\omega}$ and $a \in (0, 1)$. Suppose $0 < \epsilon < 1$. There exist $\rho, \tau \in 2^{<\omega}$ such that:*

- (1) $\sigma \prec \rho$
- (2) $a[\tilde{\rho}] \subset [\tilde{\tau}]$
- (3) $K(\tau)/\ell(\tau) < \epsilon$

Proof. Let σ, a and ϵ be given. Consider $a[\tilde{\sigma}]$. Since $[\tilde{\sigma}]$ is open, so is $a[\tilde{\sigma}]$, and thus it contains a closed dyadic interval. Take the largest (in diameter) such interval I , and pick $\tau' \in 2^{<\omega}$ such that $\tilde{\tau}'$ is the left end-point of I . By closedness, $\tilde{\tau}' \in a[\tilde{\sigma}]$. By standard results on Kolmogorov complexity, there exists a smallest $s < \omega$ such that $\tau = \tau'0^s$ satisfies

$$\frac{K(\tau)}{\ell(\tau)} < \epsilon.$$

In particular, $\tilde{\tau}' = \tilde{\tau} \in I$. Now consider $[\tilde{\tau}]$, which is open and hence so is $a^{-1}[\tilde{\tau}]$. Let J denote the largest closed dyadic interval contained in $a^{-1}[\tilde{\tau}]$, and call its left end-point d . Again by closedness, $d \in a^{-1}[\tilde{\tau}]$. Let $\rho \in 2^{<\omega}$ be such that $\tilde{\rho} = d$.

Now $\sigma \prec \rho$: by construction, $\tilde{\rho} = d \in J \subset a^{-1}[\tilde{\tau}]$. The string τ properly extends τ' , thus $[\tilde{\tau}] \subset [\tilde{\tau}']$. Since $\tilde{\tau}'$ is the left end-point of I , the interior of I equals $[\tilde{\tau}']$. Hence $[\tilde{\tau}] \subset [\tilde{\tau}'] \subset I \subset a[\tilde{\sigma}]$, and so $\tilde{\rho} \in a^{-1}[\tilde{\tau}] \subset a^{-1}(a[\tilde{\sigma}]) = [\tilde{\sigma}]$ as needed. Further, $a[\tilde{\rho}] \subset [\tilde{\tau}]$, since $[\tilde{\rho}] \subset J \subset a^{-1}[\tilde{\tau}]$ with all inclusions proper. This completes the argument. \square

In order to achieve cofinality in the Turing degrees when constructing a suitable $r \in (0, 1)$, we need to satisfy each condition (as per item (2) in section 5.1) while coding a given oracle $A \in 2^\omega$ into r . Let

$$\nu(k) = 2^{2^k}$$

determine at which bits of r to code A . We will use the gaps in between the range of ν to satisfy the conditions. We call ν the *folding map*.

5.2.1. *The construction of r .* Suppose (a_i) is the set of conditions, where $a_i \in (0, 1)$ for all $i < \omega$. We construct $r \in (0, 1)$ in stages, by determining its binary expansion, which is given by successive extensions $x_0 \prec x_1 \prec x_2 \prec \dots$ with $x_i \in 2^{<\omega}$. We argue by induction on ω .

- (1) Let $A \in 2^\omega$ be given.
- (2) Let $x_0 = \emptyset$, the empty string.
- (3) Let x_k be given. At stage $k+1$, decode $k+1 = \langle i, n \rangle$ via Cantor's pairing function, for instance, and attend to requirement i . Hence we attend to each requirement infinitely often.
- (4) Apply lemma 5.6 with $a = a_i$ and $\epsilon = \frac{1}{k}$ to obtain a suitable extension $\rho_k \succ x_k$.
- (5) Let $t = \nu(k+1) - \ell(\rho_k) - 1$ and $d = A(k)$ and define

$$x_{k+1} = \begin{cases} \rho_k 0^t d & \text{if } \ell(\rho_k) < \nu(k+1) \\ (\rho_k \upharpoonright (\nu(k+1) - 1))d & \text{otherwise.} \end{cases}$$

Therefore, if $k > 0$ then $\ell(x_k) = \nu(k)$ by induction.

- (6) Define $x = \bigcup_{k < \omega} x_k$, and hence let $r = \tilde{x}$.
- (7) Observe that A is computably *folded into x* : for all $k < \omega$, we have $x(\nu(k+1)-1) = A(k)$.

In order to complete the proof of proposition 5.3, we need to ensure that the second case in the equation in item (5) only occurs finitely often for each requirement a_i . The following lemma assures us that this is indeed the case.

Before we proceed with the proof, another couple of useful facts about intervals follow. Let $(x, y) \subset (0, 1)$.

- (i) By $\text{diam}((x, y)) = y - x$ we denote the diameter of (x, y) . If $\sigma \in 2^{<\omega}$ then $\text{diam}([\tilde{\sigma}]) = 2^{-\ell(\sigma)}$. In particular, $-\log(\text{diam}([\tilde{\sigma}])) = \ell(\sigma)$.
- (ii) If $k < \omega$ is such that $k \geq -\log(\text{diam}((x, y))) + 2$ then there exists $j < \omega$ such that the closed dyadic interval $[j/2^k, (j+1)/2^k] \subset (x, y)$.

