

## General interpolation on the lattice $h\mathbb{Z}^s$ : Compactly supported fundamental solutions \*

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[the date of receipt and acceptance should be inserted later]

**Summary.** In this paper we combine an earlier method developed with K. Jetter on general cardinal interpolation with constructions of compactly supported solutions for cardinal interpolation to gain compactly supported fundamental solutions for the general interpolation problem. The general interpolation problem admits the interpolation of the functional and derivative values under very weak restrictions on the derivatives to be interpolated. In the univariate case, some known general constructions of compactly supported fundamental solutions for cardinal interpolation are discussed together with algorithms for their construction that make use of MAPLE. Another construction based on nite decomposition and reconstruction for spline spaces is also provided. Ideas used in the latter construction are lifted to provide a general construction of compactly supported fundamental solutions for cardinal interpolation in the multivariate case. Examples are provided, several in the context of some general interpolation problem to illustrate how easy is the transition from cardinal interpolation to general interpolation.

*Mathematics Subject Classification (1991):* AMS: Primary 42C05 41A63 41A30 Secondary 41A15

### 1. Introduction

This paper follows Jetter, Riemenschneider, and Shen (1994) where a general method of interpolation for functional and derivative data on the integer lattice was introduced. The goal of that paper was to provide a construction that would have exponentially decaying fundamental solutions from the space generated by the shifts (integer translates) of one or more functions. This could be accomplished for example, by beginning with the space spanned by shifts of some compactly supported function,

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*Offprint requests to:* Sherman D. Riemenschneider

\* Research supported in part by NSERC Canada under Grant # A7687 and NATO CRG 901018

$$(1.1) \quad \mathcal{S}(\varphi) := \{S(x) = \varphi * c \mid c : \mathbb{Z}^s \rightarrow \mathbb{C}\},$$

for which exponentially decaying fundamental solutions exist for cardinal interpolation (interpolation of functional values) and then augmenting by that space multiplied by appropriate shift-invariant (1-periodic) trigonometric polynomials,  $\sigma_j$  ( $j = 1, \dots, r$ ), (one for each of the  $r$  derivatives to be interpolated)

$$(1.2) \quad \mathcal{S} := \mathcal{S}(\varphi) + \sum_{j=1}^r \sigma_j \mathcal{S}(\varphi).$$

Exponentially decaying fundamental solutions were found in the shift-invariant space  $\mathcal{S}$ . Further, it was shown that the approximation order of the interpolation operator defined via these fundamental solutions is the same as the approximation order for cardinal interpolation from the space  $\mathcal{S}(\varphi)$ . An alternative approach to the general interpolation problem has been given by Narcowich and Ward (1993).

The paper (Jetter, Riemenschneider, and Shen 1994) was motivated in part by the failure of the usual method of Hermite interpolation to yield fundamental solutions of the desired type in the case of box splines. The idea of modifying the underlying approximation space to gain additional desirable properties of the approximations is certainly not new and has been used several times in the past. One such desirable property is the existence of a compactly supported fundamental solution for the interpolation of function values. The present paper is motivated by the paper of Micchelli (1991) which contains interesting ideas and general results about how to obtain compactly supported fundamental solutions. That paper was preceded by earlier contributions; the ones most pertinent to our discussion are Schoenberg (1946), Qi (1981), Dahmen, Goodman, and Micchelli (1988, 1989), Chui (1991), and Chui and Diamond (1991) (additional references may be found in those papers). Obtainment of compactly supported fundamental solutions is an apparent step beyond what was accomplished in Jetter, Riemenschneider, and Shen (1994) (although the latter emphasized interpolation of derivative values as well), but now we realize that once a compactly supported fundamental solution for cardinal interpolation is available in the context of shift invariant spaces, then the theory in Jetter, Riemenschneider, and Shen (1994) applies directly to give compactly supported fundamental solutions for the general interpolation problems. This paper is organized as follows: In Sect. 2, we describe the method of Jetter, Riemenschneider, and Shen (1994) and the relevant results from Micchelli (1992). In Sect. 3, some the known general univariate constructions of compactly supported fundamental solutions for cardinal interpolation from the papers mentioned above are discussed, together with algorithms for their construction that make use of MAPLE. Another construction based on nite decomposition and reconstruction of spline spaces important for wavelet theory (Micchelli 1991, Cohen, Daubechies, and Feauveau 1992) is also discussed. Section 4 is devoted to constructions for the multivariate case, including a general construction preserving approximation order. In both sections concrete examples illustrating the transition from cardinal interpolation to a more general interpolation problem are given.

The paper Jetter, Riemenschneider, and Shen (1994) did not require the function  $\varphi$  appearing in Eq.(1.1) to be compactly supported, but we find it convenient to make that restriction here since all of our examples have this property and since we are free to take (as we will) the compactly supported fundamental solution of the cardinal interpolation problem as the generator of the space.

## 2. The Interpolation Method

We now recall in more detail the method presented in Jetter, Riemenschneider, and Shen (1994) and the relevant material from Micchelli (1992). We are interested in interpolation at points on the scaled (by  $h$ ) integer lattice  $h\mathbb{Z}^s$ , and the approximation power of such interpolants as  $h \rightarrow 0$ . The interpolation will be from spaces of the type Eq.(1.1), generated by the shifts of a given compactly supported function dened via convolution with sequences dened on the lattice. We adopt the notation in the book de Boor, Hollig, and Riemenschneider (1993): The **convolution** of the function  $\varphi$  with the sequence  $c$  is

$$\varphi * c := \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot - \alpha) c(\alpha),$$

while the **semi-discrete convolution** of the function  $\varphi$  with a function  $f$  dened on  $\mathbb{R}^s$  is

$$\varphi *' f := \varphi * f|_1 = \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot - \alpha) f_1(\alpha),$$

where  $f|_1 := f|_{\mathbb{Z}^s}$  is the sequence obtained by the restriction of  $f$  to the lattice. Cardinal interpolation from the space  $\mathcal{S}(\varphi)$  is deemed **correct** if for an arbitrary bounded sequence  $d$  on  $\mathbb{Z}^s$ , there is a unique element  $S = \varphi * c$  from  $\mathcal{S}(\varphi)$  with  $c \in \ell_\infty(\mathbb{Z}^s)$  such that

$$S(\beta) = d(\beta), \quad \text{for all } \beta \in \mathbb{Z}^s.$$

For correct interpolation, the interpolant,  $L$ , to the Dirac delta sequence  $\delta_0$  is called the **fundamental solution** for cardinal interpolation, and the interpolation can be described via the **cardinal interpolation operator** in **Lagrange form** by

$$\mathcal{L} : d \mapsto \mathcal{L}d := L * d.$$

The "correctness" of the interpolation from  $\mathcal{S}(\varphi)$  is governed by the (periodic) **symbol** of the function  $\varphi$ , dened on  $\mathbb{R}^s$  by

$$(2.1) \quad \tilde{\varphi}(z) := \sum_{\alpha \in \mathbb{Z}^s} \varphi(\alpha) z^\alpha, \quad z = \exp(-i\alpha y), \quad y \in \mathbb{R}^s.$$

Cardinal interpolation from  $\mathcal{S}(\varphi)$  is correct if and only if the symbol  $\tilde{\varphi}$  does not vanish on  $\mathbb{R}^s$ ; moreover, when interpolation is correct, the fundamental solution is given by

$$L := \varphi * \lambda \quad \text{with} \quad \lambda(\alpha) := \frac{1}{(2\pi)^s} \int_{[-\pi, \pi]^s} \frac{\exp(i\alpha y)}{\tilde{\varphi}(\exp(-i\alpha y))} dy, \quad \alpha \in \mathbb{Z}^s,$$

and has exponential decay (see, for example, de Boor, Hollig, and Riemenschneider 1993). Unfortunately, for most of the important common examples, the procedure just described leads to innitely supported fundamental solutions.

Next, we describe the solution to the problem of interpolating derivative data on the integer lattice  $\mathbb{Z}^s$  as formulated in Jetter, Riemenschneider, and Shen (1994). The derivatives to be interpolated are given by a linearly independent set of real constant coeicient linear (partial) differential operators. Let

$$p_j = \sum_{m=0}^{n_j} f_{j,m}, \quad f_{j,m} \text{ homogeneous of degree } m, \quad j = 1, \dots, r,$$

be nonconstant linearly independent polynomials on  $\mathbb{R}^s$  with distinct leading terms  $f_{j,n_j}$ ,  $j = 1, \dots, r$ , for which we may assume that for all  $k < j$ ,  $0 < n_k \leq n_j$  and  $f_{j,n_j}$  contains a monomial that does not appear in  $f_{k,n_k}$ . This places no restriction on the generality since we are dealing with a linear process and the assumptions may be achieved by taking linear combinations and relabelling the polynomials. With each polynomial  $p_j$  we associate a homogeneous polynomial  $q_j$  as follows: Choose  $q_j$  to be that portion of the leading homogeneous term  $f_{j,n_j}$  containing those monomials not in  $p_k$ ,  $k < j$ . Dene

$$(2.2) \quad \sigma_j(x) := q_j(e^{2\pi i x} - 1), \quad j = 1, \dots, r,$$

where we have adopted the multivariate notation

$$(2.3) \quad (e^{2\pi i x} - 1)^\alpha := \prod_{m=1}^s (e^{2\pi i x(m)} - 1)^{\alpha(m)}.$$

The trigonometric polynomials  $\sigma_j$ ,  $j = 1, \dots, r$  are **shift invariant**; i.e.,  $\sigma_j(\cdot - \alpha) = \sigma_j$  for all  $\alpha \in \mathbb{Z}^s$ . For classical Hermite interpolation,  $q_j = p_j$  are the monomials associated with the derivatives to be interpolated. In general, this leads to a complex-valued approximation process. There are other possible choices here: For example, we may discard some monomials from  $q_j$ , and/or we may replace  $e^{2\pi i x} - 1$  in the denition of  $\sigma_j$  by  $\sin(2\pi x)$  and sometimes by  $\sin(\pi x)$  (when all the monomials contain only even powers as in the case of the Laplacian). For the interpolation of real functions, one may wish to use  $\sin(2\pi x)$  in the denition of  $\sigma_j$  (with  $(\sin(2\pi x))^\alpha := \prod (\sin(2\pi i x(m)))^{\alpha(m)}$ ).

