Co-analytic Counterexamples to Marstrand's Projection Theorem

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Victoria University of Wellington

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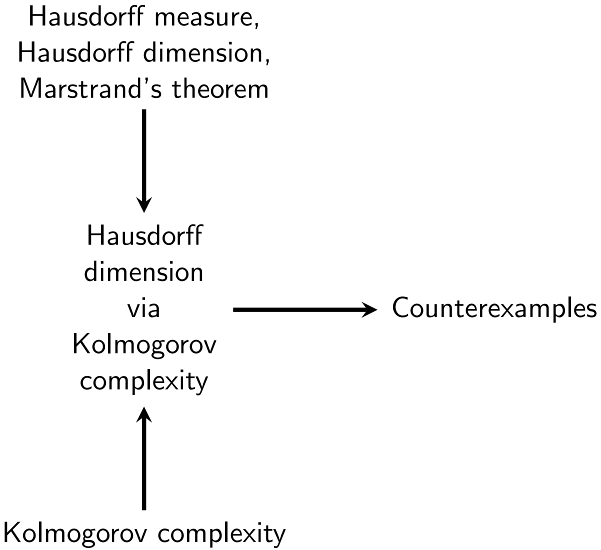
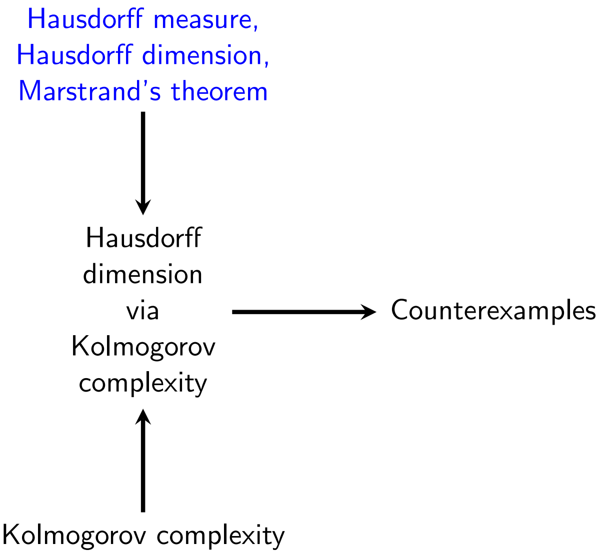

Hausdorff measure, Hausdorff dimension, Marstrand's theorem
Hausdorff dimension via Kolmogorov complexity
Counterexamples
Kolmogorov complexity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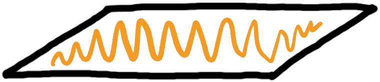
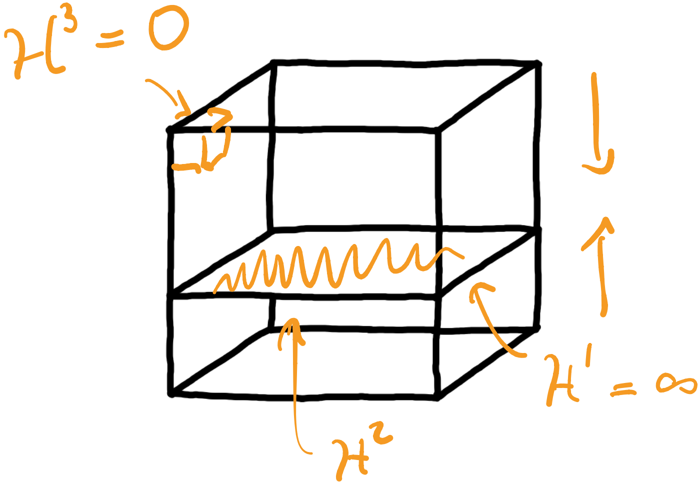
Hausdorff dimension: motivation





$\mathcal{H}^3 = 0$
$\mathcal{H}^2$
$\mathcal{H}^1 = \infty$

Definition (Hausdorff dimension)

