Augmented Hamiltonian Formulation and Energy-Based Control Design of Uncertain Mechanical Systems

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***Abstract*—This paper mainly investigates augmented Hamiltonian formulation for both fully actuated and underactuated uncertain mechanical systems. First, a high-order partial derivative operator, called the unified partial derivative operator (UPDO), is given, and its properties are investigated, which plays a very important role in presenting the main results of this paper. Secondly, using the tool UPDO, the idea of shaping potential energy, and the pre-feedback technique, an augmented Hamiltonian structure with dissipation is provided for both fully actuated and underactuated uncertain mechanical systems. It is shown that the augmented Hamiltonian formulation has some nice properties for further analysis and control, and at the same time, its matching condition in the underactuated case becomes a set of algebraic equations, which are much easier to solve in comparison with solving a set of partial differential equations. Finally, as an application, the energy-based robust adaptive control is studied by using the augmented Hamiltonian formulation, and a new energy-based adaptive L_2 disturbance attenuation controller is designed for the uncertain mechanical systems. Study of an illustrative example with simulations shows that the controller obtained in this paper works very well in handling disturbances and uncertainties in the systems.**

***Index Terms*—Augmented Hamiltonian formulation, energy-based robust adaptive control, potential energy shaping, pre-feedback, uncertain mechanical system, unified partial derivative operator (UPDO).**

I. Introduction

MECHANICAL systems are highly nonlinear and dynamically coupled systems, which presents many challenging control problems. Since traditional linear control approaches do not easily apply, increasingly sophisticated tools from nonlinear control theory have been developed for the control of mechanical systems [1], [8]–[17], [19], [24], [25], [27]–[33], [39]. To facilitate the stability analysis and control design for mechanical systems, a commonly used approach is to rewrite the dynamics in terms of energy transfer [2]–[4], [7], [9], [23], [34]. In [34], the idea of shaping potential energy was introduced, and an important feedback method was proposed for dynamic control of manipulators. This shaping idea was then extended in [2] and [3] by shaping both potential and kinetic energies with the so-called controlled Lagrangian technique in a Lagrangian formulation. The method in [2] and [3] was Lyapunov based and could yield large and computable basins of stability, which became asymptotically stable when dissipative controls were added. In [23], the passivity-based control (PBC) design methodology, known as interconnection and damping assignment (IDA-PBC) [22], was applied to study the Hamiltonian formulation of mechanical systems, and a nice port-controlled Hamiltonian (PCH) structure was provided for underactuated mechanical systems. One of the main advantages of the PCH structure is that its Hamiltonian function can directly serve as a Lyapunov function for the systems. To obtain the PCH structure, one needs to be very skillful in solving a set of matching partial differential equations (PDEs), i.e., the so-called matching condition, as demonstrated in [23] for several benchmark problems.

However, in mechanical systems, there always exist uncertainties such as parametric uncertainties, payload uncertainties, inadequate modeling, etc. When there are uncertainties in mechanical systems, the Hamiltonian/Lagrangian formulations mentioned above can hardly work for the systems. A question naturally arises: How does one provide a suitable Hamiltonian/Lagrangian structure for mechanical systems in the case of uncertainties? Although many results on the Hamiltonian/Lagrangian formulation were obtained for certain mechanical systems, there are, to the authors' best knowledge, few studies on the problem for uncertain mechanical systems up to now.

In this paper, we investigate augmented Hamiltonian formulation for both fully actuated and underactuated uncertain mechanical systems. Using the shaping idea [34] and the pre-feedback technique [37], we provide the systems an augmented Hamiltonian structure with dissipation, which possesses some nice properties for further analysis and control design. In our new Hamiltonian formulation, the matching condition in the underactuated case becomes a set of algebraic equations, which are much easier to solve in comparison with solving a set of PDEs. As an application, we then utilize the augmented Hamiltonian formulation to study energy-based robust adaptive control design for the uncertain mechanical systems, and propose a new energy-based adaptive L_2 disturbance attenuation control scheme. The study of an example with simulations shows that the controller obtained in this paper has the robust

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property and built-in capability in handling disturbances and uncertainties in the systems.

The derivation of the augmented Hamiltonian formulation needs a new mathematical tool to be developed in this paper. The new tool is a kind of high-order partial derivative operator, called the unified partial derivative operator (UPDO) here, which plays an essential role in presenting the main results of this paper. After the UPDO is defined, its properties are then investigated, through which we can obtain a very useful matrix operator, which plays the same role as the swap matrix (square case) defined in [5] and [18] for semitensor products [5]. Using the UPDO's properties presented in this paper, we can reveal one underlying property of mechanical systems, which is fundamental to the augmented Hamiltonian formulation, and at the same time, with the help of UPDO, we can show that the structure matrix of mechanical systems in the augmented Hamiltonian formulation can be expressed as the form of a skew-symmetric matrix minus a positive semidefinite one, i.e., the form of a dissipative Hamiltonian structure. In addition, it is well worth pointing out that, as a mathematical tool, the UPDO can also be used in other engineering or mathematical problems, although it is developed in this paper only for the augmented Hamiltonian formulation of mechanical systems.

This paper is organized as follows. Section II develops the new mathematical tool UPDO. In Section III, we utilize the new tool to investigate the augmented Hamiltonian formulation for both fully actuated and underactuated uncertain mechanical systems. In Section IV, the energy-based robust adaptive control is investigated by using the augmented Hamiltonian formulation, and a new adaptive L_2 disturbance attenuation controller is proposed for uncertain mechanical systems. Section V studies an illustrative example with numerical simulations to support our new results, which is followed by a conclusion in Section VI.

II. UPDO

In this section, we develop the new mathematical tool UPDO, which plays an essential role in the derivation of the augmented Hamiltonian formulation in this paper.

Assume that $f(x), x \in \mathbb{R}^n$ is a scalar function. It is well known that the first-order partial derivative of $f(x)$ can be defined as gradient $\nabla f(x) := \partial f/\partial x$ and the second partial derivative can be given by the Hessian matrix Hess $(f(x))$. A natural extension is the definition of partial derivatives of arbitrarily any order, say, n.

Motivated by [6], we now define the UPDO, which can express arbitrary order partial derivatives for a scalar function, a vector field, or a function matrix easily.

Definition 1: Let

$$\frac{\partial}{\partial x} = \left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]^T, \qquad x \in \mathbb{R}^n.$$

The UPDO is defined recursively as

$$\frac{\partial^m}{\partial x^m} = \frac{\partial^{m-1}}{\partial x^{m-1}} \otimes \frac{\partial}{\partial x}, \qquad m \geq 1 \tag{1}$$

where $\otimes$ is the Kronecker product, and the products between elements are defined as

$$\begin{aligned}&\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots \partial x_n^{i_n}} \cdot \frac{\partial}{\partial x_j}\\ &\quad := \frac{\partial^{i_1+\cdots+(i_j+1)+\cdots+i_n}}{\partial x_1^{i_1}\cdots \partial x_j^{i_j+1}\cdots\partial x_n^{i_n}}, \qquad i_1+\cdots+i_n = m-1;\\ &\quad j = 1, 2, \ldots, n\end{aligned}$$

in which $(\partial^{i_1+\cdots+i_n})/(\partial x_1^{i_1}\cdots\partial x_n^{i_n})(i_1+\cdots+i_n = m-1)$ and $(\partial)/(\partial x_j)$ are arbitrary components of $(\partial^{m-1})/(\partial x^{m-1})$ and $(\partial)/(\partial x)$, respectively. For completeness, we further define $(\partial^0)/(\partial x^0) := \mathcal{I}$, which is called the identity operator and satisfies

$$\mathcal{I} \otimes \frac{\partial^s}{\partial x^s} = \frac{\partial^s}{\partial x^s} \otimes \mathcal{I} = \frac{\partial^s}{\partial x^s} \qquad \forall s \geq 1.$$

From the above definition, it is easy to know that $(\partial^m)/(\partial x^m), m \geq 1$ is an n^m-dimensional column vector operator, e.g.,

$$\begin{aligned}\frac{\partial^2}{\partial x^2} = \Bigg[&\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2}{\partial x_1 \partial x_n}, \ldots,\\ &\frac{\partial^2}{\partial x_n \partial x_1}, \frac{\partial^2}{\partial x_n \partial x_2}, \ldots, \frac{\partial^2}{\partial x_n^2}\Bigg]^T\end{aligned}$$

is an n^2-dimensional column vector.

