THE BOREL-DEFINABLE GROUP COHOMOLOGY OF $\mathbb{R}^n$ IS (HOPEFULLY) SIMPLE

Abstract. I will motivate group extensions, which allow us to decompose any finite group into a set of finite simple groups, its building blocks. Using descriptive set theory, we can study the special class of *definable* group extensions: those induced by Borel functions. Kanovei and Reeken have made inroads into classifying these Borel extensions—I will outline their result for abelian Borel extensions of $(\mathbb{R}, +)$ by any countable abelian group, and present work in progress on the case $\mathbb{R}^n$. This is joint work with Dan Turetsky.

Introduction

Definability (especially the first-order kind) plays an important role in mathematical logic: classically from Russell's paradox, over the contemporary pillars of logic in **Computability Theory** (Post's theorem, and extensions to higher computability theory on admissible ordinals), **Set Theory** (absoluteness, $V{=}L$, the definability of forcing), to **Model Theory**, arguably the study itself of definable objects.

Applications to classical mathematics can also easily be found: one active of area concerns **Descriptive Set Theory**: in particular, Borel equivalence relations. There, we augment objects with a natural Polish structure. This allows us to think of definable objects as being those that can be obtained from Borel considerations: either, they are Borel sets themselves, or they are induced by Borel functions in some way (similar approaches can be applied to Baire-measurability).

One reason why definable examples are important (and useful) is because they often sidestep the trivial existence by AC—those usually do not inform about the inherent structure of an object at all; their existence is given straight from the axiom. The definable examples lie on the other end of this spectrum: not only are they normally constructive (in a general sense), they also spell out *why* they satisfy a given property.

Building Groups From Simple Groups

In this talk, we consider the context of **group extensions**. By way of analogy, consider the primes in $\mathbb{Z}$. Each non-zero $k \in \mathbb{Z}$ can be written as a product of primes (and units), which up to order of the factors, is unique. Further, the decomposition of k into integers can be carried out in stages: first divide k by a prime appearing in its prime decomposition; then take the divisor and divide by the next prime, and carry on in this manner. Since the prime decomposition of k is finite, after finitely many steps we arrive at a prime p; and one more division yields 1, and we are done. For instance, for $k = 12$ we have

$$\begin{tikzcd} 12 \arrow[d, dotted, "12/6"'] & > & 6 \arrow[d, dotted, "6/3"'] & > & 3 \arrow[d, dotted, "3/1"'] & > & 1 \\ 2 & & 2 & & 3 & & \end{tikzcd}$$

Figure 1.

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which shows us immediately that the set of prime factors of $k = 12$ is given by $\{2, 2, 3\}$, counting multiplicity. Further, observe that the order of decomposition does not matter. Indeed, we can define the following finer notion of maximality:

Definition 1. *We call $m < k$ **maximal in** k if $m|k$ and for all $m < m' < k$, if $m' \mid k$ then we have $m \nmid m'$.*

For example, 4 is maximal in 12. From this definition, we recover the prime decomposition of 12 in a second, slightly different way: namely it is clear that we can choose *any* maximal integer to obtain the prime factors; clearly, the following sequence also works

$$\begin{tikzcd}
12 \arrow[d, dotted, "12/4"'] & > & 4 \arrow[d, dotted, "4/2"'] & > & 2 \arrow[d, dotted, "2/1"'] & > & 1 \\
3 & & 2 & & 2 & &
\end{tikzcd}$$

FIGURE 2.

which recovers, importantly, the same decomposition, albeit in a different order. This importance of *maximal* notions is a recurring theme.

Assuming we do not know the prime decomposition a priori (in which case the above process is trivial), how de we actually decompose an integer into primes? Practically, how exactly do we find the factors? One way of doing so involves searching: given k find the largest $l < k$ that divides it. Then divide; e obtain the factors.

