International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

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Online publication date: 21 September 2010

To cite this Article Yang, Chenguang , Li, Yanan , Ge, Shuzhi Sam and Lee, Tong Heng(2010) 'Adaptive control of a class of discrete-time MIMO nonlinear systems with uncertain couplings', International Journal of Control, 83: 10, 2120 — 2133

To link to this Article: DOI: 10.1080/00207179.2010.508092

URL: http://dx.doi.org/10.1080/00207179.2010.508092

International Journal of Control
Vol. 83, No. 10, October 2010, 2120–2133

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(*Received 3 November 2009; final version received 8 July 2010*)

In this article, adaptive control is investigated for a class of discrete-time multi-input-multi-output nonlinear systems in block-triangular form with uncertain couplings of delayed states among subsystems. Future states prediction is carried out to facilitate adaptive control design and auxiliary outputs are introduced to develop a novel compensation mechanism for the uncertain nonlinear couplings. By using Lyapunov method and ordering signals growth rate, it is rigorously proved that all the signals in the whole closed-loop systems are globally bounded and the output tracking errors asymptotically converge to zeros. The effectiveness of the proposed control is demonstrated in the simulation study.

Keywords: adaptive control; MIMO discrete-time systems; couplings

1. Introduction

Control theory of multi-input–multi-output (MIMO) nonlinear systems has attracted an ever increasing interest in recent years. Practically, most systems are of nonlinear and multi-variable characteristics. However, the control problem of MIMO nonlinear systems is very complicated, and it is generally non-trivial to extend the control designs from single-input–single-output (SISO) systems to MIMO systems due to the interactions among various inputs, outputs and states. When the couplings among each subsystems are uncertain, the closed-loop stability analysis becomes much more complex.

It is well known that adaptive control has been developed for decades with major concern on the system parametric uncertainties. Adaptive controls designed via backstepping have been studied for MIMO systems in parametric-strict-feedback form (Krstic, Kanellakopoulos, and Kokotovic 1995). Later, robust adaptive control has been developed in Yao and Tomizuka (2001) for MIMO nonlinear systems in semi-strict-feedback forms. It is noted that no uncertainties are considered in the input coupling in these results. As a matter of fact, the system interconnections are assumed to be either known functions or bounded by known functions in most of the nonlinear control results on MIMO system (Schwartz, Isidori, and Tarn 1999; Lin and Qian 2001). In Ge, Hang, and Zhang (2000), Ge and Wang (2004), Chen and Li (2008), Chen and Li (2010) and Li, Chen, and Li (2010), by using neural network (NN) approximation and exploitation of the structural properties of triangular and block-triangular form systems, adaptive NN control has been designed for nonlinear MIMO system with unknown couplings.

In this article, adaptive control of discrete-time nonlinear MIMO system with unknown interactions are to be investigated. It is noted that the aforementioned control designs are mainly restricted in the continuous-time, while for the ease of control design, sometimes it is convenient to model processes in discrete-time because the process data are typically available only at discrete-time instants. It should be mentioned that there are also remarkable differences between adaptive control design for continuous-time system and discrete-time systems. One notable fact is that for a quadratic Lyapunov function, in continuous-time its derivative is linear in the state derivative while in discrete-time its difference is still quadratic in the state first difference (Song and Grizzle 1993). Thus, many Lyapunov-based adaptive control design methodologies in continuous-time, such as adaptive backstepping in Krstic et al. (1995) and Yao and Tomizuka (2001), are not directly applicable to discrete-time systems.

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ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207179.2010.508092
http://www.informaworld.com

It is interesting and challenging to investigate adaptive control design for discrete-time MIMO nonlinear systems, especially for systems with nonlinear uncertain couplings. In most of the existing results for adaptive control design of MIMO system in discrete-time, the systems under study are linear (Goodwin and Sin 1984; Ossaman and Kamen 1987; Corradini and Orlando 1997; Chen 2006). In this article we will study adaptive control for a class of discrete-time MIMO nonlinear systems composed of a number of strict-feedback form nonlinear subsystems coupled with each other. The system under study in this article is more general than the system studied in Ge, Zhang, and Lee (2004) by NN control, e.g. there are interconnections in every equation of each subsystem rather than only in the last equation of each subsystem. In addition to parametric uncertainties, there are also nonlinear function uncertainties of coupled states with time delays. It makes adaptive control design more challenging as most of the conventional adaptive control design only concerns parametric uncertainties. To deal with the nonlinear function uncertainties, the compensation techniques proposed for SISO system in our previous work (Ge, Yang, Dai, Jiao, and Lee 2009) will be further developed in this article for MIMO system while no approximation tools will be employed. In addition, the compensation technique will be combined with augmented states vector to solve the problem of unknown time delays, which is frequently encountered in engineering systems to be controlled (Kolmanovskii and Myshkis 1992). The augmented states information at previous steps will be used to compensate for the uncertain couplings at current step. It should be mentioned that the powerful tool of Lyapunov–Krasovskii functional (Kharitonov and Melchor-Aguilar 2003) which plays a pivotal role in Lyapunov-based stability analysis for time-delay systems in continuous-time does not have a counterpart in discrete-time.

Each subsystem in the MIMO system under study is assumed to be in strict-feedback form similar to the SISO system we studied before (Ge, Yang, and Lee 2008; Ge et al. 2009). The future state prediction-based adaptive control method developed for SISO system in Ge et al. (2008) and Ge et al. (2009) will be further developed in this article. However, it is not straightforward to apply the method from SISO to MIMO system under study, as the state variables of one subsystem are embedded in another subsystem inside and outside of the control range, and even in the uncertain coupling nonlinearities. Therefore, the relation among various states, inputs and outputs has been fully studied for the open-loop system such that the growth rate of various closed-loop signals can be sorted for stability analysis using Lyapunov approach.

2. Problem formulation and preliminaries

2.1 *System representation*

The nonlinear MIMO system under study consists of n subsystems interacting with each other in the following manner:

$$\Sigma : \begin{cases} \Sigma_1 & \begin{cases} \xi_{1,i_1}(k+1) = \Theta_{1,i_1}^T \Phi_{1,i_1}(\bar{\xi}_{1,i_1-m_{11}}(k), \bar{\xi}_{2,i_1-m_{12}}(k), \ldots, \bar{\xi}_{n,i_1-m_{1n}}(k)) \\ \qquad + g_{1,i_1}\xi_{1,i_1+1}(k), \quad i_1 = 1,2,\ldots,n_1-1 \\ \xi_{1,n_1}(k+1) = \Theta_{1,n_1}^T \Phi_{1,n_1}(\Xi(k)) + g_{1,n_1}u_1(k) + \nu_1(\Xi_{\tau_1}(k)) \\ y_1(k) = \xi_{1,1}(k) \end{cases} \\ \vdots & \\ \Sigma_j & \begin{cases} \xi_{j,i_j}(k+1) = \Theta_{j,i_j}^T \Phi_{j,i_j}(\bar{\xi}_{1,i_j-m_{j1}}(k), \bar{\xi}_{2,i_j-m_{j2}}(k), \ldots, \bar{\xi}_{n,i_j-m_{jn}}(k)) \\ \qquad + g_{j,i_j}\xi_{j,i_j+1}(k), \quad i_j = 1,2,\ldots,n_j-1 \\ \xi_{j,n_j}(k+1) = \Theta_{j,n_j}^T \Phi_{j,n_j}(\Xi(k), \bar{u}_{j-1}(k)) + g_{j,n_j}u_j(k) + \nu_j(\Xi_{\tau_j}(k)) \\ y_j(k) = \xi_{j,1}(k) \end{cases} \\ \vdots & \\ \Sigma_n & \begin{cases} \xi_{n,i_n}(k+1) = \Theta_{n,i_n}^T \Phi_{n,i_n}(\bar{\xi}_{1,i_n-m_{n1}}(k), \bar{\xi}_{2,i_n-m_{n2}}(k), \ldots, \bar{\xi}_{n,i_n-m_{nn}}(k)) \\ \qquad + g_{n,i_n}\xi_{n,i_n+1}(k), \quad i_n = 1,2,\ldots,n_n-1 \\ \xi_{n,n_n}(k+1) = \Theta_{n,n_n}^T \Phi_{n,n_n}(\Xi(k), \bar{u}_{n-1}(k)) + g_{n,n_n}u_n(k) + \nu_n(\Xi_{\tau_n}(k)) \\ y_n(k) = \xi_{n,1}(k), \end{cases} \end{cases} \tag{1}$$

where $\xi_{j,i_j}(k)$ is the i_jth state variable of subsystem Σ_j, $\bar{\xi}_{j,i_j}(k) = [\xi_{j,1}(k), \xi_{j,2}(k), \ldots, \xi_{j,i_j}(k)]^T$ is a vector from the first to the i_jth state variables of subsystem Σ_j, $i_j = 1,2,\ldots,n_j$, and $\Xi(k) = [\bar{\xi}_{1,n_1}(k), \bar{\xi}_{2,n_2}(k), \ldots,$

$\bar{\xi}_{n,n_n}(k)]^T$ is a vector of all the states in the whole system, which is assumed to be measurable. The delayed state vectors, $\Xi_{\tau_j}(k)$, are defined as

$$\Xi_{\tau_j}(k) = [\bar{\xi}_{1,n_1}(k-\tau_{j,1}), \bar{\xi}_{2,n_2}(k-\tau_{j,2}), \ldots, \bar{\xi}_{n,n_n}(k-\tau_{j,n})]^T,$$
$$j = 1, 2, \ldots, n, \qquad (2)$$

where the unknown delays $\tau_{j,l}$ satisfy $0 \le \tau_{\min} \le \tau_{j,l} \le \tau_{\max}$, $l = 1, 2, \ldots, n$.

The notation $m_{jl} = n_j - n_l$ represents the order difference between the jth and the lth subsystem as introduced in Ge and Wang (2004). When $i_j - m_{jl} \le 0$, it means the states vector $\bar{\xi}_{j,i_j-m_{jl}}(k)$ is not included in the i_jth equation of subsystem Σ_j in (1). It is noted that when $l = j$, we have $m_{jl} = 0$ and $\bar{\xi}_{j,i_j-m_{jl}}(k) = \bar{\xi}_{j,i_j}(k)$, and when $i_j = n_j$, $j = 1, 2, \ldots, n$, we have $[\bar{\xi}_{1,n_j-m_{j1}}(k), \bar{\xi}_{2,n_j-m_{j2}}(k), \ldots, \bar{\xi}_{n,n_j-m_{jn}}(k)] = \Xi(k)$. This is the reason we use notation $\Xi(k)$ in the last equations of every subsystem Σ_j.

The system functions $\Phi_{j,i_j}(\cdot)$, $j = 1, 2, \ldots, n$, are known, but system parameters $\Theta_{j,i_j}^T \in R^{p_{j,i_j}}$ and $g_{j,i_j} \in R$ are unknown as well as the uncertain coupling terms $\nu_j(\Xi_{\tau_j}(k))$. The notations $u_j(k)$ and $y_j(k)$ represent system inputs and outputs, respectively, $j = 1, 2, \ldots, n$.

