

# Robust Adaptive Sliding Mode Control for Uncertain Time-Delay Systems

Yuanqing Xia, S. S. Ge, G. P. Liu, P. Shi and D. Rees

**Abstract**—This paper is devoted to robust adaptive sliding mode control for time-delay systems with mismatched parametric uncertainties. Sufficient conditions for the existence of linear sliding surfaces is given in terms of matrix inequalities, by which the corresponding adaptive reaching motion controller is also designed.

## I. INTRODUCTION

The control problem of time-delay systems has received considerable attention over the past years, and different design approaches have been proposed, see for example [1], [2]. However, the existing results are sensitive to the uncertainty, which directly affects the control systems performance.

On the other hand, the sliding mode control has attractive features to keep the systems insensitive to the uncertainties on the sliding surface [3]. The existence condition of a linear sliding surface for systems with mismatched uncertainties was given in [4], [5], but the developed methods cannot be applied to systems with time-delay. In [6], a new robust stability criterion for uncertain time-delay systems is given and sliding model control (SMC) is proved to be applicable. By using matrix norm, the result is more or less conservative and complicated, which is inconvenient in designing the sliding surface for uncertain time-delay systems.

In this paper, we consider how to design a sliding surface and reaching motion controller for a class of time-delay systems with mismatched uncertainties and exogenous disturbance, where the parameters of unmatched uncertainties are known to lie in a priori specified intervals and the matched uncertainties and exogenous disturbance are assumed to be bounded with unknown bound. The aim of this paper is to

Yuanqing Xia, G. P. Liu, D. Rees and P. Shi are with Faculty of Advanced Technology, University of Glamorgan, Pontypridd, CF37 1DL, UK xia\_yuanqing@163.net; gpliu@glam.ac.uk; drees@glam.ac.uk; pshi@glam.ac.uk. Yuanqing Xia is also with the Department of Automatic Control, Beijing Institute of Technology, Beijing 100081, China.

S. S. Ge is with Faculty of Department of Electrical & Computer Engineering, National University of Singapore, 117576 Singapore Tel. (+65) 6874-6821, Fax (+65) 6779-1103 eleges@nus.edu.sg

combine adaptive control and variable structure control to realize their individual advantages. The use of variable structure control improves the transient response of the overall system significantly. As a result, the adaptive law proposed is able to reduce the chattering due to the implementation of variable structure controller, and possesses the desired characteristics of robustness and good performance. Sufficient conditions for the existence of linear sliding surfaces are derived. The solution to the condition can be used to characterize linear sliding surfaces, and by using smooth projection method for adaptive control law the reaching motion controller is designed.

## II. PROBLEM FORMULATION

Consider the uncertain time-delay system of the form

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t - \tau) \\ &\quad + B(u(t) + w(x, x(t - \tau), t)) \\ x(t) &= \varphi(t), t \in [-\tau, 0] \end{aligned} \quad (1)$$

where  $x(t) \in R^n$  is the state,  $w \in R^l$  are the matched uncertainties and disturbance.  $\Delta A$  and  $\Delta A_d$  are unmatched uncertainties.  $u(t) \in R^m$  is the control input,  $A, A_d$  and  $B$  are real constant matrices with appropriate dimensions and  $\text{rank}(B) = m$ .

*Assumption 1:* The uncertainties  $\Delta A$  and  $\Delta A_d$  in (1) are assumed to have the following form

$$\begin{aligned} \Delta A &= \sum_{i=1}^p \theta_i A_i \\ \theta &= [\theta_1 \ \theta_2 \ \dots \ \theta_p]^T \in \Omega = \{\theta \mid |\theta_i| \leq 1\} \\ \Delta A_d &= \sum_{i=1}^q \beta_i A_{di} \\ \beta &= [\beta_1 \ \beta_2 \ \dots \ \beta_q]^T \in \Omega_d = \{\beta \mid |\beta_i| \leq 1\} \end{aligned} \quad (2)$$

where the matrices  $A_i$  and  $A_{di}, i = 1, \dots, p$  are known with  $\text{rank}(A_i) = a_i$  and  $\text{rank}(A_{di}) = a_{di}$ ,  $\theta_i$  and  $\beta_i$  are unknown constants. Suppose that  $A_i$  and  $A_{di}$  have full rank factorization of  $A_i = G_i H_i$  and  $A_{di} = G_{di} H_{di}$ , respectively, where  $G_i \in R^{n \times a_i}$ ,  $H_i \in R^{a_i \times n}$ ,  $G_{di} \in R^{n \times a_{di}}$  and  $H_{di} \in R^{a_{di} \times n}$ . Then

$$\begin{aligned}\Delta A &= \sum_{i=1}^p \theta_i A_i = \begin{bmatrix} G_1 & G_2 & \cdots & G_p \end{bmatrix} D \times \\ &\quad \begin{bmatrix} H_1 & H_2 & \cdots & H_p \end{bmatrix}^T = GDH \\ \Delta A_d &= \sum_{i=1}^q \beta_i A_{di} = \begin{bmatrix} G_{d1} & G_{d2} & \cdots & G_{dq} \end{bmatrix} D_d \times \\ &\quad \begin{bmatrix} H_{d1} & H_{d2} & \cdots & H_{dq} \end{bmatrix}^T = G_d D_d H_d\end{aligned}\quad (3)$$

where

$$\begin{aligned}D &= \text{blockdiag} \begin{bmatrix} \theta_1 I_{a_1 \times a_1} & \theta_2 I_{a_2 \times a_2} & \cdots & \theta_p I_{a_p \times a_p} \end{bmatrix} \\ D_d &= \text{blockdiag} \begin{bmatrix} \beta_1 I_{a_{d1} \times a_{d1}} & \beta_2 I_{a_{d2} \times a_{d2}} \\ \cdots & \beta_q I_{a_{dq} \times a_{dq}} \end{bmatrix}\end{aligned}\quad (4)$$

*Remark 1:* The parameters  $\theta_i$  and  $\beta_i$  are unknown, which will be taken as uncertainties when sliding surfaces are designed, and will be estimated by an adaptive control law when the reaching mode control is proposed.