Lemma 5.7. *For each $a_i \in (0, 1)$ there exists $M_i < \omega$ such that if $k+1 > M_i$ and $k+1 = \langle i, n \rangle$ attends to requirement a_i , then $\ell(\rho_k) < \nu(k+1)$.*

Proof. Fix some $a_i = a$ and suppose we are at stage $k+1 = \langle i, n \rangle$. Let $\rho = \rho_k$. Recall that $\tilde{\rho} \in J \subset a^{-1}[\tilde{\tau}]$. Observe that $\text{diam}(a^{-1}[\tilde{\tau}]) = a^{-1}2^{-\ell(\tau)}$. Now, since J is defined to be the maximal (in diameter) closed dyadic interval inside $a^{-1}[\tilde{\tau}]$, and since $\tilde{\rho}$ is the left end-point of J , items (i) and (ii) imply

$$\begin{aligned} \ell(\rho) &\leq -\log(\text{diam}(a^{-1}[\tilde{\tau}])) + 2 \\ &= \log(a) - \log(2^{-\ell(\tau)}) + 2 = \log(a) + \ell(\tau) + 2. \end{aligned}$$

Recall that $\tau = \tau'0^s$, and hence

$$\ell(\rho) \leq \log(a) + \ell(\tau') + s + 2.$$

Recall that ρ is an extension of x_k (so $x_k = \sigma$ in lemma 5.6). By construction, $\tilde{\tau}' \in I \subset a[\tilde{x}_k]$, where I is dyadic maximal in $a[\tilde{x}_k]$. Therefore

$$\ell(\tau') \leq -\log(\text{diam}(a[\tilde{x}_k])) + 2 = -\log(a) + \ell(x_k) + 2$$

from which we obtain via item (5) that

$$\ell(\rho) \leq \ell(x_k) + s + 4 = \nu(k) + s + 4$$

Consider some stage $k + 1$. We are building $x_{k+1} \succ x_k$, where $\ell(x_{k+1}) = \nu(k + 1)$. Our construction is successful if we need not truncate ρ (as in the latter case in item (5)). In such a case, $\ell(\rho) < \ell(x_{k+1}) = \nu(k + 1)$. Hence it suffices to show that, eventually, $s < \nu(k + 1) - \nu(k) - 4$.

Recall that s is chosen so that $\frac{K(\tau)}{\ell(\tau)} = \frac{K(\tau'0^s)}{\ell(\tau')+s} < \frac{1}{k}$. Simplify this as follows:

$$\begin{aligned} \frac{K(\tau'0^s)}{\ell(\tau') + s} &\leq \frac{K(\tau') + K(0^s) + c'}{s} \\ &\leq \frac{K(\tau')}{s} + \frac{K(s)}{s} + \frac{c''}{s} \\ &\leq \frac{\ell(\tau') + 2 \log(\ell(\tau'))}{s} + \frac{\log(s) + 2 \log(\log(s) + 1)}{s} + \frac{c}{s} \end{aligned}$$

for a sum of machine constants c .

These terms are easily bounded. Clearly, $\frac{c}{s} < \frac{1}{3k}$ if $s > 3kc$. For the middle term, observe that if $s \geq 2$ then $\log(s) + 2 \log(\log(s) + 1) < 3 \log(s)$. Hence,

$$\frac{\log(s) + 2 \log(\log(s) + 1)}{s} < \frac{3 \log(s)}{s}.$$

Since $\log(s)/s$ is monotonically decreasing, if $s > 2^k$ then

$$\frac{3 \log(s)}{s} < \frac{3 \log(2^k)}{2^k} = \frac{3k}{2^k}.$$

Then $\frac{3k}{2^k} < \frac{1}{3k}$ if $9k^2 < 2^k$ which holds for $k \geq 10$. Hence, for large enough k , the bound $s > 2^k$ suffices.

For the first term, recall that $\ell(\tau') \leq -\log(a) + \nu(k) + 2$. Since $a \in (0, 1)$ we know $-\log(a) > 0$. So, for large enough k , it follows that

$$\begin{aligned} \frac{\ell(\tau') + 2 \log(\ell(\tau'))}{s} &\leq \frac{-\log(a) + \nu(k) + 2 + 2 \log(-\log(a) + \nu(k) + 2)}{s} \\ &\leq \frac{-\log(a) + 3\nu(k)}{s} \end{aligned}$$

Since a is fixed we have, for large enough k , that

$$\frac{\ell(\tau') + 2 \log(\ell(\tau'))}{s} \leq \frac{-\log(a) + 3\nu(k)}{s} \leq \frac{4\nu(k)}{s}.$$

Now observe that $\frac{4\nu(k)}{s} \leq \frac{1}{3k}$ if $s > 12k\nu(k)$. Choosing a large enough k we hence see that $s > \max\{3kc, 2^k, 12k\nu(k)\}$ suffices, which, again, reduces to $s > 12k\nu(k)$ once k is large enough.

Finally, note that $12k\nu(k) + 1 < \nu(k + 1) - \nu(k) - 4$ for $k \geq 3$. Thus, once k is sufficiently large to satisfy all conditions above, $s = 12k\nu(k) + 1$ satisfies $\frac{K(\tau'0^s)}{s} < \frac{1}{k}$ while $s < \nu(k + 1) - \nu(k) - 4$. So, eventually, $\ell(\rho)$ is small enough, as required. \square

Proof of Proposition 5.3. Let $A \in 2^\omega$ be given, and suppose (a_i) is the countable sequence of requirements. Construct $x = \bigcup_{k < \omega} x_k$ as in section 5.2.1. Let $r = \tilde{x}$. From section 2.4 and definition 4.2 it is easily seen that A can be obtained computably from the binary expansion of r . Hence we only need to show that the dimension of $a_i r$ is minimal. Fix $i < \omega$ and consider a_i . By lemma 5.7 there exists M such that if $k > M$ and $k = \langle i, n \rangle$ then $\rho_k \prec x$. For each such k , let τ_k be as obtained from lemma 5.6 alongside ρ_k . Now

$$\dim(a_i r) = \liminf_{s \rightarrow \infty} \frac{K(\overline{a_i r}[s])}{s}$$

by corollary 2.15. By construction, $a_i[\tilde{\rho}_k] \subset [\tilde{\tau}_k]$ and $a_i \tilde{\rho}_k \in [\tilde{\tau}_k]$. Thus $a_i r = a_i \tilde{x} \in [\tilde{\tau}_k]$. Further, $K(\tau_k)/\ell(\tau_k) < 1/k$. Let $D = \{k > M \mid k = \langle i, n \rangle \text{ for some } n\}$. Then

$$\begin{aligned} \dim(a_i r) &\leq \liminf_{k \rightarrow \infty, k \in D} \frac{K(\overline{a_i r}[\ell(\tau_k)])}{\ell(\tau_k)} \\ &\leq \liminf_{k \rightarrow \infty, k \in D} \frac{K(\tau_k) + c}{\ell(\tau_k)} \\ &\leq \liminf_{k \rightarrow \infty, k \in D} \frac{1}{k} \\ &= 0 \end{aligned}$$

where c is the machine constant obtaining τ_k from $\overline{a_i r}[\ell(\tau_k)]$ (as per section 2.4). This completes the proof. \square

6. THE PROOF OF THEOREM 6.1

In this section provide a proof to the following result, which is optimal by corollary 5.5.