Remark 1: It is interesting to note that though the first-order UPDO is the same as the gradient operator ∇, the second-order UPDO $(\partial^2)/(\partial x^2)$ is no longer equal to the operator Hess.

With the above definition, given any scalar function $f(x)$

$$f(x) \otimes \frac{\partial^m}{\partial x^m} := \frac{\partial^m f(x)}{\partial x^m}, \qquad m \geq 1 \tag{2}$$

is well defined. Based on (2), given a vector field $X(x) \in \mathbb{R}^p$, we define

$$\frac{\partial^m X(x)}{\partial x^m} := X(x) \otimes \frac{\partial^m}{\partial x^m}, \qquad m \geq 1. \tag{3}$$

From (3), it is easy to show that the following recursive formula holds for the higher order derivative operators:

$$\frac{\partial^m X(x)}{\partial x^m} = \frac{\partial}{\partial x}\left(\frac{\partial^{m-1}X(x)}{\partial x^{m-1}}\right), \qquad m \geq 1 \tag{4}$$

where $(\partial^0 X(x))/(\partial x^0) := X(x) \otimes \mathcal{I} = X(x)$.

In fact, from (1) and (3), we have

$$\begin{aligned}\frac{\partial^m X(x)}{\partial x^m} &= X(x) \otimes \frac{\partial^m}{\partial x^m}\\ &= X(x) \otimes \left(\frac{\partial^{m-1}}{\partial x^{m-1}} \otimes \frac{\partial}{\partial x}\right)\\ &= \left(X(x) \otimes \frac{\partial^{m-1}}{\partial x^{m-1}}\right) \otimes \frac{\partial}{\partial x}\\ &= \left(\frac{\partial^{m-1}X(x)}{\partial x^{m-1}}\right) \otimes \frac{\partial}{\partial x}\\ &= \frac{\partial}{\partial x}\left(\frac{\partial^{m-1}X(x)}{\partial x^{m-1}}\right).\end{aligned}$$

Thus, (4) holds.

Similarly, given any function matrix $A(x) \in \mathbb{R}^{p\times q}$, from (2), we define

$$\frac{\partial^m A(x)}{\partial x^m} := A(x) \otimes \frac{\partial^m}{\partial x^m}, \qquad m \geq 1. \tag{5}$$

It can be shown that

$$\frac{\partial^m A(x)}{\partial x^m} = \frac{\partial}{\partial x}\left(\frac{\partial^{m-1} A(x)}{\partial x^{m-1}}\right), \qquad m \geq 1 \tag{6}$$

where $(\partial^0 A(x))/(\partial x^0) := A(x) \otimes \mathcal{I} = A(x)$.

In the following, we study the properties of the UPDO, as they play an essential role in the derivation of the our Hamiltonian formulation. The following study needs the so-called row-swap matrix (operator) $E_n(i,j) \in \mathbb{R}^{n\times n}$, which is obtained from the $n\times n$ identity matrix I_n by swapping the ith row with the jth row (*note*: $E_n(i,j)$ can be find in any linear algebra textbooks).

Proposition 1: Assume that $A(x) \in \mathbb{R}^{n\times n}$ $(x \in \mathbb{R}^n)$ is a function matrix, and that $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n$ are constant vectors. Then,

$$\frac{\partial}{\partial x}(\alpha^T A(x)\beta) = (I_n \otimes \alpha^T)\left(\Gamma_n \cdot \frac{\partial A(x)}{\partial x}\right)\beta \tag{7}$$

where $(\partial A(x))/(\partial x) \in \mathbb{R}^{n^2\times n}$ is defined by (5) and

$$\Gamma_n := \prod_{i=1}^{n-1}\prod_{j>i}^{n} E_{n^2}((i-1)n+j,(j-1)n+i). \tag{8}$$

Proof: Let $h(x) := \alpha^T A(x)\beta$. Then

$$\begin{aligned}\frac{\partial h}{\partial x_s} &= \alpha^T \frac{\partial A(x)}{\partial x_s}\beta \\ &= \alpha^T \left[\frac{\partial a_{ij}(x)}{\partial x_s}\right]_{n\times n} \cdot \beta, \qquad s = 1,2,\ldots,n\end{aligned} \tag{9}$$

from which we obtain

$$\begin{aligned}\frac{\partial h}{\partial x} &= \begin{bmatrix}\alpha^T \frac{\partial A(x)}{\partial x_1}\beta \\ \vdots \\ \alpha^T \frac{\partial A(x)}{\partial x_n}\beta\end{bmatrix} \\ &= (I_n \otimes \alpha^T)\begin{bmatrix}\frac{\partial A(x)}{\partial x_1} \\ \vdots \\ \frac{\partial A(x)}{\partial x_n}\end{bmatrix}\beta \\ &:= (I_n \otimes \alpha^T)\Phi(x)\beta\end{aligned} \tag{10}$$

where $\Phi(x) = [(\partial A^T(x))/(\partial x_1),\ldots,(\partial A^T(x))/(\partial x_n)]^T \in \mathbb{R}^{n^2\times n}$.

Comparing $\Phi(x)$ with $(\partial A(x))/(\partial x) \in \mathbb{R}^{n^2\times n}$, we know that $\Phi(x)$ can be obtained from $(\partial A(x))/(\partial x)$ through the following swapping transformations of rows in turn.

1. The second row with the $[(2-1)n+1]$th row; the third row with the $[(3-1)n+1]$th row, ..., the nth row with the $[(n-1)n+1]$th row.

2. The $(n+3)$th row with $[(3-1)n+2]$th row, the $(n+4)$th row with the $[(4-1)n+2]$th row, ..., the $(n+n)$th row with the $[(n-1)n+2]$th row.

......

n-1. The $[(n-2)n+n]$th row with the $[(n-1)n+(n-1)]$th row.

Thus, we obtain

$$\begin{aligned}\Phi(x) &= \left\{\prod_{i=n-1}^{1}\prod_{j=n}^{j>i} E_{n^2}((i-1)n+j,(j-1)n+i)\right\}\frac{\partial A(x)}{\partial x} \\ &= \left\{\prod_{i=1}^{n-1}\prod_{j>i}^{n} E_{n^2}((i-1)n+j,(j-1)n+i)\right\}\frac{\partial A(x)}{\partial x}.\end{aligned} \tag{11}$$

Therefore, the proposition follows from (10) and (11). □

Γ_n, defined in (8), is a very useful matrix, having the following properties, which will be used in our derivation of the augmented Hamiltonian formulation.

Proposition 2: Γ_n defined in (8) has the following properties.

i) Γ_n is nonsingular and $\Gamma_n^{-1} = \Gamma_n$.

ii) For arbitrary matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n}$,

$$\Gamma_n(A\otimes B)\Gamma_n = B \otimes A. \tag{12}$$

Proof:

i) The nonsingularity of Γ_n can be shown from the nonsingularity of each $E_{n^2}((i-1)n+j,(j-1)n+i)$, $i = 1,\ldots,n-1, i<j\leq n$,

$$\begin{aligned}\Gamma_n^{-1} &= \left\{\prod_{i=1}^{n-1}\prod_{j>i}^{n} E_{n^2}((i-1)n+j,(j-1)n+i)\right\}^{-1} \\ &= \prod_{i=n-1}^{1}\prod_{j=n}^{j>i} E_{n^2}^{-1}((i-1)n+j,(j-1)n+i) \\ &= \prod_{i=n-1}^{1}\prod_{j=n}^{j>i} E_{n^2}((i-1)n+j,(j-1)n+i) \\ &= \prod_{i=1}^{n-1}\prod_{j>i}^{n} E_{n^2}((i-1)n+j,(j-1)n+i) \\ &= \Gamma_n.\end{aligned}$$

ii) Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ and $B = [b_{ij}] \in \mathbb{R}^{n\times n}$. Then, $B\otimes A$ can be expressed as

$$B\otimes A = [b_{ij}A] \in \mathbb{R}^{n^2\times n^2}, \qquad i,j = 1,2,\ldots,n. \tag{13}$$

Thus, the $[(i-1)n+k,(j-1)n+s]$th element of $B\otimes A$, i.e., the ksth element of the ijth block of $B\otimes A$, is as follows:

$$b_{ij}\cdot a_{ks}, \qquad i,j = 1,2\ldots,n; \quad k,s = 1,2,\ldots,n. \tag{14}$$

Similarly, $A\otimes B$ can be rewritten as

$$A\otimes B = [a_{ij}B] \in \mathbb{R}^{n^2\times n^2}, \qquad i,j = 1,2,\ldots,n \tag{15}$$

and the $[(k-1)n+i,(s-1)n+j]$th element of $A\otimes B$, i.e., the ijth element of the ksth block of $A\otimes B$ should be

$$a_{ks}\cdot b_{ij}, \qquad k,s=1,2,\dots n; \quad i,j=1,2\dots,n. \tag{16}$$

On the other hand, from the definition of Γ_n, we know that the $[(i-1)n+k,(j-1)n+s]$th element of $\Gamma_n[A\otimes B]\Gamma_n$ should be the $[(k-1)n+i,(j-1)n+s]$th element of $[A\otimes B]\Gamma_n$, while the $[(k-1)n+i,(j-1)n+s]$th element of $[A\otimes B]\Gamma_n$ is the $[(k-1)n+i,(s-1)n+j]$th element of $A\otimes B$, which is given in (16). That is to say, the $[(i-1)n+k,(j-1)n+s]$th element of $\Gamma_n[A\otimes B]\Gamma_n$ is given in (16).