*Assumption 2:* The matched uncertainties  $w(x, x(t-\tau), t)$  are assumed to satisfy the following condition

$$\|w(x, x(t-\tau), t)\| \leq c + k\|x(t)\| = \rho \quad (5)$$

where  $c$  and  $k$  are constants, but it may not be easily obtained due to the complexity of the structure of the uncertainty.

To obtain a regular form of the systems (1), a nonsingular matrix  $T$  can be chosen such that

$$TB = \begin{bmatrix} 0_{(n-m) \times m} \\ B_2 \end{bmatrix}$$

where  $B_2 \in R^{m \times m}$  is nonsingular. For convenience, choose

$$T = \begin{bmatrix} U_2^T \\ U_1^T \end{bmatrix}$$

where  $U_1 \in R^{n \times m}$  and  $U_2 \in R^{n \times (n-m)}$  are two sub-blocks of a unitary matrix resulting from the singular value decomposition of  $B$ , i.e.,

$$B = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma \\ 0_{(n-m) \times m} \end{bmatrix} V^T$$

where  $\Sigma \in R^{m \times m}$  is a diagonal positive-definite matrix and  $V \in R^{m \times m}$  is a unitary matrix. By state transformation  $z = Tx$ , system (1) has the regular form

$$\begin{aligned}\dot{z}(t) &= (\bar{A} + \Delta\bar{A})z(t) + (\bar{A}_d + \Delta\bar{A}_d)z(t-\tau) \\ &\quad + \begin{bmatrix} 0_{(n-m) \times m} \\ B_2 \end{bmatrix} (u(t) + w(z(t), z(t-\tau), t)) \\ z(t) &= \bar{\varphi}(t), t \in \begin{bmatrix} -\tau, & 0 \end{bmatrix}\end{aligned}\quad (6)$$

where  $\bar{A} = TAT^{-1}$ ,  $\bar{A}_d = TA_dT^{-1}$ ,  $\Delta\bar{A} = T\Delta AT^{-1}$ ,  $\Delta\bar{A}_d = T\Delta A_dT^{-1}$  and  $\bar{\varphi}(t) = T\varphi(t)$ ,  $w(z(t), z(t-\tau), t) = w(T^{-1}z(t), T^{-1}z(t-\tau), t)$ . It is obvious that  $\|w(z(t), z(t-$

$\tau), t)\| \leq c + k\|T^{-1}z(t)\| = \rho$ . System (6) can be written as:

$$\begin{aligned}\dot{z}_1(t) &= (\bar{A}_{11} + \Delta\bar{A}_{11})z_1(t) + (\bar{A}_{d11} + \Delta\bar{A}_{d11})z_1(t-\tau) \\ &\quad + (\bar{A}_{12} + \Delta\bar{A}_{12})z_2(t) + (\bar{A}_{d12} + \Delta\bar{A}_{d12})z_2(t-\tau) \\ \dot{z}_2(t) &= (\bar{A}_{21} + \Delta\bar{A}_{21})z_1(t) + (\bar{A}_{d21} + \Delta\bar{A}_{d21})z_1(t-\tau) \\ &\quad + (\bar{A}_{22} + \Delta\bar{A}_{22})z_2(t) + (\bar{A}_{d22} + \Delta\bar{A}_{d22})z_2(t-\tau) \\ &\quad + B_2(u + w(z(t), z(t-\tau), t)) \\ z_1(t) &= \bar{\varphi}_1(t), t \in \begin{bmatrix} -\tau, & 0 \end{bmatrix} \\ z_2(t) &= \bar{\varphi}_2(t), t \in \begin{bmatrix} -\tau, & 0 \end{bmatrix}\end{aligned}\quad (7)$$

where  $z_1 \in R^{n-m}$ ,  $z_2 \in R^m$ ,  $B_2 = \Sigma V^T$ ,  $\bar{A}_{11} = U_2^T A U_2$ ,  $\bar{A}_{12} = U_2^T A U_1$ ,  $\bar{A}_{d11} = U_2^T A_d U_2$ ,  $\bar{A}_{d12} = U_2^T A_d U_1$ ,  $\Delta\bar{A}_{11} = U_2^T G D H U_2$ ,  $\Delta\bar{A}_{d11} = U_2^T G_d D_d H_d U_2$ ,  $\Delta\bar{A}_{12} = U_2^T G D H U_1$ ,  $\Delta\bar{A}_{d12} = U_2^T G_d D_d H_d U_1$ ,  $\bar{\varphi}_1(t) \in R^{(n-m)}$  and  $\bar{\varphi}_2(t) \in R^m$  are the sub-blocks of  $\bar{\varphi}(t)$ .

It is obvious that the first equation of system (7) represents the sliding motion dynamics of system (6), hence the corresponding sliding surface can be chosen as follows:

$$\delta(t) = \begin{bmatrix} C & I \end{bmatrix} z(t) = C z_1(t) + z_2(t) = 0 \quad (8)$$

where  $C \in R^{m \times (n-m)}$ . Substituting  $z_2 = -C z_1$  into the first equation of system (7) gives the sliding motion

$$\begin{aligned}\dot{z}_1(t) &= (\bar{A}_{11} + \Delta\bar{A}_{11} - \bar{A}_{12}C - \Delta\bar{A}_{12}C)z_1(t) \\ &\quad + (\bar{A}_{d11} + \Delta\bar{A}_{d11} - \bar{A}_{d12}C - \Delta\bar{A}_{d12}C)z_1(t-\tau) \\ z_1(t) &= \bar{\varphi}_1(t), t \in \begin{bmatrix} -\tau, & 0 \end{bmatrix}.\end{aligned}\quad (9)$$

