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Delay-Independent Sliding Mode Control of Nonlinear Time-Delay Systems

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Abstract— **In this paper, adaptive sliding mode control is presented for a class of parametric-strict-feedback nonlinear systems with unknown time delays. Using appropriate Lyapunov-Krasovskii functionals, the uncertainties from unknown time delays are compensated for so that the proposed control scheme is delay-independent. Controller singularity problems are solved by employing practical control and regrouping unknown parameters. By using novel differentiable approximation, sliding mode control and practical control can be carried out in the backstepping design. It is proved that the proposed systematic backstepping design method is able to guarantee global uniformly ultimately boundedness of all the signals in the closed-loop system and the tracking error is proven to converge to a small neighborhood of the origin.**

I. INTRODUCTION

The sliding mode control (SMC) has a large historical background, belonging to the framework of variable structure systems. In a finite-dimension framework (without delay), it is known that if a complex system can be written in a so-called "normal form", then an appropriate sliding mode strategy can be designed so as to dominate the nonlinear terms and the disturbances, provided the disturbance satisfies appropriate "matching condition". The combination of delay phenomena with relay actuators or sensors makes the situation much more complex.

Robust control of systems with time delays has attracted much attention due to the great challenge in the mathematical complexity and the application demand in real-time control systems. The existence of time delays may make the stabilization problem become more difficult. Lyapunov-Krasovskii functionals [1] combined with the LMI technique [2] has been used to establish a framework for the stability and control of time-delay systems. For the tremendous work contributed to this area, please refer to the two latest monograph [3] [4] and the references therein. For nonlinear systems with delay in the state, few results are reported. In [5] and [6], the authors have studied a class of nonlinear time-delay systems in strict-feedback form and systematic and practical backstepping design has been presented. Under the mild assumption on the upper bound of the unknown time-delay, the proposed design based on the Lyapunov stability is delay-independent in the sense that the design is totally free of unknown delays. The controller singularity problem is solved by introducing the practical design. The problem of differentiation of the control functions at the discontinuous points can be solved by simply setting the finite value at these points. However, the control functions are not smooth at all. In this paper, we present a practical adaptive sliding mode control for a class of nonlinear time-delay systems in parametric-strict-feedback form. A novel smooth approximation is introduced to solve the differentiability problem for the control functions, while the stability of the closed-loop systems remains.

In this paper, decoupled backstepping design [7], [5], [6] is utilized in the practical design, where the coupling term $s_i s_{i+1}$ in each step is decoupled by elegantly using the Young's inequality rather than leaving to it to be cancelled in the next step as classic backstepping design usually does. The residual set of each state in s_i coordinate can be iteratively determined individually. The last but never the least, as the practical control (intermediate and final control) will be only activated in the unbounded region $\Omega_{s_i}^0$ and the compact region $\Omega_{s_i}^I$, while there is no control effort in the other bounded region, i.e., Ω_{s_i}. The stability of the proposed control methodology can only be guaranteed using decoupled backstepping design, which will be remarked later.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a class of single-input-single-output (SISO) nonlinear time-delay systems

$$\begin{aligned}\dot{x}_i(t) &= g_i x_{i+1}(t) + \theta_i^T f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t-\tau_i)) \\ \dot{x}_n(t) &= g_n u(t) + \theta_n^T f_n(x(t)) + h_n(x(t-\tau_n)) \quad (1)\end{aligned}$$

where $1 \le i \le n-1$, $\bar{x}_i = [x_1, x_2, \cdots, x_i]^T$, $x = [x_1, x_2, \cdots, x_n]^T \in R^n$, $u \in R$ are the state variables and system input respectively, $f_i(\cdot) \in R^{m_i}$ are known vector fields, $\theta_i \in R^{m_i}$ are unknown constant vectors, $h_i(\cdot)$ are unknown smooth functions, g_i are unknown constants, and τ_i are unknown time delays of the states, $i = 1, \cdots, n$. The control objective is to design a controller for system (1) such that the state $x_1(t)$ follows a desired reference signal $y_d(t)$, while all signals in the closed-loop system are bounded. Define the desired trajectory $\bar{x}_{d(i+1)} = [y_d, \dot{y}_d, \cdots, y_d^{(i)}]^T$, $i = 1, \cdots, n-1$, which is a vector of y_d up to its ith time derivative $y_d^{(i)}$. We have the following assumptions for the system functions, unknown time delays and reference signals.

Assumption 1: The signs of g_i are known, and there exist constants $g_{\max} \ge g_{\min} > 0$ such that $g_{\min} \le |g_i| \le g_{\max}$.

Assumption 2: The unknown functions $h_i(\cdot)$ satisfy the following properties

$$|h_i(\bar{x}_i(t))| \le p_i^* \phi_i(\bar{x}_i(t)) \quad (2)$$

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where $\phi_i(\cdot)$ are known positive functions, p_i^* are unknown constants.

Assumption 3: The size of the unknown time delays is bounded by a known constants, i.e., $\tau_i \leq \tau_{\max}$, $i = 1, \cdots, n$.

Assumption 4: The desired trajectory vectors $\bar{x}_{di} \in \Omega_{di} \subset R^i$, $i = 2, \cdots, n$ are continuous and available with Ω_{di} known compact set.

Lemma 1: Even function $q_i(x) : R \to R$

$$q_i(x) = \begin{cases} 1, & |x| \geq \lambda_{ai} + \lambda_{bi} \\ c_{qi} \int_{\lambda_{ai}}^{x} [(\frac{\lambda_{bi}}{2})^2 - (\sigma - \lambda_{ai} - \frac{\lambda_{bi}}{2})^2]^{n-i} d\sigma, & \lambda_{ai} < x < \lambda_{ai} + \lambda_{bi} \\ c_{qi} \int_{x}^{-\lambda_{ai}} [(\frac{\lambda_{bi}}{2})^2 - (\sigma + \lambda_{ai} + \frac{\lambda_{bi}}{2})^2]^{n-i} d\sigma, & -(\lambda_{ai} + \lambda_{bi}) < x < -\lambda_{ai} \\ 0, & |x| \leq \lambda_{ai} \end{cases} \quad (3)$$

where $c_{qi} = \frac{[2(n-i)+1]!}{\lambda_{bi}^{2(n-i)+1} [(n-i)!]^2}$ [8], $\lambda_{ai}, \lambda_{bi} > 0$ and integer $i \in R^+$, is $(n-i)$th differentiable, i.e., $q_i(x) \in C^{n-i}$ and bounded by 1.