For $E \subset \mathbb{R}^n$

$$\dim_H(E) = \sup\{s \mid \mathcal{H}^s(E) = \infty\}$$

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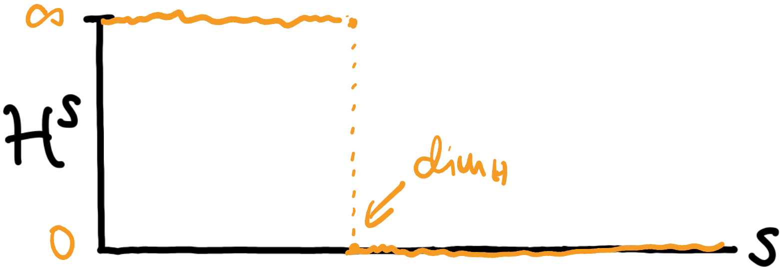

Lemma

$\dim_H$ is invariant under isometries.

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Marstrand's Projection Theorem (J. Marstrand (1954), P. Mattila (1975))

Let $E \subset \mathbb{R}^2$ be analytic. For almost all θ we have

$$\dim_H(p_\theta(E)) = \min\{\dim_H(E), 1\}.$$

This also holds for $\mathbb{R}^n$ and projections onto $\mathbb{R}^m$.

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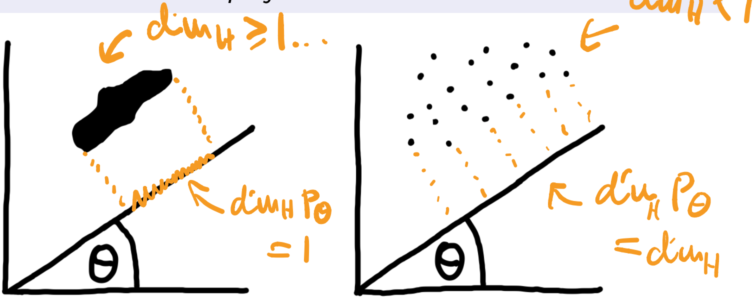

What do we know?

Theorem (N. Lutz and Stull (2018))

If $E \subset \mathbb{R}^2$ and $\dim_H(E) = \dim_P(E)$ then Marstrand's theorem applies.

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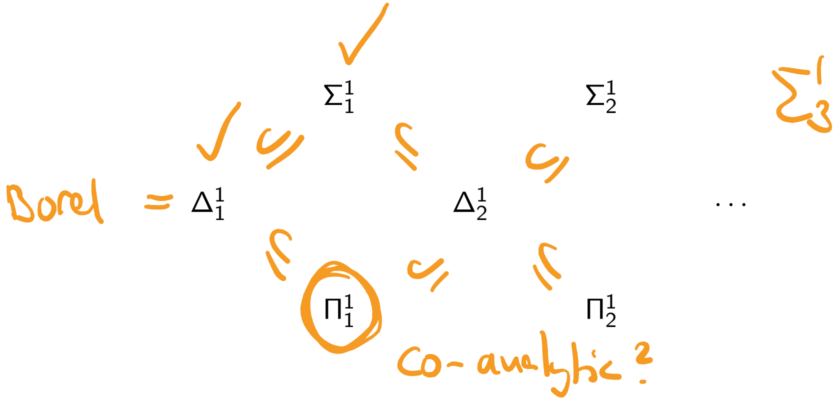
If $E \subset \mathbb{R}^2$ *and* $\dim_H(E) = \dim_P(E)$ *then Marstrand's theorem applies.*

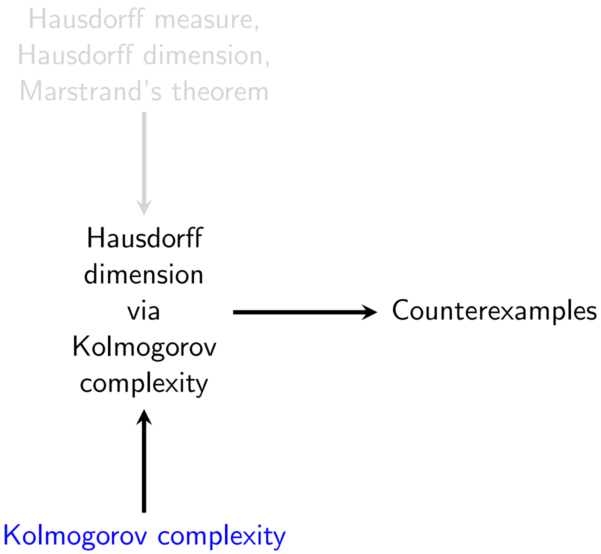
Theorem (Davies (1979))

(CH) *There exists* $E \subset \mathbb{R}^2$ *such that* $\dim_H(E) = 1$ *while* $\dim_H(p_\theta(E)) = 0$ *for all* θ.