For simplicity, the sliding motion can be written as

$$\begin{aligned}\dot{z}_1(t) &= \tilde{A}z_1(t) + \tilde{A}_d z_1(t-\tau) \\ z_1(t) &= \bar{\varphi}_1(t), t \in \begin{bmatrix} -\tau, & 0 \end{bmatrix}.\end{aligned}\quad (10)$$

where

$$\begin{aligned}\tilde{A} &= \bar{A}_{11} + \Delta\bar{A}_{11} - \bar{A}_{12}C - \Delta\bar{A}_{12}C \\ \tilde{A}_d &= \bar{A}_{d11} + \Delta\bar{A}_{d11} - \bar{A}_{d12}C - \Delta\bar{A}_{d12}C\end{aligned}\quad (11)$$

To facilitate control design, scaling matrices and structural properties of the system are introduced below.

In accordance to the structures of  $D$  and  $D_d$  defined in (4), the following scaling matrices are defined and used as design parameters later:

$$\begin{aligned}S_D &= \{Y | Y = \text{blockdiag}[Y_1, \cdots, Y_p] \\ &\quad 0 < Y_i = Y_i^T \in R^{n_{a_i} \times n_{a_i}}\}\end{aligned}\quad (12)$$

$$\begin{aligned}S_{D_d} &= \{Y_d | Y_d = \text{blockdiag}[Y_{d1}, \cdots, Y_{dq}] \\ &\quad 0 < Y_{di} = Y_{di}^T \in R^{n_{a_{di}} \times n_{a_{di}}}\}\end{aligned}\quad (13)$$

To this end, the following lemma is recalled.

*Lemma 1:* [5]: Let  $D \in S_D$ ,  $D_d \in S_{D_d}$ . Then for any  $X \in S_D$  and  $X_d \in S_{D_d}$ , the following inequalities

$$GDH + (GDH)^T \leq GXG^T + H^T X^{-1} H \quad (14)$$

$$G_d D_d H_d + (G_d D_d H_d)^T \leq G_d X_d G_d^T + H_d^T X_d^{-1} H_d \quad (15)$$

hold.

### III. MAIN RESULTS

The objective in this paper is to a design constant gain  $C \in R^{m \times (n-m)}$  and a reaching motion control law  $u(t)$  such that

- 1) The sliding motion (10) is quadratically stable; and
- 2) The trajectory of the closed-loop system (7) is convergent into a residual set of the origin with the reaching control law  $u(t)$ .

*Definition 1:* [5]: The uncertain sliding motion (10) is said to be robustly stable if the equilibrium solution  $z_1(t) = 0$  of the functional differential equation associated to sliding motion (10) is globally uniformly asymptotically stable for all admissible uncertainty  $\Delta \bar{A}_1$  and  $\Delta \bar{A}_{d1}$ .

Now, we present our first result in this paper.

**Theorem 1** The reduced order system (10) is quadratically stable if there exist symmetric positive-definite matrices  $J \in R^{m \times m}$ ,  $Z \in R^{m \times m}$ ,  $X \in S_D$ ,  $X_d \in S_{D_d}$  and general matrix  $Y \in R^{m \times (n-m)}$  such that

$$\Theta < 0 \quad (16)$$

and

$$\Omega \geq 0 \quad (17)$$

where

$$\begin{aligned} \Theta_{11} &= \bar{A}_{11}P - \bar{A}_{12}V + P\bar{A}_{11}^T - V^T\bar{A}_{12}^T + \tilde{M} + \tilde{M}^T + \tilde{Q} \\ &\quad + \bar{\tau}\tilde{X} + U_2^T GXG^T U_2 + U_2^T G_d X_d U_2 \\ \Theta_{12} &= \Theta_{21}^T = \bar{A}_{d11}\tilde{P} - \bar{A}_{d12}^T V - \tilde{M} + \tilde{N}^T + \bar{\tau}\tilde{Y} \\ \Theta_{13} &= \Theta_{31}^T = \bar{\tau}\lambda(\tilde{P}\bar{A}_{11}^T - V^T\bar{A}_{12}^T + U_2^T GXG^T U_2 \\ &\quad + U_2^T G_d X_d U_2) \\ \Theta_{14} &= \Theta_{41}^T = PU_2^T H^T - V^T U_1^T H \\ \Theta_{22} &= -\tilde{N} - \tilde{N}^T - (1-d)\tilde{Q} + \bar{\tau}\tilde{Z} \\ \Theta_{23} &= \Theta_{32}^T = \bar{\tau}\lambda(\tilde{P}\bar{A}_{d11}^T - V^T\bar{A}_{d12}^T) \\ \Theta_{25} &= \Theta_{52}^T = PU_2^T H_d^T - V^T U_1^T H_d^T \\ \Theta_{33} &= -\bar{\tau}\lambda\tilde{P} + \bar{\tau}^2\lambda^2(U_2^T GXG^T U_2 + U_2^T G_d X_d U_2) \\ \Theta_{44} &= -X \\ \Theta_{55} &= -X_d \end{aligned}$$

$$\Omega = \begin{bmatrix} X & Y & M \\ Y^T & Z & N \\ M^T & N^T & \lambda P \end{bmatrix}$$

and the rest of entries are zero.