Proof: The derivative of $q_i(x)$ w.r.t. x is

$$\frac{dq_i(x)}{dx} = \begin{cases} 0, & |x| \geq \lambda_{ai} + \lambda_{bi} \\ c_{qi}[(x - \lambda_{ai})(\lambda_{ai} + \lambda_{bi} - x)]^{n-i}, & \lambda_{ai} < x < \lambda_{ai} + \lambda_{bi} \\ -c_{qi}[-(x + \lambda_{ai})(x + \lambda_{ai} + \lambda_{bi})]^{n-i}, & -(\lambda_{ai} + \lambda_{bi}) < x < -\lambda_{ai} \\ 0, & |x| \leq \lambda_{ai} \end{cases} \quad (4)$$

From (4), we have that

$$\frac{d}{dx} q_i(x) \Big|_{|x| = \lambda_{ai}} = 0, \quad \frac{d}{dx} q_i(x) \Big|_{|x| = \lambda_{ai} + \lambda_{bi}} = 0$$

Similarly, we have

$$\frac{d^j}{dx^j} q_i(x) \Big|_{|x| = \lambda_{ai}} = 0, \qquad \frac{d^j}{dx^j} q_i(x) \Big|_{|x| = \lambda_{ai} + \lambda_{bi}} = 0$$

$$\frac{d^{n-i+1}}{dx^{n-i+1}} q_i(x) \Big|_{|x| = \lambda_{ai}} \neq 0, \frac{d^{n-i+1}}{dx^{n-i+1}} q_i(x) \Big|_{|x| = \lambda_{ai} + \lambda_{bi}} \neq 0$$

from which we know that $\frac{d^j}{dx^j} q_i(x)$, $j = 1, \cdots, n-i$, is continuous while $\frac{d^j}{dx^j} q_i(x)$, $j > n-i$ is not continuous. Thus, $q_i(x) \in C^{n-i}$. ■

Corollary 1: The function $q_i^s(x) : R \to R$

$$q_i^s(x) = q_i(x) \mathrm{sgn}(x) \quad (5)$$

with $\mathrm{sgn}(\cdot)$ being the sign function, is $(n-i)$th differentiable, i.e., $q_i^s(x) \in C^{n-i}$.

III. SMC DESIGN FOR FIRST-ORDER SYSTEMS

To illustrate the design methodology clearly, let us first consider the tracking problem of a first-order system

$$\dot{x}_1(t) = g_1 u(t) + \theta_1^T f_1(x_1(t)) + h_1(x_1(t - \tau_1)) \quad (6)$$

with $u(t)$ being the control input. Defining the sliding surface as $s_1(t) = x_1 - y_d$, we have

$$s_1(t)\dot{s}_1(t) \leq g_1 s_1(t)[u(t) + \frac{1}{g_1}\theta_1^T f_1(x_1(t)) - \frac{1}{g_1}\dot{y}_d(t)] + \frac{1}{2}s_1^2(t) p_1^{*^2} + \frac{1}{2}\phi_1^2(x_1(t - \tau_1))$$

Though the function $\phi_1(\cdot)$ is known, it could not be used for the construction of control laws as $x_1(t - \tau_1)$ is not obtainable due to the unknown time-delay τ_1. To tackle this problem, let us introduce the following Lyapunov-Krasovskii functional

$$V_{U_1}(t) = \frac{1}{2} \int_{t-\tau_1}^{t} \phi_1^2(x_1(\tau)) d\tau \quad (7)$$

The time derivative of $V_{U_1}(t)$ is

$$\dot{V}_{U_1}(t) = \frac{1}{2}\phi_1^2(x_1(t)) - \frac{1}{2}\phi_1^2(x_1(t - \tau_1))$$

in which, the non-positive term $-\frac{1}{2}\phi_1^2(x_1(t - \tau_1))$ can be used to compensate for the terms with unknown time delay. Accordingly, we have

$$s_1(t)\dot{s}_1(t) + \dot{V}_{U_1}(t) \leq g_1 s_1(t)[u(t) + \theta_{a,1}^T F_{a,1}(t)] \quad (8)$$

where $\theta_{a,1} \in R^{m_1+2}$ is unknown parameter vector and $F_{a,1} \in R^{m_1+2}$ is known function vector defined as

$$\theta_{a,1} = [\frac{1}{g_1}\theta_1^T, \frac{1}{g_1}, \frac{p_1^{*^2}}{g_1}]^T \quad (9)$$

$$F_{a,1}(t) = [f_1(x_1(t)), -\dot{y}_d(t) + \frac{1}{2s_1}\phi_1^2(x_1(t)), \frac{1}{2}s_1]^T \quad (10)$$

It can be seen from $F_{a,1}(t)$ that singularity problems may occur when $s_1(t) = 0$. Thus, the boundedness of the control should be taken into account. For convenience of notation, let us define the compact set $\Omega_{s_1} = \{s_1 \in R \mid |s_1| \leq \lambda_{a1}\}$.