Let  $\varphi$  be a compactly supported function in  $C^\kappa(\mathbb{R}^s)$  with  $\kappa = \max_j n_j$  and dene the functions

$$(2.4) \quad \varphi_j := \sigma_j \varphi, \quad j = 1, \dots, r.$$

For convenience of notation, let  $p_0 = 1$  and  $\varphi_0 := \varphi$ . The space  $\mathcal{S}$  in Eq.(1.2) then takes the form

$$(2.5) \quad \mathcal{S} = \sum_{j=0}^r \mathcal{S}(\varphi_j).$$

Now the general interpolation problem may be stated as follows: Given sequences  $d_j$ ,  $j = 0, \dots, r$ , dened on  $\mathbb{Z}^s$ , nd a function  $S \in \mathcal{S}$  such that

$$(2.6) \quad (p_j(D)S)(\alpha) = d_j(\alpha), \quad \text{for all } \alpha \in \mathbb{Z}^s \text{ and } j = 0, \dots, r.$$

This interpolation problem is said to be **correct** if for arbitrary bounded sequences  $d_j$ ,  $j = 0, \dots, r$ , there is a unique solution  $S \in \mathcal{S}$  of the form

$$S = \sum_{j=0}^r \varphi_j * c_j,$$

with bounded coefficient sequences  $c_j$ . In terms of the symbols for the associated difference equations, the system Eq.(2.6) for the coefficients  $c_j$ ,  $j = 0, \dots, r$ , becomes a system of linear equations

$$(2.7) \quad \sum_{j=1}^r A_{k,j} C_j = D_k, \quad k = 0, \dots, r,$$

for the unknown functions  $C_j$ , where

$$C_j = \sum_{\alpha \in \mathbb{Z}^s} c_j(\alpha) \exp(-i\alpha \cdot) \quad \text{and} \quad D_k = \sum_{\alpha \in \mathbb{Z}^s} d_k(\alpha) \exp(-i\alpha \cdot)$$

are formal Fourier series associated with the sequences  $c_j$  and  $d_k$ , and  $A_{k,j}$  are the (periodic) **symbols** of the function  $p_k(D)\varphi_j$ . The non-vanishing of  $\det A$  on  $\mathbb{R}^s$  for the **symbol matrix** of the interpolation problem,  $A := (A_{k,j})_{k,j=0}^r$ , guarantees the unique solution of Eq.(2.7) for the unknown functions  $C_j$ . The space  $\mathcal{S}$  was chosen precisely to make the non-vanishing of  $\det A$  a triviality

**Theorem 2.8** *The symbol matrix  $A = (A_{k,j})_{k,j=0}^r$  for the system Eq.(2.7) is lower triangular with diagonal entries of the form*

$$A_{j,j} = \text{const}_j \tilde{\varphi}, \quad j = 0, \dots, r,$$

where  $\text{const}_j \neq 0$ . In particular,  $\det A(\exp(-iy)) \neq 0$  for all  $y \in \mathbb{R}^s$  if and only if the symbol  $\tilde{\varphi}(\exp(-iy)) \neq 0$  for all  $y \in \mathbb{R}^s$ .

Let  $\mathcal{E}$  be the  $2\pi$ -periodic functions with exponentially decaying Fourier coefficients.

**Corollary 2.9** *Under the assumptions of Theorem 2.8, if  $\varphi$  is compactly supported and  $|\tilde{\varphi}(z)| > 0$  for  $|z| = 1$ , then the functions*

$$L_j := \sum_{m=j}^r \varphi_m * c_{m,j}, \quad j = 0, \dots, m,$$

with coefficient sequences from the functions  $C_{m,j} = \sum_{\alpha} c_{m,j} \exp(-i\alpha \cdot) \in \mathcal{E}$  defined by

$$C_{m,j} = 0, \quad m = 1, \dots, j-1,$$

$$C_{m,j} = \text{const}_m (\delta_0(m-j) - \sum_{\mu=j}^{m-1} A_{j,\mu} C_{\mu,j}) / \tilde{\varphi}, \quad m = j, \dots, r,$$

are fundamental solutions to the problem Eq.(2.6) from  $\mathcal{S}$ ; that is,  $p_j(D)L_k(\alpha) = \delta_0(j-k)\delta_0(\alpha)$  for all  $\alpha \in \mathbb{Z}^s$ . When the symbol  $\tilde{\varphi}$  is a non-zero constant, then the solutions  $C_{m,j}$  are trigonometric polynomials and the functions  $L_j$  are compactly supported.

Theorem 2.8 and Corollary 2.9 outline perhaps the most useful special case of the results in Jetter, Riemenschneider, and Shen (1994). If we want compactly supported fundamental solutions, then we must look further at the choice of  $\varphi$  used in the definition of the spaces  $\mathcal{S}$ ; a quest aided by the results of Micchelli (1992) which we describe next.

For a compactly supported function  $\varphi$ , the symbol dened in Eq.(2.1) is a Laurent polynomial in its complex arguments. Notice that the interpolation matrix  $A$  for the problem described above becomes a matrix of Laurent polynomials when we use the complex symbols. The theorem of interest here was used by Micchelli (1992) in the context of interpolation:

**Theorem 2.10** *Let  $A(z)$ ,  $z \in (\mathbf{C} \setminus \{0\})^s$ , be a  $k \times \ell$  matrix of Laurent polynomials such that  $k \leq \ell$  and*

$$\{z : z \in (\mathbf{C} \setminus \{0\})^s, \text{ all } k \times k \text{ minors of } A \text{ vanish at } z\} = \emptyset.$$

*Then there is an  $\ell \times k$  matrix of Laurent polynomials,  $C(z)$ , such that*

$$A(z)C(z) = I_k(z), \quad z \in (\mathbf{C} \setminus \{0\})^s,$$

*with  $I_k(z)$  the identity matrix.*

Actually, we shall use this nice theorem only in the case  $k = 1$  in which case the condition on the minors reduces to the Laurent polynomials in  $A$  having no common roots in  $(\mathbf{C} \setminus \{0\})^s$ . Micchelli (1992) provides several examples where this simple case is used to find compactly supported fundamental solutions for cardinal interpolation in  $\mathbb{Z}^s$ . He considers linear combinations of nitely many compactly supported functions; i.e.,  $L$  in the form

$$(2.11) \quad L = \psi_1 * c_1 + \dots + \psi_\ell * c_\ell.$$

If the functions  $\psi_j$  are chosen to have complex symbols (necessarily Laurent polynomials) that have no common roots on  $(\mathbf{C} \setminus \{0\})^s$ , then nitely supported sequences  $c_j$  can be found for which Eq.(2.11) is a fundamental solution for cardinal interpolation. Indeed, if the matrix  $A(z)$  in Theorem 2.10 is the  $1 \times \ell$  matrix of the complex symbols  $\tilde{\psi}_j(z)$ , then the sequences  $c_j$  are the coefficients of the Laurent polynomials comprising the matrix  $C(z)$  of that Theorem (cf. Corollary 2.9). A natural choice here is a combination of B-splines (Qi 1981, Dahmen, Goodman and Micchelli 1988), or box splines (Dahmen, Goodman and Micchelli 1989, Micchelli 1992).