Moreover, gain  $C = YZ^{-1}$  and the sliding surface of system (7) is

$$\delta(t) = YZ^{-1}z_1(t) + z_2(t) = 0. \quad (18)$$

*Proof:* Taking a scalar  $\lambda$ , symmetric positive-definite matrix variables  $P, Q \in R^{(n-m) \times (n-m)}$  and choosing the Lyapunov function candidate as

$$V(t) = x^T(t)Px(t) + \int_{t-\tau(t)}^t x^T(s)Qx(s)ds + \lambda \int_{-\bar{\tau}}^0 \int_{t+\theta}^t \dot{x}^T(s)P\dot{x}(s)dsd\theta \quad (19)$$

it follows that the Lyapunov derivative corresponding to system (10) is given by

$$\begin{aligned} \dot{V}(t) &= x^T(t)[P\tilde{A} + \tilde{A}^T P]x(t) + 2x^T(t)P\tilde{A}_d x(t - \tau(t)) \\ &\quad + x^T(t)Qx(t) - (1 - \dot{\tau}(t))x^T(t - \tau(t))Qx(t - \tau(t)) \\ &\quad + \lambda\bar{\tau}\dot{x}^T(t)P\dot{x}(t) - \lambda \int_{t-\bar{\tau}}^t \dot{x}^T(s)P\dot{x}(s)ds \\ &\leq x^T(t)[P\tilde{A} + \tilde{A}^T P]x(t) + 2x^T(t)P\tilde{A}_d x(t - \tau(t)) \\ &\quad + x^T(t)Qx(t) \\ &\quad - (1-d)x^T(t - \tau(t))Qx(t - \tau(t)) + \lambda\bar{\tau}\dot{x}(t)P\dot{x}(t) \\ &\quad - \lambda \int_{t-\tau(t)}^t \dot{x}^T(s)P\dot{x}(s)ds \end{aligned} \quad (20)$$

It is easy to see that

$$x(t - \tau(t)) = x(t) - \int_{t-\tau(t)}^t \dot{x}(s)ds \quad (21)$$

then, for any matrices  $M$  and  $N$  with appropriate dimensions, the following equation is derived:

$$2[x^T(t)M + x^T(t - \tau(t))N][x(t) - \int_{t-\tau(t)}^t \dot{x}(s)ds] = 0. \quad (22)$$

For any semi-positive definite matrices  $X, Z$  and general matrix  $Y$  such that  $W = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0$ , the following result is obvious

$$\begin{bmatrix} x(t) \\ x(t - \tau(t)) \end{bmatrix}^T W \begin{bmatrix} x(t) \\ x(t - \tau(t)) \end{bmatrix} \geq 0 \quad (23)$$

then,

$$\begin{aligned} &\bar{\tau} \begin{bmatrix} x(t) \\ x(t - \tau(t)) \end{bmatrix}^T W \begin{bmatrix} x(t) \\ x(t - \tau(t)) \end{bmatrix} \\ &- \int_{t-\tau(t)}^t \begin{bmatrix} x(t) \\ x(t - \tau(t)) \end{bmatrix}^T W \begin{bmatrix} x(t) \\ x(t - \tau(t)) \end{bmatrix} ds \\ &= (\bar{\tau} - \tau(t)) \begin{bmatrix} x(t) \\ x(t - \tau(t)) \end{bmatrix}^T W \begin{bmatrix} x(t) \\ x(t - \tau(t)) \end{bmatrix} \geq 0 \end{aligned}$$

For simplicity, let

$$\alpha(t) = \begin{bmatrix} x(t) \\ x(t - \tau(t)) \end{bmatrix}, \beta(t, s) = \begin{bmatrix} x(t) \\ x(t - \tau(t)) \\ \dot{x}(s) \end{bmatrix}. \quad (24)$$

Then, adding nonnegative terms into the right side of (20) results in

$$\begin{aligned} \dot{V}(t) &\leq x^T(t)[P\tilde{A} + \tilde{A}^T P]x(t) + 2x^T(t)P\tilde{A}_d x(t - \tau(t)) + \\ &x^T(t)Qx(t) - (1 - d)x^T(t - \tau(t))Qx(t - \tau(t)) + \\ &\lambda \bar{\tau} \dot{x}^T(t)P\dot{x}(t) - \lambda \int_{t-\tau(t)}^t \dot{x}^T(s)P\dot{x}(s)ds + 2[x^T(t)M \\ &+ x^T(t - \tau(t))N][x(t) - \int_{t-\tau(t)}^t \dot{x}(s)ds] \\ &+ \alpha^T(t)W\alpha(t) - \int_{t-\tau(t)}^t \alpha^T(s)W\alpha(s)ds \\ &= \alpha^T(t)\Phi\alpha(t) - \int_{t-\tau(t)}^t \beta^T(t, s)\Omega\beta(t, s)ds \end{aligned} \quad (25)$$

where

$$\begin{aligned} \Phi &= \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} \\ \Phi_{11} &= P\tilde{A} + \tilde{A}^T P + \bar{\tau}\lambda\tilde{A}^T P\tilde{A} + M + M^T + Q + \bar{\tau}X \\ \Phi_{12} &= P\tilde{A}_d + \bar{\tau}\lambda A^T P\tilde{A}_d - M + N^T + \bar{\tau}Y \\ \Phi_{22} &= \bar{\tau}\lambda\tilde{A}_d^T P\tilde{A}_d - N - N^T - (1 - d)Q + \bar{\tau}Z \\ \Omega &= \begin{bmatrix} X & Y & M \\ Y^T & Z & N \\ M^T & N^T & \lambda P \end{bmatrix} \geq 0 \end{aligned} \quad (27)$$

Then, if  $\Phi < 0$  and  $\Omega \geq 0$ ,  $\dot{V}(t) < 0$  for any  $\alpha(t) \neq 0$ . By Schur complement,  $\Phi < 0$  is equivalent to the following inequality:

$$\begin{bmatrix} \Pi_{11} - \bar{\tau}\lambda\tilde{A}^T P\tilde{A} & \Phi_{12} - \bar{\tau}\lambda A^T P\tilde{A}_d & \bar{\tau}\lambda\tilde{A}^T P \\ (\Phi_{12} - \bar{\tau}\lambda A^T P\tilde{A}_d)^T & \Phi_{22} - \bar{\tau}\lambda\tilde{A}_d^T P\tilde{A}_d & \bar{\tau}\lambda\tilde{A}_d^T P \\ \bar{\tau}\lambda P\tilde{A} & \bar{\tau}\lambda P\tilde{A}_d & -\bar{\tau}\lambda P \end{bmatrix} < 0 \quad (28)$$