The notations used in system (1) are summarised in the following table:

m_{ij}	$n_i - n_j$, the difference of orders between subsystem Σ_i and Σ_j;
$\bar{u}_j(k)$	$[u_1(k), u_2(k), \ldots, u_j(k)]^T$, vector of inputs;
$\Phi_{j,i_j}(\cdot)$	known vector-valued function of subsystem Σ_j;
$\nu_j(\cdot)$	uncertain function of subsystem Σ_j;
τ_j	unknown time delay of subsystem Σ_j;
$\bar{\xi}_{j,i_j}(k)$	$[\xi_{j,1}(k), \xi_{j,2}(k), \ldots, \xi_{j,i_j}(k)]^T$, vector of state variables of subsystem Σ_j;
$\Xi(k)$	$[\bar{\xi}_{1,n_1}(k), \bar{\xi}_{2,n_2}(k), \ldots, \bar{\xi}_{n,n_n}(k)]^T$, vector of all state variables;
$\Xi_{\tau_j}(k)$	$[\bar{\xi}_{1,n_1}(k-\tau_{j,1}), \bar{\xi}_{2,n_2}(k-\tau_{j,2}), \ldots, \bar{\xi}_{n,n_n}(k-\tau_{j,n})]^T$, vector of states with time delays, $\tau_{\min} \le \tau_{j,i} \le \tau_{\max}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$;
Θ_{j,i_j}	unknown vector parameter of subsystem Σ_j, $1 \le i_j \le n_j$;
g_{j,i_j}	unknown control gain of subsystem Σ_j;

Assumption 2.1: *The uncertain nonlinear coupling terms* $\nu_j(\cdot)$, *are Lipschitz functions, i.e.* $\|\nu_j(\varepsilon_1) - \nu_j(\varepsilon_2)\| \le L_j^\nu \|\varepsilon_1 - \varepsilon_2\|$ $\forall \varepsilon_1, \varepsilon_2 \in R^p$, *with Lipschitz coefficient* L_j^ν *satisfy* $L_j^\nu < \lambda^*$, *where* λ^* *is defined later in* (43). *The system functions,* $\Phi_{j,i_j}(\cdot)$, $1 \le j \le n$, $1 \le i_j \le n_j$ *are also Lipschitz functions with Lipschitz coefficients* L_{j,i_j}.

Assumption 2.2: *The signs of control gains* g_{j,i_j}, $(1 \le j \le n)$ *are known and satisfy* $|g_{j,i_j}| \ge \underline{g}_{j,i_j} > 0$, *where* $\underline{g}_{j,i_j}$ *is the lower bound. Without loss of generality, it is assumed that* g_{j,i_j} *are positive.*

The control objective is to drive the outputs, $y_j(k)$, to follow the given desired reference trajectories $y_j^*(k)$, respectively, and guarantee the boundedness of all the closed-loop signals.

Remark 2.1: In the following section of this article, we will design adaptive control for the nonlinear MIMO discrete-time system (1) aiming at asymptotic tracking performance. It is noted that system (1) is of uncertain parameters in the couplings among every subsystem as well as uncertain nonlinearities, such that the adaptive control design without resort to any approximation tools is of great challenge and should be interesting to the adaptive control community.

2.2 *Useful definitions and lemmas*

Definition 2.1 (Chen and Narendra 2001)**:** Let $x_1(k)$ and $x_2(k)$ be two discrete-time scalars or vector signals $\forall k \in Z_t^+$ $\forall t$, where Z_t^+ represents the set of all integers which are not less than a given integer t.

- We denote $x_1(k) = O[x_2(k)]$, if there exist positive constants m_1, m_2 and k_0 such that $\|x_1(k)\| \le m_1 \max_{k' \le k} \|x_2(k')\| + m_2$ $\forall k > k_0$.
- We denote $x_1(k) = o[x_2(k)]$, if there exists a discrete-time function $\alpha(k)$ satisfying $\lim_{k\to\infty} \alpha(k) = 0$ and a constant k_0 such that $\|x_1(k)\| \le \alpha(k) \max_{k' \le k} \|x_2(k')\|$ $\forall k > k_0$.
- We denote $x_1(k) \sim x_2(k)$ if they satisfy $x_1(k) = O[x_2(k)]$ and $x_2(k) = O[x_1(k)]$.

For convenience, in the following, we use $O[1]$ and $o[1]$ to denote bounded sequences and sequences converging to zero, respectively. According to Definition 2.1, it is straightforward to verify that the following properties hold.

Proposition 2.1: *Consider the signals* $x_1(k)$ *and* $x_2(k)$ *in Definition* 2.1, *then we have*

(i) $O[x_1(k+\tau)] + O[x_1(k)] \sim O[x_1(k+\tau)]$ $\forall \tau \ge 0$.
(ii) $x_1(k+\tau) + o[x_1(k)] \sim x_1(k+\tau)$ $\forall \tau \ge 0$.
(iii) $o[x_1(k)] + o[x_2(k)] \sim o[|x_1(k)| + |x_2(k)|]$.
(iv) *if* $x_1(k) \sim x_2(k)$ *and* $\lim_{k\to\infty} \|x_2(k)\| = 0$, *then* $\lim_{k\to\infty} \|x_1(k)\| = 0$.
(v) *if* $x_1(k) + o[x_1(k)] = o[1]$, *then* $\lim_{k\to\infty} \times \|x_1(k)\| = 0$.

Proof: See Appendix A. □

Lemma 2.1: *Under Assumption* 2.1, *the states and inputs of system* (1) *satisfy*

$$\sum_{l=1}^{n} O[\bar{\xi}_{l,i_j-m_{jl}}(k)] \sim \sum_{l=1}^{n} O[y_l(k+i_j-m_{jl}-1)],$$
$$u_j(k) = O[\Xi(k+1)], \quad \text{where we let}$$
$$\bar{\xi}_{l,i_j-m_{jl}}(k) = y_l(k+i_j-m_{jl}-1) = 0,$$
$$\text{if } i_j - m_{jl} \leq 0 \tag{3}$$

for $j=1,2,\ldots,n$ *and* $i_j=1,2,\ldots,n_j$.

Proof: See Appendix B. □

Lemma 2.2: *Consider sequences* $x_j(k)$, $j=1,2,\ldots,n$, *which satisfy* $x_j(k) = \sum_{i=1}^{n} o[x_i(k-m_{ji})] + o[1]$. *Then, we have* $\lim_{k\to\infty} x_j(k) = 0, \quad j=1,2,\ldots,n$.

Proof: See Appendix C. □

Lemma 2.3: *Given a bounded sequence* $X(k) \in R^p$. *Define* $l_k = \arg\min_{l \leq k-q} \|X(k) - X(l)\|$. *Then, we have*

$$\lim_{k\to\infty} \|X(k) - X(l_k)\| = 0$$

Proof: The proof has been given in Xie and Guo (2000) for $p=1$ and $q=1$. It is easy to extend the proof when p and q are larger than one and is thus omitted here. □

2.3 *Future states prediction*

By utilising the block-triangular structure property of system (1), future states up to $(k+n_j-1)$ step ahead for subsystem Σ_j are to be predicted at the kth step. To proceed, let us denote the estimates of Θ_{j,i_j} and g_{j,i_j} at the kth step as $\hat{\Theta}_{j,i_j}(k)$ and $\hat{g}_{j,i_j}(k)$, respectively, and $\tilde{\Theta}_{j,i_j}(k) = \hat{\Theta}_{j,i_j}(k) - \Theta_{j,i_j}$ and $\tilde{g}_{j,i_j}(k) = \hat{g}_{j,i_j}(k) - g_{j,i_j}$ as estimate errors. For convenience, the following notations will be used in the later discussion:

$$\Lambda_{j,i_j}(k) = [\hat{\Theta}^T_{j,i_j}(k), \hat{g}_{j,i_j}(k)]^T \in R^{p_{j,i_j}+1},$$
$$\tilde{\Lambda}_{j,i_j}(k) = [\tilde{\Theta}^T_{j,i_j}(k), \tilde{g}_{j,i_j}(k)]^T \in R^{p_{j,i_j}+1}. \tag{4}$$

Based on the states prediction for SISO system in Ge et al. (2008), we propose the following states prediction for MIMO system (1) as follows. By using the estimates of unknown system parameters, the one-step ahead future states of subsystem Σ_j can be straightforwardly predicted in the following manner:

$$\hat{\xi}_{j,i_j}(k+1|k) = \Lambda^T_{j,i_j}(k-n_j+2)\Psi_{j,i_j}(k),$$
$$i_j = 1,2,\ldots,n_j-1, \; j=1,2,\ldots,n,$$
$$\Psi_{j,i_j}(k) = [\Phi^T_{j,i_j}(\bar{\xi}_{1,i_j-m_{j1}}(k),\ldots,\bar{\xi}_{j,i_j}(k),\ldots,\bar{\xi}_{n,i_j-m_{jn}}(k)),$$
$$\xi_{j,i_j+1}(k)]^T, \tag{5}$$

where $\xi_{j,i_j}(k+1|k)$ is one-step ahead prediction of $\xi_{j,i_j}(k)$ at step k. Similarly, in the following, $\xi_{j,i_j}(k+l|k)$ is l step ahead prediction of $\xi_{j,i_j}(k)$ at step k. It is noted that the prediction is only proceeded for the first (n_j-1) states because the n_jth state $\xi_{j,n_j}(k+1|k)$ involves control input and thus is not predictable at the kth step.