Theorem 6.1. *For every $0 < \epsilon < 1$, there exists a co-analytic set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1 + \epsilon$ while, for every $\theta \in [0, 2\pi)$ we have $\dim_H(\text{proj}_\theta(E)) = \epsilon$.*

Observe that this is in fact a generalisation of theorem 5.1: the case $\epsilon = 0$ is covered there. Secondly, the case $\epsilon = 1$ is trivial: if $E \subset \mathbb{R}^2$ satisfies $\dim_H(E) = 2$ then corollary 5.5 implies $1 \leq \dim_H(\text{proj}_\theta(E)) \leq 1$ for each θ . Hence our theorems exhaust all cases.

6.1. Roadmap towards a proof. Let $0 < \epsilon < 1$. Assuming $V=L$, we argue as follows.

- (1) Fix an enumeration $\{\varphi_\alpha \mid \alpha < \omega_1\}$ of $[0, \pi/2]$.
- (2) At stage α , let $A_\alpha = \{(r_i, \theta_i) \mid i < \omega\}$, the set of all points already enumerated into our set.
- (3) Let $X \in 2^\omega$ be the sequence whose bits are made up of the binary expansion of φ_α . In particular, X is $\overline{\varphi_\alpha}$ with its first four bits removed (cf. section 2.4).
- (4) We will *not* satisfy condition φ_α by enumerating a point on L_{φ_α} into our set. Instead, we recover the already satisfied conditions by first coding them into r using a suitable folding map: if (r_i, θ_i) was enumerated into our set at stage β , then φ_β is folded into r_i , and can hence be recovered computably. Let $\{\varphi_i \mid i < \omega\}$ be the set of the conditions already satisfied.
- (5) Pick $\theta \in [0, \pi/2]$ such that $\bar{\theta}$ is random relative to X .

- (6) Let (a_i) be an enumeration of all $|\cos(\theta - \varphi_i)|$ and $|\cos(\theta + \pi/2 - \varphi_i)|$. Let Y be the join of X and all \bar{a}_i . We also assume that Y can compute ϵ .
- (7) Construct $r \in (0, 1)$ such that:
- (a) the binary expansion of φ_α is folded computably into the binary expansion of r ;
 - (b) $\dim(ra_i) = \epsilon$ for all $i < \omega$;
 - (c) $\dim^{Y, \bar{\theta}}(r) = \epsilon$.

Remark 6.2. At first, it appears difficult how to control items **7b** and **7c**. In practice, construct r so that $\dim(ra_i) \leq \epsilon$ and $\dim^{Y, \bar{\theta}}(r) \geq \epsilon$. Equality then follows immediately from corollary **5.5**.

We will give some insight into the verification below. Let (a_i) be an enumeration of all $|\cos(\theta - \varphi_i)|$ and $|\cos(\theta + \pi/2 - \varphi_i)|$.

In our construction of a suitable r , we adapt the methods used in the proof of theorem **5.1**. However, instead of inserting long strings of zeroes into the binary expansions of ra_i , we pick a suitable oracle $T \in 2^\omega$ and fold it into ra_i .

The oracle T is suitable if it is random relative to $Y \oplus \bar{\theta}$ (and hence all \bar{a}_i). Now suppose r is as constructed. Then Y (which computes all \bar{a}_i) can compute an initial segment of T from an initial segment of r : just compute an initial segment of ra_i for the correct i . Since T is random relative to $Y \oplus \bar{\theta}$, we can force $\dim^{Y, \bar{\theta}}(r)$ to not dip too low by coding T not too sparsely.

The details can be found in section **6.5**, and the theorem then follows by corollary **5.5**. Further, we use the following simple result, which follows from symmetry of information.

Lemma 6.3. *Let $A \in 2^\omega$ be an oracle. For any $x, y \in \mathbb{R}$ we have $\dim^A(x, y) \geq \dim^A(x) + \dim^{A, \bar{x}}(y)$.*

For an in-depth account of the interplay between relativised dimension and *conditional dimension* of elements of \mathbb{R}^n see Lutz and Lutz [25, 4.3, 4.4], who introduced the latter notion ibidem. The previous lemma is also a consequence of their arguments, and a proof can be found there, too; it follows from the fact that $K(x | y) \geq K^{\bar{y}}(x)$.

Recall that Y computes X , and hence $\dim^X(r) \geq \dim^Y(r)$ for all r . In particular, we now see that

$$\dim^X(r, \theta) \geq \dim^X(\theta) + \dim^{X, \bar{\theta}}(r) \geq \dim^X(\theta) + \dim^{Y, \bar{\theta}}(r) \geq 1 + \epsilon$$

since $\bar{\theta}$ is random relative to X , and by our construction of r . The final steps of this high-level verification are then as follows: suppose we construct E broadly as in the proof of theorem **5.1**. By the same argument as in said proof, using theorem **2.13** we see that

$$\dim_H(\text{proj}_\theta(E)) = \dim_H(E(\theta)) \leq \epsilon$$

since allowing oracles can only decrease the dimension of points. On the other hand, every oracle X appears through some $\varphi_\alpha \in [0, \pi/2]$. Hence there exists a point $(r_\alpha, \theta_\alpha)$ for which $\bar{\theta}$ is random relative to X . Since such a point exists for *every* oracle, theorem **2.13** implies

$$\dim_H(E) \geq \dim^X(r_\alpha, \theta_\alpha) \geq \dim^X(\theta) + \dim^{X, \bar{\theta}}(r) \geq 1 + \epsilon$$

by lemma **6.3**. Then the conclusion follows from corollary **5.5**.

We flesh out the details in the subsequent sections.