Question

What is the "simplest" set failing Marstrand's theorem?

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String complexity $\longleftrightarrow$ description length

Definition

For any p.c. function f, define

$$C_f(\tau) = \begin{cases} \min\{\ell(\sigma) \mid f(\sigma) = \tau\} & \text{if such } \sigma \text{ exists;} \\ \infty & \text{otherwise.} \end{cases}$$

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Definition (Solomonoff (1964); Kolmogorov (1965); Chaitin (1966))

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1. C is within a constant of every C_f
2. $C(\sigma\tau) \leq C(\sigma) + C(\tau) + 2\log(C(\sigma)) + c$

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b	1
c	01

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Definition (Levin (1973); Chaitin (1975))

$K(\tau) = \min\{\ell(\sigma) \mid h'(\sigma) = \tau\}$ where h' is universal for prefix-free machines

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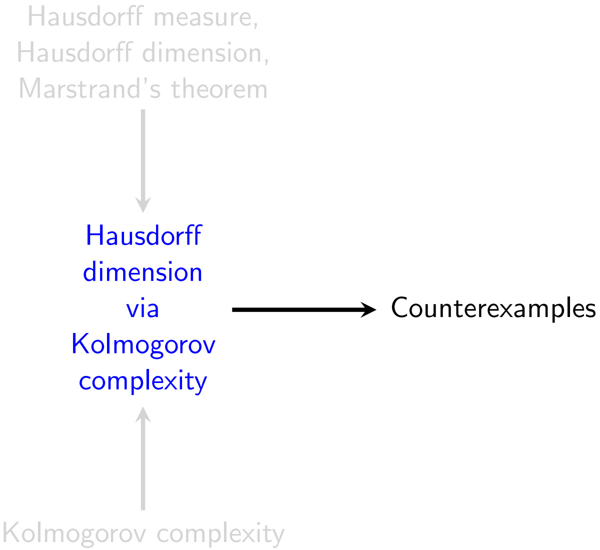
2. $K(\sigma\tau) \leq K(\sigma) + K(\tau) + c$

Definition (Chaitin (1975); Levin (1976))

$f \in 2^\omega$ is *Kolmogorov random* if there exists a constant c for which $K(f[n]) \geq n - c$.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$$\dim(f) = \liminf_{n\to\infty} \frac{K(f[n])}{n}$$

A ∈ 2^ω

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Lemma

- *If* $f \in 2^\omega$ *is computable then* $\dim(f) = 0$.
- *If* $f \in 2^\omega$ *is Kolmogorov random then* $\dim(f) = 1$.

Theorem (Hitchcock (2003))

If $X \subseteq 2^\omega$ is a union of Π^0_1-sets then

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Two questions

Can this characterisation be extended:

- to other spaces ($\mathbb{R}^n$, for instance)?
- beyond Π^0_1 sets?

“Definition”

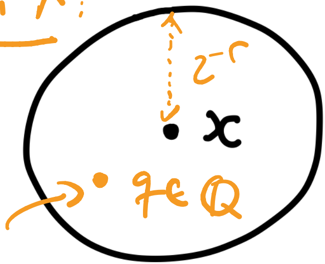
$$K_r(x) = \min\{K(q) \mid q \in \mathbb{Q} \cap B_{2^{-r}}(x)\}$$

and so

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

at precision r:

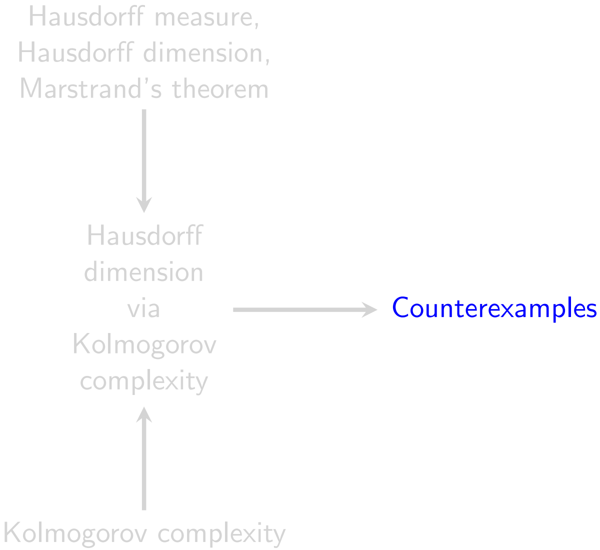
minimal complexity rational



Point-to-set Principle (J. Lutz, N. Lutz (2018))

For $E \subset \mathbb{R}^n$ we have

$$\dim_H(E) = \min_{A \in 2^\omega} \sup_{x \in E} \dim^A(x).$$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Recall Marstrand's theorem ❶

If E is analytic and $\dim_H(E) \geq 1$ then $\dim_H(p_\theta(E)) = 1$ for almost all θ.

The first counterexample

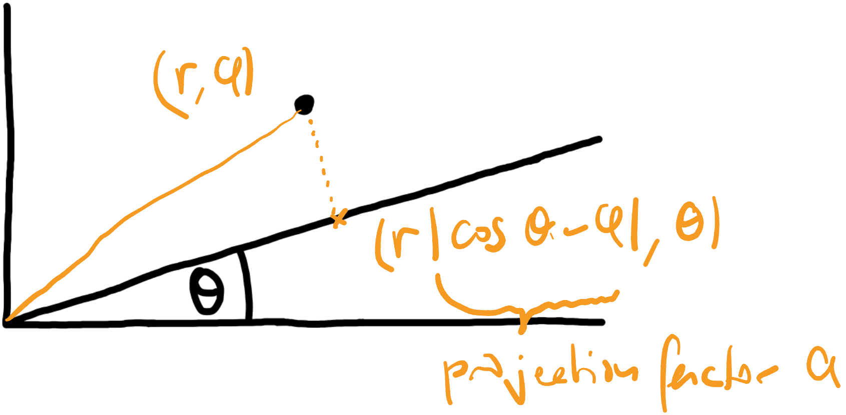
Recall Marstrand's theorem ❶

If E is analytic and $\dim_H(E) \geq 1$ then $\dim_H(p_\theta(E)) = 1$ for almost all θ.

Theorem (R)

($V{=}L$) *There exists a co-analytic* $E \subset \mathbb{R}^2$ *such that* $\dim_H(E) = 1$ *and* $\dim_H(p_\theta(E)) = 0$ *for all* θ.

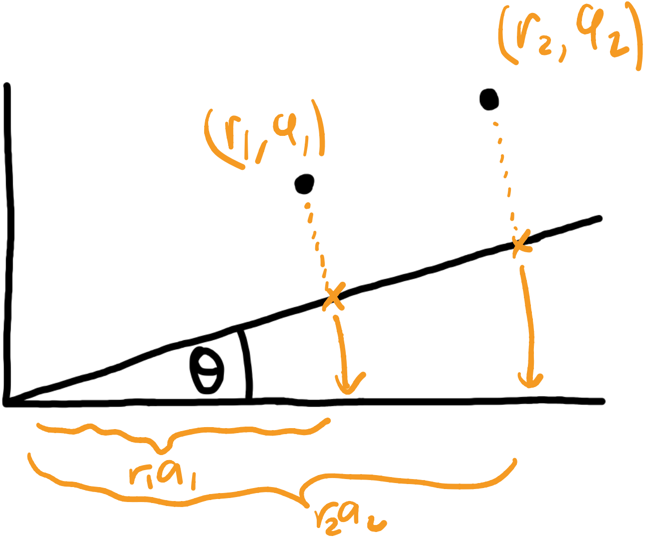
Recall: $\dim_H(E) = \min_{A \in 2^\omega} \sup_{x \in E} \dim^A(x)$



Idea

Ensure all projections have dimension 0

Recall: $\dim_H$ is invariant under isometries.