Lemma 2: For the first-order system (6), if the practical adaptive control law is chosen as

$$u(t) = \begin{cases} -\hat{\theta}_{a,1}^T F_{a,1} - \mathrm{sgn}(s_1)\epsilon, & s_1 \notin \Omega_{s_1} \\ 0, & s_1 \in \Omega_{s_1} \end{cases} \quad (11)$$

where $\epsilon > 0$ is a design constant, and the parameters are updated by

$$\dot{\hat{\theta}}_{a,1} = s_1 \Gamma_{a,1} F_{a,1} \quad (12)$$

with $\Gamma_{a,1} = \Gamma_{a,1}^T > 0$, then for any initial conditions $x_1(t) = \varphi(t) = 0$, $\forall t \in [-\tau_1, 0]$ and $\hat{\theta}_{a,1}(0)$, all signals in the closed-loop systems are bounded, and the tracking error $s_1 = x_1 - y_d$ will converge to the bounded compact set $\Omega_{s_1} = \{s_1 \in R \mid |s_1| \leq \lambda_{a,1}\}$.

Proof: To show Ω_{s_1} to be a domain of attraction, let us first consider $s_1 \notin \Omega_{s_1}$, i.e., $|s_1| > \lambda_{a1}$. Consider the Lyapunov function candidate $V_1(t)$ as

$$V_1(t) = \frac{1}{2}s_1^2(t) + V_{U_1}(t) + \frac{1}{2}\tilde{\theta}_{a,1}^T(t)\Gamma_1^{-1}\tilde{\theta}_{a,1}(t)$$

with V_{U_1} given in (7) and $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$. The time derivative of $V_1(t)$ along (8), (11) and (12) is

$$\dot{V}_1(t) \leq -g_1|s_1(t)|\epsilon < 0 \quad (13)$$

which establishes that domain Ω_{s_1} is attractive in the sense that s_1 will be driven onto compact set Ω_{s_1} in a finite time, and then after stay within. In fact, from (13), we know that $\frac{1}{2}s_1^2(t) \leq V_1(t) \leq V_1(0)$ and $V_1(t) \leq V_1(0) -$

$\int_0^t g_1|s_1(\tau)|\epsilon d\tau$, i.e., s_1 and $\hat{\theta}_{a,1}$ are bounded. For $s_1 \in \Omega_{s_1}$, i.e., $|s_1| \le \lambda_{a1}$, since $s_1 = x_1 - x_d$ and x_d are bounded, we know that x_1 is also bounded. In addition, as $\dot{\hat{\theta}}_{a,1} = 0$ for $|s_1| \le \lambda_{a1}$, i.e., $\hat{\theta}_{a,1}$ is kept unchanged in a bounded value, it is bounded as well. ■

The key point of the proposed design lies in two aspects. Firstly, the Lyapunov-Krasovskii functional is utilized such that the design difficulties from unknown time delay has been removed. Secondly, the practical sliding mode control scheme has employed to avoid possible controller singularity. It is well known in [9][10] that the above discontinuous control scheme should be avoided as it will cause chattering phenomena and excite high-frequency unmodeled dynamics. Furthermore, we would like to extend the methodology described in this section from first-order systems to more general nth-order systems. To achieve this objective, the iterative backstepping design can be used, which requires the differentiation of the control u at each step. Therefore, appropriate smooth control functions shall be used, and at the same time the controller should guarantee the boundedness of all the signals in the closed-loop and s_1 will still stay in certain domain of attraction.

IV. SMC Design for Nth-Order Systems

The backstepping design of both the control laws and the adaptive laws are based on the following change of coordinates: $s_1 = x_1 - y_d$, $s_i = x_i - \alpha_{i-1}$, $i = 2, ..., n$, where s_i is also the sliding surface in the sliding mode control framework, and $\alpha_i(t)$ is an intermediate control function. It is noted that the extension is not straightforward and is very much involved as the SMC scheme proposed in Section III is discontinuous and does not satisfy the differentiable condition required by the backstepping design. Thus, the discontinuous control scheme shall be modified accordingly, while the closed-loop stability should be guaranteed as well.

For ease of notation, the following sets are defined $\Omega_{s_i} := \{s_i \in R \mid |s_i| \le \lambda_{ai}\}$, $\Omega^I_{s_i} := \{s_i \in R \mid \lambda_{ai} < |s_i| < \lambda_{ai} + \lambda_{bi}\}$, and $\Omega^O_{s_i} := \{s_i \in R \mid |s_i| \ge \lambda_{ai} + \lambda_{bi}\}$ with $\lambda_{ai}, \lambda_{ai} >$ being small design constants. Note that in each design step, the stability analysis is carried out in the three regions defined by these sets respectively.

Define the quadratic function $V_{s_i}(t)$, the Lyapunov-Krasovskii functional $V_{U_i}(t)$, and the Lyapunov function candidate $V_i(t)$ as

$$V_{s_i}(t) = \frac{1}{2g_i} s_i^2(t) \tag{14}$$

$$V_{U_i}(t) = \frac{1}{2g_i} \sum_{j=1}^{i} \int_{t-\tau_i}^{t} \phi_i^2(\bar{x}_i(\tau)) d\tau \tag{15}$$

$$V_i(t) = V_{s_i}(t) + V_{U_i}(t) + \frac{1}{2}\tilde{\theta}^T_{a,i}(t)\Gamma_i^{-1}\tilde{\theta}_{a,i}(t) \tag{16}$$

where $\tilde{\theta}_{a,i} = \hat{\theta}_{a,i} - \theta_{a,i}$ is the parameter estimation error of constant vector $\theta_{a,i} \in R^{\bar{m}_i}$ with $\bar{m}_i = \sum_{j=1}^{i} m_j + 2i$, which is defined by

$$\theta_{a,i} := [\frac{\theta_i^T}{g_i},\ \frac{p_i^{*^2}}{g_i},\ \frac{g_{i-1}}{g_i},\ \frac{g_{i-1}}{g_i}\theta^T_{i-1}]^T \tag{17}$$