6.2. Folding a suitable oracle into r . Fix $\epsilon \in (0, 1)$, and let

$$Z = Y \oplus \bar{\theta}$$

recalling that Y already contains all a_i . Instead of constructing a $T \in 2^\omega$ random relative to Z and then coding it sparsely to obtain dimension ϵ , we use a result first proved by Athreya, Hitchcock, Lutz, and Mayordomo [1, Thm. 6.5]: for every $0 \leq \alpha \leq 1$ there exists $x \in \mathbb{R}$ such that $\dim(x) = \text{Dim}(x) = \alpha$, which they obtain precisely by sparsely coding a random sequence, interleaved with strings of zeroes. Their result relativises by choosing a sequence random to the oracle we desire. Even more so, their construction shows that we have $\dim^Z(\bar{x}) = \dim(\bar{x}) = \epsilon$, so letting $T = \bar{x}$ for a suitable $x \in \mathbb{R}$ is as required.

In the construction of theorem 5.1 we focused on satisfying requirements: we demanded a particular number of consecutive zeroes to appear in the image in order to push the complexity down sufficiently far; and in our verification, we showed that, eventually, the gap between conditions will be large enough so that enough zeroes (read, a sufficiently large s) can be accommodated. In the present argument, we need to be more careful as we must at all times be able to give a good bound on how many bits of T can be computed. Hence we fix the number of bits to be appended so that there is no “overspill”. The following lemma yields such a bound.

Lemma 6.4. *In the argument of lemma 5.6, if $s = \nu(k+1) - \nu(k) - 5$ then $\ell(\rho_k) < \nu(k+1)$.*

Proof. This follows from the proof of lemma 5.7: with a, ρ, τ' as in said argument, we have

$$\ell(\rho_k) \leq \log(a) + \ell(\tau') + s + 2 \leq \nu(k) + s + 4.$$

Equating the term to $\nu(k+1)$ and demanding strict inequalities yields the result. \square

In particular, the following corollary shows that, if we have space for s bits to encode, we can code $s - 5$ bits into the image.

Corollary 6.5. *In the argument of lemma 5.6, with $a \in (0, 1)$: if $\ell(\rho) = m$ and $n > m$ then if $s = n - m - 5$ we have that $\ell(\rho') < n$, where $\ell(\rho')$ is the extension of ρ that codes s bits into $r\bar{\rho}'$.*

We choose the folding map

$$\nu(k) = 2^{2^k} + k.$$

We introduce the shift summand k so as to make sure that the gaps between $\nu(k)$ and $\nu(k+1)$ have length $2^{2^{k+1}} + k + 1 - 2^{2^k} - k = 2^{2^{k+1}} - 2^{2^k} + 1$; the last bit is reserved to code a bit of φ_α into ra_i (as per item 7a). Now, the gap we have available to extend is exactly of length

$$(6.1) \quad \nu(k+1) - \nu(k) - 1 = 2^{2^{k+1}} - 2^{2^k} = 2^{2^k} (2^{2^k} - 1)$$

which is divisible by $2^{(2^k - k)}$. This fact will be useful in the following subsection.

6.3. Coding and saving blocks. Naïvely, our argument should work as follows: at each stage, we construct a radius r that, together with a suitable angle θ , satisfies the requirement at hand. In order to preserve the high dimension of r (relative to Z) and of the points ra_i , we code segments of T into each ra_i . In particular: if a_j is attended to right after a_i , and the last bit of T coded into ra_i is $T(k)$ for some $k < \omega$, then the first bit of T coded into ra_j at that stage is $T(k+1)$. Hence, with a long enough initial segment of r , the oracle Z can compute a long initial segment of T by just picking the correct a_i (which Z computes), computing ra_i , and picking out the coded bits of T .

For the sake of exposition, suppose $T \in 2^\omega$ and consider

$$T^{(m,n)} = \langle T(m), T(m+1), \dots, T(n-1) \rangle.$$

In particular, observe that $\ell(T^{(m,n)}) = n - m$, and that $T(n)$ does not appear in $T^{(m,n)}$. Now, the dimension of ra_i is bounded above by the dimension of T : taking a sufficiently long initial segment $\overline{ra_i}[t]$ of ra_i , we easily find a long string of the form $T^{(m,n)}$ coded into it. Provided that $\ell(T^{(m,n)}) = n - m$ is large enough compared to t , this will force the dimension down—this latter condition is easily ensured by choosing a sparse enough folding map.

However, it is now difficult to show that the dimension of ra_i does not drop properly below the dimension of T . The problem is that it is in general hard to tell how many bits in the multiplication of reals are determined by a single bit: e.g. if $a = 1/\pi$ and $\tilde{\sigma} = 0.\sigma$ for some $\sigma \in 2^{<\omega}$, and $\tau \succ \sigma$, there is no bound on how many bits of the product $a\tilde{\tau}$ are correct in the sense that every extension yields the same initial segment.¹

We circumvent this issue as follows: as we extend r , we *save blocks* of bits that are coded into ra_i throughout the stage. We do this by pulling back the interval, as seen in lemma 5.6. Hence we define the *block map* $\mu: \omega \rightarrow \omega$ by

$$\mu(k) = 2^{(2^k - k)}.$$

Recall that our folding map is given by $\nu(k) = 2^{2^k} + k$. Hence, at stage k with r_k at hand, we have $\nu(k+1) - \nu(k)$ many bits to extend r_k . In particular, the *number of blocks fitting into the gap of stage $k+1$* is given by

$$(6.2) \quad \xi(k) = \frac{\nu(k+1) - \nu(k) - 1}{\mu(k)} = \frac{2^{2^k} (2^{2^k} - 1)}{2^{(2^k - k)}} = 2^k (2^{2^k} - 1).$$

Note that, by our choices, we hence have $\xi(k)\mu(k) = \nu(k+1) - \nu(k) - 1$.

Now, a few lemmas are needed. Firstly, we need to have a good bound on how many bits we can code into ra_i at each stage k , and in each block. And secondly, it is not clear that saving blocks does not cost too many bits. The first is not an issue due to corollary 6.5. We resolve the second later in the *cost lemma 6.6*, after introducing the construction in detail.