Whatever the method of arriving at a compactly supported fundamental solution,  $L$ , we can choose the function  $\varphi = L$  as the function that generates the space  $\mathcal{S}$ . Since  $\tilde{\varphi} = \tilde{L} = 1$  in this case, the symbol matrix for the general interpolation problem has constants on its diagonal, and all the fundamental solutions in Corollary 2.9 will be compactly supported. Hence, we quite easily arrive at our objective of compactly supported fundamental solutions. There is a further advantage in the computation of the coefficient sequences in Corollary 2.9. All of the symbols  $A_{j,\mu}$  are determined by nitely supported sequences  $(p_j(D)\varphi_j)_\mu$  and the coefficients  $c_{m,j}$  are generated by a recurrence relation involving discrete convolution of nite sequences:

$$\begin{aligned} c_{m,j} &= 0, \quad m = 1, \dots, j-1; \\ c_{m,j} &= \text{const}_m(\delta_0(m-j) - \sum_{\mu=j}^{m-1} (p_j(D)\varphi_j)_\mu * c_{\mu,j}). \end{aligned}$$

For the approximation properties of the interpolation, recall that a compactly supported function  $\varphi \in C^\kappa$  satisfies the Strang-Fix conditions of order  $\eta \leq \kappa$  if

$$\widehat{\varphi}(0) = 1, \text{ and } D^\beta \widehat{\varphi}(2\pi\alpha) = 0 \text{ for } |\beta| < \eta, \quad \alpha \in \mathbb{Z}^s \setminus \{0\}.$$

It is well-known that if  $\varphi$  satisfies the Strang-Fix conditions of order  $\eta$ , or reproduces polynomials of order  $\eta$ , then for all sufficiently smooth functions, the cardinal interpolation operator  $\mathcal{L}$  for the space  $\mathcal{S}(\varphi)$  provides approximation order  $\eta$ : If

$$\mathcal{L}_h := \Sigma_{1/h} \mathcal{L} \Sigma_h, \quad \Sigma_h : f \rightarrow f(h \cdot),$$

then for sufficiently smooth  $f$  with support in the compact set  $\Omega$ ,

$$(2.12) \quad \|f - \mathcal{L}_h f\|_{\infty, \Omega} = O(h^\eta).$$

Any scaled general interpolation operator associated with the differential operators  $p_j(D)$  of order  $\leq \kappa$  given by

$$\mathcal{J}_h := \Sigma_{1/h} \mathcal{J} \Sigma_h, \quad \Sigma_h : f \rightarrow f(h \cdot),$$

with

$$\mathcal{J}f := \sum_{j=0}^r L_j *'(p_j(D)f),$$

has the same approximation power as the cardinal interpolant:

$$\|f - \mathcal{J}_h f\|_{\infty, \Omega} = O(h^\eta)$$

for all sufficiently smooth  $f$  with support in the compact set  $\Omega$  (Jetter, Riemenschneider, and Shen (1994)).

### 3. Univariate Examples

Several of the examples occurring in this section date back to Schoenberg (1946), and as mentioned in there, some particular examples date back even earlier. Others have rediscovered some of them or have placed them in broader contexts. Here we mention them in light of the previous section and indicate possible algorithms for their generation by the symbolic software MAPLE which makes their use feasible on modest computing equipment.

#### 3.1 Fundamental functions of Dahmen, Goodman and Micchelli

There have been some cases where authors have introduced extra knots in the spline space and have used the freedom gained for additional properties of the interpolation. We shall mention here only the work of Qi (1981) and Dahmen, Goodman, and Micchelli (1988) on cardinal interpolation. Let  $\mathcal{S}_n(T)$  denote the splines of order  $n$  spanned by the B-splines with knots

$$(3.1) \quad T = \mathbb{Z} \cup (\alpha + \mathbb{Z}) = (t_\alpha)_{\alpha \in \mathbb{Z}}, \quad t_\alpha := (\alpha + (\lambda - 1/2)(1 - (-1)^\alpha)), \quad \alpha \in \mathbb{Z},$$

for some fixed  $\lambda$ ,  $0 < \lambda < 1$ . (The case  $\lambda = 1/2$  was done in Qi (1981)). The simplest theorem on cardinal interpolation in Dahmen, Goodman and Micchelli (1988) is

**Theorem 3.2** *For any  $n \geq 3$ , there exists  $L \in \mathcal{S}_n(T)$  such that*

$$(3.3) \quad \text{supp } L = [-(n-1), \dots, n-1], \quad L(\alpha) = \delta_0(\alpha) \forall \alpha \in \mathbb{Z},$$

and for which the cardinal interpolation operator  $\mathcal{L}f = L * f$  reproduces polynomials of order  $n-1$ .

The  $L$  obtained in Theorem 3.2 is found as the unique spline of the form

$$L = \sum_{j=0}^{2n-4} \gamma_j M_n(\cdot + n - 1 - t_j)$$

satisfying Eq.(3.3) (guaranteed to exist by the Schoenberg-Whitney Theorem, see Dahmen, Goodman and Micchelli (1988) for details). More generally, Dahmen, Goodman and Micchelli (1988) studied the trade off between the size of the support of  $L$  and the degree of the polynomials reproduced by the interpolation operator. For fixed  $\ell, \nu$  with  $-1 \leq \ell \leq n-1$  and  $1 \leq \nu \leq n-2+\ell$ , they prove the existence of a unique fundamental solution  $L \in \mathcal{S}_n(T)$  with  $\text{supp } L = [-\nu, -\nu+n+\ell-1]$  for which the corresponding cardinal interpolation operator reproduces polynomials of degree  $\ell$ . Their comprehensive study included much more general knot sequences as well.

By the remarks of Sect. 2, the fundamental solutions of Theorem 3.2 give rise to fundamental solutions of the general interpolation problem retaining the full approximation order. Examples are the fundamental solutions corresponding to B-splines of order 4 and 5 respectively and  $\lambda = 1/2$ :

$$\begin{aligned} L^{\{4\}} &= \frac{1}{6}M_4(\cdot + 3) - \frac{8}{6}M_4(\cdot + 5/2) + \frac{20}{6}M_4(\cdot + 2) \\ &\quad - \frac{8}{6}M_4(\cdot + 3/2) + \frac{1}{6}M_4(\cdot + 1), \\ L^{\{5\}} &= -\frac{2}{45}M_5(\cdot + 4) + \frac{32}{45}M_5(\cdot + 7/2) - \frac{59}{18}M_5(\cdot + 3) + \frac{56}{9}M_5(\cdot + 5/2) \\ &\quad - \frac{59}{18}M_5(\cdot + 2) + \frac{32}{45}M_5(\cdot + (3/2)) - \frac{2}{45}M_5(\cdot + 1). \end{aligned}$$

### 3.2 Combinations of $M_n$ and $M_{n+1}$

It is well known that the roots of the Euler-Frobenius polynomials, and therefore the nonzero roots of the complex symbol  $\widetilde{M}_n$  for the B-spline  $M_n$ , are simple and negative. Moreover, the nonzero roots of  $\widetilde{M}_n$  and  $\widetilde{M}_{n+1}$  strictly interlace. Therefore, Theorem 2.10 applies and there is a fundamental solution of the form

$$L = M_n * c_1 + M_{n+1} * c_2$$

with finite sequences  $c_1, c_2$ . Such a fundamental function will satisfy the Strang-Fix conditions of order at least  $n$ , and consequently, will give rise to interpolation obtaining approximation order  $n$ . However, there are many possibilities here and we would like to choose  $L$  of small support, say in  $[-(n-1), n-1]$  as in Theorem 3.2, and to be symmetric about the origin.

The only common root of  $\widetilde{M}_n$  and  $\widetilde{M}_{n+1}$  is  $z = 0$  and there exist polynomials  $C_1, C_2$  such that



$$C_1(z)\widetilde{M}_n(z) + C_2(z)\widetilde{M}_{n+1}(z) = \gcd(\widetilde{M}_n, \widetilde{M}_{n+1}) = z^\alpha, \quad \alpha = 1.$$

But then also any nonzero integer power  $\alpha$  of  $z$  (as a multiple of the gcd) can be obtained in this way, and once found, division by that monomial will yield Laurent polynomials. That's the motivation for the following algorithm based on the MAPLE 'gcdex' command that leads to the desired fundamental solutions, at least for low order splines:

**Algorithm 3.4** Given  $\widetilde{M}_n$  and  $\widetilde{M}_{n+1}$ .

i. If  $n$  is odd, set

$$p_1(z) = \widetilde{M}_n(z), \quad p_2(z) = \widetilde{M}_{n+1}(z), \quad p_3(z) = z^{n-1},$$

else if  $n$  is even, set

$$p_1(z) = \widetilde{M}_n(z^2), \quad p_2(z) = \widetilde{M}_{n+1}(z^2), \quad p_3(z) = z^{2n-1}.$$

ii. Execute the MAPLE command

$$\text{gcdex}(p_1, p_2, p_3, z, 's', 't').$$

(Finds  $s, t$  so that  $sp_1 + tp_2 = p_3$  when  $p_3$  is a multiple of  $\gcd(p_1, p_2)$ .)

iii. If  $n$  is odd, set

$$C_1(z) = z^{-(n-1)}s(z), \quad \text{and} \quad C_2(z) = z^{-(n-1)}t(z),$$

else if  $n$  is even

$$C_1(z^2) = z^{-(2n-1)}s(z), \quad \text{and} \quad C_2(z^2) = z^{-(2n-1)}t(z),$$

to obtain Laurent polynomials  $C_1$  and  $C_2$  for which

$$C_1(z)\widetilde{M}_n(z) + C_2(z)\widetilde{M}_{n+1}(z) = 1.$$

Table 1 shows the output of this algorithm for  $3 \leq n \leq 7$ . The resulting fundamental functions all have support in  $[-(n-1), n-1]$  and are symmetric about the origin. The fundamental solutions corresponding to  $n = 3, \dots, 8$  are plotted together in Fig. 1. Notice the very rapid decay (the graphs of  $L^{\{7,8\}}$  and  $L^{\{8,9\}}$  are not visible on their full support even though the vertical direction was scaled by a factor of 6).