Let us consider the following adaptive sliding mode control scheme

$$\alpha_i = q_i(s_i)\big[-k_i(t)s_i - \hat{\theta}^T_{a,i}F_{a,i} - \mathrm{sgn}(s_i)\epsilon_i\big] \tag{18}$$

$$k_i(t) = k_{i0} + \frac{1}{s_i^2}\sum_{j=1}^{i}\int_{t-\tau_{\max}}^{t} \phi_j^2(\bar{x}_j(\tau))d\tau \tag{19}$$

$$\dot{\hat{\theta}}_{a,i} = q_i(s_i)\Gamma_i(F_{a,i}s_i - \sigma_i\hat{\theta}_{a,i}) \tag{20}$$

where $q_i(\cdot)$ is defined in (3), constants $k_{i0}, \epsilon_i > 0$ are design parameters, constant matrix $\Gamma_i = \Gamma_i^{-1} > 0$, $\sigma_i > 0$ is a small constant to introduce the $\sigma-$modification for the closed-loop system, and $F_{a,i} \in R^{\bar{m}_i}$ is a known function vector defined as

$$\begin{aligned} F_{\theta i} := [&f_i^T(\bar{x}_i),\ \frac{1}{2}s_i,\ -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}x_i,\ -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}f^T_{i-1}, \\ &\frac{1}{2}s_i(\frac{\partial\alpha_{i-1}}{\partial x_{i-1}})^2,\ -\frac{\partial\alpha_{i-1}}{\partial x_{i-2}}x_{i-1},\ -\frac{\partial\alpha_{i-1}}{\partial x_{i-2}}f^T_{i-2}, \\ &\frac{1}{2}s_i(\frac{\partial\alpha_{i-1}}{\partial x_{i-2}})^2,\ \cdots,\ -\frac{\partial\alpha_{i-1}}{\partial x_1}x_2,\ -\frac{\partial\alpha_{i-1}}{\partial x_1}f^T_1, \\ &\frac{1}{2}s_i(\frac{\partial\alpha_{i-1}}{\partial x_1})^2,\ \frac{1}{2s_i}\sum_{j=1}^{i}\psi_j^2(\bar{x}_j) - \omega_{i-1}]^T \end{aligned} \tag{21}$$

with $\omega_{i-1} = \frac{\partial\alpha_{i-1}}{\partial\bar{x}_{di}}\dot{\bar{x}}_{di} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j}$. In addition, let us define the positive constants $c_i := \min\{\frac{3}{2}g_{\min}k_{i0}, 2g_{\min}, \frac{\sigma_i}{\lambda_{\max}(\Gamma_i^{-1})}\}$ and $\lambda_i := \frac{1}{2}\sigma_i\|\theta_{a,i}\|^2$.

Step 1: Let us firstly consider the s_1-subsystem as

$$\begin{aligned} \dot{s}_1(t) = &g_1[s_2(t) + \alpha_1(t)] + \theta_1^T f_1(x_1(t)) + h_1(x_1(t-\tau_1)) \\ &-\dot{y}_d(t) \end{aligned} \tag{22}$$

Following the same procedure as in Section III and considering $V_{s_1}(t)$ and V_{U_1} given in (14) and (15) respectively, we obtain

$$\dot{V}_{s_1}(t) + \dot{V}_{U_1}(t) \le s_1 s_2 + s_1(\alpha_1 + \theta^T_{a,1}F_{a,1}) \tag{23}$$

where $\theta_{a,1}$ and $F_{a,1}(t)$ are defined in (9) and (10) respectively.

As stated in Section III, to avoid the control singularity problem, the control objective is now relaxed to show the convergence of $s_1(t)$ to certain domain of attraction rather than the origin. At the same time, the control functions shall be smooth or at least differentiable to certain degree required by the backstepping design.

Considering the Lyapunov function candidate $V_1(t)$ given in (16) and the adaptive SMC scheme (18)-(20), the stability analysis is carried out in the following three regions defined by the compact sets Ω_{s_1}, $\Omega^I_{s_1}$ and $\Omega^O_{s_1}$ respectively.

(i) *Region 1:* $s_1 \in \Omega_{s_1}^O$. As $q_1(s_1) = 1$ when $s_1 \in \Omega_{s_1}^O$, the time derivative of $V_1(t)$ along (23), (18)-(20) is

$$\dot{V}_1(t) \le s_1 s_2 - k_{10} s_1^2 - \int_{t-\tau_{\max}}^{t} \phi_1^2(x_1(\tau))d\tau \\ -|s_1|\epsilon_1 - \sigma_1 \tilde{\theta}_{a,1}^T \hat{\theta}_{a,1}$$

Using the inequalities $-\frac{1}{4}k_{10}s_1^2 + s_1 s_2 \le \frac{1}{k_{10}} s_2^2$ and $-\sigma_1 \tilde{\theta}_{a,1}^T \hat{\theta}_{a,1} \le -\frac{1}{2}\sigma_1 \|\tilde{\theta}_{a,1}\|^2 + \frac{1}{2}\sigma_1\|\theta_{a,1}\|^2$, we have

$$\dot{V}_1(t) \le -\frac{3}{4}k_{10}s_1^2 - \int_{t-\tau_{\max}}^{t} \phi_1^2(x_1(\tau))d\tau + \frac{1}{k_{10}} s_2^2 \\ -\frac{1}{2}\sigma_1\|\tilde{\theta}_{a,1}\|^2 + \frac{1}{2}\sigma_1\|\theta_{a,1}\|^2 \quad (24)$$

Since $\tau_1 \le \tau_{\max}$ according to Assumption 3, it holds that $\int_{t-\tau_1}^t \phi_1^2(x_1(\tau))d\tau \le \int_{t-\tau_{\max}}^t \phi_1^2(x_1(\tau))d\tau$. Accordingly, (24) becomes

$$\dot{V}_1(t) \le -c_1 V_1(t) + \lambda_1 + \frac{1}{k_{10}} s_2^2 \quad (25)$$

where constants $c_1 > 0$ and $\lambda_1 > 0$ are defined before. In this case, if s_2 can be regulated as bounded, i.e., $|s_2| \le s_{2\max}$ with $s_{2\max}$ being finite, we have

$$\dot{V}_1(t) \le -c_1 V_1(t) + \bar{\lambda}_1$$

with $\bar{\lambda}_1 = \lambda_1 + \frac{1}{k_{10}} s_{2\max}^2$, then s_1, x_1, and $\hat{\theta}_{a,1}$ are guaranteed to be bounded. The regulation of s_2 will be conducted in the next steps.

