Co-analytic Counterexamples to Marstrand's Projection Theorem

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Victoria University of Wellington

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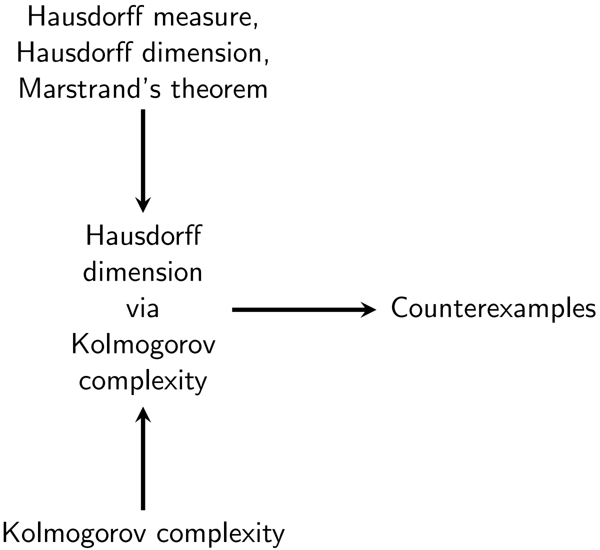
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Hausdorff dimension,
Marstrand's theorem
Hausdorff dimension via Kolmogorov complexity
Counterexamples
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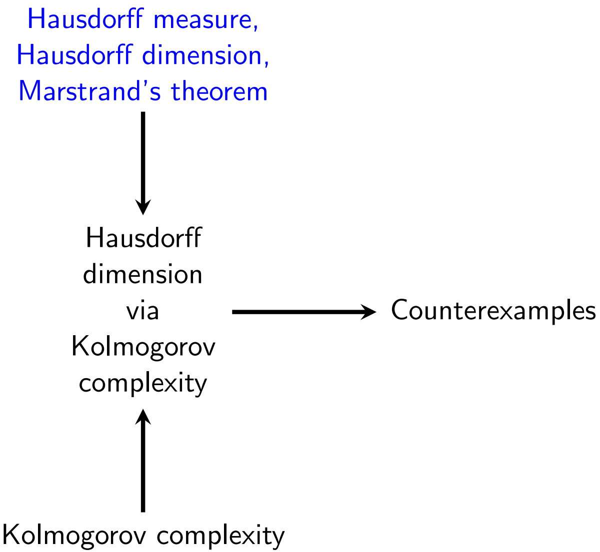
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Hausdorff dimension: motivation

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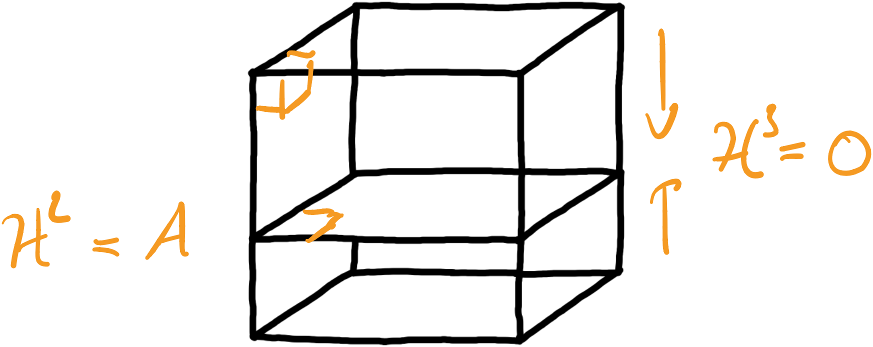

Definition (Hausdorff dimension)