The rest of the examples of each type ( $n$ -odd, even) in Table 1 are:

$$\begin{aligned} L^{\{3,4\}} &= 3M_4(\cdot + 2) - M_3(\cdot + 1) - M_3(\cdot + 2) \\ L^{\{4,5\}} &= \frac{1}{2}(M_4(\cdot + 1) + 8M_4(\cdot + 2) + M_4(\cdot + 3)) \\ &\quad - 2(M_5(\cdot + 2) + M_5(\cdot + 3)). \end{aligned}$$

To illustrate the theory of Sect. 2, we use each of these to solve a simple general interpolation problem: Since  $L^{\{3,4\}}$  is  $C^1$ , it can be used to solve Hermite interpolation of a function  $f$  and its derivative on  $\mathbb{Z}$ . We take the two functions

$$\varphi_0 = L^{\{3,4\}}, \quad \text{and} \quad \varphi_1 = \sin(2\pi \cdot) L^{\{3,4\}},$$

**Table 1.** Pairs of successive  $\widetilde{M}_n$  and the associated Laurent polynomials  $c_1, c_2$  found via Algorithm 3.4 for which  $c_1 \widetilde{M}_n + c_2 \widetilde{M}_{n+1} = 1$ .

$n$	$\widetilde{M}_n$	
3	$\frac{z}{2} + \frac{z^2}{2}$	
4	$\frac{z}{6} + \frac{2z^2}{3} + \frac{z^3}{6}$	
5	$\frac{z}{24} + \frac{11z^2}{24} + \frac{11z^3}{24} + \frac{z^4}{24}$	
6	$\frac{z}{120} + \frac{13z^2}{60} + \frac{11z^3}{20} + \frac{13z^4}{60} + \frac{z^5}{120}$	
7	$\frac{z}{720} + \frac{19z^2}{240} + \frac{151z^3}{360} + \frac{151z^4}{360} + \frac{19z^5}{240} + \frac{z^6}{720}$	
8	$\frac{z}{5040} + \frac{z^2}{42} + \frac{397z^3}{1680} + \frac{151z^4}{315} + \frac{397z^5}{1680} + \frac{z^6}{42} + \frac{z^7}{5040}$	
$(n, n+1)$	$c_1$	$c_2$
(3, 4)	$-\frac{1+z}{z^2}$	$\frac{3}{z^2}$
(4, 5)	$\frac{z^2+8z+1}{2z^3}$	$-\frac{2+2z}{z^3}$
(5, 6)	$-\frac{7z^3+167z^2+167z+7}{32z^4}$	$\frac{35z^2+310z+35}{32z^4}$
(6, 7)	$\frac{9z^4+496z^3+1814z^2+496z+9}{112z^5}$	$-\frac{27z^3+651z^2+651z+27}{56z^5}$
	$(n, n+1) = (7, 8)$	
$c_1$	$-\frac{1079z^5+127653z^4+1071484z^3+1071484z^2+127653z+1079}{41616z^6}$	
$c_2$	$\frac{7553+417732z+1591478z^2+417732z^3+7553z^4}{41616z^6}$	

as the two generators of the interpolation space. The corresponding symbol matrix will have nonzero entries

$$A_{1,1}(z) = A_{2,2}(z)/(2\pi) = 1, \quad \text{and}$$

$$A_{2,1}(z) = D\widetilde{L}^{\{3,4\}}(z) = (z^{-1} - z)/2,$$

so that the corresponding fundamental functions are

$$L_0 = L^{\{3,4\}}(\cdot) - (4\pi^2)^{-1} \sin(2\pi \cdot) (L^{\{3,4\}}(\cdot + 1) - L^{\{3,4\}}(\cdot - 1))$$

$$L_1 = -(2\pi)^{-1} \sin(2\pi \cdot) L^{\{3,4\}}(\cdot).$$

For  $L^{\{4,5\}}$  we choose the lacunary interpolation problem of interpolating  $f$  and  $f''$  on  $\mathbb{Z}$ . This time the functions are

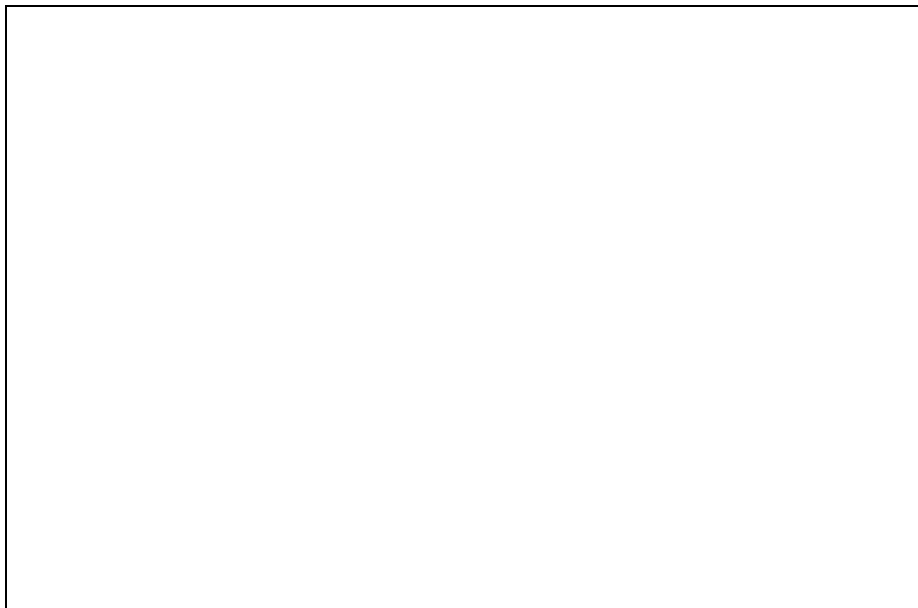
$$\varphi_0 = L^{\{4,5\}}, \quad \text{and} \quad \varphi_1 = \sin(2\pi \cdot)^2 L^{\{4,5\}},$$

and the symbol matrix has nonzero entries

$$A_{1,1}(z) = A_{2,2}(z)/(2\pi)^2 = 1, \quad \text{and}$$

$$A_{2,1}(z) = D^2 \widetilde{L}^{\{4,5\}}(z) = -z^{-2}/2 + 3z^{-1} - 5 + 3z - z^2/2.$$

Hence, the fundamental functions are



**Fig.1.** Fundamental solutions for cardinal interpolation from a linear combination of translates of  $\widetilde{M}_n$  and  $\widetilde{M}_{n+1}$  for  $n = 3, \dots, 8$ .

$$L_0 = L^{\{4,5\}}(\cdot) - (8\pi^2)^{-1}(\sin(2\pi\cdot))^2 \left( - (1/2)L^{\{4,5\}}(\cdot + 2) + 3L^{\{4,5\}}(\cdot + 1) \right. \\ \left. - 5L^{\{4,5\}}(\cdot) + 3L^{\{4,5\}}(\cdot - 1) - (1/2)L^{\{4,5\}}(\cdot - 2) \right);$$

$$L_1 = (8\pi^2)^{-1}(\sin(2\pi\cdot))^2 L^{\{4,5\}}(\cdot).$$

The success of Algorithm 3.4 encourages us to test whether a variation might apply to other situations. In fact, the following algorithm generates the fundamental solutions of Theorem 3.2 as linear combinations of  $M_n$  and the shifted  $B$ -spline  $M_n(\cdot + 1/2)$  for all cases tested. First, we need the symbol of  $M_n(\cdot + 1/2)$ :

$$\widetilde{M}_{n, \frac{1}{2}}(z) := \sum_{j \in \mathbb{Z}} M_n(j + 1/2)z^j,$$

**Algorithm 3.5** Given  $\widetilde{M}_n$  and  $\widetilde{M}_{n, \frac{1}{2}}$ .

i. Set

$$p_1(z) = \widetilde{M}_n(z), \quad p_2(z) = \widetilde{M}_{n, \frac{1}{2}}(z), \quad p_3(z) = z^{n-1}.$$

ii. Execute the MAPLE command

$$\text{gcdex}(p_1, p_2, p_3, z, 's', 't').$$

iii. Set

$$C_1(z) = z^{-(n-1)}s(z), \quad \text{and} \quad C_2(z) = z^{-(n-1)}t(z),$$

to obtain Laurent polynomials  $C_1$  and  $C_2$  for which

$$C_1(z)\widetilde{M}_n(z) + C_2(z)\widetilde{M}_{n, \frac{1}{2}}(z) = 1.$$

Table 2 lists the results of this algorithm for  $n = 4, \dots, 7$ . By comparing the polynomials  $C_1$  and  $C_2$  in Tables 1 and 2 with the examples of fundamental solutions given in this subsection and the last, it is easy to write down the missing fundamental solutions.

**Table 2.** The symbols  $\tilde{M}_{n, \frac{1}{2}}$  and the associated Laurent polynomials  $c_1, c_2$  found via Algorithm 3.5 which satisfy  $c_1 \tilde{M}_n + c_2 \tilde{M}_{n, \frac{1}{2}} = 1$  and give rise to the fundamental solutions in Theorem 3.2.