This idea is very important, and can be applied to groups in the following way. Let G be a group. Recall that G is **simple** if G has no normal subgroups save G and 1 itself. Importantly:

Lemma 2. *If $N \lhd G$ is a maximal subgroup (under inclusion) then G/N is simple.*

So we can do the following: fix a finite group G and take any maximal normal subgroup of G; consider its quotient, and continue. In finitely many steps we will arrive at the trivial subgroup 1. By lemma 2, the quotients (or **factors** of course) we obtain along the way are all *simple* (compare with *prime* in the ring-case $\mathbb{Z}$ above).

Again, an example helps: consider the finite group S_4. A maximal normal subgroup is A_4, and the quotient has order 2, and hence is (isomorphic to) $\mathbb{Z}_2$—which is simple, as promised. Now, A_4 has as maximal normal subgroup K_4, the Klein Four Group, with quotient of order 3, hence isomorphic to $\mathbb{Z}_3$—again simple. Finally, K_4 has a normal subgroup of order 2, and their quotient has order 2 itself, which completes the decomposition.

$$\begin{tikzcd}
S_4 \arrow[d, dotted, "S_4/A_4"'] & \rhd & A_4 \arrow[d, dotted, "A_4/K_4"'] & \rhd & K_4 \arrow[d, dotted, "K_4/\mathbb{Z}_2"'] & \rhd & \mathbb{Z}_2 \arrow[d, dotted, "\mathbb{Z}_2/1"'] & \rhd & 1 \\
\mathbb{Z}_2 & & \mathbb{Z}_3 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & &
\end{tikzcd}$$

FIGURE 3.

An instructive analogy with the ring $\mathbb{Z}$ from earlier: consider the group $\mathbb{Z}_{12}$: its decomposition as groups yields exactly the composition of prime factors of the integer 12 in the ring $\mathbb{Z}$.

$$\begin{tikzcd}
\mathbb{Z}_{12} \arrow[d, dotted, "\mathbb{Z}_{12}/\mathbb{Z}_6"'] & \rhd & \mathbb{Z}_6 \arrow[d, dotted, "\mathbb{Z}_6/\mathbb{Z}_3"'] & \rhd & \mathbb{Z}_3 \arrow[d, dotted, "\mathbb{Z}_3/1"'] & \rhd & 1 \\
\mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_3 & &
\end{tikzcd}$$

Carrying the analogy further, comparing figs. 1 and 2, we would hope that the order of decomposition of groups not matter. Luckily, this is in fact the case:

Theorem 3 (The Jordan-Hölder-Theorem)**.** *The set of composition factors is uniquely determined, up to isomorphism.*

What about going the *other* way? Suppose we have a set of decomposition factors, i.e. a sequence of simple groups. How easy is it to find the original group? **Not easy at all**.

Firstly, in $\mathbb{Z}$ multiplication of primes (or of integers in general for that matter) is well-defined, it of course yields a single integer (up to associates). This is not as clear here; there is no reason to believe that there should only exist a single group which can be composed from a given sequence of composition factors. Indeed, here is an example: consider the sequence $(\mathbb{Z}_3, \mathbb{Z}_2)$ of decomposition factors. We aim to find groups G_i such that

$$\begin{tikzcd}
G_0 \arrow[d, dotted, "G_0/G_1"'] & \rhd & G_1 \arrow[d, dotted, "\mathbb{Z}_3/1"'] & \rhd & 1 \\
\mathbb{Z}_2 & & \mathbb{Z}_3 & &
\end{tikzcd}$$

FIGURE 4.

It is evident that $G_1 \cong \mathbb{Z}_3$; there is no choice to be made. But what about G_0? The conditions we need G_0 to satisfy are the following:

- G_0 has a normal subgroup of order 3
- $G_0/G_1 \cong G_0/\mathbb{Z}_3 \cong \mathbb{Z}_2$

We call them the **extension conditions**. In this case, G_0 is **an extension of** $\mathbb{Z}_2$ **by** G_1.

In our example, there is not *much* choice, but there is some: clearly, $|G_0| = 6$, but there are two groups of order 6 (up to isomorphism), and both have a normal subgroup of order 3. These are

(1) $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$
(2) S_3

which are, importantly, not isomorphic as groups. Hence, while *decomposition* is unique (up to isomorphism), *composition* is not! And it isn't easy either!