Therefore, comparing (14) with (16), we have $\Gamma_n(A\otimes B)\Gamma_n=B\otimes A$. □

Equation (12) shows that Γ_n can play the role of the swap matrix $W[n,n]\in\mathbb{R}^{n^2\times n^2}$, defined in [5] and [18]. In [5], $W[m,n]\in\mathbb{R}^{mn\times mn}$ is constructed in the following way: label its columns by $(11,12,\dots,1n,\dots,m1,m2,\dots,mn)$ and its rows by $(11,21,\dots,m1,\dots,1n,2n,\dots,mn)$, then its element at the position $[(IJ),(ij)]$ is assigned as

$$w_{(IJ),(ij)}=\delta_{ij}^{IJ}=\begin{cases}1, & I=i \text{ and } J=j\\ 0, & \text{otherwise.}\end{cases}$$

Although the definitions are quite different from each other, Γ_n and $W[n,n]$ play the same role of the swap operator, i.e., for arbitrary matrices $A,B\in\mathbb{R}^{n\times n}$

$$\begin{cases}\Gamma_n(A\otimes B)\Gamma_n=B\otimes A\\ W[n,n](A\otimes B)W[n,n]=B\otimes A.\end{cases} \tag{17}$$

For more general results about the swap operator $W[m,n]$, please refer to [5].

Proposition 3: Assume that $A(x)\in\mathbb{R}^{n\times n}$ $(x\in\mathbb{R}^n)$ is a nonsingular function matrix. Then

$$\frac{\partial}{\partial x}[A^{-1}(x)]=-\Gamma_n\left(I_n\otimes A^{-1}(x)\right)\left(\Gamma_n\frac{\partial A(x)}{\partial x}\right)A^{-1}(x). \tag{18}$$

Proof: From the equation $A(x)A^{-1}(x)=I_n$, we obtain

$$\frac{\partial A(x)}{\partial x_i}A^{-1}(x)+A(x)\frac{\partial A^{-1}(x)}{\partial x_i}=0, \qquad i=1,2,\dots,n. \tag{19}$$

Thus, we have

$$\frac{\partial A^{-1}(x)}{\partial x_i}=-A^{-1}(x)\frac{\partial A(x)}{\partial x_i}A^{-1}(x), \qquad i=1,2,\dots,n \tag{20}$$

from which it can be seen that

$$\begin{bmatrix}\frac{\partial A^{-1}(x)}{\partial x_1}\\ \vdots\\ \frac{\partial A^{-1}(x)}{\partial x_n}\end{bmatrix}=-(I_n\otimes A^{-1}(x))\begin{bmatrix}\frac{\partial A(x)}{\partial x_1}\\ \vdots\\ \frac{\partial A(x)}{\partial x_n}\end{bmatrix}A^{-1}(x). \tag{21}$$

From the Proof of Proposition 1, we have

$$\Gamma_n\frac{\partial A^{-1}(x)}{\partial x}=-(I_n\otimes A^{-1}(x))\left(\Gamma_n\frac{\partial A(x)}{\partial x}\right)A^{-1}(x). \tag{22}$$

Notice that Γ_n is nonsingular and $\Gamma_n^{-1}=\Gamma_n$ (see Proposition 2). The proposition then follows from (22).

With Propositions 2 and 3, we have the following result. □

Corollary 1: Assume that $A(x)\in\mathbb{R}^{n\times n}$ $(x\in\mathbb{R}^n)$ is a nonsingular function matrix. Then,

$$\frac{\partial}{\partial x}[A^{-1}(x)]=-(A^{-1}(x)\otimes I_n)\frac{\partial A(x)}{\partial x}A^{-1}(x). \tag{23}$$

Proof: From (12) and (18), we obtain

$$\begin{aligned}\frac{\partial}{\partial x}&[A^{-1}(x)]\\ &=-\Gamma_n(I_n\otimes A^{-1}(x))\left(\Gamma_n\frac{\partial A(x)}{\partial x}\right)A^{-1}(x)\\ &=-\Gamma_n(I_n\otimes A^{-1}(x))\Gamma_n\Gamma_n^{-1}\left(\Gamma_n\frac{\partial A(x)}{\partial x}\right)A^{-1}(x)\\ &=-(A^{-1}(x)\otimes I_n)\frac{\partial A(x)}{\partial x}A^{-1}(x).\end{aligned}$$

□

Remark 2: Although the UPDO is developed in the above to investigate the augmented Hamiltonian formulation of mechanical systems, as a mathematical tool, the UPDO can also be used in other engineering or mathematical problems.

III. Augmented Hamiltonian Formulation of Uncertain Mechanical Systems

In this section, using the UPDO developed in Section II, we investigate the augmented Hamiltonian formulation for both fully actuated and underactuated uncertain mechanical systems, and provide the systems with an augmented Hamiltonian framework, which possesses some nice properties for further analysis and control design.

Consider an uncertain mechanical system described as [11], [27]

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=B(q)\tau \tag{24}$$

where $q=[q_1,q_2,\dots,q_n]^T\in\mathbb{R}^n$ is the position vector (the generalized coordinate), $\dot{q}\in\mathbb{R}^n$ is the velocity vector, $\tau\in\mathbb{R}^m$ is the control torque vector, $B(q)\in\mathbb{R}^{n\times m}$ has full column rank $(m\le n)$, $M(q)\in\mathbb{R}^{n\times n}$ is the inertia matrix, which is symmetric positive definite, $C(q,\dot{q})\in\mathbb{R}^{n\times n}$ is the centripetal and Coriolis matrix, $G(q)\in\mathbb{R}^n$ is the gravitational force vector, and $M(q)$, $C(q,\dot{q})$ and $G(q)$ are assumed to have unknown constant parameters.

Our study is completed under the following assumption.

Assumption 1: Assume that the unknown part in $G(q)$ depends linearly on a constant vector $\theta\in\mathbb{R}^l$, which is a function of the unknown parameters, i.e., there exists a matrix $\Phi(q)\in\mathbb{R}^{n\times l}$ such that

$$G(q)=G_0(q)+\Phi(q)\theta \tag{25}$$

where $G_0(q) \in \mathbb{R}^n$ stands for the separable known or nominal part of $G(q)$.

To provide system (24) with a Hamiltonian structure, we rewrite the system in another form first. Let $M(q) = [m_{ij}]_{n\times n}, C(q,\dot{q}) = [c_{ij}(q,\dot{q})]_{n\times n}$, where

$$c_{ij} = \sum_{k=1}^{n} c_{kji}\dot{q}_k = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i}\right)\dot{q}_k \tag{26}$$

and c_{kji} is the Christoffel symbol [11].