$n$	$\tilde{M}_{n, \frac{1}{2}}$	
4	$\frac{1}{48} + \frac{23z}{48} + \frac{23z^2}{48} + \frac{z^3}{48}$	
5	$\frac{1}{384} + \frac{19z}{96} + \frac{115z^2}{192} + \frac{19z^3}{96} + \frac{z^4}{384}$	
6	$\frac{1}{3840} + \frac{79z}{1280} + \frac{841z^2}{1920} + \frac{841z^3}{1920} + \frac{79z^4}{1280} + \frac{z^5}{3840}$	
7	$\frac{1}{46080} + \frac{361z}{23040} + \frac{10543z^2}{46080} + \frac{5887z^3}{15360} + \frac{10543z^4}{46080} + \frac{361z^5}{23040} + \frac{z^6}{46080}$	
$n$	$c_1$	$c_2$
4	$\frac{z^2+20z+1}{6z^3}$	$-\frac{4z+4}{3z^2}$
5	$-\frac{4z^3+295z^2+295z+4}{90z^4}$	$\frac{32z^2+280z+32}{45z^3}$
6	$\frac{61z^4+14340z^3+75358z^2+14340z+61}{6240z^5}$	$-\frac{61z^3+1469z^2+1469z+61}{195z^4}$
	$n = 7$	
$c_1$	$-\frac{3377z^5+2432377z^4+31422342z^3+31422342z^2+2432377z+3377}{1844640z^6}$	
$c_2$	$\frac{6754z^4+373344z^3+1413580z^2+373344z+6754}{57645z^5}$	

For the proper implementation of 'gcdex' in the above algorithms, the coefficients in  $p_1, p_2$  should be given as rational numbers. The coefficients of the symbols are most easily generated via the recurrence relations for  $B$ -splines:

$$nM_{n+1}(x) = xM_n(x) + (n+1-x)M_n(x-1).$$

### 3.3 Interpolation related to nite decomposition and reconstruction

The idea in the preceding subsection is useful for nite decomposition and reconstruction in spline spaces important to wavelet theory (see for example, Micchelli 1991, Cohen, Daubechies, and Feauveau 1992). Next, we interpret some of the ideas in these papers to obtain compactly supported fundamental functions for cardinal interpolation. Let  $M := M_n$  be the  $B$ -spline of order  $n$ . Then  $M$  generates a **multiresolution approximation** of  $L_2(\mathbb{R})$ ; i.e., the cardinal spline space

$$V_0 := \text{closure}_{L_2(\mathbb{R})} \{M * a : a \text{ nitely supported} \}$$

gives rise to spaces  $V_\nu := \{f : f(2^{-\nu} \cdot) \in V_0\}$  which are nested,  $V_\nu \subset V_{\nu+1}$ , have common intersection equal to the zero function, and whose union is dense in  $L_2(\mathbb{R})$ . We want to write  $V_1$  as a direct sum of  $V_0$  and another space  $W_0$ , generated by a

function  $\psi$ . The sequences  $m$  and  $m_\psi$  for the representation of the generators of  $V_0$  and  $W_0$  as elements in the space  $V_1$ ,

$$(3.6) \quad M = \sum_{\alpha \in \mathbb{Z}} M(2 \cdot -\alpha)m(\alpha), \quad \psi = \sum_{\alpha \in \mathbb{Z}} M(2 \cdot -\alpha)m_\psi(\alpha),$$

are called the **reconstruction sequences**, while sequences  $d_M$  and  $d_\psi$  such that

$$(3.7) \quad M(2 \cdot) = M * d_M + \psi * d_\psi$$

are called **decomposition sequences**. The sequence  $m$  is the well-known nitely supported **renement mask** for the spline  $M$ . In terms of Fourier transforms and symbols, Eq.(3.6) reads

$$(3.8) \quad \begin{aligned} 2\widehat{M}(2y) &= \tilde{m}(z)\widehat{M}(y), & \tilde{m}(z) &= \sum_{\alpha \in \mathbb{Z}} m(\alpha)z^\alpha = \frac{(1+z)^n}{2^{n-1}}, \\ 2\widehat{\psi}(2y) &= \tilde{m}_\psi(z)\widehat{M}(y), & \tilde{m}_\psi(z) &= \sum_{\alpha \in \mathbb{Z}} m_\psi(\alpha)z^\alpha, \quad z = \exp(-iy). \end{aligned}$$

For a construction to make  $m_\psi$ ,  $d_M$  and  $d_\psi$  nitely supported, we rst turn to  $m_\psi$ . With the complex symbol for  $\tilde{m}$ , it follows from Eq.(3.8) that

$$(3.9) \quad \tilde{m}(z) = G_0(z^2) + zG_1(z^2),$$

where  $G_0, G_1$  are Laurent polynomials with no common roots (since such a root would be common to  $(1-z)^n$  and  $(1+z)^n$ ). Hence, there exist Laurent polynomials  $Q_0$  and  $Q_1$  such that

$$(3.10) \quad G_0Q_0 + G_1Q_1 = 1.$$

Dene

$$(3.11) \quad \tilde{m}_\psi(z) := z^{-1}(Q_0(z^2) - z^{-1}Q_1(z^2)), \quad z = \exp(-iy),$$

and then dene  $\widehat{\psi}$  by Eq.(3.8). Clearly,  $m_\psi$ , the sequence of coefficients for  $\tilde{m}_\psi$ , is nitely supported. The decomposition sequences may be taken as the coefficients of the complex symbols

$$\tilde{d}_M(z) := Q_0(z), \quad \tilde{d}_\psi(z) := -zG_1(z),$$

for then, from Eq.(3.9), Eq.(3.11), and Eq.(3.10),

$$\begin{aligned} \tilde{d}_M(z^2)\tilde{m}(z) + \tilde{d}_\psi(z^2)\tilde{m}_\psi(z) &= Q_0(z^2)(G_0(z^2) + zG_1(z^2)) \\ &\quad - z^2G_1(z^2)(z^{-1}Q_0(z^2) - z^{-2}Q_1(z^2)) \\ &= Q_0(z^2)G_0(z^2) + Q_1(z^2)G_1(z^2) = 1; \end{aligned}$$

hence,

$$2\tilde{d}_M(\exp(-i2y))\widehat{M}(2y) + 2\tilde{d}_\psi(\exp(-i2y))\widehat{\psi}(2y) = \widehat{M}(y),$$

which is equivalent to Eq.(3.7).

**Proposition 3.12** The collection  $\{M(\cdot - \alpha), \psi(\cdot - \beta) : \alpha, \beta \in \mathbb{Z}\}$  forms a Riesz basis for the space  $V_1$  with nitely supported decomposition and reconstruction sequences.

*Proof* It only remains to check the assertion about Riesz basis. According to (Riemenschneider and Shen 1992, Prop. 3.6) we only need to check that the vectors  $(\tilde{m}(z), \tilde{m}(-z))$ ,  $(\tilde{m}_\psi(z), \tilde{m}_\psi(-z))$  are linearly independent for each  $z = \exp(-iy)$ ,  $y \in \mathbb{R}$ . But this follows from Eq.(3.9), Eq.(3.10), Eq.(3.11) and

$$\begin{bmatrix} \tilde{m}(z) & \tilde{m}_\psi(z) \\ \tilde{m}(-z) & \tilde{m}_\psi(-z) \end{bmatrix} = \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix} \begin{bmatrix} G_0(z^2) & -z^{-2}Q_1(z^2) \\ G_1(z^2) & z^{-2}Q_0(z^2) \end{bmatrix}, \quad z = \exp(-iy).$$

Table 3 contains some examples with  $Q_0 = \tilde{d}_M$  and  $Q_1$  obtained from the MAPLE 'gcdex' command. Notice that in these examples,  $\psi$  will be of one sign and  $\hat{\psi}(0) \neq 0$ . Consequently, in general,  $\{\psi(2^\nu \cdot -\alpha) : \nu, \alpha \in \mathbb{Z}\}$  does not constitute a Riesz bases for  $L_2(\mathbb{R})$  (see p. 63 in Daubechies (1992)).

**Table 3.** Functions and symbols useful in the nite decomposition and reconstruction of spline spaces, and in the derivation of related fundamental solutions for cardinal interpolation.