How do we **construct co-analytic sets?**

$X = \{x_\alpha \mid \alpha < \omega_1\}$

$B = \{p_\alpha \mid \alpha < \omega_1\}$

$\forall \alpha$: $x_\alpha \in F(X \restriction \alpha, p_\alpha)$.

Z. Vidnyánszky's co-analytic recursion principle (2014)

($V{=}L$) Recursion on co-analytic subsets of Polish spaces with sufficiently nice candidates produces co-analytic sets.

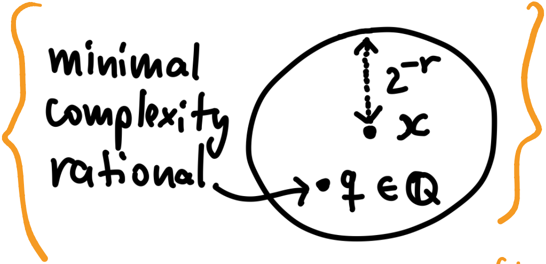
co-analytic

$$F \subseteq M^{<\omega} \times B \times M$$

cofinal

$$F_{(A,p)} = \{x \in M \mid (A,p,x) \in F\}$$

How do we **construct reals?**



$$= \quad (\text{modulo } K(r))$$

$$\left\{ x = 0.\underbrace{x_1 x_2 \ldots x_r}_{\in 2^{<\omega}} \ldots \right\}$$

How do we **construct reals?**

Lemma (N. Lutz, Stull (2020))

If $x \in \mathbb{R}$ *and* $\overline{x} \in 2^\omega$ *is* x *coded in its binary expansion, then* $\dim(x) = \dim(\overline{x})$*. This also works in* $\mathbb{R}^n$*.*

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Also works in polar coordinates!

How do we **control dimension?**

Recall: $\dim_H(E) = \min_{A \in 2^\omega} \sup_{x \in E} \dim^A(x)$

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Proof.

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A is arbitrary, so PTS completes the argument. □

Constructing E by recursion

- use co-analytic recursion on lines θ
- at step θ, take all previous lines $\theta_0, \theta_1, \theta_2, \ldots$ ← by $V=L$, so CH
- find r so that $\dim(p_{\theta_i}(r)) = \dim(a_i r) = 0$
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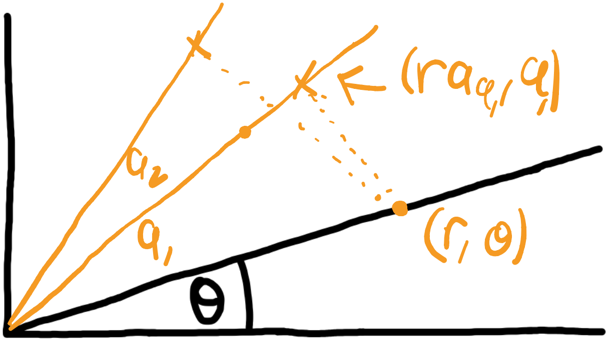
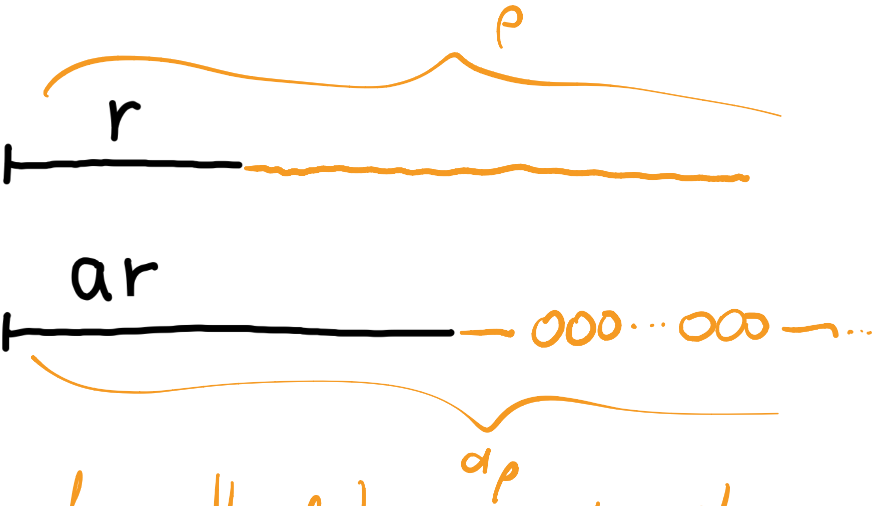


Constructing r

$$\dim(\alpha r) = \liminf_{n\to\infty} \frac{K(\alpha r[n])}{n}$$



for all extensions of ρ!