For $E \subset \mathbb{R}^n$

$$\dim_H(E) = \sup\{s \mid \mathcal{H}^s(E) = \infty\}$$

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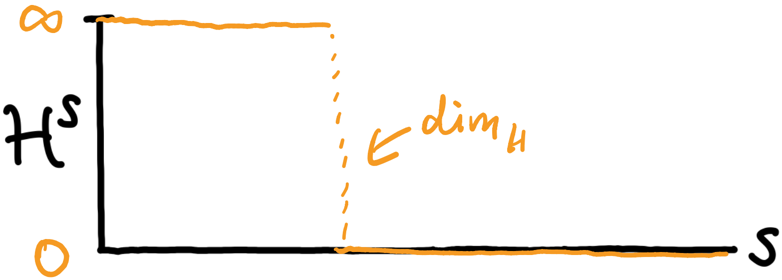
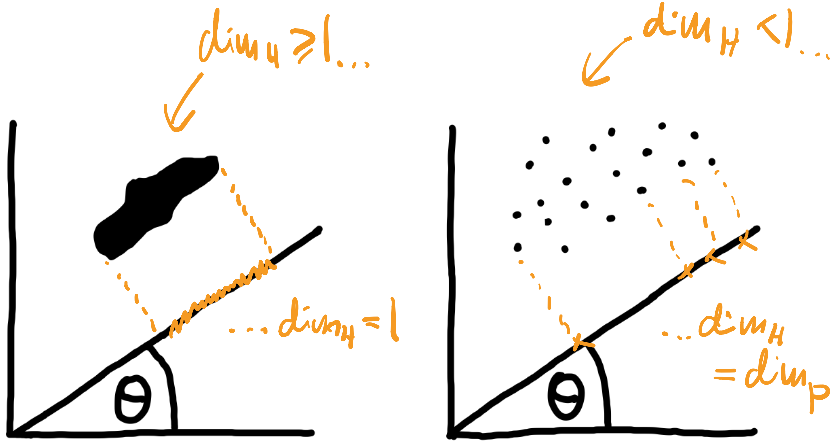


Lemma

$\dim_H$ is invariant under isometries.


dim_H ≥ 1...
...dim_H = 1
θ
dim_H < 1...
...dim_H
= dim_P
θ

Marstrand's Projection Theorem (J. Marstrand (1954))

Let $E \subset \mathbb{R}^2$ be analytic. For almost all θ

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where p_θ is the orthogonal projection onto the line θ.

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Theorem (N. Lutz and Stull (2018))

If $E \subset \mathbb{R}^2$ *and* $\dim_H(E) = \dim_P(E)$ *then Marstrand's theorem applies.*

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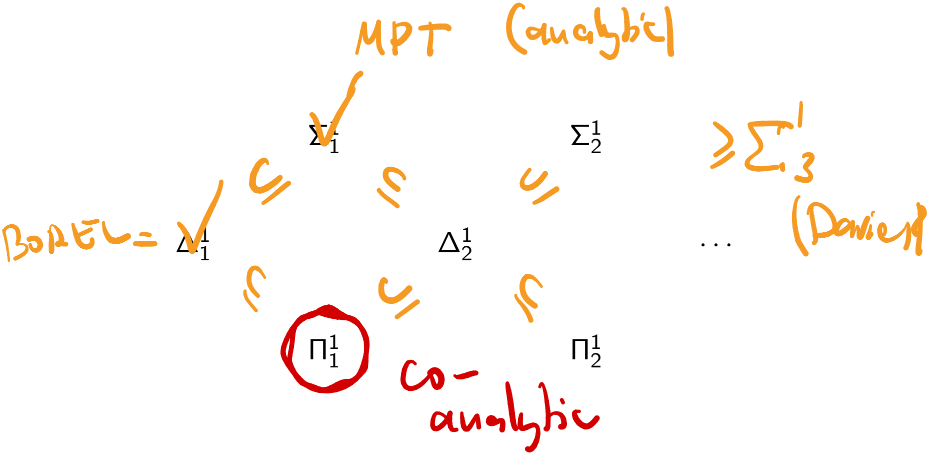
If $E \subset \mathbb{R}^2$ and $\dim_H(E) = \dim_P(E)$ then Marstrand's theorem applies.

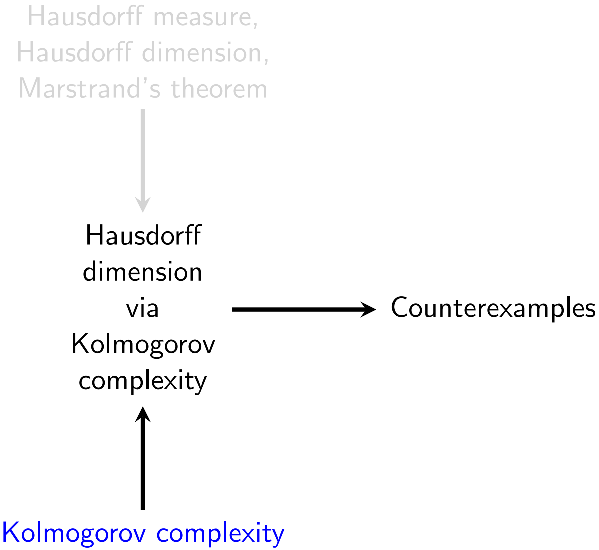
Theorem (Davies (1979))

(CH) *There exists $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1$ while $\dim_H(p_\theta(E)) = 0$ for all θ.*

Question

What is the "simplest" set failing Marstrand's theorem?