$n$	3	4
$\tilde{m}$	$\frac{(1+z)^3}{4}$	$\frac{(1+z)^4}{8}$
$G_0$	$\frac{1+3z}{4}$	$\frac{1+6z+z^2}{6}$
$\tilde{d}_M$	$-\frac{1}{2}$	$\frac{2}{z}$
$\frac{\tilde{d}_\psi(z)}{z}$	$-\frac{3+z}{4}$	$-\frac{1+z}{2}$
$Q_1$	$\frac{3}{2}$	$-\frac{1+z}{2}$
$\tilde{m}_\psi$	$-\frac{3+z}{2z^2}$	$\frac{1+4z+z^2}{2z^4}$
$\pm\psi$	$\frac{3M_3(2 \cdot +2)+M_3(2 \cdot +1)}{2}$	$\frac{M_4(2 \cdot +4)+4M_4(2 \cdot +3)+M_4(2 \cdot +2)}{2}$
$\psi_0$	$\frac{\psi_3(\cdot) - 3\psi_3(\cdot - 1) + 3\psi_3(\cdot - 2) - \psi_3(\cdot - 3)}{4}$	$\frac{\psi_4(\cdot) - 4\psi_4(\cdot - 1) + 6\psi_4(\cdot - 2) - 4\psi_4(\cdot - 3) + \psi_4(\cdot - 4)}{8}$
$C_1^*$	$\frac{-z^2+4z+1}{4z^2}$	$\frac{z^4-6z^3+2z^2-6z+1}{12z^4}$
$C_2^*$	$\frac{1}{2z}$	$-\frac{z+1}{6z^2}$
$n$	5	
$\tilde{m}$	$\frac{(1+z)^5}{16}$	
$G_0$	$\frac{1+10z+5z^2}{16}$	
$\tilde{d}_M$	$-\frac{3z+25}{8z}$	
$\frac{\tilde{d}_\psi(z)}{z}$	$\frac{5+10z+z^2}{16}$	
$Q_1$	$-\frac{5+15z}{8z}$	
$\tilde{m}_\psi$	$\frac{5+25z+15z^2+3z^3}{8z^4}$	
$\pm\psi$	$\frac{5M_5(2 \cdot +4)+25M_5(2 \cdot +3)+15M_5(2 \cdot +2)+3M_5(2 \cdot +1)}{8}$	
$\psi_0$	$\frac{\psi_5(\cdot) - 5\psi_5(\cdot - 1) + 10\psi_5(\cdot - 2) - 10\psi_5(\cdot - 3) + 5\psi_5(\cdot - 4) - \psi_5(\cdot - 5)}{16}$	
$C_1^*$	$\frac{5}{144} + \frac{z}{60} - \frac{1}{2z} + \frac{95}{72z^2} + \frac{1}{9z^3} + \frac{11}{240z^4} - \frac{1}{36z^5}$	
$C_2^*$	$\frac{2}{45z} + \frac{7}{18z^2} + \frac{2}{45z^3}$	

The functions  $M$  and  $\psi$  do however allow for the construction of a compactly supported fundamental solution for cardinal interpolation. Observe that by grouping the even and odd powers, any symbol  $\tilde{\varphi}$  with coefficients  $\{\varphi(\alpha)\}$  can be written as

$$\tilde{\varphi}(z) = \tilde{\varphi}_e(z^2) + z^{-1}\tilde{\varphi}_o(z^2),$$

where

$$\tilde{\varphi}_e(z) = \sum_{\alpha \in \mathbb{Z}} \varphi(2\alpha)z^\alpha, \quad \tilde{\varphi}_o(z) = \sum_{\alpha \in \mathbb{Z}} \varphi(2\alpha - 1)z^\alpha.$$

Since the symbol,  $\tilde{M}$ , of the  $B$ -spline has only negative or zero real roots and non-negative coefficients, it follows that the symbols  $\tilde{M}_e$  and  $\tilde{M}_o$  can have no common roots. Moreover, from Eq.(3.6),

$$\begin{aligned} \tilde{M}(z) &= \tilde{m}_e(z)\tilde{M}_e(z) + z^{-1}\tilde{m}_o(z)\tilde{M}_o(z), \\ \tilde{\psi}(z) &= \tilde{m}_{\psi,e}(z)\tilde{M}_e(z) + z^{-1}\tilde{m}_{\psi,o}(z)\tilde{M}_o(z). \end{aligned}$$

Writing this in another way, we find

$$\begin{aligned} \begin{bmatrix} \tilde{M}(z) \\ \tilde{\psi}(z) \end{bmatrix} &= \begin{bmatrix} \tilde{m}_e(z) & z^{-1}\tilde{m}_o(z) \\ \tilde{m}_{\psi,e}(z) & z^{-1}\tilde{m}_{\psi,o}(z) \end{bmatrix} \begin{bmatrix} \tilde{M}_e(z) \\ \tilde{M}_o(z) \end{bmatrix} \\ &= \begin{bmatrix} G_0(z) & zG_1(z) \\ -z^{-2}Q_1(z) & z^{-1}Q_0(z) \end{bmatrix} \begin{bmatrix} \tilde{M}_e(z) \\ \tilde{M}_o(z) \end{bmatrix}. \end{aligned}$$

The nonsingularity of the last  $2 \times 2$  matrix for  $z$  in  $\mathbb{C} \setminus \{0\}$  (cf Eq.(3.10)) guarantees that  $\tilde{M}$  and  $\tilde{\psi}$  have no common roots in that region.

Further, since  $\tilde{M}(1) = 1$  ( $z = 1$ ), the Laurent polynomials  $\tilde{M}$  and  $(\tilde{m}_-)\tilde{\psi}$ ,  $\tilde{m}_-(z) := \tilde{m}(-z) = (1-z)^n/2^{n-1}$ , have no common roots in  $\mathbb{C} \setminus \{0\}$ . Hence, there exist Laurent polynomials  $C_1^*$  and  $C_2^*$  so that

$$C_1^*\tilde{M} + C_2^*(\tilde{m}_-)\tilde{\psi} = 1.$$

If  $c_1^*, c_2^*, m_-$  are the coefficient sequences of  $C_1, C_2, \tilde{m}_-$  respectively, it follows that

$$(3.13) \quad L = M * c_1^* + \psi_0 * c_2^*, \quad \psi_0 := \psi * m_-,$$

satisfies  $L(\alpha) = \delta_0(\alpha)$ ,  $\alpha \in \mathbb{Z}$ . Finally,

$$\begin{aligned} \hat{L}(y) &= C_1^*(z)\hat{M}(y) + C_2^*(z)\hat{\psi}_0(y) = C_1^*(z)\tilde{M}(z) + C_2^*(z)\tilde{m}_-(z)\tilde{\psi}(z) = 1 \\ &\text{for } y = 0, \quad z = \exp(-iy) = 1, \end{aligned}$$

while both  $\hat{M}$  and  $\hat{\psi}_0 = (\tilde{m}_-)\hat{\psi}$  vanish  $n$ -fold at  $y = 2\pi\alpha$ ,  $\alpha \in \mathbb{Z} \setminus \{0\}$  ( $z = 1$ ). We have shown

**Theorem 3.14** *The function  $L$  in Eq.(3.13) is a fundamental solution for cardinal interpolation from the space  $V_1$  corresponding to the  $B$ -spline  $M$  order  $n$ . The corresponding (general) cardinal interpolation operator obtains approximation order  $n$ .*

Included in Table 3 are functions  $\psi_0$ , derived from the  $\psi$  taken as nonnegative, and corresponding Laurent polynomials giving rise to the following fundamental solutions of Theorem 3.14:

$$\begin{aligned}
L_3 &= -M_3(\cdot)/4 + M_3(\cdot + 1) + M_3(\cdot + 2)/4 - \psi_{0,3}(\cdot + 1)/2; \\
L_4 &= M_4(\cdot)/12 - M_4(\cdot + 1)/2 + 11M_4(\cdot + 2)/6 - M_4(\cdot + 3)/2 \\
&\quad + M_4(\cdot + 4)/12 - \psi_{0,4}(\cdot + 1)/6 - \psi_{0,4}(\cdot + 2)/6; \\
L_5 &= 5M_5(\cdot)/144 + M_5(\cdot - 1)/60 - M_5(\cdot + 1)/2 + 95M_5(\cdot + 2)/72 \\
&\quad + M_5(\cdot + 3)/9 + 11M_5(\cdot + 4)/240 - M_5(\cdot + 5)/36 \\
&\quad + 2\psi_{0,5}(\cdot + 1)/45 + 7\psi_{0,5}(\cdot + 2)/18 + 2\psi_{0,5}(\cdot + 3)/45.
\end{aligned}$$

A word on their derivation. The MAPLE 'gcdex' command was used in two places, to obtain the  $Q_0, Q_1$  hence the  $\psi$ , and to obtain the  $C_1^*, C_2^*$ . In each place, there is a choice of solving for the gcd or a multiple of it (a power of  $z$ ). Many different solutions could be obtained in this way, but some are not desirable. It turned out that for each choice in the rst step, there was one choice in the second step that led to symmetric fundamental solutions with the general appearance of those in Fig. 1. Our choices above were selected at each stage to get functions as symmetric as possible and of smallest  $L_\infty$  norm, and all our fundamental solutions are symmetric and of norm 1; in fact,  $L_4 = L^{\{4\}}$  and  $L_5 = L^{\{5\}}$ . But in the case  $n = 5$  for example, other choices would lead to fundamental solutions with  $L_\infty$  norm greater than 2000! Of course one can symmetrize the fundamental solution by taking  $(L(x) + L(-x))/2$ , but this spreads out the support and does not eliminate the norm problem in general.