Multiplying both sides of inequalities (27) and (28) with  $\text{diag}[P^{-1} P^{-1} P^{-1}]$ , and let  $\tilde{P} = P^{-1}$ ,  $\tilde{M} = P^{-1}MP^{-1}$ ,  $\tilde{N} = P^{-1}NP^{-1}$ ,  $\tilde{X} = P^{-1}XP^{-1}$ ,  $\tilde{Q} = P^{-1}QP^{-1}$ ,  $\tilde{Y} = P^{-1}YP^{-1}$ , yields

$$\begin{bmatrix} \tilde{X} & \tilde{Y} & \tilde{M} \\ \tilde{Y}^T & \tilde{Z} & \tilde{N} \\ \tilde{M}^T & \tilde{N}^T & \lambda\tilde{P} \end{bmatrix} \geq 0 \quad (29)$$

$$\begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} & \tilde{\Phi}_{13} \\ \tilde{\Phi}_{12}^T & \tilde{\Phi}_{22} & \tilde{\Phi}_{23}^T \\ \tilde{\Phi}_{13}^T & \tilde{\Phi}_{23}^T & \tilde{\Phi}_{33} \end{bmatrix} < 0 \quad (30)$$

where  $\tilde{\Phi}_{11} = \tilde{A}\tilde{P} + \tilde{P}\tilde{A}^T + \tilde{M} + \tilde{M}^T + \tilde{Q} + \bar{\tau}\tilde{X}$ ,  $\tilde{\Phi}_{12} = \tilde{A}_d\tilde{P} - \tilde{M} + \tilde{N}^T + \bar{\tau}\tilde{Y}$ ,  $\tilde{\Phi}_{13} = \bar{\tau}\lambda\tilde{P}\tilde{A}^T$ ,  $\tilde{\Phi}_{22} = -\tilde{N} - \tilde{N}^T - (1 - d)\tilde{Q} + \bar{\tau}\tilde{Z}$ ,  $\tilde{\Phi}_{23} = \bar{\tau}\lambda\tilde{P}\tilde{A}_d^T$ ,  $\tilde{\Phi}_{33} = -\bar{\tau}\lambda\tilde{P}$ .

Note that (11), and let  $V = C\tilde{P}$ ,  $\Omega_{111} = \bar{A}_{11}\tilde{P} - \bar{A}_{12}V + \tilde{P}\bar{A}_{11}^T - V^T\bar{A}_{12}^T + \tilde{M} + \tilde{M}^T + \tilde{Q} + \bar{\tau}\tilde{X}$ ,  $\Omega_{121} = \tilde{P}\bar{A}_{d11} - V^T\bar{A}_{d12}^T - \tilde{M}^T + \tilde{N} + \bar{\tau}\tilde{Y}^T$ ,  $\Omega_{122} = -\tilde{N} - \tilde{N}^T - (1 -$

$d)\tilde{Q} + \bar{\tau}\tilde{Z}$ ,  $\Omega_{131} = \bar{\tau}\lambda(\bar{A}_{11}\tilde{P} - \bar{A}_{12}V)$ ,  $\Omega_{132} = \bar{\tau}\lambda(\bar{A}_{d11}\tilde{P} - \bar{A}_{d12}V)$ ,  $\Omega_{133} = -\bar{\tau}\lambda\tilde{P}$  and

$$\begin{aligned} \Omega_1 &= \begin{bmatrix} \Omega_{111} & \Omega_{121}^T & \Omega_{131}^T \\ \Omega_{121} & \Omega_{122} & \Omega_{132}^T \\ \Omega_{131} & \Omega_{132} & \Omega_{133} \end{bmatrix} \\ \Omega_2 &= \begin{bmatrix} U_2^T G & U_2^T G_d \\ 0 & 0 \\ \bar{\tau}\lambda U_2^T G & \bar{\tau}\lambda U_2^T G_d \end{bmatrix} \\ \Omega_3 &= \begin{bmatrix} HU_2 P - HU_1 V & 0 \\ 0 & H_d U_2 P - H_d U_1 V \end{bmatrix} \\ \bar{D} &= \begin{bmatrix} D & 0 \\ 0 & D_d \end{bmatrix} \end{aligned} \quad (31)$$

then, (30) can be written as

$$\Omega_1 + \Omega_2 \bar{D} \begin{bmatrix} \Omega_3 & 0 \end{bmatrix} + \begin{bmatrix} \Omega_3 & 0 \end{bmatrix}^T \bar{D}^T \Omega_2^T < 0 \quad (32)$$

By Lemma 1, equation (32) holds if there exist  $X \in S_D$  and  $X_d \in S_{D_d}$  such that

$$\begin{aligned} \Omega_1 + \Omega_2 \begin{bmatrix} X & 0 \\ 0 & X_d \end{bmatrix} \Omega_2^T + \begin{bmatrix} \Omega_3 & 0 \end{bmatrix}^T \begin{bmatrix} X^{-1} & 0 \\ 0 & X_d^{-1} \end{bmatrix} \\ \begin{bmatrix} \Omega_3 & 0 \end{bmatrix} < 0 \end{aligned} \quad (33)$$

By Schur Complement formula, inequality (33) is equivalent to

$$\begin{bmatrix} \Omega_1 + \Omega_2 \begin{bmatrix} X & 0 \\ 0 & X_d \end{bmatrix} \Omega_2^T & \begin{bmatrix} \Omega_3 & 0 \end{bmatrix}^T \\ \begin{bmatrix} \Omega_3 & 0 \end{bmatrix} & - \begin{bmatrix} X & 0 \\ 0 & X_d \end{bmatrix} \end{bmatrix} < 0 \quad (34)$$

**Remark 2:** When  $\lambda = 0$ , inequalities (16) and (17) are in the form of LMI which is very efficient in computation because of the efficient LMI algorithm [7]. When  $\lambda \neq 0$ , inequalities (16) and (17) are nonlinear. They can also be solved using LMI toolbox by fixing  $\lambda$  first, then searching for feasible solutions using different values of  $\lambda$ . The introduction equality (22) can give less conservative results than the method in [8], the details can be found in [9] and [10].