Moving one step ahead in the equations of subsystem Σ_j in (1), we see that the two-step ahead predictions can be constructed by substituting the one-step future states with one-step predicted states. Because there is no prediction for $\bar{\xi}_{j,n_j}(k+1|k)$, the two-step ahead prediction can only be proceeded up to the (n_j-2)th state, i.e. $\bar{\xi}_{j,i_j}(k+2)$, $i_j=1,2,\ldots,n_j-2$. To continue the procedure, the l-step ahead states prediction, $\xi_{j,i_j}(k+l|k)$, $l=2,3,\ldots,n_j-1$, can be constructed in the same manner.

$$\hat{\xi}_{j,i_j}(k+l|k) = \Lambda^T_{j,i_j}(k-n_j+l+1)\hat{\Psi}_{j,i_j}(k+l-1|k),$$
$$i_j = 1,2,\ldots,n_j-l, \tag{6}$$

$$\hat{\Psi}_{j,i_j}(k+l-1|k)$$
$$= [\Phi^T_{j,i_j}(\hat{\bar{\xi}}_{1,i_j-m_{j1}}(k+l-1|k)),\ldots,\hat{\bar{\xi}}_{j,i_j}(k+l-1|k)),\ldots,$$
$$\times \hat{\bar{\xi}}_{n,i_j-m_{jn}}(k+l-1|k)), \; \hat{\xi}_{j,i_j+1}(k+l-1|k)]^T, \tag{7}$$

$$\hat{\bar{\xi}}_{j,i_j}(k+l-1|k)$$
$$= [\hat{\xi}_{j,1}(k+l-1|k), \hat{\xi}_{j,2}(k+l-1|k),\ldots,\hat{\xi}_{j,i_j}(k+l-1|k)]^T. \tag{8}$$

Remark 2.2: Unlike the prediction of SISO system developed in Ge et al. (2008), for MIMO systems, the prediction of future states of subsystem Σ_j involves the predicted future states of other systems. For one-step ahead predicted state vectors of subsystem Σ_j, $\bar{\xi}_{j,i_j}(k+1|k)$, $i_j=1,2,\ldots,n_j-1$, they involve state vectors of subsystem Σ_l, $\bar{\xi}_{l,i_j-m_{jl}}(k)$, $l=1,2,\ldots,n$, and $\xi_{j,i_j+1}(k)$, which are available at kth step. For two-step ahead predicted state vectors $\hat{\bar{\xi}}_{j,i_j}(k+2|k)$, $i_j=1,2,\ldots,n_j-2$, they involve one-step ahead predicted state vectors of subsystem Σ_l, $\hat{\bar{\xi}}_{l,i_j-m_{jl}}(k+1|k)$, $l=1,2,\ldots,n$ and $\hat{\xi}_{j,i_j+1}(k+1|k)$, which are also available at kth step because $i_j - m_{jl} \leq n_l - 2$ and $i_j + 1 \leq n_j - 1$ and for each subsystem the one-step prediction is proceeded up to the (n_l-1)th state. Continuing the analysis, we see that the prediction method developed above is well defined without any noncausal problem.

The parameter estimates are obtained by the following update law:

$$\Lambda_{j,i_j}(k+1) = \Lambda_{j,i_j}(k-n_j+2) - \frac{\tilde{\xi}_{j,i_j}(k+1|k)\Psi_{j,i_j}(k)}{D_{j,i_j}(k)},$$
$$D_{j,i_j}(k) = 1 + \Psi^T_{j,i_j}(k)\Psi_{j,i_j}(k),$$
$$\tilde{\xi}_{j,i_j}(k+1|k) = \hat{\xi}_{j,i_j}(k+1|k) - \xi_{j,i_j}(k+1),$$
$$j = 1,2,\ldots,n, \quad i_j = 1,2,\ldots,n_j-1. \tag{9}$$

Lemma 2.4: *The parameter estimates* $\Lambda_{j,i_j}(k)$, $j=1,2,\ldots,n$, $i_j=1,2,\ldots,n_j-1$, *in* (9) *are bounded and the prediction errors satisfy*

$$\bar{\tilde{\xi}}_{j,i_j}(k+n_j-i_j|k) = \sum_{l=1}^{n} o[O[y_l(k+n_j-m_{jl}-1)]] \quad \text{with}$$
$$\bar{\tilde{\xi}}_{j,i_j}(k+n_j-i_j|k) = \bar{\hat{\xi}}_{j,i_j}(k+n_j-i_j|k) - \bar{\xi}_{j,i_j}(k+n_j-i_j) \quad \text{and}$$
$$\bar{\hat{\xi}}_{j,i_j}(k+n_j-i_j|k) = [\hat{\xi}_{j,1}(k+n_j-1|k), \hat{\xi}_{j,2}(k+n_j-2|k),\ldots, \hat{\xi}_{j,i_j}(k+n_j-i_j|k)]^T. \tag{10}$$

Proof: See Appendix D. □

3. Adaptive control design

Instead of the nested step-by-step backstepping design, in this section, adaptive control will be synthesised directly based on the predicted future states. First, subsystem Σ_j is transformed into the the following compact form by combining all the equations together using iterative substitution. For the first subsystem, we have

$$y_1(k+n_1) = \Theta_1^T \Phi_1(k+n_1-1) + \Theta^{gT}_{1,n_1}\Phi_{1,n_1}(\Xi(k)) + \nu_1(\Xi_{\tau_1}(k)) + g_1u_1(k). \tag{11}$$

Similarly, for subsystems Σ_j, $j=2,3,\ldots,n$, we have

$$y_j(k+n_j) = \Theta_j^T \Phi_j(k+n_j-1) + \Theta^{gT}_{j,n_j}\Phi_{j,n_j}(\Xi(k),\bar{u}_{j-1}(k)) + \nu_j(\Xi_{\tau_j}(k)) + g_ju_j(k), \tag{12}$$

where for all the subsystems Σ_j, $j=1,2,\ldots,n$, we have

$$\Phi_j(k+n_j-1) = [\Phi^T_{j,1}(k+n_j-1), \Phi^T_{j,2}(k+n_j-2),\ldots,\Phi^T_{j,n_j-1}(k)]^T, \tag{13}$$

where $\Phi^T_{j,i_j}(k+n_j-i_j)$, $i_j=1,2,\ldots,n_j-1$, is the abbreviation of $\Phi^T_{j,i_j}(\bar{\xi}_{1,i_j-m_{j1}}(k+n_j-i_j),\ldots, \bar{\xi}_{n_j,i_j-m_{jn}}(k+n_j-i_j))$ and

$$\Theta_j = [\Theta^{gT}_{j,1}, \Theta^{gT}_{j,2},\ldots,\Theta^{gT}_{j,n_j-1}],$$
$$\Theta^g_{j,i_j} = \left(\prod_{l=1}^{i_j-1} g_{j,l}\right)\Theta_{j,i_j}\in R^{p_{j,i_j}}, \quad g_j = \prod_{i_j=1}^{n_j} g_{j,i_j} \ge \underline{g}_j = \prod_{i_j=1}^{n_j}\underline{g}_{j,i_j}, \tag{14}$$

where $\underline{g}_{j,i_j}$ is defined in Assumption 2.2. In the following, we utilise previous states information to compensate for the effect of nonlinear uncertainties $\nu_j(\cdot)$ including states with unknown time delays. To start with, let us introduce the following notations.

The augmented state vector is defined as

$$\bar{\Xi}(k) = [\Xi^T(k-\tau_{\min}),\ldots,\Xi^T(k-\tau_j),\ldots,\Xi^T(k-\tau_{\max})]^T. \tag{15}$$

According to Lemma 2.3, we define

$$l_k = \arg \min_{l\le k-\max_{1\le j\le n}\{n_j\}} \|\bar{\Xi}(k) - \bar{\Xi}(l)\| \tag{16}$$

such that $l_k + n_j \le k$ and

$$\bar{\Xi}(l_k) = [\Xi^T(l_k-\tau_{\min}),\ldots,\Xi^T(l_k-\tau_j),\ldots,\Xi^T(l_k-\tau_{\max})]^T. \tag{17}$$

It should be noted that if $\|\bar{\Xi}(k)-\bar{\Xi}(l_k)\| \to 0$, then $\|\Xi_{\tau_j}(k)-\Xi_{\tau_j}(l_k)\| \to 0$ according to the definition of $\Xi_{\tau_j}(k)$ in (2). This is the key underlying idea to compensate for the effect of state time delays in the nonlinear uncertainties.

In the following part, we use notation $\Phi_{1,n_1}(\Xi(k),\bar{u}_0(k))$ to denote $\Phi_{1,n_1}(\Xi(k))$ for convenience without any confusion. To exploit the input and output data, let us introduce an auxiliary output $y^a_j(k)$ for each subsystem Σ_j, $j=1,2,\ldots,n$, defined as follows:

$$y^a_j(k+n_j-1) = \Theta_j^T\Phi_j(k+n_j-1) + \Theta^{gT}_{j,n_j}\Phi_{j,n_j}(\Xi(k),\bar{u}_{j-1}(k)) + \nu_j(\Xi_{\tau_j}(k)) \tag{18}$$

such that (12) can be rewritten as

$$y_j(k+n_j) = y^a_j(k+n_j-1) + g_ju_j(k), \tag{19}$$

which implies the auxiliary output $y^a_j(k+n_j-1)$ can be directly obtained from inputs $u_j(k)$ and outputs $y_j(k+n_j)$ if g_j is known. In the following part, by utilising the predicted future states and the notation l_k defined in (16), we seek a proper way for prediction of $y^a_j(k+n_j-1)$. First, from (18) and (19), the following equality can be obtained:

$$\begin{aligned} y^a_j(k+n_j-1) &= y^a_j(k+n_j-1) - y^a_j(l_k+n_j-1) + y^a_j(l_k+n_j-1), \\ &= \Theta_j^T[\Phi_j(k+n_j-1) - \Phi_j(l_k+n_j-1)] \\ &\quad + \Theta^{gT}_{j,n_j}[\Phi_{j,n_j}(\Xi(k),\bar{u}_{j-1}(k)) - \Phi_{j,n_j}(\Xi(l_k),\bar{u}_{j-1}(l_k))] \\ &\quad + y_j(l_k+n_j) - g_ju_j(l_k) + \nu_j(\Xi_{\tau_j}(k)) - \nu_j(\Xi_{\tau_j}(l_k)). \end{aligned} \tag{20}$$

Remark 3.1: According to Assumption 2.1, if $\|\Xi_{\tau_j}(k)-\Xi_{\tau_j}(l_k)\|\to 0$, then $\|\nu_j(\Xi_{\tau_j}(k))-\nu_j(\Xi_{\tau_j}(l_k))\|\to 0$, so the effect of the uncertain function $\nu_j(\cdot)$ with time delayed states will be eliminated in (20).

Let us predict $\hat{y}_j^a(k+n_j-1|k)$ based on (20) in a straightforward manner by ignoring the nonlinear uncertainty terms of $\nu_j(\cdot)$ and only dealing with the parametric uncertainty. Denote $\hat{\Theta}_j(k)$, $\hat{\Theta}^g_{j,n_j}(k)$ and $\hat{g}_j(k)$ as the estimates of unknown parameters Θ_j, Θ^g_{j,n_j} and g_j, respectively. Then, let us predict $y_j^a(k+n_j-1)$ as follows:

$$\begin{aligned}\hat{y}_j^a(k+n_j-1|k) &= \hat{\Theta}_j^T(k)[\hat{\Phi}_j(k+n_j-1|k)-\Phi_j(l_k+n_j-1)]\\ &\quad+\hat{\Theta}^{gT}_{j,n_j}(k)[\Phi_{j,n_j}(\Xi(k),\bar{u}_{j-1}(k))-\Phi_{j,n_j}(\Xi(l_k),\bar{u}_{j-1}(l_k))]\\ &\quad+y_j(l_k+n_j)-\hat{g}_j(k)u_j(l_k),\end{aligned} \tag{21}$$

where $l_k\le k-n_j$ is defined in (16) and $\hat{\Phi}_j(k+n_j-1|k)$ is defined as

$$\begin{aligned}&\hat{\Phi}_j(k+n_j-1|k)\\ &\quad=[\hat{\Phi}^T_{j,1}(k+n_j-1|k),\hat{\Phi}^T_{j,2}(k+n_j-2|k),\ldots,\\ &\qquad\hat{\Phi}^T_{j,n_j-1}(k+1|k)]^T,\ \text{with}\\ &\hat{\Phi}_{j,i_j}(k+n_j-i_j|k)\\ &\quad=\Phi_{j,i_j}(\hat{\bar{\xi}}_{1,i_j-m_{j1}}(k+n_j-i_j|k),\ldots,\hat{\bar{\xi}}_{n_j,i_j-m_{jn}}(k+n_j-i_j|k))\end{aligned} \tag{22}$$

for $i_j=1,2,\ldots,n_j-1$ and the predicted future states are obtained from Section 2.3.