#### 4. Multivariate Examples

We take it as understood that tensor products of functions  $\varphi$  given in the last section can be used where appropriate to generate the spaces  $\mathcal{S}$  for any general interpolation problem in  $\mathbb{R}^s$ . Some other interesting multivariate examples of compactly supported fundamental solutions have appeared in the literature; for example, Dahmen, Goodman and Micchelli (1992), Chui (1992), and Chui and Diamond (1991) (where a general method based on quasiinterpolants and blending is given). Here we shall concentrate on functions  $\varphi$  that cannot be given as tensor products and are similar to the theory discussed in the previous section. In particular, we give a general method of construction that will ensure full approximation order. The use of symbolic packages would again be a help in generating fundamental solutions for these constructions, but the packages we used are less well developed and this clearly presents opportunity for future work.

##### 4.1 Examples of Micchelli

Micchelli (1992) considered in some detail two special cases when the functions  $\psi_j$  in Eq.(2.11) are derived from a single (nitely) renable function  $\psi$ ; i.e., there is a nite sequence  $m$  such that  $\psi = \sum_\alpha \psi(2 \cdot - \alpha)m(\alpha)$ . Let  $\mathbb{Z}_2^s = \{0, 1\}^s$  denote the extreme points of the unit cube in  $\mathbb{R}^s$ . The rst choice is to consider  $L$  of the form

$$\begin{aligned}
(4.1) \quad L &= \sum_{\alpha \in \mathbb{Z}^s} \psi(2 \cdot - \alpha)c(\alpha) = \sum_{\nu \in \mathbb{Z}_2^s} \sum_{\beta \in \mathbb{Z}^s} \psi(2(\cdot - \beta) + \nu)c_\nu(\beta) \\
&:= \sum_{\nu \in \mathbb{Z}_2^s} \psi_\nu * c_\nu,
\end{aligned}$$



where  $\psi_\nu = \psi(2 \cdot + \nu)$ ,  $c_\nu = c(2 \cdot - \nu)$ . Micchelli (1992) gave necessary and sufficient conditions on the symbol  $\tilde{\psi}$  in order that a compactly supported fundamental function would exist:

**Theorem 4.2** *For a continuous compactly supported function  $\psi$ , there is a fundamental function for cardinal interpolation of the form Eq.(4.1) with nitely supported  $c$  if and only if the Laurent polynomials  $\tilde{\psi}((-1)^\nu z)$ ,  $\nu \in \mathbb{Z}_2^s$ , have no common roots in  $(\mathbb{C} \setminus \{0\})^s$ .*

Micchelli showed that Theorem 4.2 could be applied to low order bivariate three- and four-direction box splines, but in this case, the approximation order is not necessarily established.

The second form considered by Micchelli was the form:

$$(4.3) \quad L = \sum_{\alpha \in \mathbb{Z}^s} \psi\left(\cdot - \frac{\alpha}{2}\right)c(\alpha) = \sum_{\alpha \in \mathbb{Z}^s} (\psi * m)(2 \cdot - \alpha)c(\alpha).$$

Hence, Theorem 4.2 applies to the function  $\psi * m$  with symbol

$$(4.4) \quad (\widetilde{\psi * m})(z) = \tilde{\psi}(z) \sum_{\beta \in \mathbb{Z}^s} m(\beta)z^\beta.$$

Low order bivariate three-direction box splines provide examples of functions  $\psi$  that yield fundamental functions of the form Eq.(4.3), and in this case, the full approximation order of the box spline space can be achieved.

#### 4.2 Combinations of Box Splines

Any combination of box splines whose symbols satisfy Theorem 2.10 will give rise to compactly supported fundamental solutions to the general interpolation problem. In the bivariate setting, only the three-direction box splines have any hope of providing correct cardinal interpolation in the classical sense, and they do so if and only if their centrally symmetric point, or center, belongs to  $\mathbb{Z}^2$  (when a three-direction box spline is centered, that is, the origin is translated to the center of the box spline, then cardinal interpolation is always correct) (see de Boor, Hollig and Riemenschneider 1993). In particular in the classical setting, four-direction box splines can never provide correct cardinal interpolation and either one must be satisfied with a slowly decaying fundamental function (see Jetter and Stockler 1991, de Boor, Hollig and Riemenschneider 1989), or one must consider interpolation on a sublattice (Jetter and Riemenschneider (1987)). An alternative is to use four-direction box splines in combination with three-direction box splines. Here is one such example:

Two box splines with nice smoothness and symmetries are the Zwart-Powell (ZP-)element, viz. the four-direction box spline  $M_{1,1,1,1}$ , and the three-direction box spline  $M_{2,2,2}$  based on the direction sets  $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_2 + \mathbf{i}_1, \mathbf{i}_2 - \mathbf{i}_1\}$ , and  $\{\mathbf{i}_1, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_2, \mathbf{i}_2 + \mathbf{i}_1, \mathbf{i}_2 + \mathbf{i}_1\}$ , respectively. The symbols of  $M_{2,2,2}$  and  $M_{1,1,1,1}$  are

$$\begin{aligned} \widetilde{M}_{2,2,2}(z) &= \frac{1}{12}(z^{(1,1)} + z^{(2,1)} + z^{(1,2)} + 6z^{(2,2)} + z^{(3,2)} + z^{(2,3)} + z^{(3,3)}), \\ \widetilde{M}_{1,1,1,1} &= \frac{1}{4}(z^{(0,1)} + z^{(1,1)} + z^{(0,2)} + z^{(1,2)}). \end{aligned}$$

Hence, if we set

$$C_1(z) = -z^{-(1,2)}(1 + z^{(1,1)}), \quad \text{and} \quad C_2(z) = 3z^{-(2,2)},$$

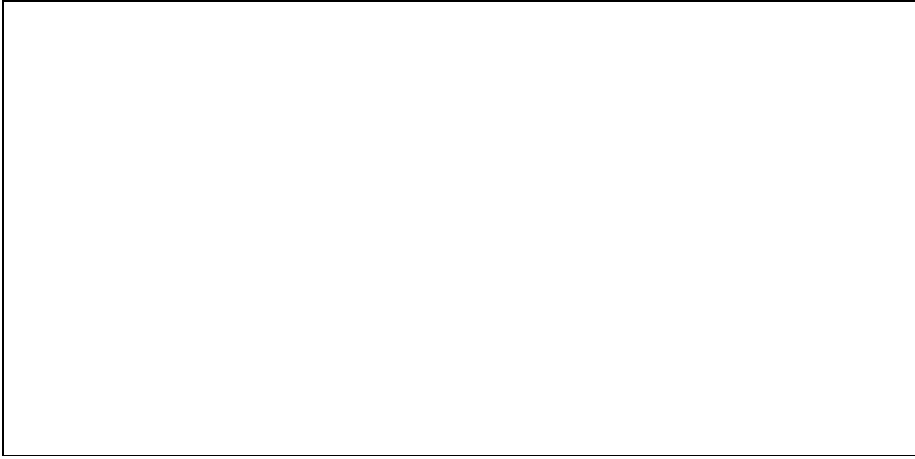
then we find

$$C_1 \widetilde{M}_{1,1,1,1} + C_2 \widetilde{M}_{2,2,2} = 1, \quad \text{on} \quad (\mathbb{C} \setminus \{0\})^s$$

*Example 4.5* The  $C^1$  quartic spline

$$L(x) = 3M_{2,2,2}(x + (2, 2)) - M_{1,1,1,1}(x + (1, 2)) - M_{1,1,1,1}(x + (0, 1))$$

on the four-direction mesh satisfies  $L(\alpha) = \delta_0(\alpha)$ ,  $\alpha \in \mathbb{Z}^2$ . It has the same support as  $M_{2,2,2}$ ,  $\{x : |x(1)| \leq 2, |x(2)| \leq 2, |x(1) - x(2)| \leq 2\}$ , and the approximation order for the general interpolation problem with  $\varphi = L$  is  $h^3$ , the same approximation order as from the spline space generated by  $M_{1,1,1,1}$ . (See, Fig. 2.)



**Fig.2.** A fundamental solution for cardinal interpolation from a linear combination of the box splines  $M_{2,2,2}$  and  $M_{1,1,1,1}$ . The supports of three terms in the combination are shown on the left.

This example appears to be very attractive for applications since it has a relatively small support (there are at most 12 translates of the support meeting any  $x$ ), it is smooth, and it provides reasonable approximation order.

This example is suitable to use for Hermite interpolation of a function and its first order partials on  $\mathbb{Z}^2$ . Here the generating functions for the interpolation space will be

$$\varphi_0 = L, \quad \varphi_1 = \sin(2\pi i_1 \cdot) L, \quad \text{and} \quad \varphi_2 = \sin(2\pi i_2 \cdot) L.$$

The nonzero entries of the symbol matrix are

$$A_{1,1}(z) = A_{2,2(z)}/(2\pi) = A_{3,3(z)}/(2\pi) = 1, \quad \text{and} \\ A_{j,1} = \widetilde{D}_{i_j} L = (z^{-i_j} - z^{i_j})/2, \quad j = 1, 2.$$

Hence, the fundamental solutions for Hermite interpolation with the spline of this example are

$$\begin{aligned} L_0 &= L(\cdot) - (4\pi)^{-1} \sin(2\pi \mathbf{i}_1 \cdot) (L(\cdot + \mathbf{i}_1) - L(\cdot - \mathbf{i}_1)) \\ &\quad - (4\pi)^{-1} \sin(2\pi \mathbf{i}_2 \cdot) (L(\cdot + \mathbf{i}_2) - L(\cdot - \mathbf{i}_2)) \\ L_j &= (4\pi)^{-1} \sin(2\pi \mathbf{i}_j \cdot) L(\cdot), \quad j = 1, 2. \end{aligned}$$

### 4.3 A general method

It is obvious that if you dilate the function enough so that it is nonzero only on one lattice point, then cardinal interpolation with the integer translates of the dilated function is correct. However, that case usually results in complete loss of approximation order for the interpolation operator. Here we take a combination of the original function and shifts of some dilate, an idea suggested by our last univariate construction. In comparison to Micchelli's combination Eq.(4.1), we do not require  $\varphi$  to be renable.