(ii) *Region 2:* $s_1 \in \Omega_{s_1}^I$. As $\lambda_{a1} < |s_1| < \lambda_{a1} + \lambda_{b1}$ in this region, s_1 is bounded. Thus, $x_1 = s_1 + y_d$ is also bounded. Considering the quadratic functions $V_{s_1}(t)$ and $V_{U_1}(t)$, we know that $V_{s_1}(t)$ and $V_{U_1}(t)$ are bounded. Let us define positive function $V_{\theta_1}(t) := \frac{1}{2}\tilde{\theta}_{a,1}^T(t)\Gamma_1^{-1}\tilde{\theta}_{a,1}(t)$. Its time derivation along (20) is

$$\dot{V}_{\theta_1}(t) = q_1(s_1)\tilde{\theta}_{a,1}^T(F_{a,1}s_1 - \sigma_1\hat{\theta}_{a,1}) \quad (26)$$

Applying the inequalities $q_1(s_1)\tilde{\theta}_{a,1}^T F_{a,1} s_1 \le \frac{1}{2k_{\theta 1}} q_1(s_1)\|\tilde{\theta}_{a,1}\|^2 + \frac{k_{\theta 1}}{2} q_1(s_1) F_{a1}^T F_{a1} s_1^2$, $k_{\theta 1} > 0$, eq. (26) becomes

$$\dot{V}_{\theta 1}(t) \le -\frac{1}{2}q_1(s_1)(\sigma_1 - \frac{1}{k_{\theta 1}})\|\tilde{\theta}_{a,1}\|^2 \\ +\frac{1}{2}q_1(s_1)(\sigma_1\|\theta_{a,1}\|^2 + k_{\theta 1}F_{a,1}^T F_{a,1} s_1^2) \quad (27)$$

For $s_1 \in \Omega_{s_1}^I$, we know that $q_1(s_1) \in (0,1)$, and $F_{a,1}$ is smooth and bounded. Choosing $k_{\theta 1}$ such that $\sigma_1^* := \sigma_1 - \frac{1}{k_{\theta 1}} > 0$, and letting $\lambda_{\theta 1} := \sup_{s_1 \in \Omega_{s_1}^I}\{\sigma_1\|\theta_{a,1}\|^2 + k_{\theta 1}F_{a,1}^T F_{a,1}s_1^2\}$, we have

$$\dot{V}_{\theta 1}(t) \le -\frac{1}{2}q_1(s_1)\sigma_1^*\|\tilde{\theta}_{a,1}\|^2 + \frac{1}{2}q_1(s_1)\lambda_{\theta 1} \\ \le -q_1(s_1)\frac{\sigma_1^*}{\lambda_{\max}(\Gamma_1^{-1})}V_{\theta 1}(t) + \frac{1}{2}q_1(s_1)\lambda_{\theta 1} \quad (28)$$

Letting $c_{\theta 1}^q := q_1(s_1)\frac{\sigma_1^*}{\lambda_{\max}(\Gamma_1^{-1})}$, $\lambda_{\theta 1}^q := \frac{1}{2}q_1(s_1)\lambda_{\theta 1}$, and

$$\rho_{\theta 1}^q := \lambda_{\theta 1}^q / c_{\theta 1}^q = \frac{1}{2}\lambda_{\theta 1}\lambda_{\max}(\Gamma_1^{-1})/\sigma_1^* \quad (29)$$

it follows from (28) that

$$0 \le V_{\theta 1}(t) \le [V_{\theta 1}(0) - \rho_{\theta 1}^q]e^{-c_{\theta 1}^q t} + \rho_{\theta 1}^q \le V_{\theta 1}(0) + \rho_{\theta 1}^q$$

from which, we can conclude that $V_{\theta 1}(t)$ is bounded, and hence $\tilde{\theta}_{a,1}$ is bounded. Consider the Lyapunov function candidate $V_1(t)$ defined in (16). As it has been already shown that $V_{s_1}(t)$, $V_{U_1}(t)$, and $\tilde{\theta}_{a,1}$ are bounded, we can conclude that $V_1(t)$ is bounded for $s_1 \in \Omega_{s_1}^I$.

Remark 1: In this region, it is noted that though the function $q_1(s_1)$ is not of fixed value, the ultimate boundedness of the closed-loop signals is independent of $q_1(s_1)$ as can be seen from the definition of $V_{\theta 1}(0)$ and $\rho_{\theta 1}^q$.

(iii) *Region 3:* $s_1 \in \Omega_{s_1}$. In this region, s_1 is already bounded as $|s_1| \le \lambda_{a1}$. For $s_1 \in \Omega_{s_1}$, we know that $q_1(s_1) = 0$, and $\dot{\hat{\theta}}_{a,1} = 0$. Hence, $x_1 = s_1 + y_d$ is bounded, and $\hat{\theta}_{a,1}$ is kept unchanged in bounded values. As $V_{s_1}(t)$ and $V_{U_1}(t)$ are smooth functions, we know that for bounded x_1 and s_1, $V_{s_1}(t)$ and $V_{U_1}(t)$ are bounded, and $V_1(t)$ is bounded.