As we code T in blocks, it is prudent to describe a suitable partitioning of T beforehand. We do this here: by recursion reconstruct T into segments T_k^j , where k denotes the last

¹For an extreme case, one considers a and σ so that $a\tilde{\sigma} = 1/2 - \epsilon$ for some very small ϵ , and $\text{diam}(a[\tilde{\sigma}]) > 1/4$. Then each bit of precision added to σ shifts the interval a little bit to the right, and halves it. So the number of bits it takes until the first bit is determined, i.e. until $1/2$ does not appear in the interval, depends on ϵ .

completed stage (so if we see T_k^j then we are in stage $k + 1$), and j the active block. In summary, at stage $k + 1$:

- we code $\xi(k) = 2^k(2^{2^k} - 1)$ -many blocks, which follows from eq. (6.2);
- and each block of T coded into the image has length $\mu(k) - 5$, as we lose 5 bits each time as per corollary 6.5.

Hence we obtain

$$T = \bigcup_{3 \leq k < \omega} \left(\bigcup_{1 \leq j \leq \xi(k)} T_k^j \right)$$

where the union operator denotes concatenation. Hence

$$T = T_3^1 \cup T_3^2 \cup \dots \cup T_3^{2040} \cup T_3^1 \cup \dots \cup T_k^{\xi(k)} \cup T_{k+1}^1 \dots$$

since $\xi(3) = 2040$. As mentioned, we lose 5 bits each time we code a block of T , hence

$$\ell(T_k^j) = \mu(k) - 5 = 2^{(2^k - k)} - 5.$$

This confirms why the outer union starts at $k = 3$: below, we have $2^{(2^2 - 2)} - 5 < 0$, so there is no space to code any bits. In particular, the first stage at which bits are coded is stage $k + 1 = 4$, with $\ell(T_k^j) = \ell(T_3^j) = 27$ and $\xi(k) = \xi(3) = 2040$.

6.4. The construction. Recall that our folding and block map are $\nu(k) = 2^{2^k} + k$ and $\mu(k) = 2^{(2^k - k)}$, respectively. Now, the radius r is constructed as follows: suppose φ_α is the active requirement.

- (1) Let $A \in 2^\omega$.
- (2) Let $x_0 = \emptyset$, the empty string.
- (3) Let x_k be given. At stage $k + 1$, decode $k + 1 = \langle i, n \rangle$; we now attend to requirement i .
- (4) We iterate over all $\xi(k)$ -many blocks. Let $0 \leq j < \xi(k) = 2^k(2^{2^k} - 1)$.
 - (a) Let $x_k^0 = x_k$.
 - (b) At block $j + 1$, suppose we have x_k^j . We apply lemma 5.6, but instead of coding zeroes, we code T_k^{j+1} into $a_i \tilde{x}_k^j$. Let ρ_k^{j+1} be the resulting extension. By filling up with s -many zeroes (courtesy of lemma 6.4), we hence find $x_k^{j+1} = \rho_k^{j+1} 0^s$ of length

$$\ell(x_k^j) + \mu(k) = \ell(x_k) + 2^{(2^k - k)}(j + 1).$$

- (5) After the last block, we have one bit left to code A or $\bar{\varphi}_\alpha$ (this follows from eq. (6.1)). By construction, $\ell(x_k^{\xi(k)}) = \nu(k + 1) - 1$; hence define

$$x_{k+1} = x_k^{\xi(k)} d$$

where

$$d = \begin{cases} A(k/2) & \text{if } k \text{ is even} \\ \bar{\varphi}_\alpha((k-1)/2) & \text{if } k \text{ is odd;} \end{cases}$$

hence $\ell(x_{k+1}) = \nu(k + 1)$, as intended. Further, we code the active line into the real we are building, so that we can recover it later.

Of course, we code A as in theorem 5.1 in order to be able to apply theorem 4.3. This completes the construction.

6.5. The verification. In the present context, we have two results to prove: that $\dim(ra_i) \leq \epsilon$ and that $\dim^Z(r) \geq \epsilon$, where $Z = Y \oplus \bar{\theta}$. Then the theorem follows from corollary 5.5. We prove both results individually.

6.5.1. *The dimension of ra_i .* Both verification arguments are “bit counting” arguments: we exhibit a piece of a complicated string coded inside ra_i , and show that said segment is long enough in a precise sense: its length dwarves the length of all non-coded bits. Let $a = a_i$.

Consider $\overline{ar}[m]$ for some m such that

$$\overline{ar}[m] = \sigma \cup \left(\bigcup_{1 \leq j \leq \xi(k)} \sigma_j T_k^j \right)$$

for some k ; hence stage $k + 1$ has just been completed.² We also know that $\ell(\sigma) \leq -\log(a) + \ell(x_k) + 2 = -\log(a) + \nu(k) + 2$, by lemma 5.7. Further, the *cost* of saving a block is given by a bound on the length of each σ_j which we give here:

Lemma 6.6 (The cost lemma). *Let $a \in (0, 1)$ and $r_m \in 2^{<\omega}$. As in lemma 5.6, find $\tilde{\tau}_m$ and I_m dyadic such that $[\tilde{\tau}_m] \subset I_m \subset a[\tilde{\tau}_m]$; let τ'_m be the left end-point of I_m . Further, let $J \subset a^{-1}[\tilde{\tau}_m]$ be dyadic, where $\tilde{\rho}_k$ is the left-endpoint of J . Let $r_{m+1} = \rho_m 0^t$ so that $\ell(r_{m+1}) = \ell(r_m) + \mu(k)$ where k denotes the current stage. Finally, suppose τ'_{m+1} is the left end-point of $I_{m+1} \subset a[\tilde{\tau}_{m+1}]$.*

Then $|\ell(\tau'_{m+1}) - \ell(\tau'_m)| \leq 7$.