At this stage, using the predicted auxiliary output, the control law is ready to be constructed. Based on Equation (19), the adaptive control is designed using certainty equivalence principle as follows:

$$u_j(k)=-\frac{1}{\hat{g}_j(k)}(\hat{y}_j^a(k+n_j-1|k)-y_j^*(k+n_j)), \tag{23}$$

where $\hat{g}_j(k)$ is the estimate of g_j at the kth step.

Next, we are to design a proper parameter estimate law for the adaptive control. Let us consider the following augmented tracking error:

$$\epsilon_j(k)=e_j(k)+\beta_j(k-1), \tag{24}$$

where $e_j(k)$ is output tracking error defined as $e_j(k)=y_j(k)-y_j^*(k)$ and $\beta_j(k)$ can be regarded as a measurement of future states prediction error, which is defined as

$$\beta_j(k)=\hat{\Theta}_j^T(k-n_j+1)[\hat{\Phi}_j(k|k-n_j+1)-\Phi_j(k)]. \tag{25}$$

According to the Lipschitz condition of $\nu_j(\cdot)$ in Assumption 2.1 and the definition of $\bar{\Xi}(k)$ in (15), we have

$$\begin{aligned}&|\nu_j(\Xi_{\tau_j}(k))-\nu_j(\Xi_{\tau_j}(l_k))|\\ &\quad\le\lambda_j\|\Xi_{\tau_j}(k)-\Xi_{\tau_j}(l_k)\|\le\lambda_j\|\bar{\Xi}(k)-\bar{\Xi}(l_k)\|,\end{aligned} \tag{26}$$

where λ_j can be any constant satisfying $L_j^\nu\le\lambda_j<\lambda^*$, with λ^* defined later in (43).

To counter the effect of nonlinear uncertainties $\nu_j(\cdot)$ in parameter estimation, we consider using a deadzone method with the deadzone indicator defined as

$$a_j(k)=\begin{cases}1-\frac{\lambda_j\|\bar{\Xi}(k-n_j)-\bar{\Xi}(l_{k-n_j})\|}{|\epsilon_j(k)|}, & \text{if } |\epsilon_j(k)|>\lambda_j\|\bar{\Xi}(k-n_j)-\bar{\Xi}(l_{k-n_j})\|\\ 0, & \text{otherwise}\end{cases} \tag{27}$$

For convenience, let us define an auxiliary tracking error as $\epsilon_j^a(k)=a_j(k)\epsilon_j(k)$. According to the definition in (27), it is easy to obtain the following inequality:

$$|\epsilon_j(k)|\le|\epsilon_j^a(k)|+\lambda_j\|\bar{\Xi}(k-n_j)-\bar{\Xi}(l_{k-n_j})\|. \tag{28}$$

The estimated parameters in the auxiliary output estimate (21) are obtained from the following update laws, $j=1,2,\ldots,n$ as shown below.

$$\begin{aligned}\hat{\Theta}_j(k)&=\hat{\Theta}_j(k-n_j)+\gamma_j\frac{\epsilon_j^a(k)[\Phi_j(k-1)-\Phi_j(l_{k-n_j}+n_j-1)]}{D_j(k-n_j)},\\ \hat{\Theta}^g_{j,n_j}(k)&=\hat{\Theta}^g_{j,n_j}(k-n_j)+\gamma_j\frac{\epsilon_j^a(k)[\Phi_{j,n_j}(k-n_j)-\Phi_{j,n_j}(l_{k-n_j})]}{D_j(k-n_j)},\\ \hat{g}_j(k)&=\begin{cases}\hat{g}_j'(k), & \text{if } \hat{g}_j'(k)>\underline{g}_j\\ \underline{g}_j, & \text{otherwise}\end{cases},\\ \hat{g}_j'(k)&=\hat{g}_j(k-n_j)+\frac{\gamma_j\epsilon_j^a(k)}{D_j(k-n_j)}[u_j(k-n_j)-u_j(l_{k-n_j})],\\ D_j(k-n_j)&=1+\|\Phi_j(k-1)-\Phi_j(l_{k-n_j}+n_j-1)\|^2\\ &\quad+\|\bar{\Xi}(k-n_j)-\bar{\Xi}(l_{k-n_j})\|^2\\ &\quad+\|\Phi_{j,n_j}(k-n_j)-\Phi_{j,n_j}(l_{k-n_j})\|^2\\ &\quad+[u_j(k-n_j)-u_j(l_{k-n_j})]^2,\end{aligned} \tag{29}$$

where $\Phi_{j,n_j}(k)$ is used to denote $\Phi_{j,n_j}(\Xi(k),\bar{u}_j(k))$ and $0<\gamma_j<2$. It is noted that $\hat{g}_j(k)$ obtained from (29) is guaranteed to be bounded away from zero such that the adaptive control defined in (23) is free of singularity problem.

By now, the adaptive control has been accomplished. To clearly demonstrate the whole structure of the developed adaptive control law, a diagram is presented in Figure 1. In addition, a control design

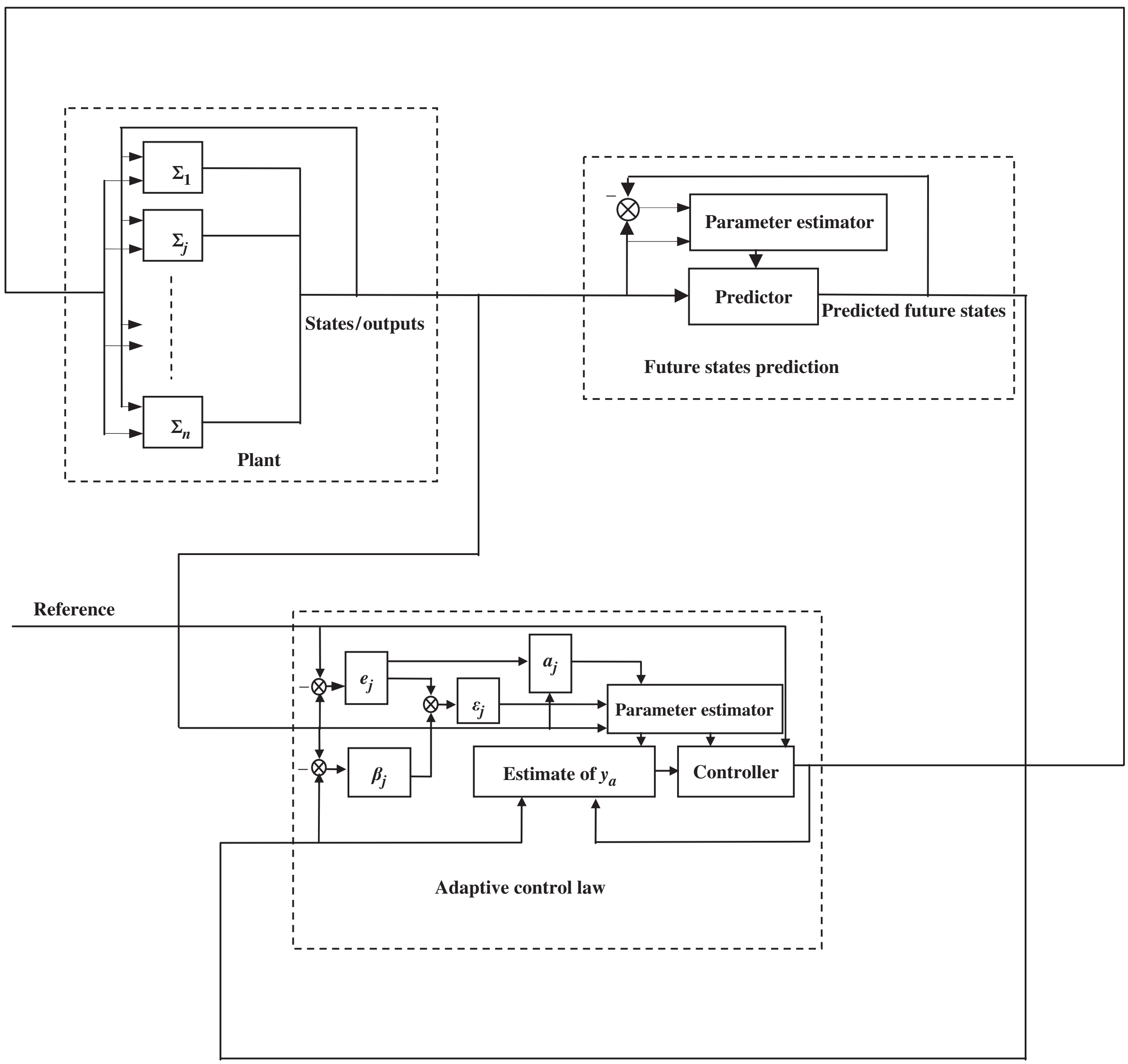

Figure 1. Adaptive control diagram.

example will be provided in Section 5 for better illustration of design details to the readers.

Remark 3.2: By employing the predicted future states of each subsystem, adaptive control design has been constructed with compensation of the uncertain couplings among each subsystem, such that asymptotical tracking performance can be achieved. The key technique of the compensation scheme lies in the time index l_k introduced in Lemma 2.3 and deadzone introduced in (27). As shown in next section, the designed controller will make $\epsilon_j^a(k) \to 0$ and guarantee boundedness of states such that $\|\bar{\Xi}(k-n_j) - \bar{\Xi}(l_{k-n_j})\|$ will converge to zero according to Lemma 2.3. Therefore, according to the inequality in (28) one can see that $\epsilon_j(k) \to 0$ and it will lead to, as shown later, tracking errors $e_j(k) \to 0$.

4. Control performance analysis

We are ready to present the main result in this article.

Theorem 4.1: *Consider the whole closed-loop adaptive system that combines all the n-coupled closed-loop subsystems, with each closed-loop subsystem consisting of subsystem* Σ_j *described in* (1), *adaptive control input* (23) *and parameter update law* (29). *All the signals* *in the whole closed-loop adaptive system are bounded. Furthermore, the output of each subsystem* Σ_j *asymptotically tracks the desired reference trajectory* $y_j^*(k)$, $j=1,2,\ldots,n$.