From the stability analysis in the three regions, we can conclude that (i) the boundedness of the closed-loop signals in Region 2 ($s_1 \in \Omega_{s_1}^I$) and Region 3 ($s_1 \in \Omega_{s_1}$) is guaranteed and independent of the signal s_2; (ii) the boundedness of the closed-loop signals in Region 1 ($s_1 \in \Omega_{s_1}^O$) is dependent on the boundedness of the signal s_2, who will be regulated in the next steps.

Remark 2: According to Lemma 1 and Corollary 1, it is noted that both the intermediate control function (18) and the updating laws (20) are differentiable to certain degree, which makes it possible to carry out the backstepping design in the next steps.

Step i ($2 \le i \le n-1$): Similar procedures are taken for each steps when $i = 2, \cdots, n-1$ as in Step 1. The time derivative of $s_i(t)$ is given by

$$\dot{s}_i(t) = g_i[s_{i+1}(t) + \alpha_i(t)] + \theta_i^T f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t-\tau_i)) \\ -\sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}\Big[g_j x_{j+1} + \theta_j^T f_j(\bar{x}_j) + h_j(\bar{x}_j(t-\tau_j))\Big] \\ -\omega_{i-1}(t)$$

Consider the sliding surface as $s_i(t)$, the quadratic function $V_{s_i}(t)$ given in (14), and the Lyapunov-Krasovskii functional $V_{U_i}(t)$ given in (15). Applying Assumption 2 and using Young's Inequality, we have

$$\dot{V}_{s_i} + \dot{V}_{U_i} \le s_i s_{i+1} + s_i(\alpha_i + \theta_{a,i}^T F_{a,i}) \quad (30)$$

where $\theta_{a,i}$ and $F_{a,i}$ are defined in (17) and (21) respectively.

Similarly, the adaptive sliding mode control law (18)-(20) is proposed. Considering the Lyapunov function candidate V_i given in (16), similar as in Steps 1 and 2, the stability analysis is carried out in the three regions defined by the compact sets Ω_{s_i}, $\Omega_{s_i}^I$ and $\Omega_{s_i}^O$ respectively and we can conclude that: (i) For $s_i \in \Omega_{s_i}^O$, we have $\dot{V}_i(t) \le -c_i V_i(t) + \lambda_i + \frac{1}{k_{i0}}s_{i+1}^2$ with c_i and λ_i defined before, from which it can be seen that the stability of s_i-subsystem in this case is dependent on s_{i+1}, which will be dealt with in the next steps. (ii) For $s_i \in \Omega^I_{s_i}$, s_i is bounded, from which, it can be derived backwards that $s_{i-1}, ..., s_i$ are all bounded so that the boundedness of $x_i, x_{i-1}, .., x_1$ can be guaranteed as well. The boundedness of $\hat{\theta}_{a,i}$ can be obtained from the similar analysis carried out in Region 1 of *Step* 1. (iii) For $s_i \in \Omega_{s_i}$, the boundedness of s_i, x_i and $\hat{\theta}_{a,i}$ directly follows.

Step n: This is the final step, since the actual control u appears in the derivative of $s_n(t)$ as given in

$$\begin{aligned}\dot{s}_n(t) &= g_n u(t) + \theta_n^T f_n(x(t) + h_n(x(t-\tau_n)) \\ &-\sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}\Big[g_j x_{j+1} + \theta_j^T f_j(\bar{x}_j) + h_j(\bar{x}_j(t-\tau_j))\Big] \\ &-\omega_{n-1}(t) \end{aligned} \tag{31}$$

Consider the quadratic function $V_{s_n}(t)$ (14) and the Lyapunov-Krasovskii functional $V_{U_n}(t)$ (15). Applying Assumption 2 and using Young's Inequality, it has

$$\dot{V}_{s_n} + \dot{V}_{U_n} \leq s_n(u + \theta_{a,n}^T F_{a,n}) \tag{32}$$

where $\theta_{a,n}$ and $F_{a,n}$ are defined in (17) and (21) respectively.

Similarly, the adaptive sliding mode control law (18)-(20) is proposed. Considering the following Lyapunov function candidate $V_n(t)$ (16), similar as in the previous steps, the stability analysis is carried out in the three regions defined by the compact sets Ω_{s_n}, $\Omega^I_{s_n}$ and $\Omega^I_{s_n}$ respectively as follows: (i) For $s_n \in \Omega^O_{s_n}$, $q_n(s_n) = 1$, the final control $u(t)$ is invoked and the time derivative of $V_n(t)$ along (32) and (18)-(20) is $\dot{V}_n(t) \leq -c_n V_n(t) + \lambda_n$ with positive constants c_n and λ_n defined before, from which we can conclude that $V_n(t)$ is bounded, hence s_n, $\hat{\theta}_{a,n}$ are bounded. (ii) For $s_n \in \Omega^I_{s_n}$, s_n is already bounded. It can be derived backwards that all the previous s_ith-subsystem, $i = 1, ..., n-1$, are stable, i.e., $s_i, \hat{\theta}_{a,i}, i = 1, ..., n-1$, are all bounded. As $x_i = s_i + \alpha_{i-1}$, $i = 2, ..., n$, $x_1 = s_1 + y_d$ and α_i, $i = 1, ..., n-1$ are smooth functions, we know that α_i are bounded, and hence x_i, $i = 1, ..., n$ are bounded. The boundedness of $\hat{\theta}_{a,n}$ can be obtained from the similar analysis carried out in Region 1 of *Step* 1. (iii) For $s_n \in \Omega_{s_n}$, s_n is bounded. Hence, s_i, x_i and $\hat{\theta}_{a,i}$, $i = 1, ..., n-1$ are bounded. As $q_n(s_n) = 0$, $\dot{\hat{\theta}}_{a,n} = 0$, $\hat{\theta}_{a,n}$ is kept fixed in a bounded value.