Now consider the ith row of $C(q,\dot{q})\dot{q}$ as follows:

$$\begin{aligned}\sum_{j=1}^{n} c_{ij}\dot{q}_j &= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i}\right)\dot{q}_k\dot{q}_j \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{kj}}{\partial q_i}\right)\dot{q}_k\dot{q}_j \\ &\quad + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2}\frac{\partial m_{ik}}{\partial q_j}\dot{q}_k\dot{q}_j \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{kj}}{\partial q_i}\right)\dot{q}_k\dot{q}_j \\ &\quad + \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{1}{2}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_j\dot{q}_k \\ &= \sum_{j=1}^{n}\left\{\sum_{k=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial m_{kj}}{\partial q_i}\right)\dot{q}_k\right\}\dot{q}_j \\ &:= \sum_{j=1}^{n} \bar{c}_{ij}\dot{q}_j, \qquad i = 1,2,\ldots,n\end{aligned} \tag{27}$$

where

$$\bar{c}_{ij} = \sum_{s=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_s} - \frac{1}{2}\frac{\partial m_{sj}}{\partial q_i}\right)\dot{q}_s. \tag{28}$$

Defining

$$\bar{C}(q,\dot{q}) := [\bar{c}_{ij}(q,\dot{q})]_{n\times n} \tag{29}$$

we have $C(q,\dot{q})\dot{q} = \bar{C}(q,\dot{q})\dot{q}$ with which system (24) can be rewritten as

$$M(q)\ddot{q} + \bar{C}(q,\dot{q})\dot{q} + G(q) = B(q)\tau. \tag{30}$$

System (30) is our desired form in this paper.

Consider the generalized momentum of system (30)

$$p = M(q)\dot{q}. \tag{31}$$

With system (30) and the help of UPDO, we can reveal the following underlying relationship property of mechanical systems, which are essential for our Hamiltonian formulation.

Proposition 4: In system (30), the following relationship holds for all $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^n$:

$$\begin{aligned}M_c(q,p) &:= \dot{M}(q) - \bar{C}(q,\dot{q}) \\ &\quad + \frac{1}{2}(I_n \otimes p^T)\left(\Gamma_n\frac{\partial M^{-1}(q)}{\partial q}\right)M(q) \\ &\equiv 0\end{aligned} \tag{32}$$

where Γ_n is given by (8).

Proof: $\dot{M}(q) = [\dot{m}_{ij}(q)]_{n\times n}$, where

$$\dot{m}_{ij}(q) = \sum_{s=1}^{n}\frac{\partial m_{ij}(q)}{\partial q_s}\dot{q}_s. \tag{33}$$

From Proposition 3 and the properties of the Kronecker product, we obtain

$$\begin{aligned}&\frac{1}{2}(I_n \otimes p^T)\left(\Gamma_n\frac{\partial M^{-1}(q)}{\partial q}\right)M(q) \\ &= -\frac{1}{2}(I_n \otimes p^T) \\ &\quad \times\left\{\Gamma_n\cdot\Gamma_n(I_n \otimes M^{-1}(q))\left(\Gamma_n\frac{\partial M(q)}{\partial q}\right)M^{-1}(q)\right\}M(q) \\ &= -\frac{1}{2}(I_n\otimes p^T)(I_n\otimes M^{-1}(q))\left(\Gamma_n\frac{\partial M(q)}{\partial q}\right) \\ &= -\frac{1}{2}\{(I_n\cdot I_n)\otimes(p^T M^{-1}(q))\}\left(\Gamma_n\frac{\partial M(q)}{\partial q}\right) \\ &= -\frac{1}{2}(I_n\otimes\dot{q}^T)\left(\Gamma_n\frac{\partial M(q)}{\partial q}\right).\end{aligned}$$

From the Proof of Proposition 1, we have

$$(I_n\otimes\dot{q}^T)\left(\Gamma_n\frac{\partial M(q)}{\partial q}\right) = \begin{bmatrix}\dot{q}^T\dfrac{\partial M(q)}{\partial q_1} \\ \vdots \\ \dot{q}^T\dfrac{\partial M(q)}{\partial q_n}\end{bmatrix}.$$

Thus, we obtain

$$\begin{aligned}\frac{1}{2}(I_n\otimes p^T)\left(\Gamma_n\frac{\partial M^{-1}(q)}{\partial q}\right)M(q) &= -\frac{1}{2}\begin{bmatrix}\dot{q}^T\dfrac{\partial M(q)}{\partial q_1} \\ \vdots \\ \dot{q}^T\dfrac{\partial M(q)}{\partial q_n}\end{bmatrix} \\ &:= [d_{ij}(q,\dot{q})] \in \mathbb{R}^{n\times n}\end{aligned}$$

where

$$d_{ij}(q,\dot{q}) = -\frac{1}{2}\sum_{s=1}^{n}\frac{\partial m_{sj}(q)}{\partial q_i}\dot{q}_s. \tag{34}$$

From (28), (33), and (34), $m_c^{ij}(q,p)$, the element located at (i,j) in matrix $M_c(q,p)$, can be expressed as

$$\begin{aligned} m_c^{ij}(q,p) &= \dot{m}_{ij}(q) - \bar{c}_{ij}(q,\dot{q}) + d_{ij}(q,\dot{q}) \\ &= \sum_{s=1}^{n} \frac{\partial m_{ij}(q)}{\partial q_s}\dot{q}_s \\ &\quad - \sum_{s=1}^{n}\left(\frac{\partial m_{ij}(q)}{\partial q_s} - \frac{1}{2}\frac{\partial m_{sj}(q)}{\partial q_i}\right)\dot{q}_s \\ &\quad - \frac{1}{2}\sum_{s=1}^{n}\frac{\partial m_{sj}(q)}{\partial q_i}\dot{q}_s \\ &\equiv 0. \end{aligned}$$

Therefore, $M_c(q,p) \equiv 0$ for $\forall q \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$. □

Remark 3: Proposition 4 holds true only when the Coriolis matrix $\bar{C}(q,\dot{q})$ is defined by (28).

Remark 4: Proposition 4 reveals yet another important underlying property of mechanical systems. It is the very property that makes it possible for us to provide a nice augmented Hamiltonian structure for the system under consideration.

Next, we investigate the augmented Hamiltonian formulation for the uncertain system (24).

Denote by $\mathcal{P}(q)$ the gravitational potential energy of system (24). Since $G(q) = (\partial\mathcal{P}(q))/(\partial q)$, from [36] and the Poincare lemma, $\mathcal{P}(q)$ can be expressed as

$$\mathcal{P}(q) = \sum_{i=1}^{n}\int_{x_i^{(0)}}^{q_i} g_i\left(x_1^{(0)},\ldots,x_{i-1}^{(0)},q_i,q_{i+1},\ldots,q_n\right)\mathrm{d}q_i \tag{35}$$

where $g_i(q_1,q_2,\ldots,q_n)$ is the ith component of $G(q), i = 1,2,\ldots,n$, and $x^{(0)} = [x_1^{(0)},\ldots,x_n^{(0)}]^T$ is an arbitrary point in $\mathbb{R}^n$.

It is easy to see from (35) that the gravitational potential energy is not positive definite. As a result, the total energy of system (24) cannot be used as a Lyapunov function for the system. Thus, we have to introduce another form of the Hamiltonian function to provide a nice Hamiltonian structure for the system.

In the following, we use the shaping idea [34] to give the system a new form of Hamiltonian function. Assume that $q^{(0)}$ is the target position (the equilibrium position) of the system to be designed. We choose

$$H(q,p,\hat{\theta}) = \mathcal{K}(q,p) + \mathcal{P}_g(q) + \frac{1}{2}(\hat{\theta}-\theta)^T\Gamma_0(\hat{\theta}-\theta) \tag{36}$$

as the new Hamiltonian function for system (24), where

$$\mathcal{K}(q,p) := \frac{1}{2}p^TM^{-1}(q)p = \frac{1}{2}\dot{q}^TM(q)\dot{q} \tag{37}$$

is the system's kinetic energy

$$\mathcal{P}_g(q) := \frac{1}{2}(q-q^{(0)})^T\Lambda\left(q-q^{(0)}\right) \tag{38}$$

is the so-called virtual potential energy, $\Lambda \in \mathbb{R}^{n\times n}$ and $\Gamma_0 \in \mathbb{R}^{l\times l}$ are two constant positive definite matrices, and $\hat{\theta}$ is the estimate of θ [see (25)], which satisfies

$$\dot{\hat{\theta}} = -\Gamma_0^{-1}\Phi^T(q)\dot{q}. \tag{39}$$

Remark 5: Since there are uncertainties in the system, we have added the term $(1/2)(\hat{\theta}-\theta)^T\Gamma_0(\hat{\theta}-\theta)$ into the Hamiltonian function. Without this term, one cannot provide the system with a nice Hamiltonian structure.

Remark 6: Unlike the gravitational potential energy $\mathcal{P}(q), \mathcal{P}_g(q)$ has a minimum at the target point $q^{(0)}$, from which it can be seen that $H(q,p,\hat{\theta})$ is a positive definite function.