The severe drawback of sliding mode control is that it is discontinuous across sliding surfaces. The discontinuity leads to control chattering in practice, which involves high-frequency dynamics. To remove control chattering, the following smooth projection will be introduced.

**Definition 2:** [11] Let  $\theta = [\theta_1 \ \theta_2 \ \dots \ \theta_p]^T \in \Omega$  be an unknown parameter vector,  $\hat{\theta}$  be the estimate, and  $\Omega \in R^p$  be a closed ball of known radius  $r_\Omega$ . The projection algorithm

$\text{Proj}(y, \hat{\theta})$  is given by

$$\text{Proj}(y, \hat{\theta}) = \begin{cases} y, & \text{if } p(\theta) \leq 0 \\ y, & \text{if } (p(\hat{\theta}) \geq \text{and } \frac{\partial p}{\partial \hat{\theta}} y \leq 0) \\ y - \frac{p(\hat{\theta}) \frac{\partial p}{\partial \hat{\theta}} y}{\|\frac{\partial p}{\partial \hat{\theta}}\|^2} \frac{\partial p}{\partial \hat{\theta}}^T, & \text{otherwise} \end{cases} \quad (35)$$

where  $p(\hat{\theta}) = \frac{\|\hat{\theta}\|^2 - r_\Omega^2}{\nu^2 + 2\epsilon r_\Omega}$ ,  $\nu$  is an arbitrary positive real scalar. From (35), if  $\hat{\theta}(0) \in \Omega$ , the following nice properties follow immediately:

- 1)  $\|\hat{\theta}(t)\| \leq r_\Omega + \nu, \forall t \geq 0$ ;
- 2)  $\text{Proj}(y, \hat{\theta})$  is Lipschitz continuous;
- 3)  $\|\text{Proj}(y, \hat{\theta})\| \leq \|y\|$ ;
- 4)  $\tilde{\theta}^T \text{Proj}(y, \hat{\theta}) \geq \tilde{\theta}^T y, (\tilde{\theta} = \theta - \hat{\theta})$ .

The result of designing of reaching motion controller is given in the next theorem.

**Theorem 2:** With the gain  $C$  obtained in Theorem 1 and the linear sliding surface is given by (18). Then the trajectory of the closed-loop system (7) can be driven onto the sliding surface in limited time with the control and evolves in a neighborhood around the sliding surface, and finally, converges into a residual set at the origin.

$$u = -B_2^{-1}[\Pi\delta + \overline{CA}z(t) + \overline{CA}_d z(t-\tau) + f_1^T(z(t))\hat{\theta} + f_2^T(z(t-\tau))\hat{\beta}] - u_N \quad (36)$$

$$u_N = \begin{cases} -\frac{B_2^T \delta}{\|B_2^T \delta\|} \hat{\rho}, & \text{if } \hat{\rho} \|B_2^T \delta\| > \epsilon \\ -\frac{B_2^T \delta}{\epsilon} \hat{\rho}^2, & \text{if } \hat{\rho} \|B_2^T \delta\| \leq \epsilon \end{cases} \quad (37)$$

and the adaptation laws as

$$\dot{\hat{\theta}} = q_1 \text{Proj}(f_1^T(z(t))\delta, \hat{\theta}) \quad (38)$$

$$\dot{\hat{\beta}} = q_2 \text{Proj}(f_2^T(z(t-\tau))\delta, \hat{\beta}) \quad (39)$$

$$\dot{\hat{c}}(t, z) = q_3(-\epsilon_0 \hat{c} + \|B_2^T \delta\|) \quad (40)$$

$$\dot{\hat{k}}(t, z) = q_4(-\epsilon_1 \hat{k} + \|B_2^T \delta\| \|z\|) \quad (41)$$

where  $\Pi$  is a positive definite matrix,  $q_1, q_2, q_3, q_4, \epsilon_0$  and  $\epsilon_1$  are design parameters, and  $\overline{C} = [C \ I]$ ,

$$\begin{aligned} \hat{\rho} &= \hat{c}(z(t), t) + \hat{k}(z(t), t) \|z(t) \\ f_1^T(z(t)) &= [(\overline{CT}A_1 T^{-1} z(t))^T \delta(t) \\ &\quad \dots (\overline{CT}A_p T^{-1} z(t))^T \delta(t)]^T \\ f_2^T(z(t-\tau)) &= [(\overline{CT}A_{d1} T^{-1} z(t-\tau))^T \delta(t) \\ &\quad \dots (\overline{CT}A_{dq} T^{-1} z(t-\tau))^T \delta(t)]^T \end{aligned} \quad (42)$$

**Proof:** We will complete the proof by showing that via the control law (36), the trajectory of the closed-loop system (7) can be driven onto the sliding surface in limited time and evolves in a neighborhood around the sliding surface. In

steady state, it is convergent into a residual set at the origin. Consider the following Lyapunov function:

$$V = \frac{1}{2}[\delta^T \delta + \frac{1}{q_1} \tilde{\theta}^T \tilde{\theta} + \frac{1}{q_2} \tilde{\beta}^T \tilde{\beta} + \frac{1}{q_3} \tilde{c}^2 + \frac{1}{q_4} \tilde{k}^2] \quad (43)$$

where  $\tilde{\theta} = \theta - \hat{\theta}$ ,  $\tilde{\beta} = \beta - \hat{\beta}$ ,  $\tilde{c} = c - \hat{c}(z(t), t)$  and  $\tilde{k} = k - \hat{k}(z(t), t)$ . Its time derivation is

$$\dot{V} = \delta^T \dot{\delta} - \frac{1}{q_1} \tilde{\theta}^T \dot{\tilde{\theta}} - \frac{1}{q_2} \tilde{\beta}^T \dot{\tilde{\beta}} - \frac{1}{q_3} \tilde{c} \dot{\tilde{c}} - \frac{1}{q_4} \tilde{k} \dot{\tilde{k}} \quad (44)$$