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String complexity $\longleftrightarrow$ description length

Definition

For any p.c. function f, define

$$C_f(\tau) = \begin{cases} \min\{\ell(\sigma) \mid f(\sigma) = \tau\} & \text{if such } \sigma \text{ exists;} \\ \infty & \text{otherwise.} \end{cases}$$

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Definition (Solomonoff (1964); Kolmogorov (1965); Chaitin (1966))

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Definition (Solomonoff (1964); Kolmogorov (1965); Chaitin (1966))

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1. C is within a constant of every C_f
2. $C(\sigma\tau) \leq C(\sigma) + C(\tau) + 2\log(C(\sigma)) + c$

What if codes should be *uniquely decodable*?

message	codeword
a	0
b	1
c	01

What does 01 decode to?

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Definition (Levin (1973); Chaitin (1975))

$K(\tau) = \min\{\ell(\sigma) \mid h'(\sigma) = \tau\}$ where h' is universal for prefix-free machines

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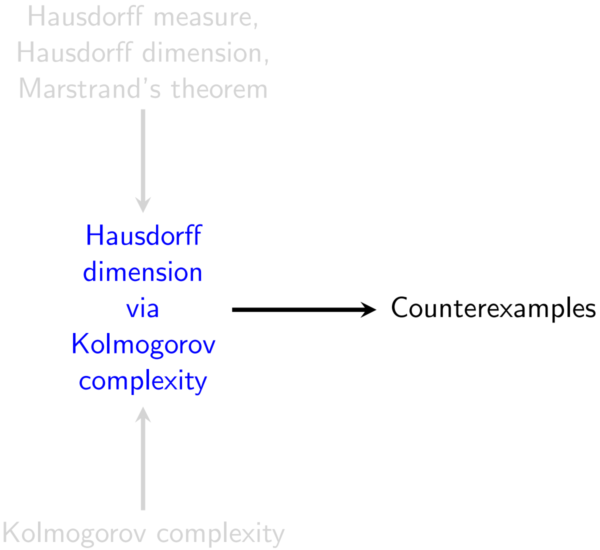
Definition (Chaitin (1975); Levin (1976))

$f \in 2^\omega$ is *Kolmogorov random* if there exists a constant c for which $K(f[n]) \geq n - c$.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Theorem (J. Lutz; Mayordomo (2003))

There exists dim *on* 2^ω *given by*

$$\dim(f) = \liminf_{n\to\infty} \frac{K(f[n])}{n}$$

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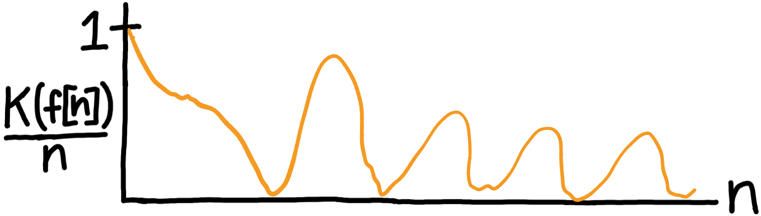
Lemma

- *If* $f \in 2^\omega$ *is computable then* $\dim(f) = 0$.
- *If* $f \in 2^\omega$ *is Kolmogorov random then* $\dim(f) = 1$.



Theorem (Hitchcock (2003))

If $X \subseteq 2^\omega$ is a union of Π^0_1-sets then

$$\dim_H(X) = \sup_{f \in X} \dim(f).$$

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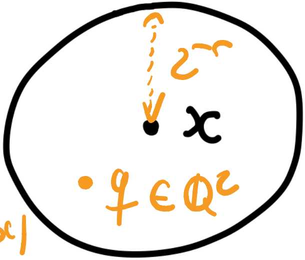
Can this characterisation be extended:

- to other spaces $(\mathbb{R}, \mathbb{R}^2, \ldots)$?
- beyond Π^0_1 sets?