In the following, we consider the Hamiltonian formulation problem in two cases, i.e., 1) the system is fully actuated $(m = n)$ and 2) the system is underactuated $(m < n)$.

a) Fully Actuated Case: In this case, $B(q)$ is nonsingular. From (36)–(38), using Proposition 1, we have

$$\frac{\partial H}{\partial p} = M^{-1}(q)p = \dot{q} \tag{40}$$

$$\frac{\partial H}{\partial q} = \frac{1}{2}(I_n\otimes p^T)\left(\Gamma_n\frac{\partial M^{-1}(q)}{\partial q}\right)p + \Lambda\left(q-q^{(0)}\right) \tag{41}$$

where $(\partial M^{-1}(q))/(\partial q)$ is defined by (5), and Γ_n is given by (8).

On the other hand, using system (30) and noting that $p = M(q)\dot{q}$, we obtain

$$\begin{aligned} \dot{p} &= \dot{M}(q)\dot{q} + M(q)\ddot{q} \\ &= \dot{M}(q)\dot{q} - \bar{C}(q,\dot{q})\dot{q} - G(q) + B(q)\tau. \end{aligned} \tag{42}$$

To provide the system with a nice Hamiltonian structure, we design a pre-feedback law as follows:

$$\begin{cases} \tau = B^{-1}(q)\left[G_0(q) + \Phi(q)\hat{\theta} - \Lambda\left(q-q^{(0)}\right) - K_D\dot{q}\right] + u \\ \dot{\hat{\theta}} = -\Gamma_0^{-1}\Phi^T(q)\dot{q} \end{cases} \tag{43}$$

where $K_D = K_D^T \in \mathbb{R}^{n\times n}$ is a constant positive definite matrix, and $u \in \mathbb{R}^n$ is the new control input.

Remark 7: Since $\dot{q}$ and q are measurable and $G_0(q)$ stands for the known part of $G(q)$, the control law (43) can be realized in practice.

With (40)–(43), we obtain

$$\begin{bmatrix}\dot{q}\\ \dot{p}\\ \dot{\hat{\theta}}\end{bmatrix} = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & K_c(q,p)-K_D & \Phi(q)\Gamma_0^{-1} \\ 0 & -\Gamma_0^{-1}\Phi^T(q) & 0\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial q}\\ \frac{\partial H}{\partial p}\\ \Gamma_0(\hat{\theta}-\theta)\end{bmatrix} + \begin{bmatrix}0\\ B(q)\\ 0\end{bmatrix}u \tag{44}$$

where

$$K_c(q,p) = \dot{M}(q) - \bar{C}(q,\dot{q}) + \frac{1}{2}(I_n \otimes p^T)\left(\Gamma_n \frac{\partial M^{-1}(q)}{\partial q}\right) M(q). \quad (45)$$

Noticing that $K_c(q,p) = M_c(q,p)$, from Proposition 4, we know that $K_c(q,p) \equiv 0$. With this and letting $X = [q^T, p^T, \hat{\theta}^T]^T \in \mathbb{R}^{2n+l}$, system (44) can be expressed as the following PCH system with dissipation:

$$\dot{X} = \left[J(X) - R(X)\right]\frac{\partial H}{\partial X} + g_c u \quad (46)$$

where

$$\begin{aligned} J(X) &= -J^T(X) \\ &= \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & \Phi(q)\Gamma_0^{-1} \\ 0 & -\Gamma_0^{-1}\Phi^T(q) & 0 \end{bmatrix} \\ &\in \mathbb{R}^{(2n+l)\times(2n+l)} \\ 0 \le R(X) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_D & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(2n+l)\times(2n+l)} \\ g_c &= \begin{bmatrix} 0 \\ B(q) \\ 0 \end{bmatrix} \in \mathbb{R}^{(2n+l)\times m}. \end{aligned}$$

Theorem 1: Assume that Assumption 1 holds for system (24). With the new Hamiltonian function (36) and the adaptive control law (43), system (24) can be expressed as an augmented Hamiltonian system given by (46), which is a PCH system with dissipation.

b) Underactuated Case: In this case, $B(q)$ is not invertible. Similar to the fully actuated case, we can also obtain (40)–(42). In the underactuated case, to provide the system with a nice Hamiltonian structure, a suitable pre-feedback law $\tau(q,p,\hat{\theta})$ should be designed for the system such that

$$B(q)\tau(q,p,\hat{\theta}) = G_0(q) + \Phi(q)\hat{\theta} - \Lambda\left(q - q^{(0)}\right) - K_D M^{-1}(q)p \quad (47)$$

where $K_D = K_D^T \in \mathbb{R}^{n\times n}$ is a constant positive definite matrix to be determined. Since $B(q)$ is not invertible and only has full column rank, τ can only influence the terms in the range space of $B(q)$ [23]. This leads to the following constraint equation:

$$B^{\perp}(q)\left(G_0(q) + \Phi(q)\hat{\theta} - \Lambda\left(q - q^{(0)}\right) - K_D M^{-1}(q)p\right) = 0 \quad (48)$$

for any choice of τ such that (47) holds, where $B^{\perp}(q)$ is a full rank left annihilator satisfying $B^{\perp}(q)\cdot B(q) = 0$.

Equation (48), the so-called matching condition, is a set of algebraic equations with respect to Λ and K_D, which can be solved by many mathematical toolboxes such as Maple and MATLAB. If a solution pair (Λ, K_D) of (48) is obtained, then it is easy to see that $G_0(q) + \Phi(q)\hat{\theta} - \Lambda(q - q^{(0)}) - K_D M^{-1}(q)p$ with this pair (Λ, K_D) can be expressed as

$$\begin{aligned} G_0(q) + \Phi(q)\hat{\theta} - \Lambda\left(q - q^{(0)}\right) - K_D M^{-1}(q)p \\ = b_1(q,p,\hat{\theta})\beta_1(q) + \cdots + b_m(q,p,\hat{\theta})\beta_m(q) \end{aligned} \quad (49)$$

where $\beta_1(q), \ldots, \beta_m(q)$ are column vectors of $B(q)$, i.e., $B(q) = [\beta_1(q) \cdots \beta_m(q)]$, and $b_i(q,p,\hat{\theta}), i = 1,2,\ldots,m$, are scalar functions.

Choosing pre-feedback law

$$\begin{cases} \tau = \tau(q,p,\hat{\theta}) = [b_1(q,p,\hat{\theta}),\ldots,b_m(q,p,\hat{\theta})]^T + u \\ := V(q,p,\hat{\theta}) + u \\ \dot{\hat{\theta}} = -\Gamma_0^{-1}\Phi^T(q)\dot{q} \end{cases} \quad (50)$$

we obtain

$$B(q)\tau = G_0(q) + \Phi(q)\hat{\theta} - \Lambda\left(q - q^{(0)}\right) - K_D M^{-1}(q)p + B(q)u$$

where u is the new control input.

With (36), (40)–(42), and (50), we have the following augmented Hamiltonian system:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\hat{\theta}} \end{bmatrix} = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & K_c(q,p) - K_D & \Phi(q)\Gamma_0^{-1} \\ 0 & -\Gamma_0^{-1}\Phi^T(q) & 0 \end{bmatrix} \times \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \Gamma_0(\hat{\theta} - \theta) \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \\ 0 \end{bmatrix} u \quad (51)$$

where $K_c(q,p)$ is given in (45).

From Proposition 4, system (51) can be rewritten as

$$\dot{X} = [J(X) - R(X)]\frac{\partial H}{\partial X} + g_c u \quad (52)$$

where $X, J(X), R(X)$, and g_c are the same as in the fully actuated case.

Theorem 2: Given a solution pair (Λ, K_D) of the constraint (48) for the underactuated case, then with the Hamiltonian function (36) and the pre-feedback (50), the underactuated system (24) can be expressed as an augmented Hamiltonian system given by (52), which is a dissipative PCH system.

Remark 8: Regardless of whether it is fully actuated or underactuated, system (24) has the same augmented Hamiltonian formulation (46) or (52).

Remark 9: The uncertainties in $M(q)$ and $C(q,\dot{q})$ seem disappeared in our Hamiltonian formulation, but it is not the case. In fact, they have been transferred into the Hamiltonian function (36).

Since system (46) [or (52)] is a dissipative Hamiltonian system and the Hamiltonian function (36) is positive definite, the system is naturally stable when $u = 0$. Summarizing this, we have the following result.

Proposition 5: System (46) [or (52)] is stable when $u = 0$, and the Hamiltonian function (36) can serve as a Lyapunov function for the system.