These sequences are called **composition series** and often denoted by

$$G = G_0 \rhd G_1 \rhd G_2 = 1.$$

Its **length** is given by its number of composition factors (so the series in fig. 4 has length 2). It should be intuitively clear that, the longer the composition series is, the more complicated is the composition of the possible groups from said sequence of composition factors.

Remark 1. *There is an important distinction between fig. 1 and fig. 3: the relation $\lhd$ is not normally transitive, while $<$ (on $\mathbb{Z}$) is! Something to be aware of.*

The problem of reconstructing groups can be broken down as follows: suppose H, A are fixed groups. We want to find a group E such that both of the following hold:

- $H \lhd E$
- $E/H \cong A$

This recovers exactly our extension conditions from above, which is clear form the following diagram:

$$\begin{tikzcd} E \arrow[d, dotted, "E/H"'] & \rhd & H \\ A & & \end{tikzcd}$$

This relation has a useful homological form in the shape of **short exact sequences**:

$$1 \xrightarrow{\alpha_1} H \xrightarrow{\alpha_2} E \xrightarrow{\alpha_3} A \xrightarrow{\alpha_4} 1$$

where all maps involved are group homomorphisms and, importantly

$$\operatorname{image}(\alpha_i) = \ker(\alpha_{i+1}).$$

This yields the following result straightforwardly.

Lemma 4. *The map α_1 is injective, α_4 is surjective, and $E/H \cong A$.*

Proof. Firstly, $\operatorname{image}(\alpha_1) = 1$, thus $\ker(\alpha_2) = 1$, so α_2 is injective. Similarly, $\operatorname{image}(\alpha_3) = \ker(\alpha_4) = A$, and so α_3 is surjective. Finally, let[1] $H \lhd E$. Then since α_3 is a group homomorphism, the first isomorphism theorem implies that

$$E/\ker(\alpha_3) \cong E/\operatorname{image}(\alpha_2) \cong E/H \cong \operatorname{image}(\alpha_3) = A$$

since α_3 is surjective, as required. □

Now plug in our example from before: we obtain the sequence

$$1 \longrightarrow \mathbb{Z}_3 \longrightarrow G_0 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

whose solutions for G_0 are exactly the *compositions* introduced above, as required. Its solutions here are S_3 and $\mathbb{Z}_2 \times \mathbb{Z}_3$, for instance. (However, observe that if we had attempted to construct a group from composition factors $(\mathbb{Z}_2, \mathbb{Z}_3)$ and started with $\mathbb{Z}_2$, then there would have been only the extension $\mathbb{Z}_2 \times \mathbb{Z}_3$ since S_3 has no normal subgroup of order 2.) Hence we finally define group extensions:

Definition 5. *Given groups A, H, the group E is* ***an extension of*** *A* ***by*** *H if the sequence $1 \longrightarrow H \longrightarrow E \longrightarrow A \longrightarrow 1$ is short exact (in the category of groups).*

Characterising Group Extensions

Recall that we can classify all finite simple groups—hence if we were able to easily solve the group extension problem then, by a process similar to the above, we could classify all finite groups, simply from their composition of simple groups.

Sadly, finding (let alone characterising) all possible extensions to a given groups is a very difficult problem. There are multiple approaches:

(1) The direct sum of the groups A and H is always an extension of A by H, with the obvious injection and projection.
(2) Some extensions can be obtained by considering the actions via automorphisms of A on H: these are the so-called **semi-direct products** [3, p. 68].
(3) One can use **factor sets**: these are related to short exact sequences, and are originally due to Otto Hölder (1893). An introduction can be found in [2, p. 255].

[1] Of course, we are being sloppy here: technically, we require finding a normal subgroup $N \lhd H$ such that $H \cong N$; but it is often much easier to conflate the isomorphic normal subgroups here. Fewer symbols is better.

(4) However, there is **no unified theory** that encompasses all approaches to the group extension problem.