Proof: For the analysis of the whole adaptive closed-loop system, we start from the dynamics of the error signals. In the following, we use $\Phi_{j,n_j}(k)$ to denote $\Phi_{j,n_j}(\Xi(k),\bar{u}_j(k))$ for convenience. By comparing (20) and (21), the prediction error of the auxiliary output $y_j^a(k+n_j-1)$ can be written as

$$\begin{aligned}\tilde{y}_j^a(k+n_j-1|k)&=\hat{y}_j^a(k+n_j-1|k)-y_j^a(k+n_j-1)\\&=\tilde{\Theta}_j^T(k)[\Phi_j(k+n_j-1)-\Phi_j(l_k+n_j-1)]\\&\quad+\tilde{\Theta}_{j,n_j}^{gT}(k)[\Phi_{j,n_j}(k)-\Phi_{j,n_j}(l_k)]\\&\quad-[\nu_j(\Xi_{\tau_j}(k))-\nu_j(\Xi_{\tau_j}(l_k))]+\beta_j(k+n_j-1)\\&\quad-\tilde{g}_j(k)u_j(l_k),\end{aligned}\quad(30)$$

where $\tilde{\Theta}_j(k)=\hat{\Theta}_j(k)-\Theta_j$, $\tilde{\Theta}_{j,n_j}^g(k)=\hat{\Theta}_{j,n_j}^g(k)-\Theta_{j,n_j}^g(k)$, $\tilde{g}_j(k)=\hat{g}_j(k)-g_j$.

Now, by combining (19), (23) and (30), the output tracking error can be written as

$$\begin{aligned}e_j(k+n_j)&=y_j^a(k+n_j-1)+\hat{g}_j(k)u(k)-\tilde{g}_j(k)u_j(k)\\&\quad-y_j^*(k+n_j)\\&=-\tilde{y}_j^a(k+n-1|k)-\tilde{g}_j(k)u_j(k),\end{aligned}\quad(31)$$

which according to (24) immediately leads to

$$\begin{aligned}\epsilon_j(k)&=-\tilde{\Theta}_j^T(k-n_j)[\Phi_j(k-1)-\Phi_j(l_{k-n_j}+n_j-1)]\\&\quad-\tilde{\Theta}_{j,n_j}^{gT}(k-n_j)[\Phi_{j,n_j}(k-n_j)-\Phi_{j,n_j}(l_{k-n_j})]\\&\quad-\tilde{g}_j(k-n_j)[u_j(k-n_j)-u_j(l_{k-n_j})]\\&\quad+\nu_j(\Xi_{\tau_j}(k-n_j))-\nu_j(\Xi_{\tau_j}(l_{k-n_j})).\end{aligned}\quad(32)$$

Then, we consider a Lyapunov function candidate

$$\begin{aligned}V_j(k)&=\sum_{l=1}^{n_j}\|\tilde{\Theta}_j(k-n_j+l)\|^2+\sum_{l=1}^{n_j}\|\tilde{\Theta}_{j,n_j}^g(k-n_j+l)\|^2\\&\quad+\sum_{l=1}^{n_j}\tilde{g}_j^2(k-n_j+l).\end{aligned}$$

Since $\tilde{g}_j'^2(k)\geq\tilde{g}_j^2(k)$ from (29), following a similar procedure in Ge et al. (2009) it is easy to show that

$$\begin{aligned}\Delta V_j(k)&=V_j(k)-V_j(k-1)\\&=\|\tilde{\Theta}_j(k)\|^2-\|\tilde{\Theta}_j(k-n_j)\|^2+\|\tilde{\Theta}_{j,n_j}^g(k)\|^2\\&\quad-\|\tilde{\Theta}_{j,n_j}^g(k-n_j)\|^2+\tilde{g}_j'^2(k)-\tilde{g}_j^2(k-n_j)\\&\leq\frac{\gamma_j^2\epsilon_j^{a2}(k)}{D_j(k-n_j)}-\frac{2\gamma_j\epsilon_j^{a2}(k)}{D_j(k-n_j)}\\&=-\frac{\gamma_j(2-\gamma_j)\epsilon_j^{a2}(k)}{D_j(k-n_j)}.\end{aligned}\quad(33)$$

Noting that $0<\gamma_j<2$ in (29), we can conclude from (33) that $\Delta V_j(k)$ is non-positive, such that the boundedness of $V_j(k)$ is obvious, and immediately the boundedness of $\hat{\Theta}_j(k)$, $\hat{\Theta}_{j,n_j}^g(k)$ and $\hat{g}_j(k)$ is guaranteed. Furthermore, the following can be obtained from (33):

$$\lim_{k\to\infty}\frac{\epsilon_j^{a2}(k)}{D_j(k-n_j)}=0,\quad\text{or}\quad\epsilon_j^a(k)=o\Big[D_j^{\frac{1}{2}}(k-n_j)\Big].\quad(34)$$

It is important to rank the growth rates of signals with respect to each other in the adaptive closed-loop systems, such that we can establish the boundedness of all the signals later. First, let us consider $\beta_j(k)$ defined in (25). Due to the boundedness of $\hat{\Theta}_j(k)$ proved above, there exists a constant C_{β_j} such that

$$\begin{aligned}|\beta_j(k+n_j-1)|&\leq C_{\beta_j}\|\hat{\Phi}_j(k+n_j-1|k)-\Phi_j(k+n_j-1)\|\\&=\sum_{l=1}^{n}o[O[y_l(k+n_j-m_{jl}-1)]],\end{aligned}\quad(35)$$

where Lemma 2.4 and Assumption 2.1 are used to establish the equality. Having $y_j(k)\sim e_j(k)$ because $y_j^*(k)$ is bounded, we are ready to show that

$$\beta_j(k+n_j-1)=\sum_{l=1}^{n}o[O[e_l(k+n_j-m_{jl}-1)]],\quad(36)$$

which, together with the definition of augmented error in (24), implies that

$$|e_j(k+n_j-1)|\sim|\epsilon_j(k+n_j-1)|+\sum_{l=1}^{n}o[O[e_l(k+n_l-2)]].\quad(37)$$

Taking summation on both hand sides of (37) and using Proposition 2.1, we have

$$\sum_{j=1}^{n}|e_j(k+n_j-1)|\sim\sum_{j=1}^{n}|\epsilon_j(k+n_j-1)|.\quad(38)$$

From Lemma 2.1, it is easy to derive

$$\begin{aligned}\sum_{j=1}^{n}O[\bar{\xi}_{j,n_j}(k)]&\sim\sum_{j=1}^{n}O[y_j(k+n_j-1)]\\&\sim\sum_{l=1}^{n}O[\bar{\xi}_{l,i_j-m_{jl}}(k+n_j-i_j)],\end{aligned}\quad(39)$$

which together with (38), $y_j(k)\sim e_j(k)$ and Proposition 2.1 leads to

$$\begin{aligned}\max_{k'\leq k}\|\Xi(k')\|&\leq\sum_{j=1}^{n}\max_{k'\leq k}\|\bar{\xi}_{j,n_j}(k')\|\\&=\sum_{j=1}^{n}O[e_j(k+n_j-1)]\\&=\sum_{j=1}^{n}O[\epsilon_j(k+n_j-1)].\end{aligned}\quad(40)$$

Then, we have

$$\max_{k'\le k}\|\bar{\Xi}(k')\| \le \sum_{\tau=\tau_{\min}}^{\tau_{\max}} \max_{k'\le k}\|\Xi(k'-\tau)\| = \sum_{j=1}^{n} O[\epsilon_j(k+n_j-1)], \quad (41)$$

which asserts the existence of constants $C_{j,1}$ and $C_{j,2}$ such that

$$\begin{aligned}\max_{k'\le k}\|\bar{\Xi}(k')\| &\le \sum_{j=1}^{n}\left\{C_{j,1}\max_{k'\le k+n_j-1}\{|\epsilon_j(k')|\}+C_{j,2}\right\}\\ &\le \sum_{j=1}^{n} C_{j,1}\max_{k'\le k+n_j-1}|\epsilon_j^a(k')|\\ &\quad+\sum_{j=1}^{n} C_{j,1}\lambda_j\max_{k'\le k-1}\|\bar{\Xi}(k')-\bar{\Xi}(l_{k'})\|\\ &\quad+\sum_{j=1}^{n} C_{j,2},\end{aligned} \quad (42)$$

where the last inequality is established in (28). This together with $\max_{k'\le k-1}\|\bar{\Xi}(k')-\bar{\Xi}(l_{k'})\|\le 2\max_{k'\le k}\|\bar{\Xi}(k')\|$, implies that there exists a constant

$$\lambda^* = 1\Big/\sum_{j=1}^{n}(2C_{j,1}) \quad (43)$$

such that $\forall \lambda_j < \lambda^*$, $j=1,2,\ldots,n$, we have

$$\max_{k'\le k}\|\bar{\Xi}(k')\| \le \frac{\sum_{j=1}^{n} C_{j,1}}{\sum_{j=1}^{n}(2C_{j,1})}\max_{k'\le k+n_j-1}\epsilon_j^a(k') + \frac{\sum_{j=1}^{n} C_{j,2}}{1-\sum_{j=1}^{n}\lambda_j C_{j,1}}, \quad (44)$$

which leads to

$$\Xi(k-n_l+1) = O[\bar{\Xi}(k-n_l+1)] = \sum_{j=1}^{n} O[\epsilon_j^a(k+n_j-n_l)]. \quad (45)$$

From definition of $\Phi_j(k+n_j-1)$ in (13), we derive the following equation according to Lemma 2.1, Equation (39) and Lipschitz condition of $\Phi_j(\cdot)$:

$$\Phi_j(k+n_j-1) = \sum_{j=1}^{n} O[\bar{\xi}_{j,n_j}(k)] = O[\Xi(k)] = O[\bar{\Xi}(k)]. \quad (46)$$

From Lemma 2.1 we also have $u_j(k-n_j)=O[\Xi(k-n_j+1)]$. According to the definition of $D_j(k-n_j)$ in (29), we have

$$\begin{aligned}D_j^{\frac{1}{2}}(k-n_j) &\le 1+\|\Phi_j(k-1)-\Phi_j(l_{k-n_j}+n_j-1)\|\\ &\quad+\|\bar{\Xi}(k-n_j)-\bar{\Xi}(l_{k-n_j})\|\\ &\quad+L_{j,n_j}\|\Xi(k)-\Xi(l_k)\|\\ &\quad+L_{j,n_j}\|\bar{u}_{j-1}(k)-\bar{u}_{j-1}(l_k)\|\\ &\quad+|u_j(k-n_j)-u_j(l_{k-n_j})|\\ &= O[\bar{\Xi}(k-n_j)]+O[\bar{\Xi}(k-n_j+1)]\\ &\sim O[\bar{\Xi}(k-n_j+1)].\end{aligned} \quad (47)$$

From (34), (45) and (47), we have the following equalities:

$$\epsilon_j^a(k) = o[O[\Xi(k-n_j+1)]] = \sum_{l=1}^{n} o[\epsilon_j^a(k-m_{jl})]+o[1]. \quad (48)$$