$\mathbb{R}^2$

least complexity rational in $B_{2^{-r}}(x)$

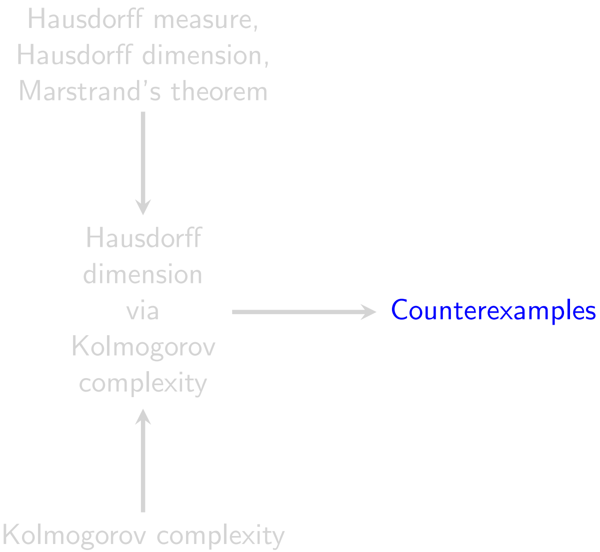


$$K_r(x) = K(q)$$

Point-to-set Principle (J. Lutz, N. Lutz (2018))

For $E \subset \mathbb{R}^n$ we have

$$\dim_H(E) = \min_{A \in 2^\omega} \sup_{x \in E} \dim^A(x).$$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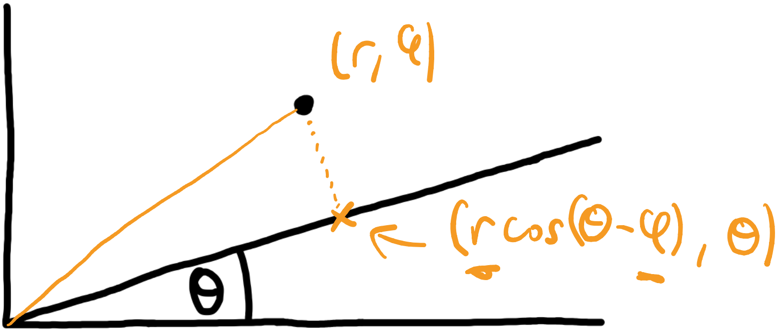
Theorem (R)

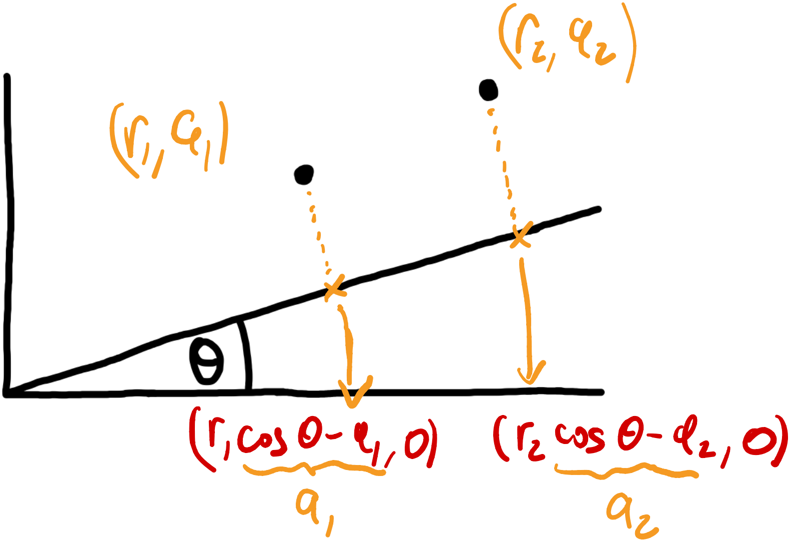
(V=L) *There exists a co-analytic $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1$ and $\dim_H(p_\theta(E)) = 0$ for all θ.*

Recall: $\dim_H(E) = \min_{A \in 2^\omega} \sup_{x \in E} \dim^A(x)$



The idea

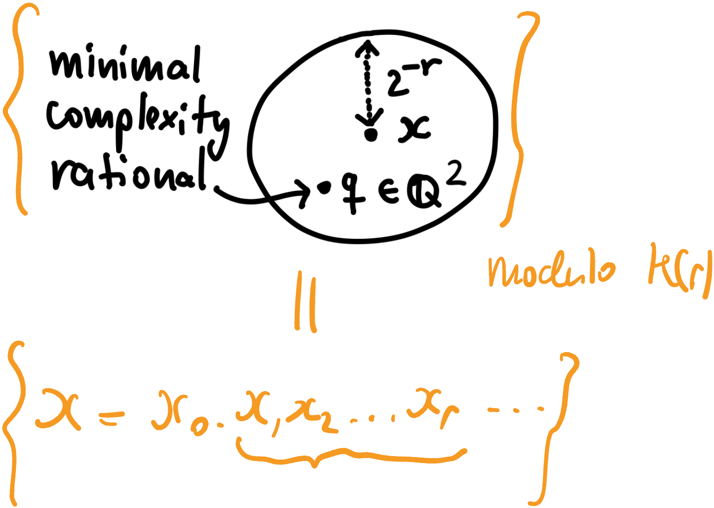
Recall: $\dim_H$ is invariant under isometries.