From the sliding surface

$$\delta(t) = \begin{bmatrix} C & I \end{bmatrix} z$$

we have

$$\dot{\delta}(t) = \begin{bmatrix} C & I \end{bmatrix} \dot{z} = \overline{C}(\overline{A} + \Delta \overline{A})z(t) + \overline{C}(\overline{A}_d + \Delta \overline{A}_d)z(t-\tau) + B_2(u(t) + w(z, z(t-\tau), t)) \quad (45)$$

If  $\|B_2^T \delta\| \hat{\rho} > \epsilon$ , with the control law defined in (36) and adaptation laws defined in (38)-(41), we have

$$\begin{aligned} \dot{V} &= \delta^T [-\Pi\delta + \overline{C}\Delta \overline{A}z(t) + \overline{C}\Delta \overline{A}_d z(t-\tau) - f_1^T(z(t))\hat{\theta} \\ &\quad - f_2^T(z(t-\tau))\hat{\beta}] - \tilde{\theta}^T \text{Proj}(f_1(z(t)), \hat{\theta}) - \\ &\quad \tilde{\beta}^T \text{Proj}(f_2(z(t-\tau)), \hat{\beta}) - \|B_2^T \delta\|(\hat{k}\|z\| + \hat{c}) + \\ &\quad \delta^T B_2 w(z(t), z(t-\tau), t) \end{aligned}$$

From (2), we have

$$\begin{aligned} \dot{V} &= -\delta^T \Pi \delta - f_1^T(z(t))\hat{\theta} - f_2^T(z(t-\tau))\hat{\beta} + \theta^T \times \\ &\quad [(\overline{CT}A_1 T^{-1} z(t))^T \delta(t) \dots (\overline{CT}A_p T^{-1} z(t))^T \delta(t)]^T \\ &\quad + \beta^T [(\overline{CT}A_{d1} T^{-1} z(t-\tau))^T \delta(t) \dots (\overline{CT}A_{dq} T^{-1} z(t-\tau))^T \delta(t)]^T - \tilde{\beta}^T \text{Proj}(f_2(z(t-\tau)), \hat{\beta}) \\ &\quad - \|B_2^T \delta\|(\hat{k}\|z\| + \hat{c}) - \tilde{\theta}^T \text{Proj}(f_1(z(t)), \hat{\theta}) \\ &\quad + \delta^T B_2 w(z(t), z(t-\tau), t) \end{aligned} \quad (46)$$

It follows from (42) that

$$\begin{aligned} \dot{V} &= -\delta^T \Pi \delta + \tilde{\theta}^T (f_1(z(t) - \text{Proj}(f_1^T(z(t), \hat{\theta}))) + \\ &\quad \tilde{\beta}^T (f_2(z(t-\tau) - \text{Proj}(f_2^T(z(t-\tau), \hat{\beta}))) \\ &\quad + \|B_2^T \delta\|(\hat{k}\|z\| + \hat{c}) - \|B_2^T \delta\|(\hat{k}\|z\| + \hat{c}) - \\ &\quad \tilde{c}(-\epsilon_0 \hat{c} + \|B_2^T \delta\|) - \tilde{k}(-\epsilon_1 \hat{k} + \|B_2^T \delta\| \|z\|) \end{aligned} \quad (47)$$

From the property 4) of the operator  $\text{Proj}(y, \hat{\theta})$  and completing the squares we obtain

$$\begin{aligned} \dot{V} &\leq -\delta^T \Pi \delta + \epsilon_0 \tilde{c} \hat{c} + \epsilon_1 \tilde{k} \hat{k} \\ &= -\delta^T \Pi \delta - \epsilon_0 (\hat{c} - \frac{1}{2} c)^2 - \epsilon_1 (\hat{k} - \frac{1}{2} k)^2 + \frac{1}{4} (\epsilon_0 c^2 + \epsilon_1 k^2) \\ &\leq -\delta^T \Pi \delta + \frac{1}{4} (\epsilon_0 c^2 + \epsilon_1 k^2) \end{aligned} \quad (48)$$

If  $\|B_2^T \delta\| \hat{\rho}^2 \leq \epsilon$ , with the control law defined in (36) and adaptation laws defined in (38)-(41), we obtain

$$\begin{aligned}
\dot{V} &\leq -\delta^T \Pi \delta - \frac{\|B_2^T \delta\|^2}{\epsilon} \hat{\rho}^2 + \|B_2^T \delta\| (k \|z\| + c) - \\
&\quad \tilde{c} (-\epsilon_0 \hat{c} + \|B_2^T \delta\|) - \tilde{k} (-\epsilon_1 \hat{k} + \|B_2^T \delta\| \|z\|) \\
&= -\delta^T \Pi \delta - \frac{\|B_2^T \delta\|^2}{\epsilon} \hat{\rho}^2 + \|B_2^T \delta\| \hat{\rho} + \epsilon_0 \tilde{c} \hat{c} + \epsilon_1 \tilde{k} \hat{k} \\
&= -\delta^T \Pi \delta - \left( \frac{\|B_2^T \delta\|^2}{\sqrt{\epsilon}} \hat{\rho} - \frac{\sqrt{\epsilon}}{2} \right)^2 + \frac{\epsilon}{4} \\
&\quad - \epsilon_0 \left( \hat{c} - \frac{1}{2} c \right)^2 - \epsilon_1 \left( \hat{k} - \frac{1}{2} k \right)^2 + \frac{1}{4} (\epsilon_0 c^2 + \epsilon_1 k^2) \\
&\leq -\delta^T \Pi \delta + \frac{\epsilon}{4} + \frac{1}{4} (\epsilon_0 c^2 + \epsilon_1 k^2) \tag{49}
\end{aligned}$$