Applying Lemma 2.2 to Equation (48), we have $\lim_{k\to\infty}\epsilon_j^a(k)=0$, which combined with (45) implies the boundedness of states vectors, $\Xi(k)$ and $\bar{\Xi}(k)$. Using Lemma 2.3, we have $\lim_{k\to\infty}\|\bar{\Xi}(k)-\bar{\Xi}(l_k)\|=0$, then we obtain $\lim_{k\to\infty}\epsilon_j(k)=0$ from (28). It, together with (38), leads to $\lim_{k\to\infty}e_j(k)=0$. Then, the boundedness of outputs $y_j(k)$ is established. According to Lemma 2.1, the boundedness of inputs $u_j(k)$ of all the subsystems are guaranteed. □

5. Simulation results

In this section, to better illustrate the adaptive control design procedure to the readers, an example of control design based on the following system will be presented. Thereafter, simulation studies are also carried out on the system.

$$\Sigma:\left\{\begin{array}{ll}\Sigma_1 & \left\{\begin{array}{l}\xi_{1,1}(k+1)=\Theta_{1,1}^T\Phi_{1,1}(\xi_{1,1}(k))+0.2\xi_{1,2}(k)\\ \xi_{1,2}(k+1)=\Theta_{1,2}^T\Phi_{1,2}(\bar{\xi}_{1,2}(k),\xi_{2,1}(k))+0.8\xi_{1,3}(k)\\ \xi_{1,3}(k+1)=\Theta_{1,3}^T\Phi_{1,3}(\bar{\xi}_{1,3}(k),\bar{\xi}_{2,2}(k),\xi_{3,1}(k))+u_1(k)+\nu_1(\Xi_{\tau_1}(k))\\ y_1(k)=\xi_{1,1}(k)\end{array}\right.\\ \Sigma_2 & \left\{\begin{array}{l}\xi_{2,1}(k+1)=\Theta_{2,1}^T\Phi_{2,1}(\bar{\xi}_{1,2}(k),\xi_{2,1}(k))+0.3\xi_{2,2}(k)\\ \xi_{2,2}(k+1)=\Theta_{2,2}^T\Phi_{2,2}(\bar{\xi}_{1,3}(k),\bar{\xi}_{2,2}(k),\xi_{3,1}(k),u_1(k))\\ \qquad\qquad +1.2u_2(k)+\nu_2(\Xi_{\tau_2}(k))\\ y_2(k)=\xi_{2,1}(k)\end{array}\right.\\ \Sigma_3 & \left\{\begin{array}{l}\xi_{3,1}(k+1)=\Theta_{3,1}^T\Phi_{3,1}(\bar{\xi}_{1,3}(k),\bar{\xi}_{2,2}(k),\xi_{3,1}(k),u_1(k),u_2(k))\\ \qquad\qquad +u_3(k)+\nu_3(\Xi_{\tau_3}(k))\\ y_3(k)=\xi_{3,1}(k),\end{array}\right.\end{array}\right. \quad (49)$$

where

$$\Theta_{1,1} = 0.2, \quad \Theta_{1,2}^T = [0.1, 0.3],$$
$$\Theta_{1,3}^T = [0.5, 0.4],$$
$$\Theta_{2,1}^T = [0.2, 0.01], \quad \Theta_{2,2}^T = [0.05, 0.1],$$
$$\Theta_{3,1} = 0.1, \quad \Phi_{1,1}(\cdot) = \xi_{1,1}(k)\cos(\xi_{1,1}(k)),$$
$$\Phi_{1,2}^T(\bar{\xi}_{1,2}(k), \xi_{2,1}(k)) = \left[0, \frac{1}{1+\xi_{1,2}^2(k)}\right],$$

and

$$\Phi_{1,3}^T(\cdot) = \left[\frac{\xi_{1,1}(k)\xi_{1,2}(k)}{1+\xi_{1,1}^2(k)+\xi_{1,3}^2(k)}, \frac{\xi_{2,2}(k)}{2+\xi_{2,1}^2(k)}\right],$$
$$\Phi_{2,1}^T(\cdot) = [0, \xi_{1,2}(k)\sin(\xi_{2,1}(k))],$$
$$\Phi_{2,2}^T(\cdot) = \left[\frac{u_1(k)\xi_{2,2}(k)(1+e^{-\xi_{1,1}^2(k)})}{1+\xi_{2,1}^2(k)+\xi_{2,1}^2(k)}, \sin(\xi_{1,3}(k))\xi_{3,1}(k)\right],$$
$$\Phi_{3,1}(\cdot) = \frac{\sin(u_2(k))\xi_{3,1}(k)}{1+\xi_{1,2}^2(k)},$$
$$\nu_1(\Xi_{\tau_1}(k)) = 0.01\cos(\xi_{1,1}(k-2))\xi_{1,3}(k-2) + 0.03\xi_{2,1}(k-1)\log(1+\xi_{3,1}^2(k-1)),$$
$$\nu_2(\Xi_{\tau_2}(k)) = 0.3(\xi_{1,1}(k-1)+\xi_{1,2}(k-1)) + 0.1\cos(\xi_{2,1}(k-2)\xi_{2,2}(k-2)),$$
$$\nu_3(\Xi_{\tau_3}(k)) = 0.01\cos(\xi_{2,1}(k-1))\xi_{2,2}(k-2).$$

We see from the above system (49) that in every equation of each subsystem there are states from the other subsystem and there are uncertain coupling terms in the last equations. In the following, the details of the control design for this system are given.

5.1 *Control design illustration*

Note that there are three subsystems in (49) with orders $n_1 = 3$, $n_2 = 2$ and $n_3 = 1$. Thus, future states prediction will be carried out only for subsystems Σ_1 and Σ_2 as follows:

$$\hat{\xi}_{1,1}(k+1|k) = \hat{\bar{\Theta}}_{1,1}^T(k-1)\Psi_{1,1}(k),$$
$$\hat{\xi}_{1,1}(k+2|k) = \hat{\bar{\Theta}}_{1,1}^T(k)\hat{\Psi}_{1,1}(k+1|k),$$
$$\hat{\xi}_{1,2}(k+1|k) = \hat{\bar{\Theta}}_{1,2}^T(k-1)\Psi_{1,2}(k),$$
$$\hat{\xi}_{2,1}(k+1|k) = \hat{\bar{\Theta}}_{2,1}^T(k)\Psi_{2,1}(k),$$

where

$$\Psi_{1,1}(k) = [\Phi_{1,1}^T(\xi_{1,1}(k)), \xi_{1,2}(k)]^T,$$
$$\hat{\Psi}_{1,1}(k+1|k) = [\hat{\Phi}_{1,1}^T(\hat{\xi}_{1,1}(k+1|k)), \hat{\xi}_{1,2}(k+1|k)]^T,$$
$$\Psi_{1,2}(k) = [\Phi_{1,2}(\bar{\xi}_{1,2}(k), \xi_{2,1}(k)), \xi_{1,3}(k)]^T,$$
$$\Psi_{2,1}(k) = [\Phi_{2,1}^T(\bar{\xi}_{1,2}(k), \xi_{2,1}(k)), \xi_{2,2}(k)]^T,$$

and the estimated parameters are obtained from the following update law according to (9):

$$\hat{\bar{\Theta}}_{1,1}(k) = \hat{\bar{\Theta}}_{1,1}(k-2) + (\xi_{1,1}(k) - \hat{\xi}_{1,1}(k|k-1)) \times \Psi_{1,1}(k-1)/(1 + \Psi_{1,1}^T(k-1)\Psi_{1,1}(k-1)),$$
$$\hat{\bar{\Theta}}_{1,2}(k) = \hat{\bar{\Theta}}_{1,2}(k-2) + (\xi_{1,2}(k) - \hat{\xi}_{1,2}(k|k-1)) \times \Psi_{1,2}(k-1)/(1 + \Psi_{1,2}^T(k-1)\Psi_{1,2}(k-1)),$$
$$\hat{\bar{\Theta}}_{2,1}(k) = \hat{\bar{\Theta}}_{2,1}(k-1) + (\xi_{2,1}(k) - \hat{\xi}_{2,1}(k|k-1)) \times \Psi_{2,1}(k-1)/(1 + \Psi_{2,1}^T(k-1)\Psi_{2,1}(k-1)).$$

For subsystem Σ_1, according to (20) and (23), the adaptive control law using predicted future states is given as below:

$$u_1(k) = \frac{1}{\hat{g}_1(k)}(y_1^*(k+3) - (\hat{\Theta}_1^T(k) \times [\hat{\Phi}_1(k+2|k) - \Phi_1(l_k+2)] + \hat{\Theta}_{1,3}^{gT}(k)[\Phi_{1,3}(\Xi(k)) - \Phi_{1,3}(\Xi(l_k))] + y_1(l_k+3) - \hat{g}_1(k)u_1(l_k))), \quad (50)$$

where

$$\Phi_1(k) = [\Phi_{1,1}^T(\xi_{1,1}(k)), \Phi_{1,2}^T(\bar{\xi}_{1,2}(k), \xi_{2,1}(k))]^T,$$
$$\hat{\Phi}_1(k+2|k) = [\Phi_{1,1}^T(\hat{\xi}_{1,1}(k+2|k)), \Phi_{1,2}^T(\bar{\xi}_{1,2}(k+1|k), \xi_{2,1}(k+1|k)]^T \quad (51)$$

and the estimated parameters $\hat{g}_1(k)$, $\hat{\Theta}_1(k)$ and $\hat{\Theta}_{1,3}^g(k)$ are obtained from the update law (29), in which the following information is used:

$$e_1(k) = \xi_{1,1}(k) - y_1^*(k), \quad \epsilon_1(k) = e_1(k) + \beta_1(k-1),$$
$$\beta_1(k-1) = \hat{\Theta}_1^T(k-3)(\hat{\Phi}_1(k-1|k-3) - \Phi_1(k-1)).$$

Similarly, the control input for subsystem Σ_2 is given as follows:

$$u_2(k) = \frac{1}{\hat{g}_2(k)}(y_2^*(k+2) - \hat{\Theta}_2^T(k)[\hat{\Phi}_2(k+1|k) - \Phi_2(l_k+1)] - \hat{\Theta}_{2,2}^{gT}(k)[\Phi_{2,2}(\Xi(k), u_1(k)) - \Phi_{2,2}(\Xi(l_k), u_1(k))] - y_1((l_k+2) + \hat{g}_2(k)u_2(l_k))), \quad (52)$$

where $\Phi_2(k) = \Phi_{2,1}(\bar{\xi}_{1,2}(k), \xi_{2,1}(k))$, $\hat{\Phi}_2(k+1|k) = \Phi_{2,1}(\bar{\xi}_{1,2}(k+1|k), \xi_{2,1}(k+1|k))$ and the estimated parameters $\hat{g}_2(k)$, $\hat{\Theta}_2(k)$ and $\hat{\Theta}_{2,2}^g(k)$ are obtained from the update law (29) based on the the following information:

$$e_2(k) = \xi_{2,1}(k) - y_2^*(k), \quad \epsilon_2(k) = e_2(k) + \beta_2(k-1),$$
$$\beta_2(k-1) = \hat{\Theta}_2^T(k-2)(\hat{\Phi}_2(k-1|k-2) - \Phi_2(k-1)).$$