Here, we make our lives easier: we assume that **all groups are abelian**. In that case, there is a nice approach to classifying the abelian extensions: via **group cohomology**.

Firstly, let us observe that in our quest to classify all group extensions, it is advisable when exactly two extensions are "the same". We say that two group extensions E, E' of groups A, H are **congruent** if the following diagram commutes:

$$\begin{tikzcd} & & E \arrow[dd] \arrow[dr] & & \\ 1 \arrow[r] & H \arrow[ur] \arrow[dr] & & A \arrow[r] & 0 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

A few facts about group extensions:

- It turns out that E and E' being isomorphic does not suffice for the extensions to be congruent [2, p. 261]. For instance, consider the following diagram:

$$\begin{tikzcd} & & \mathbb{Z}_9 \arrow[dd, "f"] \arrow[dr, "q"] & & \\ 1 \arrow[r] & \mathbb{Z}_3 \arrow[ur, "i"] \arrow[dr, "i'"'] & & \mathbb{Z}_3 \arrow[r] & 1 \\ & & \mathbb{Z}_9 \arrow[ur, "q"'] & & \end{tikzcd}$$

  where $i(1) = 3$ and $i'(1) = 6$. Clearly, $\mathrm{image}(i) = \mathrm{image}(i') = \{0, 3, 6\} = \ker(q)$. But what is f? Observe that $6 = i'(1) = f(i(1)) = f(3) = 3f(1)$ implies that $f(1) \in \{2, 5, 8\}$. But then $q(1) = 1 \neq q(2) = q(5) = q(8) = 2$, a contradiction.
- It is further not true that all extensions of A, H are pairwise isomorphic: we saw above that both S_3 and $\mathbb{Z}_2 \times \mathbb{Z}_3$ extend $\mathbb{Z}_2$ by $\mathbb{Z}_3$, but they are clearly not isomorphic.
- It also follows immediately that extensions of abelian groups by abelian groups need not be abelian themselves.

Fix abelian groups A, H as before, and let $C\colon A^2 \to H$ be a function satisfying the following conditions:

- $C(x, y) = C(y, x)$
- $C(x, y) + C(x + y, z) = C(x, z) + C(x + z, y)$

In such a case, we call C a **symmetric cocycle**. If there exists a function $f\colon A \to H$ such that

$$C(x, y) = f(x) + f(y) - f(x + y)$$

then we call C a **coboundary**. These notions are important elements of **group cohomology**—and they play an important part here for the following reason.

Theorem 6. *Let A, H be abelian groups, and suppose C is a symmetric cocycle. Then there exists an abelian group E_C defined on $A \times H$ such that E_C is an extension of A by H. Equally, if E is an abelian extension of A and H, then there exists a symmetric cocycle C for which $E = E_C$.*

Further, cohomology comes with its own structure—and that structure transfers nicely into the theory of groups extensions: two cocycles C, C' are said to be **cohomologous** if $C - C'$ is a coboundary. It turns out that cohomologous cocycles induce congruent group extensions! Hence, in this simplified context, understanding group extensions reduces to characterising all cocycles (up to cohomology).

This will be the focus of the remainder of this talk.

Introducing the Polish Context

The maps $C\colon A^2 \to H$ (as well as the maps from A to H which induce coboundaries) are defined on A and H as groups. Given the introduction of this talk, it is hoped that after augmenting them with additional structure we might learn more about the definability of present group extensions.

Definition 7. *Let A, H be Borel groups (i.e. Borel subsets of some Polish space, whose group operations are Borel maps themselves). Then E is a **Borel group extension** of A by H if and only if $E = E_C$ for some Borel cocycle C. A Borel coboundary is a coboundary with a Borel witness.*

Here it is also clear that if C is a Borel cocycle then E_C is a Borel extension of A by H—it is defined on $A \times H$, and its group operation can be shown to be Borel, too, by the fact that A, H are Borel groups themselves.

This opens up the following question: **can we easily classify all Borel group extensions?** This is clearly a rather more logical than algebraic context, and one would hope that the tools from descriptive set theory provide new tools here, for instance.