Before we proceed with the proof, a few comments are in order. Firstly, consulting fig. 3 alongside the statement and proof of the above lemma is useful, as the figure serves as its motivation. Conceptually, one can think of the hypotheses of this lemma as the intermediate step between moving from one block to the next within a given stage in our construction: r_m is the available string in block m inside some stage, and $\rho \succ \sigma$ is its computed extension. Importantly, $a[\tilde{\tau}_{m+1}]$ contains τ_m as a substring. We are asking: after saving τ_m in $a[\tilde{\tau}_{m+1}]$, how many bits are lost before we begin coding the next block? In particular, if we construct a real r by such approximations r_m and we have established that

$$ar \succ \tau_m \lambda \tau_{m+1}$$

for some $\lambda \in 2^{<\omega}$ by successive block saving, then how long can λ be at most?

Proof. By assumption we have $[\tilde{\tau}_m] \subset I_m \subset [\tilde{r}_m]$, and so $\text{diam}([\tilde{\tau}_m]) \leq \text{diam}(I_m) \leq \text{diam}([\tilde{r}_m])$. Applying $-\log$ and by item (i) we have $-\log(\text{diam}(I_m)) \in [-\log(a) + \ell(r_k), \ell(\tau_k)]$. Since $\tilde{\tau}'_k$ is the left end-point of I_k we have in particular that $\ell(\tau'_m) \in [-\log(a) + \ell(r_m), \ell(\tau_m)]$. We can give an even better bound: by item (ii), we see that $\ell(\tau'_m) \leq -\log(\text{diam}(a[\tilde{r}_m])) + 2 = -\log(a) + \ell(r_m) + 2$, and hence

$$\ell(\tau'_m) \in [-\log(a) + \ell(r_m), -\log(a) + \ell(r_m) + 2].$$

²Considering the strings at the end of stages is prudent as we easily have access to a long consecutive segment of T (albeit interrupted).

By construction, at stage $k + 1$ we code $\mu(k) - 5$ bits into the image for each block (we lose 5 bits each block, as per corollary 6.5). Hence $\ell(\tau_m) = \ell(\tau'_m) + (\mu(k) - 5)$. Therefore, observing by construction that $\ell(\tau'_{m+1}) \geq \ell(\tau_m)$, we see that

$$\begin{aligned} \ell(\tau'_{m+1}) - \ell(\tau_m) &= \ell(\tau'_{m+1}) - \ell(\tau'_m) - (\mu(k) - 5) \\ &\leq -\log(a) + \ell(r_{m+1}) + 2 + \log(a) - \ell(r_m) - (\mu(k) - 5) \\ &= (\ell(r_{m+1}) - \ell(r_m)) - (\mu(k) - 5) + 2 \\ &= \mu(k) - (\mu(k) - 5) + 2 \\ &= 7 \end{aligned}$$

where we use that the block size is $\mu(k)$, and hence $\ell(r_{m+1}) - \ell(r_m) = \mu(k)$. \square

Hence $\ell(\sigma_j) \leq 7$. For simplicity, we let

$$T_k = T_k^1 \cup \dots \cup T_k^{\xi(k)};$$

hence $\ell(T_k) = \xi(k)(\mu(k) - 5)$. The next lemma provides the final technical detail in this half of our verification. For simplicity of notation, let

$$S_k = \bigcup_{1 \leq j \leq \xi(k)} \sigma_j T_k^j.$$

Lemma 6.7. *For $k < \omega$ and $\sigma, (\sigma_j)$ as above, we have*

$$|K(T_k) - K(\sigma S_k)| \leq O(2^{2^k}).$$

Proof. This is an easy “bit counting” argument: the number of bits by which T_k and σS_k differ is given by $\ell(\sigma) + \sum_j \ell(\sigma_j)$. If we also know where the σ_j 's are located, then we can construct each string from the other. Thus,

$$|K(T_k) - K(\sigma S_k)| \leq K(\sigma) + \sum_{1 \leq j \leq \xi(k)} K(\sigma_j, m_j)$$

omitting constants, where m_j denotes the index at which σ_j begins inside S_k . We know $\ell(\sigma_j) \leq 7$ and $\ell(\sigma) \leq -\log(a) + \ell(x_k) + 2 = -\log(a) + 2^{2^k} + k + 2$. Further,

$$\begin{aligned} \ell(\sigma S_k) &= \ell(\sigma) + \sum_{1 \leq j \leq \xi(k)} \ell(\sigma_j) + \ell(T_k) \\ &\leq -\log(a) + \ell(x_k) + 2 + 7\xi(k) + \xi(k)(\mu(k) - 5) \\ &= -\log(a) + \ell(x_k) + 2 + \xi(k)(\mu(k) + 2) \end{aligned}$$

since each of the $\xi(k)$ -many blocks codes $\mu(k) - 5$ -many bits. Observe that

$$\xi(k)\mu(k) = 2^{2^k}(2^{2^k} - 1)$$

and hence is of order $2^{2^{k+1}}$. As $m_j \leq \ell(S_k)$ we see that m_j is thus at most of order $2^{2^{k+1}}$. But now $K(m_j)$ is at most of order 2^{k+1} . It is now easily seen that $\sum_j K(\sigma_j, m_j)$ is of order at most $\xi(k)2^{k+1}$, which is $O(2^{2^k})$, as required. \square

We can now complete the argument: using the previous lemma, we see

$$K(\overline{ar}[m]) = K(\sigma S_k) = K(T_k) + O(2^{2^k}).$$

Further, observe that $\ell(T_k)$ is of order $2^{2^{k+1}}$, since $\ell(T_k) = \xi(k)(\mu(k) - 5)$. As before, $\lim_{k \rightarrow \infty} \frac{2^{2^k}}{2^{2^{k+1}}} = 0$, and so we may ignore terms of order at most 2^{2^k} . This allows us to simplify: let \mathcal{D} be the set of $m < \omega$ at which requirement $a = a_i$ has just been attended to. (In other words, $\overline{ar}[m] = \sigma S_k$ for some k .) Then

$$\dim(ar) \leq \liminf_{m \in \mathcal{D}} \frac{K(\overline{ar}[m])}{m} \leq \liminf_{m \in \mathcal{D}} \frac{K(T_k)}{m} = \epsilon$$

by definition of T .