How do we **construct co-analytic sets?**

Z. Vidnyánszky's co-analytic recursion principle (2014)

($V=L$) Recursion on co-analytic subsets of Polish spaces with sufficiently nice candidates produces co-analytic sets.


minimal
complexity
rational
2^{-r}
x
$q \in \mathbb{Q}^2$
modulo K(r)
$x = x_0 . x_1 x_2 \ldots x_r \ldots$

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A is arbitrary, so PTS completes the argument. $\square$

Constructing E by recursion

V=L ⇒ CH

- use co-analytic recursion on lines θ
- at step θ, take all previous lines $\theta_0, \theta_1, \theta_2, \ldots$
- find r so that $\dim(p_{\theta_i}(r, \theta)) = \dim(a_i r) = 0$
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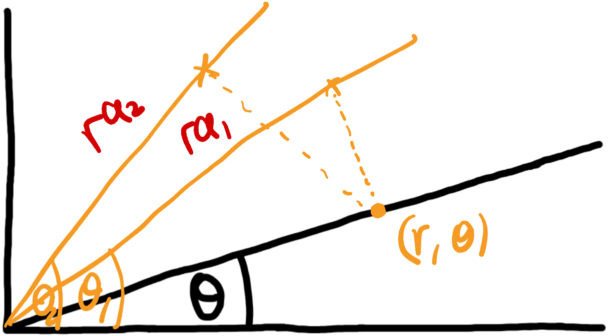
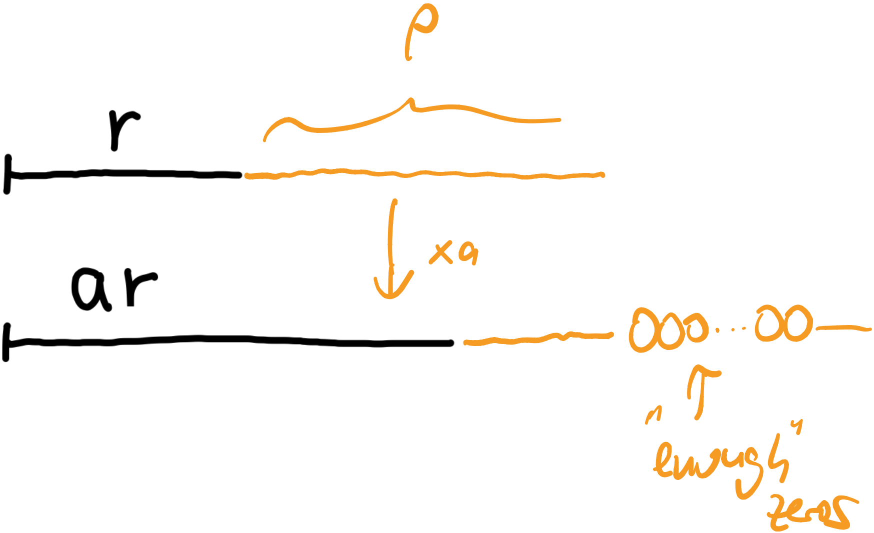