IV. Energy-Based Robust Adaptive Control Design

As an application, this section applies the augmented Hamiltonian formulation (46) [or (52)] to investigate the energy-based robust adaptive control of uncertain mechanical systems, and designs an adaptive L_2 disturbance attenuation controller for the systems.

Consider the uncertain system (24) with unknown disturbances, which is described as follows:

$$\begin{cases} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)\tau + w \\ z = W(q,\dot{q})B^T(q)\dot{q} \end{cases} \tag{53}$$

where $w \in \mathbb{R}^n$ is the disturbance, $z \in \mathbb{R}^s$ is the chosen penalty signal, $W(q,\dot{q}) \in \mathbb{R}^{s\times m}$ is a weighting matrix with full column rank, and $M(q), C(q,\dot{q})$ and $G(q)$ are still assumed to have unknown parameters.

Assume that Assumption 1 holds for $G(q)$, and $q^{(0)}$ is the target position (equilibrium) to be designed. According to Theorems 1 and 2 (in the underactuated case, we assume that a solution pair (Λ, K_D) of (48) can be obtained), under the following control law:

$$\begin{cases} \tau = \begin{cases} V(q,p,\hat{\theta}) + u, & \text{for } m < n \\ B^{-1}(q)\left[G_0(q) + \Phi(q)\hat{\theta} - \Lambda\left(q - q^{(0)}\right) - K_D\dot{q}\right] + u, & \text{for } m = n \end{cases} \\ \dot{\hat{\theta}} = -\Gamma_0^{-1}\Phi^T(q)\dot{q} \end{cases} \tag{54}$$

system (53) can be rewritten as

$$\begin{cases} \dot{X} = [J(X) - R(X)]\dfrac{\partial H}{\partial X} + g_c u + g_d w \\ z = W_1(X) g_c^T \dfrac{\partial H}{\partial X} \end{cases} \tag{55}$$

where $V(q,p,\hat{\theta})$ is given in (50), $g_d = [0, I_n, 0]^T \in \mathbb{R}^{(2n+l)\times n}, W_1(X) = W(q, M(q)p)$, and $X, J(X), R(X), H(X), \Lambda, g_c$ and K_D are the same as those in Section III.

Now, based on system (55) and [38, Th. 3.1], we design an L_2 disturbance attenuation controller, which can attenuate the disturbance w such that the L_2 gain from w to z is bounded by $\gamma > 0$, where γ is the given disturbance attenuation level. First, we check the condition of [38, Th. 3.1].

Let

$$\gamma^* = \inf\left\{\gamma > 0 \,\middle|\, K_D + \frac{1}{2\gamma^2}\left[B(q)B^T(q) - I_n\right] \geq 0\right\}. \tag{56}$$

It is easy to check that

$$R(X) + \frac{1}{2\gamma^2}\left[g_c g_c^T - g_d g_d^T\right] \geq 0$$

holds when $\gamma > \gamma^*$, which means that the condition of [38, Th. 3.1] is satisfied.

Thus, from [38, Th. 3.1], the desired robust control law for system (55) can be designed as

$$\begin{aligned} u &= -\frac{1}{2}\left[W_1^T(X)W_1(X) + \frac{1}{\gamma^2}I_m\right] g_c^T \frac{\partial H}{\partial X} \\ &= -\frac{1}{2}\left[W^T(q,\dot{q})W(q,\dot{q}) + \frac{1}{\gamma^2}I_m\right] B^T(q)\dot{q} \end{aligned} \tag{57}$$

and the γ-dissipation inequality

$$\dot{H} + \nabla^T H\left[R(X) + \frac{1}{2\gamma^2}(g_c g_c^T - g_d g_d^T)\right]\nabla H \leq \frac{1}{2}\{\gamma^2\|w\|^2 - \|z\|^2\}$$

holds along trajectories of the closed-loop system consisted of (55) and (57), where $\gamma > \gamma^*$ and $\nabla H := (\partial H)/(\partial X)$.

Substituting (57) into (54), we obtain a complete adaptive L_2 disturbance attenuation controller for system (53) as follows:

$$\begin{cases} \tau = \begin{cases} V(q,p,\hat{\theta}) - \dfrac{1}{2}\left[W^T(q,\dot{q})W(q,\dot{q}) + \dfrac{1}{\gamma^2}I_m\right] B^T(q)\dot{q}, \\ \quad \text{for } m < n \\ B^{-1}(q)\left[G_0(q) + \Phi(q)\hat{\theta} - \Lambda\left(q - q^{(0)}\right) - K_D\dot{q}\right] \\ \quad -\dfrac{1}{2}\left[W^T(q,\dot{q})W(q,\dot{q}) + \dfrac{1}{\gamma^2}I_m\right] B^T(q)\dot{q}, \\ \quad \text{for } m = n \end{cases} \\ \dot{\hat{\theta}} = -\Gamma_0^{-1}\Phi^T(q)\dot{q}. \end{cases} \tag{58}$$

Summarizing the above, we have the following theorem.

Theorem 3: Assume that Assumption 1 holds for the uncertain system (53) with $q^{(0)}$ as its target equilibrium, and that a solution pair (Λ, K_D) of the matching (48) can be obtained when the system is underactuated. Given a disturbance attenuation level $\gamma > 0$, if $\gamma > \gamma^*$, then an energy-based adaptive L_2 disturbance attenuation controller of the system can be designed as (58).

V. Illustrative Example

This section gives an illustrative example to show: 1) how to apply the new Hamiltonian formulation to rewrite uncertain mechanical systems as an augmented Hamiltonian system with dissipation and 2) how to use Theorem 3 and the achieved Hamiltonian structure to design an energy-based robust adaptive controller for uncertain mechanical systems.

Example 1: Consider a planar two-link manipulator with two revolute joints in the vertical plane, as shown in Fig. 1, where we suppose that the robot is holding a payload of unknown mass m_p, m_i and l_i are the mass and length of link i, l_{ci} is the distance from joint $(i-1)$ to the center of mass of link i, as indicated in the figure, and I_i is the moment of inertia of link i about an axis coming out to page through the center of the mass of link $i, i = 1, 2$ [11].

From [11], the dynamics of the planar two-link robot can be described as follows:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau + w \tag{59}$$

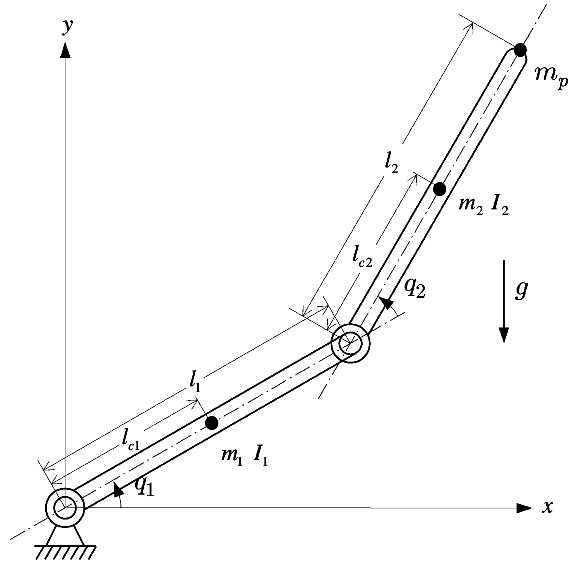

Fig. 1. Planar two-link manipulator with payload.

where $q = [q_1, q_2]^T \in \mathbb{R}^2$ is the angular position vector, $\tau = [\tau_1, \tau_2]^T \in \mathbb{R}^2$ is the control torque, $w \in \mathbb{R}^2$ is the disturbance

$$\begin{aligned} M(q) &= \begin{bmatrix} \bar{m}_1 + \bar{m}_2 + 2\bar{m}_3 \cos q_2 & \bar{m}_2 + \bar{m}_3 \cos q_2 \\ \bar{m}_2 + \bar{m}_3 \cos q_2 & \bar{m}_2 \end{bmatrix} \\ C(q, \dot{q}) &= \begin{bmatrix} -\bar{m}_3 \dot{q}_2 \sin q_2 & -\bar{m}_3(\dot{q}_1 + \dot{q}_2) \sin q_2 \\ \bar{m}_3 \dot{q}_1 \sin q_2 & 0 \end{bmatrix} \\ G(q) &= \begin{bmatrix} \bar{m}_4 g \cos q_1 + \bar{m}_5 g \cos(q_1 + q_2) \\ \bar{m}_5 g \cos(q_1 + q_2) \end{bmatrix} \\ \bar{m}_1 &= m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + m_p l_1^2 \\ \bar{m}_2 &= m_2 l_{c2}^2 + I_2 + m_p l_2^2 \\ \bar{m}_3 &= m_2 l_1 l_{c2} + m_p l_1 l_2 \\ \bar{m}_4 &= m_1 l_{c2} + m_2 l_1 + m_p l_1 \\ \bar{m}_5 &= m_2 l_{c2} + m_p l_2. \end{aligned}$$

Since the payload's mass m_p is unknown, it can be seen that none of $M(q), C(q,p)$, and $G(q)$ is exactly known. In the following, we use Theorem 1 to rewrite system (59) as an augmented Hamiltonian system.