6.5.2. *The dimension of r with respect to Z .* Recall that

$$Z = Y \oplus \bar{\theta}$$

and that Y contains all a_i . As we need to show that $\dim^Z(r) \geq \epsilon$, it does not suffice to exhibit a set of favourable elements, such as our set \mathcal{D} in the previous lemma. Instead, we show we can decode enough elements of T from *any* initial segment of r .

Suppose

$$\bar{r}[m] = \sigma_1 \cdots \sigma_{k+1} b_1 \cdots b_n \tau$$

where

- σ_i denotes the initial segment of r that satisfied the stage i ;
- b_j denotes the substring of r that satisfied block j of stage $k + 2$; and
- τ is the initial segment of the substring satisfying block $n + 1$.

Hence, observe we are at stage $k + 2$, and n blocks have already been satisfied.

Inside stage $k + 1$, the substring T_k has been coded into ar . Hence, using the oracle Z which computes all a_i , we can recover T_k from \overline{ar} . Recall that

$$\ell(T_k) = \xi(k)(\mu(k) - 5).$$

Observe that since $\lim_{k \rightarrow \infty} \frac{2^{2^k}}{2^{2^{k+1}}} = \lim_{k \rightarrow \infty} \frac{1}{2^k} = 0$, the length of T_k already dwarves the lengths of $T_1 + \dots + T_{k-1}$; hence, it suffices to compute the blocks saved at stage $k + 1$.

Secondly, the worst case to consider above is the case where $n = 0$: in that case the initial segment σ_{k+1} needs to carry enough information to survive against τ , where τ is at most of length $\mu(k + 1) - 1$. This is not an issue, since $\ell(T_k) = \xi(k)(\mu(k) - 5)$ and

$$\lim_{k \rightarrow \infty} \frac{\mu(k + 1) - 1}{\xi(k)(\mu(k) - 5)} = 0.$$

Hence, the information provided in T_k dwarves the unfinished block τ . It now suffices to show that T_k and the completely coded substrings $T_{k+1}^1, \dots, T_{k+1}^n$ can be easily recovered from $\overline{ar}[m]$. This follows from an argument analogous to lemma 6.7:

- take a machine that trims \bar{r} to length $\nu(k + 1) - 1$, and denote the resultant string by ρ (this is where stage $k + 1$ has just been completed);
- compute the correct projection factor $a_i = a$ for stage $k + 1$ using Z (and from the Cantor pairing function);

- compute the largest dyadic interval in $a[\bar{\rho}]$, and let d denote its left end-point. Now

$$\bar{d} = \sigma S_k \sigma'$$

where $\ell(\sigma') \leq 7$, by the cost lemma 6.6.

- By the previous lemma 6.7, we know that the complexity of isolating T_k from $\sigma S_k \sigma'$ is not significant, as required. An identical argument recovers the n blocks.

It now follows from lemma 6.7 that

$$K^Z(T_k \cup T_{k+1}^1 \cup \dots \cup T_{k+1}^n) \leq K^Z(\bar{r}[m]) + O(2^{2^k}) + O(n2^{k+2})$$

where $n < \xi(k+1)$. Thus, in particular

$$(6.3) \quad \frac{K^Z(T_k \cup T_{k+1}^1 \cup \dots \cup T_{k+1}^n)}{m} \leq \frac{K^Z(\bar{r}[m])}{m} + \frac{O(2^{2^k}) + O(n2^{k+2})}{m}$$

where $m = \nu(k+1) + n\mu(k+1) + \ell(\tau)$ and $n < \xi(k+1)$. Next, we verify that the length of T computed on the left-hand side of eq. (6.3) is sufficiently long: indeed,

$$|m - \ell(T_k \cup T_{k+1}^1 \cup \dots \cup T_{k+1}^n)| = \ell(\tau) + \nu(k) + 1 + 5\xi(k) + 5n.$$

Now, $m = \nu(k+1) + n\mu(k+1) + \ell(\tau)$ and $n < \xi(k+1)$ imply

$$\frac{\ell(\tau) + \nu(k) + 5\xi(k) + 5n}{m} \leq \frac{\ell(\tau) + \nu(k) + 5(\xi(k) + \xi(k+1))}{\ell(\tau) + \nu(k+1) + \xi(k+1)\mu(k+1)},$$

applying limits as k goes to infinity shows that the term vanishes. Going back and applying \liminf to both sides of eq. (6.3) now proves that its left-hand side equals ϵ .

Finally, since m is of order $\nu(k+1) + n\mu(k+1)$, i.e. of order at least $2^{2^{k+1}}$, the right-hand side of eq. (6.3) simplifies to its first term. Putting it all together and applying \liminf we hence obtain

$$\epsilon = \liminf_{k \rightarrow \infty} \frac{K^Z(T_k \cup T_{k+1}^1 \cup \dots \cup T_{k+1}^n)}{m} \leq \frac{K^Z(\bar{r}[m])}{m} = \dim^Z(r)$$

as required. Theorem 6.1 now follows immediately from the same arguments as the proof of theorem 5.1, and the overview we gave at the start of this section.

REFERENCES

- [1] K. B. Athreya, J. M. Hitchcock, J. H. Lutz, and E. Mayordomo. Effective strong dimension in algorithmic information and computational complexity. *SIAM Journal on Computing*, 37(3):671–705, 2007.
- [2] V. Beresnevich, K. Falconer, S. Velani, and A. Zafeiropoulos. Marstrand's theorem revisited: Projecting sets of dimension zero. *Journal of Mathematical Analysis and Applications*, 472(2):1820–1845, 2019.
- [3] V. Beresnevich, K. Falconer, S. Velani, and A. Zafeiropoulos. Marstrand's theorem revisited: projecting sets of dimension zero. *J. Math. Anal. Appl.*, 472(2):1820–1845, 2019. With an appendix by David Simmons, Han Yu and Zafeiropoulos.
- [4] G. Birkhoff and G.-C. Rota. *Ordinary differential equations*. John Wiley & Sons, New York-Chichester-Brisbane, third edition, 1978.
- [5] J.-Y. Cai and J. Hartmanis. On Hausdorff and topological dimensions of the Kolmogorov complexity of the real line. *J. Comput. System Sci.*, 49(3):605–619, 1994.
- [6] A. Case and J. H. Lutz. Mutual dimension. *ACM Trans. Comput. Theory*, 7(3):Art. 12, 26, 2015.