Stage α: constructing r on line θ

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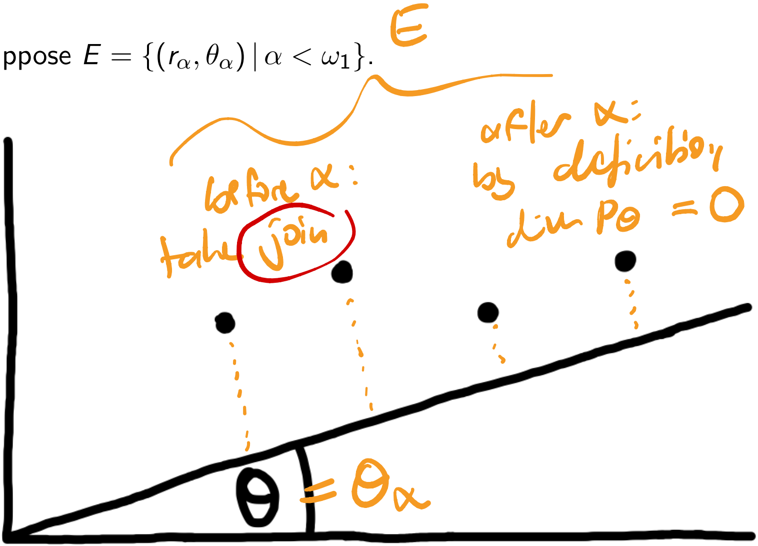
How many zeroes are enough? Ensure $\ell(\rho_k) = 2^{2^{k+1}}$.

Suppose $E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\}$.



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Theorem (R)

($V{=}L$) *For every* $\epsilon \in (0,1)$ *there exists a co-analytic* $E_\epsilon \subset \mathbb{R}^2$ *such that* $\dim_H(E_\epsilon) = 1 + \epsilon$ *and* $\dim_H(p_\theta(E_\epsilon)) = \epsilon$ *for all* θ*.*

Similar ideas, but obstacles

Fix $\epsilon > 0$.

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Problems

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Instead, find a complicated $T \in 2^{\omega}$, code pieces into all projections!

A few open questions

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$$\dim_P(E) = \min_{A \in 2^\omega} \sup_{x \in E} \mathrm{Dim}^A(x)$$

where

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- Extensions of point-to-set principle?
- Other applications: Kakeya sets, Furstenberg sets (applications to harmonic analysis)...

Thank you

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Thm 1: verification details $\dim_H(E)$

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Proof.

The line φ appeared in the induction: suppose $\varphi_1, \varphi_2, \varphi_3, \ldots$ appeared before φ.

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If $\dim^Y(r) = \epsilon$ then

$$\dim^\theta(r, \varphi) \geq \dim^\theta(\varphi) + \dim^{\theta,\varphi}(r) \geq 1 + \epsilon$$

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The construction of r (sketch)

Stage -1: find T with $\dim(T) = \dim^Y(T) = \epsilon$.

Stage 0: $r_0 = \langle\rangle$

Stage $k+1$: decode $k+1 = \langle i, n\rangle$; find $\rho_k \succ r_k$ such that $a_n[\rho_k]$ contains long substrings of T

Are coded strings of T long enough?

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How many bits of r are needed to *determine* 1 bit of ra_i?

Depends on a_i! Can be fixed by *saving* blocks.

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Given E we have:

- $\dim(ra_i) = \epsilon$, so as in counterexample 1,

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Bringing it all together

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So PTS and $\dim_H(p_\theta(E)) \geq \dim_H(E) - 1$ imply

$$\dim_H(E) = 1 + \epsilon.$$