Let $\theta := m_p$ denote the unknown parameter, then $G(q)$ can be decomposed as

$$G(q) = G_0(q) + \Phi(q)\theta \tag{60}$$

where

$$G_0(q) = \begin{bmatrix} (m_1 l_{c2} + m_2 l_1) g \cos q_1 + m_2 l_{c2} g \cos(q_1 + q_2) \\ m_2 l_{c2} g \cos(q_1 + q_2) \end{bmatrix} \tag{61}$$

$$\Phi(q) = \begin{bmatrix} l_1 g \cos q_1 + l_2 g \cos(q_1 + q_2) \\ l_2 g \cos(q_1 + q_2) \end{bmatrix} := \begin{bmatrix} \phi_1(q) \\ \phi_2(q) \end{bmatrix}. \tag{62}$$

Assume that $q^{(0)} = [q_1^{(0)}, q_2^{(0)}]^T \in \mathbb{R}^2$ is the target position of the system to be designed. Let $p = [p_1, p_2]^T = M(q)\dot{q}, \Lambda = \mathrm{Diag}\{\lambda_1, \lambda_2\} > 0$ and $\Gamma_0 = \lambda_3 > 0$. Choose

$$\begin{aligned} H(X) &= \mathcal{K}(q,p) + \mathcal{P}_g(q) + \frac{1}{2}(\hat{\theta} - \theta)^T \Gamma_0 (\hat{\theta} - \theta) \\ &= \frac{1}{2} p^T M^{-1}(q) p + \frac{1}{2}\left(q - q^{(0)}\right)^T \Lambda \left(q - q^{(0)}\right) \\ &\quad + \frac{\Gamma_0}{2}(\hat{\theta} - \theta)^2 \end{aligned} \tag{63}$$

as the Hamiltonian function, where $X = [q_1, q_2, p_1, p_2, \hat{\theta}]^T \in \mathbb{R}^5$, and $\hat{\theta}$ is the estimate of θ, which is determined by

$$\dot{\hat{\theta}} = -\Gamma_0^{-1} \Phi^T(q) \dot{q}. \tag{64}$$

Design a pre-feedback law as follows:

$$\begin{cases} \tau = G_0(q) + \Phi(q)\hat{\theta} - \Lambda\left(q - q^{(0)}\right) - K_D \dot{q} + u \\ \quad = \bar{G} + \Phi(q)\hat{\theta} + u, \\ \dot{\hat{\theta}} = -\Gamma_0^{-1}\Phi^T(q)\dot{q} \\ \quad = -\dfrac{1}{\lambda_3}(l_1 g \dot{q}_1 \cos q_1 + l_2 g(\dot{q}_1 + \dot{q}_2)\cos(q_1 + q_2)) \end{cases} \tag{65}$$

where we let $K_D = \mathrm{Diag}\{k_{d1}, k_{d2}\} > 0$, and

$$\bar{G} := \begin{bmatrix} (m_1 l_{c2} + m_2 l_1) g \cos q_1 + m_2 l_{c2} g \cos(q_1 + q_2) \\ -\lambda_1 \left(q_1 - q_1^{(0)}\right) - k_{d1}\dot{q}_1 \\ m_2 l_{c2} g \cos(q_1 + q_2) - \lambda_2 \left(q_2 - q_2^{(0)}\right) - k_{d2}\dot{q}_2 \end{bmatrix}.$$

From Theorem 1, with the Hamiltonian function (63) and the pre-feedback law (65), system (59) can be expressed as the following augmented Hamiltonian system:

$$\dot{X} = [J(X) - R(X)]\frac{\partial H}{\partial X} + g_c u + g_d w \tag{66}$$

where

$$J(X) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & \dfrac{\phi_1(q)}{\lambda_3} \\ 0 & -1 & 0 & 0 & \dfrac{\phi_2(q)}{\lambda_3} \\ 0 & 0 & -\dfrac{\phi_1(q)}{\lambda_3} & -\dfrac{\phi_2(q)}{\lambda_3} & 0 \end{bmatrix}$$

$$R(X) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{d1} & 0 & 0 \\ 0 & 0 & 0 & k_{d2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$g_c = g_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

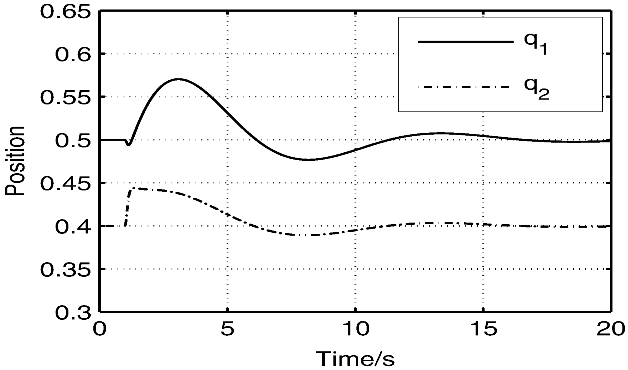

Fig. 2. Response of q under initial condition $X_0^{(1)}$.

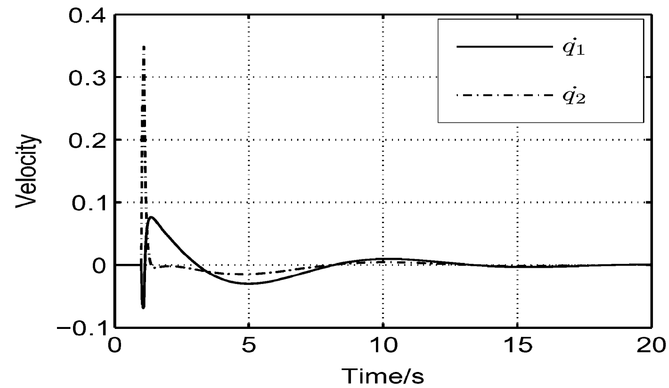

Fig. 3. Response of $\dot{q}$ under initial condition $X_0^{(1)}$.

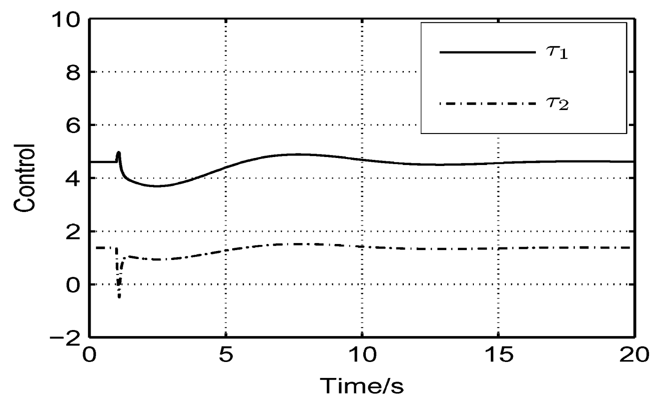

Fig. 4. Control τ under initial condition $X_0^{(1)}$.

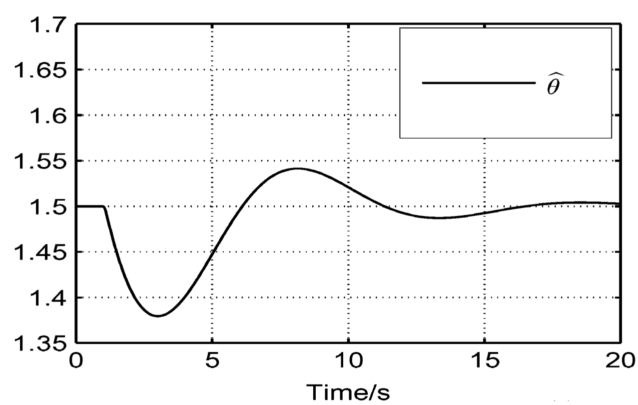

Fig. 5. Estimate $\hat{\theta}$ under initial condition $X_0^{(1)}$.