As subsystem Σ_3 is only a first order system, there are no predicted future states. Accordingly, the control

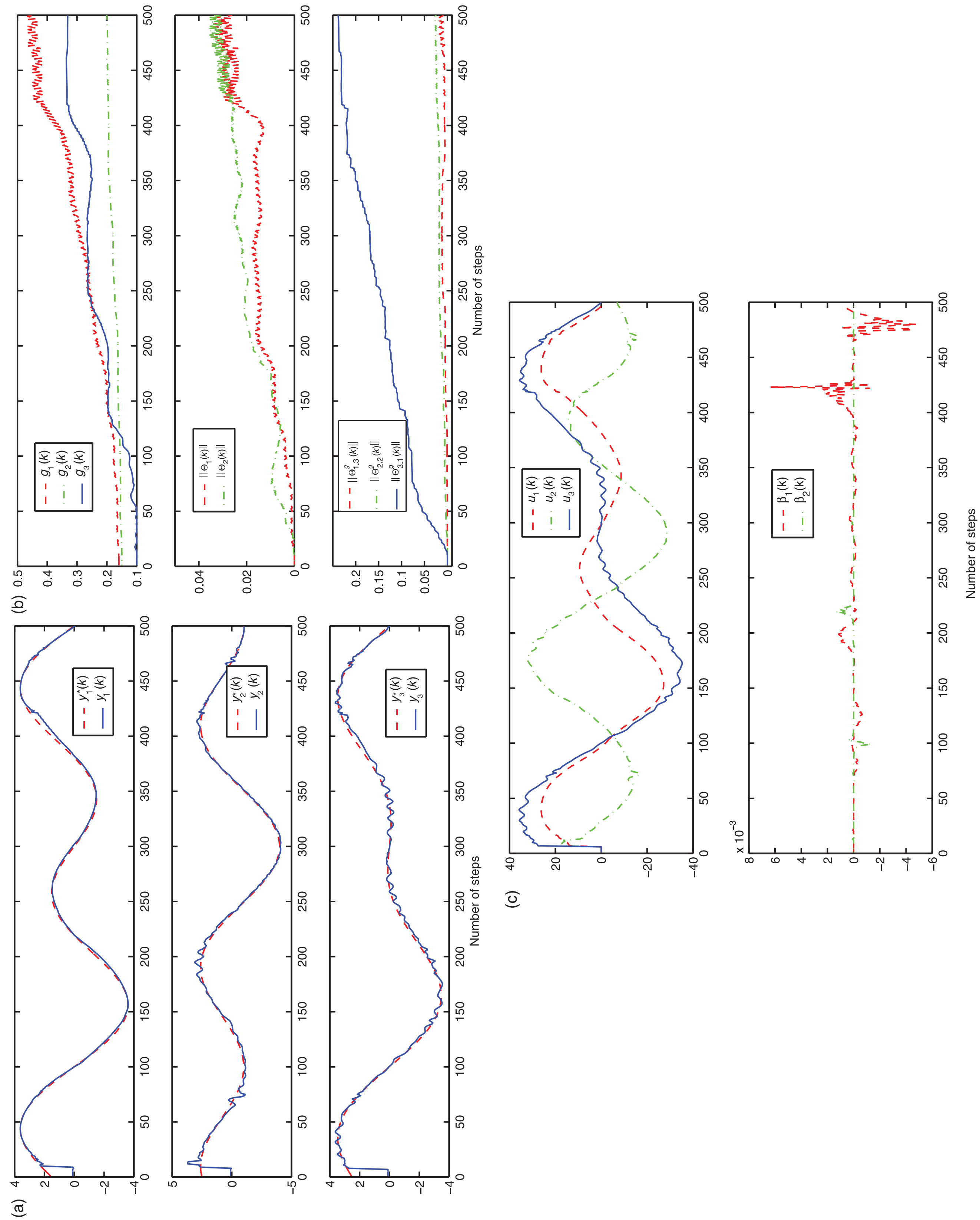

Figure 2. Simulation results. (a) System outputs and reference trajectories; (b) parameter estimates; (c) control signals and prediction error terms.

input is given as

$$u_3(k) = \frac{1}{\hat{g}_3(k)}\big(y_3^*(k+1) - \hat{\Theta}_{3,1}^{gT}(k)[\Phi_{3,1}(\Xi(k), \bar{u}_2(k)) - \Phi_{3,1}(\Xi(l_k), \bar{u}_2(k))] - y_3(l_k+1) + \hat{g}_3(k)u_3(l_k)\big), \quad (53)$$

where estimated parameters $\hat{g}_3(k)$ and $\hat{\Theta}_{3,1}^{g}(k)$ are obtained from (29) with

$$e_3(k) = \xi_{3,1}(k) - y_3^*, \quad \beta_3(k-1) = 0, \quad \epsilon_3(k) = e_3(k). \quad (54)$$

5.2 *Simulation results*

Now let us proceed to the simulation study of system (49), the desired reference trajectories are chosen as

$$y_1^*(k) = 2.5\sin\left(\frac{\pi}{2}kT\right) + 1.5\cos\left(\frac{\pi}{4}kT\right),$$
$$y_2^*(k) = 2.5\cos\left(\frac{\pi}{2}kT\right) + 1.5\sin\left(\frac{\pi}{4}kT\right) \text{ and} \quad (55)$$
$$y_3^*(k) = 1.5\cos\left(\frac{\pi}{2}kT\right) + 2.5\sin\left(\frac{\pi}{4}kT\right)$$

for each subsystem respectively, where $T = 0.02$. The initial system states are chosen as

$$\Xi(0) = [0.1, 0.1, 0.1, 0.1, 0.1, 0.1]^T.$$

The control parameters are chosen as $\underline{g}_1 = 0.16$, $\underline{g}_2 = 0.15$, $\underline{g}_3 = 0.1$, $\gamma_1 = 0.2$, $\gamma_2 = 0.03$, $\gamma_3 = 0.05$, and $\lambda_1 = 0.001$, $\lambda_2 = 0.001$, $\lambda_3 = 0.001$.

The simulation results are presented in Figure 2(a)–(c). The asymptotic tracking performances are shown in Figure 2(a). The boundedness of estimated parameters is shown in Figure 2(b). The boundedness of control signals and prediction errors is presented in Figure 2(c).

6. Conclusion

In this article, adaptive control has been investigated for a class of discrete-time MIMO nonlinear system with uncertain couplings of interconnected inputs and states carrying unknown time delays. For each subsystem, an auxiliary output and its estimate have been introduced in the adaptive control law to compensate for the nonlinear uncertainties. A novel deadzone method has been designed to make the parameter update law robust. Under the proposed adaptive control law, the outputs asymptotically track the desired reference trajectories and all other signals in the whole closed-loop system remain bounded.

Acknowledgements

The authors thank Dr Hongbin Ma for his helpful comments. This work was partially supported by the National Nature Science Foundation of China under Grants 61004059 and 60904086.

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Appendix A: Proof of Proposition 2.1

Proof: Only proof of properties (ii) and (v) are given below. Proofs of other properties are easy and are thus omitted here.

(ii) From Definition 2.1, we see that $\|o[x(k)]\| \leq \alpha(k)\max_{k'\leq k+\tau}\|x(k')\|\quad \forall k>k_0,\quad \tau\geq 0,$ where $\lim_{k\to\infty}\alpha(k)\to 0$. It implies that there exist constants k_1 and $\bar{\alpha}_1$ such that $\alpha(k)\leq\bar{\alpha}_1<1\ \forall k>k_1$. Then, we have

$$\begin{aligned}\|x(k+\tau)+o[x(k)]\| &\leq \|x(k+\tau)\|+\|o[x(k)]\|\\ &\leq (1+\bar{\alpha}_1)\max_{k'\leq k+\tau}\|x(k')\| \quad \forall k>k_1,\end{aligned}$$

which leads to $x(k+\tau)+o[x(k)]=O[x(k+\tau)]$. On the other hand, we have

$$\begin{aligned}\max_{k_1<k'\leq k+\tau}\|x(k')\| &\leq \left\|\max_{k_1<k'\leq k+\tau}x(k')+o[x(k)]\right\|+\|o[x(k)]\|\\ &\leq \left\|\max_{k_1<k'\leq k+\tau}x(k')+o[x(k)]\right\|\\ &\quad+\bar{\alpha}_1\max_{k_1<k'\leq k+\tau}\{\|x(k')\}\end{aligned}$$

and

$$\max_{k_1<k'\leq k+\tau}\|x(k')\|\leq\frac{1}{1-\bar{\alpha}_1}\left\|\max_{k_1<k'\leq k}x(k')+o[x(k')]\right\|\quad\forall k>k_1,$$

which implies $x(k+\tau)=O[x(k)+o[x(k)]]$. Then, it is obvious that $x(k+\tau)+o[x(k)]\sim x(k)$.

(v) Denote $x_2(k)=x_1(k)+o[x_1(k)]$ and let us suppose that $x_1(k)$ is unbounded and define $i_k=\arg\max_{i\leq k}\|x_1(i)\|$. Then, it is easy to see that $i_k\to\infty$ as $k\to\infty$. Due to $\lim_{k\to\infty}\alpha(k)\to 0$, there exist a k_2 such that $\alpha(i_k)\leq\frac{1}{2}$ and $\|o[x_1(k)]\|\leq\frac{1}{2}\max_{k'\leq k}\|x_1(k')\|\quad\forall k>k_2$. Considering $x_2(k)=x_1(k)+o[x_1(k)]$, we have $\|x_2(i_k)\|=\|x_1(i_k)+o[x_1(i_k)]\|\geq\|x_1(i_k)\|-\|o[x_1(i_k)]\|\geq\frac{1}{2}\|x_1(i_k)\|\ \forall k>k_2$ which leads to $\|x_1(i_k)\|\leq 2\|x_2(i_k)\|\ \forall k\geq k_2$. Then, the unboundedness of $x_1(k)$ conflicts with $\lim_{k\to\infty}\|x_2(k)\|=0$. Therefore, $x_1(k)$ must be bounded. Considering that $\alpha(k)\to 0$, we have

$$\begin{aligned}0 &\leq \|x_1(k)\|\leq\|x_1(k)+o[x_1(k)]\|+\|o[x_1(k)]\|\\ &\leq\|x_2(k)\|+\alpha(k)\max_{k'\leq k}\|x_1(k')\|\to 0,\end{aligned}$$

which implies that $\lim_{k\to\infty}\|x_1(k)\|=0$. □

Appendix B: Proof of Lemma 2.1

Proof: Denote $\bar{n}=\max_{1\leq j\leq n}\{n_j\}$ and define $s_i=\{l\,|\,n_l=\bar{n}+1-i\}$, $i=1,2,\ldots,\bar{n}$, such that all the subsystems Σ_l, $l=1,2,\ldots,n$, are divided into $\bar{n}$ groups, with each group defined by a set $S_i=\{\Sigma_l|l\in s_i\}$, $1\leq i\leq\bar{n}$. The set S_i may be empty if there is no subsystem of order $(\bar{n}+1-i)$.