Constructing r



because V=L → CH

Stage α: constructing r on line θ

← projection factor

1. Suppose $E \upharpoonright \alpha = \{(r_i, \theta_i) \mid i < \omega\}$, $A_\alpha = \{a_i \mid i < \omega\}$

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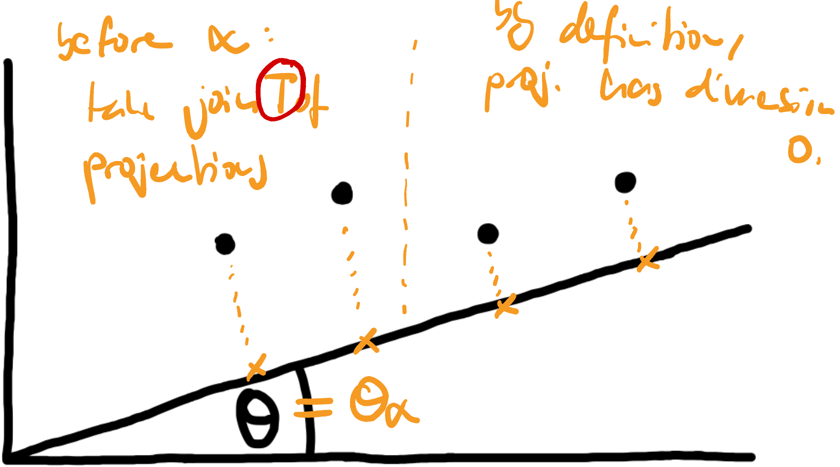
How many zeroes are enough? Ensure $\ell(\rho_k) = 2^{2^{k+1}}$.

Suppose $E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\}$.



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If $E \subset \mathbb{R}^2$ is analytic and for some $\epsilon \in (0, 1)$ we have $\dim_H(E) = 1 + \epsilon$ then $\dim_H(p_\theta(E)) = 1$ for almost all θ.

The second counterexample

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Theorem (R)

(V=L) *For every $\epsilon \in (0,1)$ there exists a co-analytic $E_\epsilon \subset \mathbb{R}^2$ such that $\dim_H(E_\epsilon) = 1 + \epsilon$ and $\dim_H(p_\theta(E_\epsilon)) = \epsilon$ for all θ.*

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Problems

- meeting every line only ensures the set has dimension at least 1, not $1 + \epsilon$
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Similar ideas, but obstacles

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Problems

- meeting every line only ensures the set has dimension at least 1, not $1 + \epsilon$
- controlling the dimension of the projection is more intricate: long zero strings do not suffice

Instead, find a complicated $T \in 2^\omega$, code pieces into all projections!

A few open questions

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PTS for packing dimension (J. Lutz, N. Lutz (2018))

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where

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- Extensions of point-to-set principle? Generalisations using gauge functions?
- Other applications: Kakeya sets, Furstenberg sets (applications to harmonic analysis)...

Thank you

Suppose $E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\}$.

Lemma

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Thm 1: verification details $\dim_H(E)$

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Now the point-to-set principle gives

$$\dim_H(p_\varphi(E)) \quad = \quad \min_{A \in 2^\omega} \sup_{\alpha<\omega_1} \dim^A(r_\alpha k_\alpha)$$

Thm 1: verification details $\dim_H(E)$

Suppose $E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\}$.

Lemma

Fix a line φ. Let k_α be the projection factor of $(r_\alpha, \theta_\alpha)$ onto φ. There exists X such that $\sup_{\alpha<\omega_1} \dim^X(r_\alpha k_\alpha) = 0$.

Proof.

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Now the point-to-set principle gives

$$\begin{aligned} \dim_H(p_\varphi(E)) &= \min_{A \in 2^\omega} \sup_{\alpha<\omega_1} \dim^A(r_\alpha k_\alpha) \\ &\leq \sup_{\alpha<\omega_1} \dim^X(r_\alpha k_\alpha) = 0. \end{aligned}$$

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Don't: find r and enumerate (r, θ)

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code complicated T into r
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If $\dim^Y(r) = \epsilon$ then

$$\dim^\theta(r, \varphi) \geq \dim^\theta(\varphi) + \dim^{\theta,\varphi}(r) \geq 1 + \epsilon$$

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The construction of r (sketch)

Stage -1: find T with $\dim(T) = \dim^Y(T) = \epsilon$.

Stage 0: $r_0 = \langle\rangle$

Stage $k+1$: decode $k+1 = \langle i, n\rangle$; find $\rho_k \succ r_k$ such that $a_n[\rho_k]$ contains long substrings of T

Are coded strings of T long enough?

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No.

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How many bits of r are needed to *determine* 1 bit of ra_i?

Depends on a_i! Can be fixed by *saving* blocks.

Recall $Y = (\bigoplus a_i) \oplus \theta \oplus \varphi$.

Given E we have:

- $\dim(ra_i) = \epsilon$, so as in counterexample 1,

$$\dim_H(p_\theta(E)) = \epsilon.$$

Bringing it all together

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So PTS and $\dim_H(p_\theta(E)) \geq \dim_H(E) - 1$ imply

$$\dim_H(E) = 1 + \epsilon.$$