It turns out that there not many Borel-definable group extensions.

Theorem 8 (Kanovei, Reeken [6])**.** *There are no non-trivial abelian Borel group extensions of $(\mathbb{R}, +)$ by countable abelian G. In symbols:*

$$H^2_{\mathrm{Bor}}(\mathbb{R}, G) = 1$$

for countable abelian G.

Remark 2. *The authors also remark that $H^2_{\mathrm{Bor}}(2^\omega, G) \neq 1$, not only for countable but even for finite G* [6, p. 265]*; and that, based on an idea of Greg Hjorth, one can construct a Borel subgroup $G \leq \mathbb{R}$ and a Borel cocycle C such that C is a coboundary in $H^2_{\mathrm{Bor}}(\mathbb{R}, \mathbb{R})$ but not in $H^2_{\mathrm{Bor}}(\mathbb{R}, G)$, showing that Theorem 8 fails in general for uncountable G. It should be noted that a similar construction works for Cantor space 2^ω* [1, 5].

What does this Theorem 8 mean in terms of cocycles? Since the abelian Borel group extensions are determined by the symmetric Borel cocycles, what Kanovei and Reeken have shown is that: **if $C\colon \mathbb{R}^2 \to G$ is a symmetric Borel cocycle then there exists a Borel map $f\colon \mathbb{R} \to G$ such that**

$$C(x, y) = f(x) + f(y) - f(x + y).$$

I will outline the proof: it uses **forcing**, but does not force over the ground model! Instead, genericity is used to simplify C as follows:

- Let M be a countable transitive model of a sufficiently large finite fragment of ZFC which also contains a code for C as well as the group G itself. Conditions are non-empty open intervals of $\mathbb{R}$, coded in M.
- C is simplified in multiple steps. This is generally done by *localising*: recall that generics do not satisfy any special properties, and thus C can be manipulated to be *invariant* on them.

- Some detail: fix an open interval $I > 0$ (which can be considered a *condition* in forcing parlance) on which C is invariant for generics. Every real greater than I can be expressed as the sum of generics from I (the set of generics is G_δ, and hence comeagre in I after all[2] [4, Appendix A]). Hence C is nice on all copies of I. Mirror this into the negative to obtain all of $\mathbb{R}$.

What about $\mathbb{R}^n$? This is joint work with Dan Turetsky, which grew out of an idea with Martino Lupini. In this context, difficulties arise even in the case $n = 2$. Emulating the argument from [6] seems reasonable but stalls early:

- A *nice* condition I in $\mathbb{R}^2$ under multiplication does not close a space under both coordinates. So the original mirroring idea of Kanovei and Reeken needs fine-tuning in $\mathbb{R}^2$. One way to do this is to fix two suitable conditions on which C is invariant *individually*.
- A nice condition in $\mathbb{R}^2$ is more complicated: the second coordinate makes it harder to prove an equivalently strong notion of invariance for M-Cohen generics (the problem being that I in the case $n = 1$ is nice on all generics, whereas $n = 2$ requires conditions which might be nice in their own right, but might not work well together).
- Some of these obstacles can be overcome, but the arguments are quite technical and don't seem to immediately simplify the subsequent arguments needed to complete the proof.

In Essence

The group extension problem is difficult. Using ideas from descriptive set theory yields insights into the definable, or *nice*, extensions between Polish spaces. Some results indicate that this investigation is a fruitful one: Kanovei and Reeken have shown [6] that

$$H^2_{\mathrm{Bor}}(2^\omega, 2^\omega) \cong \mathsf{Hom}_C(2^\omega, 2^\omega)$$

where the latter denotes the group of continuous homomorphisms. It is also not clear how the topological properties affect the Borel cohomology groups—or what happens *generally* when G is not countable. Plenty of open questions to consider in the future!

References

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[2] Since M is countable, there are only countably many dense open sets coded in M; by the Baire Category Theorem, since $\mathbb{R}$ is Polish and hence completely metrisable, their intersection is a dense G_δ, hence comeagre.