Theorem 1: Consider the closed-loop system consisting of the plant (1) under Assumptions 1-3. If we apply the controller (18)-(20), we can guarantee the following properties under bounded initial conditions: (i) s_i, $\hat{\theta}_{a,i}$ and x_i, $i = 1, ..., n$, are bounded; (ii) the signal $s(t) = [s_1, ..., s_n]^T \in R^n$ converges to the compact set defined by

$$\Omega_s := \left\{ s \;\middle|\; \|s\| \leq \mu \right\}$$

where $\mu = \max\{\sqrt{2g_{\max}\sum_{i=1}^n \rho_i}, \sqrt{\sum_{j=1}^n (\lambda_{ai} + \lambda_{bi})^2}\}$ with $\rho_n = \lambda_n/c_n$, $\rho_i = (\lambda_i + \frac{2g_{i+1}\rho_{i+1}}{k_{i0}})/c_i$, $i = 1, ..., n-1$, and the compact set Ω_s can be made as small as desired by an appropriate choice of the design parameters.

Proof: Consider the following Lyapunov function candidate

$$V(t) = \sum_{i=1}^n [V_{s_i}(t) + V_{U_i}(t) + \frac{1}{2}\tilde{\theta}_{a,i}^T(t)\Gamma_i^{-1}\tilde{\theta}_{a,i}(t)] \tag{33}$$

where $V_{s_i}(t)$, $V_{U_i}(t)$, $i = 1, \cdots, n$, and $\tilde{\theta}_{a,i}$ are defined before. The following three cases are considered.

Case 1): All $s_i \in \Omega^O_{s_i}$, $i = 1, ..., n$. In this case, $q_i(s_i) = 1$, as all the control effort including $\alpha_i(t)$, $i = 1, ..., n-1$ and $u(t)$ are invoked, from the previous analysis, we have

$$\dot{V}_i(t) \leq -c_i V_i(t) + \lambda_i + \frac{1}{k_{i0}} s_{i+1}^2, i = 1, ..., n-1 \tag{34}$$

$$\dot{V}_n(t) \leq -c_n V_n(t) + \lambda_n \tag{35}$$

where c_i, λ_i, $i = 1, ..., n$ have been defined before. Let $\rho_n := \lambda_n / c_n$, it follows from (35) that

$$0 \leq V_n(t) \leq [V_n(0) - \rho_n]e^{-c_n t} + \rho_n \leq V_n(0) + \rho_n \tag{36}$$

from which we know that $V_n(t)$ is bounded, so are s_n and $\hat{\theta}_{a,n}$ by noting the definition of $V_n(t)$. In addition, it directly follows from (36) that

$$s_n^2(t) \leq 2g_n[V_n(0) - \rho_n]e^{-c_n t} + 2g_n\rho_n \tag{37}$$

$$\lim_{t\to\infty} |s_n(t)| \leq \sqrt{2g_n\rho_n} \tag{38}$$

As s_n is bounded, using (34) backwards from $n-1$ to 1, we can conclude that s_i and $\hat{\theta}_{a,i}$, $i = 1, ..., n-1$ are all bounded.

Substituting (37) into (34) for $i = n-1$ yields

$$\begin{aligned}\dot{V}_{n-1}(t) &\leq -c_{n-1}V_{n-1}(t) + \lambda_{n-1} \\ &+ \frac{1}{k_{n-1,0}}\Big\{2g_n[V_n(0) - \rho_n]e^{-c_n t} + 2g_n\rho_n\Big\}\end{aligned}$$

If $c_{n-1} \neq c_n$, we have

$$\begin{aligned} V_{n-1}(t) &\leq V_{n-1}(0)e^{-c_{n-1}t} + \frac{\lambda_{n-1} + \frac{2g_n\rho_n}{k_{n-1,0}}}{c_{n-1}}(1 - e^{-c_{n-1}t}) \\ &+ \frac{2g_n}{k_{n-1,0}(c_{n-1} - c_n)}[V_n(0) - \rho_n](e^{-c_n t} - e^{-c_{n-1}t}) \end{aligned} \tag{39}$$

Otherwise, if $c_{n-1} = c_n$, we have

$$\begin{aligned} V_{n-1}(t) &\leq V_{n-1}(0)e^{-c_{n-1}t} \\ &+ \frac{\lambda_{n-1} + \frac{2g_n\rho_n}{k_{n-1,0}}}{c_{n-1}}(1 - e^{-c_{n-1}t}) \end{aligned} \tag{40}$$

Both (39) and (40) can lead to $\lim_{t\to\infty}|s_{n-1}(t)| \leq \sqrt{2g_{n-1}\rho_{n-1}}$ with $\rho_{n-1} \triangleq (\lambda_{n-1} + \frac{2g_n\rho_n}{k_{n-1,0}})/c_{n-1}$.

Similarly, we have $\lim_{t\to\infty}|s_i(t)| \leq \sqrt{2g_i\rho_i}$ with $\rho_i \triangleq (\lambda_i + \frac{2g_{i+1}\rho_{i+1}}{k_{i0}})/c_i$, $i = 1, ..., n-1$. Thus, $\lim_{t\to\infty}\|s\| = \sqrt{2g_{\max}\sum_{i=1}^n \rho_i}$. Since the above analysis is carried out for all $s_i \in \Omega^O_{s_i}$, i.e., $|s_i| \geq \lambda_{ai} + \lambda_{bi}$, $i = 1, ..., n$, we have that

$$\lim_{t\to\infty}\|s\| = \max\ \{\sqrt{2g_{\max}\sum_{i=1}^n \rho_i}, \sqrt{\sum_{i=1}^n (\lambda_{ai} + \lambda_{bi})^2}\}$$