- [7] G. J. Chaitin. A theory of program size formally identical to information theory. *J. Assoc. Comput. Mach.*, 22:329–340, 1975.
- [8] R. O. Davies. Two counterexamples concerning Hausdorff dimensions of projections. *Colloq. Math.*, 42:53–58, 1979.
- [9] R. G. Downey and D. R. Hirschfeldt. *Algorithmic randomness and complexity*. Theory and Applications of Computability. Springer, New York, 2010.
- [10] G. Edgar. *Measure, topology, and fractal geometry*. Undergraduate Texts in Mathematics. Springer, New York, second edition, 2008.
- [11] P. Erdős, K. Kunen, and R. D. Mauldin. Some additive properties of sets of real numbers. *Fund. Math.*, 113(3):187–199, 1981.
- [12] K. Falconer, J. Fraser, and X. Jin. Sixty years of fractal projections. In *Fractal geometry and stochastics V*, volume 70 of *Progr. Probab.*, pages 3–25. Birkhäuser/Springer, Cham, 2015.
- [13] K. Falconer and P. Mattila. Strong Marstrand theorems and dimensions of sets formed by subsets of hyperplanes. *J. Fractal Geom.*, 3(4):319–329, 2016.
- [14] L. Fortnow. Kolmogorov complexity. In *Aspects of complexity (Kaikoura, 2000)*, volume 4 of *De Gruyter Ser. Log. Appl.*, pages 73–86. de Gruyter, Berlin, 2001.
- [15] J. M. Hitchcock. *Effective fractal dimension: foundations and applications*. PhD thesis, Iowa State University, USA, 2003.
- [16] J. M. Hitchcock. Correspondence principles for effective dimensions. *Theory Comput. Syst.*, 38(5):559–571, 2005.
- [17] M. Järvenpää. On the upper Minkowski dimension, the packing dimension, and orthogonal projections. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (99):34, 1994.
- [18] R. Kaufman. On Hausdorff dimension of projections. *Mathematika*, 15:153–155, 1968.
- [19] A. N. Kolmogorov. Three approaches to the definition of the concept “quantity of information”. *Problemy Peredači Informacii*, 1(vyp. 1):3–11, 1965.
- [20] L. A. Levin. The concept of a random sequence. *Dokl. Akad. Nauk SSSR*, 212:548–550, 1973.
- [21] M. Li and P. Vitányi. *An introduction to Kolmogorov complexity and its applications*. Texts in Computer Science. Springer, Cham, 2019. Fourth edition of [MR1238938].
- [22] J. H. Lutz. Gales and the constructive dimension of individual sequences. In *Automata, languages and programming (Geneva, 2000)*, volume 1853 of *Lecture Notes in Comput. Sci.*, pages 902–913. Springer, Berlin, 2000.
- [23] J. H. Lutz. Dimension in complexity classes. *SIAM J. Comput.*, 32(5):1236–1259, 2003.
- [24] J. H. Lutz. The dimensions of individual strings and sequences. *Inform. and Comput.*, 187(1):49–79, 2003.
- [25] J. H. Lutz and N. Lutz. Algorithmic information, plane Kakeya sets, and conditional dimension. *ACM Trans. Comput. Theory*, 10(2):Art. 7, 22, 2018.
- [26] J. H. Lutz, N. Lutz, and E. Mayordomo. Extending the reach of the point-to-set principle. In *39th International Symposium on Theoretical Aspects of Computer Science*, volume 219 of *LIPICs. Leibniz Int. Proc. Inform.*, pages Art. No. 48, 14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2022.
- [27] J. H. Lutz and E. Mayordomo. Dimensions of points in self-similar fractals. *SIAM J. Comput.*, 38(3):1080–1112, 2008.
- [28] N. Lutz and D. M. Stull. Projection theorems using effective dimension. In *43rd International Symposium on Mathematical Foundations of Computer Science*, volume 117 of *LIPICs. Leibniz Int. Proc. Inform.*, pages Art. No. 71, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.
- [29] N. Lutz and D. M. Stull. Bounding the dimension of points on a line. *Inform. and Comput.*, 275:104601, 15, 2020.
- [30] J. M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions. *Proc. London Math. Soc. (3)*, 4:257–302, 1954.
- [31] P. Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.
- [32] P. Mattila. Hausdorff dimension, orthogonal projections and intersections with planes. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 1(2):227–244, 1975.
- [33] E. Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inform. Process. Lett.*, 84(1):1–3, 2002.

- [34] A. Medini and Z. Vidnyánszky. Zero-dimensional σ -homogeneous spaces, 2021.
- [35] A. W. Miller. Infinite combinatorics and definability. *Ann. Pure Appl. Logic*, 41(2):179–203, 1989.
- [36] B. Y. Ryabko. Coding of combinatorial sources and Hausdorff dimension. *Dokl. Akad. Nauk SSSR*, 277(5):1066–1070, 1984.
- [37] B. Y. Ryabko. Noise-free coding of combinatorial sources, Hausdorff dimension and Kolmogorov complexity. *Problemy Peredachi Informatsii*, 22(3):16–26, 1986.
- [38] C.-P. Schnorr. A unified approach to the definition of random sequences. *Math. Systems Theory*, 5:246–258, 1971.
- [39] R. I. Soare. *Turing computability. Theory and Applications of Computability*. Springer-Verlag, Berlin, 2016. Theory and applications.
- [40] R. J. Solomonoff. A preliminary report on a general theory of inductive inference. Zator Company Cambridge, MA, 1960.
- [41] L. Staiger. Kolmogorov complexity and Hausdorff dimension. *Inform. and Comput.*, 103(2):159–194, 1993.
- [42] Z. Vidnyánszky. Transfinite inductions producing coanalytic sets. *Fund. Math.*, 224(2):155–174, 2014.

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