Step 1: Consider the first equations of subsystems $\Sigma_{j_1}\in S_1$ $(j_1\in s_1)$, i.e. $i_{j_1}=1$. Because $i_{j_1}-m_{j_1 l}=1+n_l-\bar{n}\leq 0\ \forall l\notin s_1$, only states vectors $\bar{\xi}_{j_1,1}(k)$ from subsystems $\Sigma_{j_1}\in S_1$ $(j_1\in s_1)$, are included in the first equations $(i_{j_1}=1)$ of subsystem Σ_{j_1}. Then, it is easy to show that

$$\begin{aligned}y_{j_1}(k+1)&=\sum_{j_1\in s_1}O[y_{j_1}(k)]+O[\xi_{j_1,2}(k)]\quad\text{and}\\ \xi_{j_1,2}(k)&=O[y_{j_1}(k+1)]+\sum_{l\in s_1}O[y_l(k)].\end{aligned}\tag{B1}$$

Together with Proposition 2.1 and $\bar{\xi}_{j_1,2}(k)\sim O[\xi_{j_1,1}(k)]+O[\xi_{j_1,2}(k)]$, we have

$$\begin{aligned}\sum_{j_1\in s_1}O[\xi_{j_1,1}(k)]&\sim\sum_{j_1\in s_1}O[y_{j_1}(k)],\\ \sum_{j_1\in s_1}O[\bar{\xi}_{j_1,2}(k)]&\sim\sum_{j_1\in s_1}O[y_{j_1}(k+1)].\end{aligned}\tag{B2}$$

Step 2: *Sub-step 1* – Consider the second equations of subsystems $\Sigma_{j_1}\in S_1$ $(j_1\in s_1)$, i.e. $i_{j_1}=2$. Because $i_{j_1}-m_{j_1 l}=2+n_l-\bar{n}\leq 0\quad\forall l\notin s_1\cup s_2$, only states vector $\xi_{j_1,2}(k)$ from subsystems $\Sigma_{j_1}\in S_1$ $(j_1\in s_1)$ and $\xi_{j_2,1}(k)$ from subsystems $\Sigma_{j_1}\in S_2$ $(j_1\in s_2)$, are included in the second equations $(i_{j_1}=2)$ of subsystems $\Sigma_{j_1}\in S_1$. Using (B1), we have

$$\begin{aligned}\xi_{j_1,2}(k+1)&=\sum_{j_1\in s_1}O[\bar{\xi}_{j_1,2}(k)]+\sum_{j_2\in s_2}O[y_{j_2}(k)]+O[\xi_{j_1,3}(k)]\\ &=\sum_{j_1\in s_1}O[y_{j_1}(k+1)]+\sum_{j_2\in s_2}O[y_{j_2}(k)]+O[\xi_{j_1,3}(k)]\quad\text{and}\\ \xi_{j_1,3}(k)&=\sum_{j_1\in s_1}O[\bar{\xi}_{j_1,2}(k)]+\sum_{j_2\in s_2}O[y_{j_2}(k)]+O[\xi_{j_1,2}(k+1)]\\ &=\sum_{j_1\in s_1}O[y_{j_1}(k+1)]+\sum_{j_2\in s_2}O[y_{j_2}(k)]+O[\xi_{j_1,2}(k+1)],\end{aligned}\tag{B3}$$

which together with (B2), Proposition 2.1 and $\bar{\xi}_{j_1,3}(k) \sim O[\xi_{j_1,3}(k)] + O[\bar{\xi}_{j_1,2}(k)]$ leads to

$$\sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,3}(k)] \sim \sum_{j_1 \in s_1} O[y_{j_1}(k+2)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)]. \quad \text{(B4)}$$

Sub-step 2 – Consider the first equations of subsystems $\Sigma_{j_2} \in S_2$ $(j_2 \in s_2)$, i.e. $i_{j_2} = 1$. Because $i_{j_2} - m_{j_2 l} = 2 + n_l - \bar{n} \le 0$ for $l \notin s_1 \cup s_2$, only state vectors $\xi_{j_1,2}(k)$ from subsystems $\Sigma_{j_1} \in S_1$ $(j_1 \in s_1)$ and $\xi_{j_2,1}(k)$ from subsystems $\Sigma_{j_2} \in S_2$ $(j_2 \in s_2)$ are included in the first equations $(i_{j_2} = 1)$ of subsystems $\Sigma_{j_2} \in S_2$. Thus, we have

$$y_{j_2}(k+1) = \sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,2}(k)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)] + O[\xi_{j_2,2}(k)] \quad \text{and}$$
$$\xi_{j_2,2}(k) = \sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,2}(k)] + \sum_{j_2 \in s_2} O[y_{j_2}(k)] + O[y_{j_2}(k+1)], \quad \text{(B5)}$$

which together with (B2), $\bar{\xi}_{j_2,2}(k) = O[\xi_{j_2,2}(k)] + O[\xi_{j_2,1}(k)]$, $\sum_{j_2 \in s_2} O[\bar{\xi}_{j_2,1}(k)] \sim \sum_{j_2 \in s_2} O[y_{j_2}(k)]$ and Proposition 2.1 implies

$$\sum_{j_2 \in s_2} O[\bar{\xi}_{j_2,2}(k)] \sim \sum_{j_1 \in s_1} O[y_{j_1}(k+1)] + \sum_{j_2 \in s_2} O[y_{j_2}(k+1)]. \quad \text{(B6)}$$

Step l, $3 \le l \le \bar{n} - 1$: Consider the lth equations of $\Sigma_{j_1} \in S_1$ $(j_1 \in s_1)$, the $(l-1)$th equations of $\Sigma_{j_2} \in S_2$ $(j_2 \in s_2), \ldots,$ and the first equations of $\Sigma_{j_l} \in S_l$, $(j_l \in s_l)$. Following the above procedure and considering $\sum_{j_l \in s_l} O[\bar{\xi}_{j_l,l}(k)] \sim \sum_{j_l \in s_l} O[y_{j_l}(k)]$, we have

$$\sum_{j_1 \in s_1} O[\bar{\xi}_{j_1,l+1}(k)] \sim \sum_{j_1 \in s_1} O[y_{j_1}(k+l)] + \sum_{j_2 \in s_2} O[y_{j_2}(k+l-2)] + \cdots + \sum_{j_l \in s_l} O[y_{j_l}(k)],$$
$$\sum_{j_2 \in s_2} O[\bar{\xi}_{j_2,l}(k)] \sim \sum_{j_2 \in s_2} O[y_{j_2}(k+l-1)] + \sum_{j_1 \in s_1} O[y_{j_1}(k+l-1)] + \sum_{j_3 \in s_3} O[y_{j_3}(k+l-3)] + \cdots + \sum_{j_l \in s_l} O[y_{j_l}(k)]$$
$$\vdots$$
$$\sum_{j_l \in s_l} O[\bar{\xi}_{j_l,2}(k)] \sim \sum_{j_l \in s_l} O[y_{j_l}(k+1)] + \sum_{j_1 \in s_1} O[y_{j_1}(k+l-1)] + \sum_{j_2 \in s_2} O[y_{j_2}(k+l-2)] + \cdots + \sum_{j_{l-1} \in s_3} O[y_{j_{l-1}}(k+1)]. \quad \text{(B7)}$$

For subsystems $\Sigma_{j_{\bar{n}}} \in S_{\bar{n}}$ $(j_{\bar{n}} \in s_{\bar{n}})$, the system order is one $(n_{j_{\bar{n}}} = 1)$ and obviously we have $\bar{\xi}_{j_{\bar{n}},1}(k) = O[y_{j_{\bar{n}}}(k)]$. In summary of the above analysis and using Proposition 2.1, we have

$$\sum_{l=1}^{\bar{n}} \sum_{j_l \in s_l} O[\bar{\xi}_{j_l, n_{j_l} - i+1}(k)] \sim \sum_{l=1}^{\bar{n}} \sum_{j_l \in s_l} O[y_{j_l}(k + n_{j_l} - i)], \quad 1 \le i \le \bar{n}, \quad \text{(B8)}$$

where we let $\bar{\xi}_{j_l, n_{j_l}-i+1}(k) = y_{j_l}(k + n_{j_l} - i) = 0$, if $n_{j_l} - i + 1 \le 0$. The above equation is equivalent to

$$\sum_{l=1}^{n} O[\bar{\xi}_{l, i_j - m_{jl}}(k)] \sim \sum_{l=1}^{n} O[y_l(k + i_j - m_{jl} - 1)], \quad 1 \le i_j \le n_j, \ 1 \le j \le n, \quad \text{(B9)}$$

where $\bar{\xi}_{l, i_j - m_{jl}}(k) = y_l(k + i_j - m_{jl} - 1) = 0$, if $i_j - m_{jl} \le 0$. In addition, from the last equation of subsystem Σ_j, we have

$$u_j(k) = \frac{1}{g_{j,n_j}}[\xi_{j,n_j}(k+1) - \Theta_{j,n_j}^T \Phi_{j,n_j}(\Xi(k)) - \nu_j(\Xi_{\tau_j}(k))],$$

which leads to $u_j(k) = O[\Xi(k+1)]$. This completes the proof. □

Appendix C: Proof of Lemma 2.2

Proof: For a given l, $l = 1, 2, \ldots, n$ from $x_j(k) = \sum_{i=1}^{n} o[x_i(k - m_{ji})] + o[1]$, we have

$$x_j(k + n_j - n_l) = \sum_{i=1}^{n} o[x_i(k + n_i - n_l)] + o[1] \sim o\Big[\sum_{i=1}^{n} |x_i(k + n_i - n_l)|\Big] + o[1], \quad \text{(C1)}$$

which further leads to $\sum_{j=1}^{n} |x_j(k + n_j - n_l)| + o[n] \sim o[\sum_{i=1}^{n} |x_i(k + n_i - n_l)|] + o[1]$, from which we can obtain $\sum_{j=1}^{n} |x_j(k + n_j - n_l)| \sim o[1] \to 0$ which completes the proof. □

Appendix D: Proof of Lemma 2.4

Proof: Following the technique developed in Ge et al. (2008), it can be established that for $i_j = 1, 2, \ldots, n_j - t$, $t = 1, 4, \ldots, n_j - 1$,

$$\tilde{\xi}_{j,i_j}(k + t|k) = \sum_{l=1}^{n} o[O[y_l(k + i_j + t - 1 - m_{jl})]],$$
$$\tilde{\bar{\xi}}_{j,i_j}(k + t|k) = \sum_{l=1}^{n} o[O[y_l(k + i_j + t - 1 - m_{jl})]], \quad i_j = 1, 2, \ldots, n_j - t. \quad \text{(D1)}$$

Let $t = n_j - i_j$, then we have $\tilde{\bar{\xi}}_{j,i_j}(k + n_j - i_j|k) = \sum_{l=1}^{n} o[O[y_l(k + n_j - m_{jl} - 1)]]$, which completes the proof. □