Let  $\varphi$  be a compactly supported function in  $C^\kappa$  which satisfies the Strang-Fix conditions of some order  $\eta$ . Suppose that  $N \in \mathbb{Z}$  and  $\beta \in \mathbb{R}^s$  are chosen such that the support of the function  $\psi := \varphi(N \cdot + \beta)$  contains exactly one lattice point  $\alpha_0$ . For this choice of  $\psi$ ,  $\tilde{\psi}(z) = \text{const} z^{\alpha_0}$ . Writing  $z = (z^{i_1}, \dots, z^{i_s})$ , for the given Strang-Fix order  $\eta$  we define the polynomials  $\tilde{m}_j = (1 - z^{i_j})^\eta$ . Since  $\hat{\varphi}(0) = 1 = \tilde{\varphi}((1, \dots, 1))$ , the Laurent polynomials

$$\tilde{\varphi}, \quad \tilde{m}_1 \tilde{\psi}, \quad \dots, \quad \tilde{m}_s \tilde{\psi},$$

have no common roots in  $(\mathbb{C} \setminus \{0\})^s$ , and there exist Laurent polynomials  $C_0, C_1, \dots, C_s$  such that

$$(4.6) \quad C_0(z) \tilde{\varphi}(z) + \sum_{j=1}^s C_j(z) \tilde{m}_j(z) \tilde{\psi}(z) = 1, \quad (\mathbb{C} \setminus \{0\})^s.$$

Let  $c_j, m_j$  be the coefficient sequences of the Laurent polynomials  $C_j, \tilde{m}_j$  respectively and define the function

$$(4.7) \quad L := \varphi * c_0 + \sum_{j=1}^s \psi * m_j * c_j.$$

**Theorem 4.8** *Let  $\varphi$  be a compactly supported function in  $C^\kappa$  which satisfies the Strang-Fix condition of order  $\eta$ . The function  $L$  in Eq.(4.7) is a compactly supported fundamental solution for cardinal interpolation from the space  $\mathcal{S}(L)$ . Further,  $L$  satisfies the Strang-Fix conditions of order  $\eta$  and the corresponding (general) cardinal interpolation operator obtains approximation order  $\eta$ .*

*Proof* That  $L$  is a compactly supported fundamental solution for cardinal interpolation follows from the compact support of  $\varphi$ , Eq.(4.6) and Eq.(4.7). Since

$$\hat{L}(y) = C_0(z) \hat{\varphi}(y) + \sum_{j=1}^s C_j(z) \tilde{m}_j(z) \psi(y), \quad \text{for } z = \exp(-iy),$$

the Strang-Fix conditions for  $L$  follow from

$$\begin{aligned}\widehat{L}(y) &= C_0(z)\widehat{\varphi}(y) + \sum_{j=1}^s C_j(z)\widehat{m}_j(z)\widehat{\psi}_j(y) \\ &= C_1(z)\widetilde{\varphi}(z) + \sum_{j=1}^s C_j(z)\widetilde{m}_j(z)\widetilde{\psi}(z) = 1, \\ &\quad \text{for } y = (0, \dots, 0), \quad z = (1, \dots, 1),\end{aligned}$$

and the fact that  $\widehat{\varphi}(y)$  and  $\widetilde{m}_j(\exp(-iy))\widehat{\psi}(y)$  vanish  $\eta$ -fold at  $y = 2\pi\alpha$ ,  $\alpha \in \mathbb{Z}^s \setminus \{0\}$ .

In the case of a centrally symmetric function,  $N$  and  $\beta$  can be chosen so that the origin is translated to the center of the function and then the function can be dilated sufficiently to exclude other integer points from its support. Box splines provide examples of centrally symmetric functions. It is beneficial (for sake of symmetry) to "center" the function on the origin so that its centrally symmetric point is on the lattice. We present the  $\mathbf{ZP}$ -element in some detail.

For brevity of notation, let  $\mathbf{ZP} = M_{1,1,1,1}(\cdot + (1, 3)/2)$  be the centered  $\mathbf{ZP}$ -element and  $\mathbf{ZP}_2 = M_{1,1,1,1}(2 \cdot + (1, 3)/2)$  denote its 2-dilation. The symbols are

$$\widetilde{\mathbf{ZP}}(z, w) = (4 + w + w^{-1} + z + z^{-1})/8; \quad \widetilde{\mathbf{ZP}}_2(z, w) = 1/2.$$

One solution to

$$c_1(z, w)\widetilde{\mathbf{ZP}}(z, w) + c_2(z, w)(1 - w)^3\widetilde{\mathbf{ZP}}_2(z, w) + c_3(z, w)(1 - z)^3\widetilde{\mathbf{ZP}}_2(z, w) = 1$$

is

$$\begin{aligned}c_1(z, w) &= 1/32 - \frac{3z}{16} - \frac{3w}{16} + \frac{z^2}{32} + \frac{13zw}{8} + \frac{w^2}{32} - \frac{3wz^2}{16} - \frac{3zw^2}{16} + \frac{w^2z^2}{32} \\ c_2(z, w) &= \frac{3w}{128z} - \frac{1}{256z} - \frac{w^2}{256z} + \frac{1}{256} - \frac{3w}{128} + \frac{w^2}{256} \\ c_3(z, w) &= \frac{3z}{128w} - \frac{1}{256w} - \frac{z^2}{256w} + \frac{1}{256} - \frac{3z}{128} + \frac{z^2}{256}.\end{aligned}$$

The resulting fundamental solution for this choice is symmetric through the origin and is uniquely dened within its support by the conditions that its values agree with the delta sequence at the lattice points inside the support:

$$L = L_1 + L_2$$

where  $L_1$  is the sum of shifts of  $\mathbf{ZP}$  with nonzero coefficients only in  $[-1 \dots 1]^2 \cap \mathbb{Z}^2$  as given by the matrix on the left in Eq.(4.9) and  $L_2$  is the sum of shifts of  $\mathbf{ZP}_2$  with nonzero coefficients in  $[-2 \dots 2]^2 \cap \mathbb{Z}^2$  as given by the matrix on the right in Eq.(4.9) (the value at the origin is in boldface):

$$(4.9) \quad \frac{1}{32} \begin{bmatrix} 1 & -6 & -6 \\ -6 & 52 & -6 \\ -6 & -6 & 1 \end{bmatrix}, \frac{1}{128} \begin{bmatrix} 0 & -1 & 6 & -1 & 0 \\ -1 & 8 & -30 & 8 & -1 \\ 6 & -30 & 72 & -30 & 6 \\ -1 & 8 & -30 & 8 & -1 \\ 0 & -1 & 6 & -1 & 0 \end{bmatrix}.$$

The resulting fundamental solution is depicted in Fig.3.

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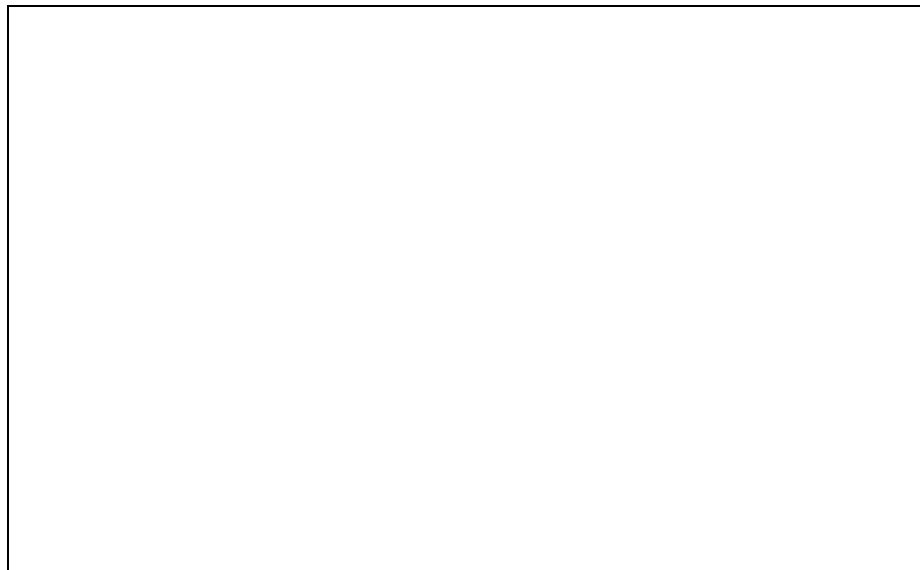


Fig. 3. A fundamental solution using the centered Zwart-Powell element and its 2-dilate and its level curves for  $-2 : .05 : 1$ .

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