It can be concluded now from (48) and (49) that all signals are uniformly ultimately bounded.  $\diamond$

In order to obtain a continuous control law (36), the unknown parameters are estimated with the smooth projection. The continuous positive scalar valued function  $\rho$  satisfying Assumption 2 is estimated by a smoothed SMC control law taking account of the boundary layer effect, that is the part,  $u_N$ . The benefits of this kind of smooth techniques have been stated in [12], [13], which offers a continuous approximation to the discontinuous sliding mode control law inside the boundary layer and guarantees the output tracking error within any neighborhood of the sliding surface. However, asymptotic stability is lost and it can guarantee the bounded motion about the sliding surface only. Therefore, we cannot analyze the stability of the dynamics of sliding mode strict to sliding surface. In the following, the stability of sliding motion will be investigated.

Consider the dynamics of the sliding motion around the sliding surface

$$\begin{aligned}
\dot{z}_1(t) &= (\bar{A}_{11} + \Delta \bar{A}_{11} - \bar{A}_{12} C - \Delta \bar{A}_{12} C) z_1(t) + \\
&\quad + (\bar{A}_{d11} + \Delta \bar{A}_{d11} - \bar{A}_{d12} C - \Delta \bar{A}_{d12} C) z_1(t - \tau) \\
&\quad + (\bar{A}_{12} + \Delta \bar{A}_{12}) \delta(t) + (\bar{A}_{d12} + \Delta \bar{A}_{d12}) \delta(t - \tau) \\
z_1(t) &= \bar{\varphi}_1(t), t \in [-\tau, 0]. \tag{50}
\end{aligned}$$

It has been proved in Theorem 2 that  $\delta(t)$  is ultimately bounded, the same holds for  $\delta(t - \tau)$  as  $t \rightarrow \infty$ , then  $(\bar{A}_{12} + \Delta \bar{A}_{12}) \delta(t) + (\bar{A}_{d12} + \Delta \bar{A}_{d12}) \delta(t - \tau)$  will be bounded as well, and can be viewed as a bounded disturbance in the dynamics of (50). As the gain  $C$  is designed to guarantee the quadratic stability of system (10), which will not converge to zero due to the existence of bounded distances, however, it will stay in a domain containing the zero within a prescribed precision.

## IV. CONCLUSION

In this paper, the problem of robust adaptive sliding mode control for a class of uncertain time-delay systems has been considered in which no matching condition is assumed for the state uncertainties. The aim of this study is to attempt to combine the advantages of adaptive control and variable control methods. The resulting combined method makes the system asymptotically stable in the ideal case, robust in the presence of uncertainties and disturbance and in addition, has a fast transient response.

## ACKNOWLEDGMENT

Yuanqing Xia would like to thank the support of the National Natural Science Foundation of China under Grant 60504020.

## REFERENCES

- [1] C. E. de Souza and X. Li, "Delay-dependent robust  $H_\infty$  control of uncertain linear state-delayed systems," *Automatica*, vol. 35, pp. 1313–1321, 1999.
- [2] E. K. Boukas, Z. Liu, and P. Shi, "Delay-dependent stability and output feedback stabilization of markov jump systems with time-delays," *IEE-Part D, Control Theory and Applications*, vol. 149, no. 5, pp. 379–386, 2002.
- [3] K. S. Narendra and J. D. Boskovic, "A combined direct, indirect, and variable structure method for robust adaptive control," *Robust stability of uncertain time-delay systems and its stabilization by variable structure control*, vol. 37, pp. 262–268, 1992.
- [4] H. H. Choi, "An explicit formula of linear sliding surfaces for a class of uncertain dynamic systems with mismatched uncertainties," *Automatica*, vol. 34, pp. 1015–1020, 1998.
- [5] K. S. Kim, Y. Park, and S. H. Oh, "Designing robust sliding surfaces for parametric uncertain systems: a riccati approach," *Automatica*, vol. 36, pp. 1041–1048, 2000.
- [6] K. Shyu and J. Yan, "Robust stability of uncertain time-delay systems and its stabilization by variable structure control," *Int. J. Control*, vol. 57, pp. 237–246, 1993.
- [7] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in systems and control theory*. Philadelphia: SIAM: Studies in Applied Mathematics, 1994.
- [8] Y. Xia and Y. Jia, "Robust sliding mode control of uncertain time-delay systems : An LMI approach," *IEEE Transactions Automatic Control*, vol. 48, no. 6, pp. 1986–1992, 2003.
- [9] Y. He, M. Wu, J. H. She, and G. P. Liu, "Parameter-dependent lyapunov functional for stability of time-delay systems with polytopic-type uncertainties," *IEEE Trans. Automat. Contr.*, vol. 49, pp. 828–832, 2004.
- [10] D. Yue, "Robust stabilization of uncertain systems with unknown input delay," *Automatica*, vol. 40(2), pp. 331–336, 2004.
- [11] S. S. Ge, F. Hong, and T. H. Lee, "Adaptive neural control of nonlinear time-delay systems with unknown virtual control coefficients," *IEEE Transactions on Systems, Man, and Cybernetics-Part B, Cybernetics*, vol. 34, no. 1, pp. 449–516, 2004.
- [12] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*. Prentice-Hall, Englewood Cliffs, NJ, 1991.
- [13] G. Wheeler, C. Su, and Y. Stepanenko, "A sliding mode controller with improved adaptation laws for the upper bounds on the norm of uncertainties," *Automatica*, vol. 12, pp. 1657–1661, 1998.