Case 2): All $s_i \in \Omega^I_{s_i}$, $i = 1, ..., n$. In this case, s_i's are already bounded in $[\pm\lambda_{a_i}, \pm(\lambda_{a_i} + \lambda_{b_i})]$, i.e., $\|s\| \leq \sqrt{\sum_{j=1}^{n}(\lambda_{ai} + \lambda_{bi})^2}$. Since s_1 and y_d are bounded, and $x_1 = s_1 + y_d$, we know that x_1 is also bounded. From the analysis for Region 2 in *Step* 1, we have that

$$\|\tilde{\theta}_{a,1}\|^2 \leq \frac{2[V_{\theta 1}(0) + \rho^q_{\theta 1}]}{\lambda_{\min}(\Gamma_1^{-1})}$$

with $V_{\theta 1}(0) = \frac{1}{2}\tilde{\theta}^T_{a,1}(0)\Gamma_1^{-1}\tilde{\theta}_{a,1}(0)$ and $\rho^q_{\theta 1}$ being defined in (29), from which we know that $\hat{\theta}_{a,1}$ is bounded. As $\alpha_1(t)$ is a smooth function of x_1, y_d, $\dot{y}_d$, and $\hat{\theta}_{a,1}$, α_1 is guaranteed as bounded. From $x_2 = s_2 + \alpha_1$, x_2 is obviously bounded. Following the same analysis for Region 2 in *Step* 1, we can show that $\hat{\theta}_{a,2}$ is bounded. Similarly, the boundedness of all the other closed-loop signals $x_3, ..., x_n$ and $\hat{\theta}_{a,3}, ..., \hat{\theta}_{a,n}$ can be shown iteratively.

Case 3): All $s_i \in \Omega_{s_i}$, $i = 1, ..., n$. In this case, as $|s_i| < \lambda_{ai}$, all s_i's are bounded in $[-\lambda_{ai}, \lambda_{ai}]$, i.e., $\|s\| \leq \sqrt{\sum_{j=1}^{n}\lambda_{ai}^2}$. In addition, as $q_i(s_i) = 0$, $\dot{\hat{\theta}}_{a,i} = 0$, $\alpha_i(t) = 0$ and $u(t) = 0$, it is known that $\hat{\theta}_{a,i}$ are kept unchanged in bounded values, and $x_i = s_i + \alpha_{i-1}$, $i = 2, ..., n$, and $x_1 = s_1 + y_d$ are all bounded.

Case 4): Some s_i's are satisfying $s_i \in \Omega^O_{s_i}$, while some s_j's are satisfying $s_j \in \Omega^I_{s_j}$ or $s_j \in \Omega_{s_j}$. For those $s_i \in \Omega^O_{s_i}$, $q_i(s_i) = 1$, the corresponding control effort $\alpha_i(t)$ or $u(t)$ and the parameter adaptation law for $\hat{\theta}_{a,i}$ are invoked and from the previous analysis, we have that

$$\dot{V}_i(t) \leq -c_i V_i(t) + \lambda_i + \frac{1}{k_{i0}}s^2_{i+1} \tag{41}$$

Let us define $V_I(t) = \sum_i V_i(t)$, and positive constants where $C_1^I = \min_i\{c_i\}$, we have that

$$\dot{V}_I(t) \leq -C_1^I V_I(t) + \sum_i \lambda_i + \sum_i \frac{1}{k_{i0}}s^2_{i+1}$$

If $s_{i+1} \in \Omega_{s_i}$, the problem becomes a subset of Case 1). If $s_{i+1} \in \Omega^I_{s_j}$ or $s_{i+1} \in \Omega_{s_j}$, then these s_i is bounded as s_{i+1} is already bounded. For those $s_j \in \Omega^I_{s_j}$ or $s_j \in \Omega_{s_j}$, as s_j is already bounded, it guarantees that the closed-loop signals in the previous steps, i.e., s_k, x_k, $\hat{\theta}_{a,k}$, $k = 1, ..., j-1$, are bounded, and the stability of the corresponding s_jth-subsystems is independent of the signals in future steps. As α_{j-1} is a smooth function of x_k, $\hat{\theta}_{a,k}$, $k = 1, ..., j-1$, α_{j-1} is bounded, hence $x_j = s_j + \alpha_{j-1}$ is bounded. The boundedness of $\hat{\theta}_{a,j}$ can be readily obtained following the similar analysis in Region 2 of *Step 1*. Or more optimal, for $s_j \in \Omega_{s_j}$, $\dot{\hat{\theta}}_{a,j} = 0$, $\hat{\theta}_{a,j}$ is kept unchanged in a bounded value.

Therefore, we can conclude from Cases 1), 2), 3), and 4) that all the closed-loop signals are bounded and there exists a compact set Ω_s such that s will eventually converge to. This completes the proof. ∎

Remark 3: Theorem 1 shows that the system tracking error converges to a domain of attraction defined by compact set Ω_s rather than the origin. This is due to the introduction of the practical control and the σ-modification for the parameter adaptation. Even though the size of the compact set is unknown due to the unknown parameters $g_{\min}$, $g_{\max}$, and $\theta_{a,i}$ $i = 1, ..., n$, it is possible to make it as small as possible by appropriately choosing the design parameters. However, parameters such as λ_{ai} or λ_{bi} cannot be made zero to void possibly control singularity and computational singularity. Therefore, in practical applications, the design parameters should be adjusted carefully for achieving suitable transient performance and control action.

Remark 4: The continuous function $q_i(\cdot)$ introduced in Lemma 1 is used to generate a sufficient smooth approximation of the practical control functions so that the backstepping design can be carried out. However, with the order of the system increasing, the good control performance may be obtained at the price of control effort in large magnitude.

V. CONCLUSION

An adaptive sliding mode control has been addressed for a class of parametric-strict-feedback nonlinear systems with unknown time delays. The uncertainty from unknown time delays has been compensated through the use of appropriate Lyapunov-Krasovskii functionals. The controller has made to be delay-independent and free from singularity problem by employing practical sliding mode control. Backstepping design has been carried out by using differentiable approximation. The proposed systematic backstepping design method has been proved to be able to guarantee global uniformly ultimately boundedness of closed-loop signals. In addition, the output of the system has been proven to converge to an arbitrarily small neighborhood of the origin.

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