In the following, based on (66) and Theorem 3, we design an energy-based robust adaptive control law for system (59). Given a disturbance attenuation level $\gamma > 0$, choose the penalty signal z as follows:

$$z = W(q,\dot{q})g_c^T \frac{\partial H}{\partial X} = [r_1\dot{q}_1, r_2\dot{q}_2]^T \tag{67}$$

where the weighting matrix is chosen as $W(q,\dot{q}) = \mathrm{Diag}\{r_1, r_2\} > 0$.

Since $g_d = g_c, \gamma^* = 0$. On the other hand, Assumption 1 holds for the system [see (60)]. Thus, all the conditions of Theorem 3 are satisfied. According to Theorem 3, an energy-based adaptive L_2 disturbance attenuation controller of system (59) can be given as

$$\begin{cases} \tau = \bar{G} + \Phi(q)\hat{\theta} - \dfrac{1}{2}\left[W^T(q,\dot{q})W(q,\dot{q}) + \dfrac{1}{\gamma^2}I_2\right]\dot{q} \\ \dot{\hat{\theta}} = -\Gamma_0^{-1}\Phi^T(q)\dot{q} \end{cases} \tag{68}$$

i.e.,

$$\begin{cases} \tau_1 = (m_1 l_{c2} + m_2 l_1)g\cos q_1 + m_2 l_{c2} g\cos(q_1+q_2) \\ \qquad -\lambda_1\left(q_1 - q_1^{(0)}\right) - \left(k_{d1} + \dfrac{1}{2}r_1^2 + \dfrac{1}{2\gamma^2}\right)\dot{q}_1 + \phi_1(q)\hat{\theta} \\ \tau_2 = m_2 l_{c2} g\cos(q_1+q_2) - \lambda_2\left(q_2 - q_2^{(0)}\right) \\ \qquad -\left(k_{d2} + \dfrac{1}{2}r_2^2 + \dfrac{1}{2\gamma^2}\right)\dot{q}_2 + \phi_2(q)\hat{\theta} \\ \dot{\hat{\theta}} = -\dfrac{1}{\lambda_3}l_1 g\dot{q}_1\cos q_1 - \dfrac{1}{\lambda_3}l_2 g(\dot{q}_1+\dot{q}_2)\cos(q_1+q_2). \end{cases} \tag{69}$$

To show the effectiveness of the controller (69), two representative numerical simulations are investigated for the system whose physical parameters are chosen to be the same as those in [11]. For the two degree-of-freedom robot, we assign $q^{(0)} = [0.5, 0.4]^T$ as the target position for both of the following different initial conditions: $X_0^{(1)} = [0.5, 0.4, 0, 0]^T, X_0^{(2)} = [0, 0, 0, 0]^T$. The former is meant to test the robustness of the closed-loop system when it is subject to external disturbances, and the latter is to verify the effectiveness of the control scheme when the target position is different from the initial condition.

To test the robustness of the proposed controller, square disturbances of amplitude $[8, 6]^T$ are added to the system in the time duration $[1s - 1.1s]$.

Figs. 2–5 are the responses of the system for the case that the initial and target positions are chosen as the same, and at the same time, the system is subjected to the above disturbances. Figs. 2 and 3 show the convergence of both positions q and velocities $\dot{q}$, while Figs. 4 and 5 depict the control signals τ used here and the estimate $\hat{\theta}$ of the payload m_p, respectively.

Figs. 6–9 are the responses of the system for the case when the initial and the target positions are chosen differently, and at

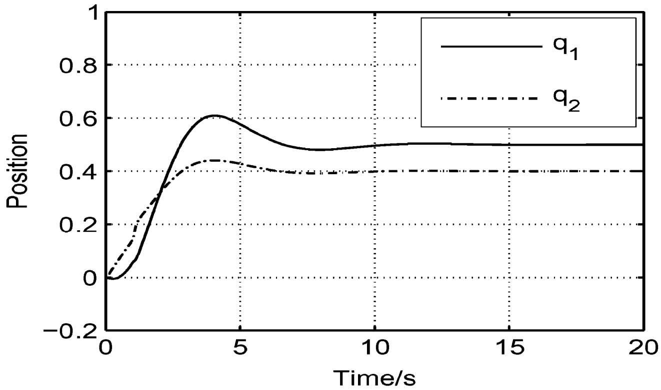

Fig. 6. Response of q under initial condition $X_0^{(2)}$.

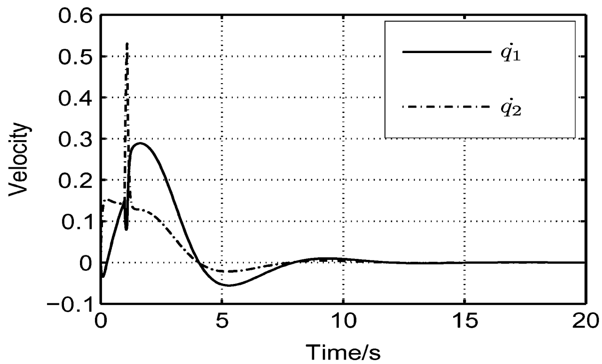

Fig. 7. Response of $\dot{q}$ under initial condition $X_0^{(2)}$.

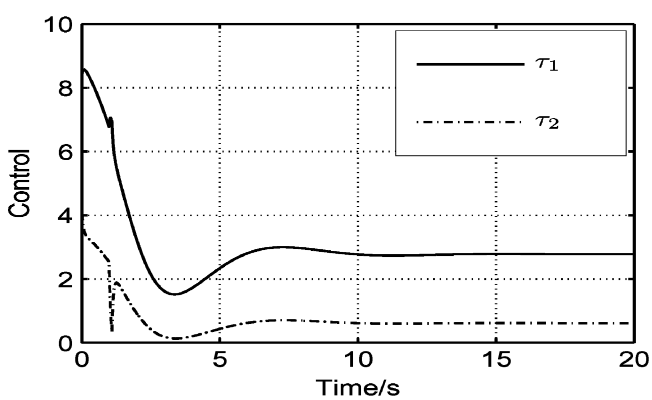

Fig. 8. Control τ under initial condition $X_0^{(2)}$.

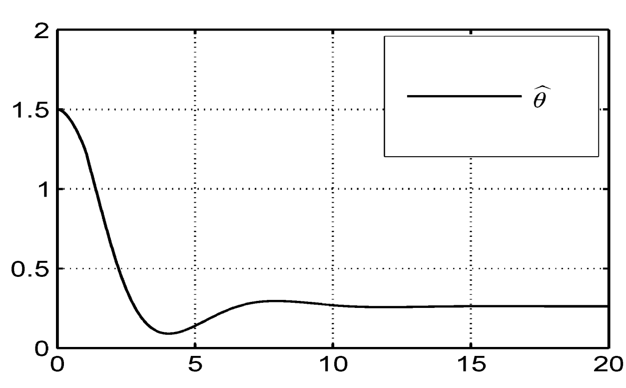

Fig. 9. Estimate $\hat{\theta}$ under initial condition $X_0^{(2)}$.

the same time the system is subjected to the above disturbances. Figs. 6 and 7 show the convergence of both positions q and velocities $\dot{q}$, while Figs. 8 and 9 depict the boundedness of the control signals τ and the parameter estimate $\hat{\theta}$ of the payload m_p, respectively.

From the numerical simulations shown in Figs. 2–9, we know that the energy-based robust adaptive controller (69) is very effective for the control of angular positions, as well as for handling both unknown parameters and external disturbances.

VI. Conclusion

In this paper, we have developed a new mathematical tool, called UPDO, which plays a very important role in our augmented Hamiltonian formulation of uncertain mechanical systems. With the help of UPDO and based on the shaping idea, an new augmented Hamiltonian structure with dissipation has been proposed for both fully actuated and underactuated uncertain mechanical systems. The new formulation has several advantages, i.e.: 1) it possesses some nice properties for further analysis and control design and 2) its matching condition is a set of algebraic equations, which are easier to solve than a set of matching PDEs. As an application, this paper has also investigated the energy-based robust adaptive control of uncertain mechanical systems by using the new Hamiltonian formulation, and presented a new energy-based adaptive L_2 disturbance attenuation controller for the systems. Study of the illustrative example with numerical simulations shows that the controller obtained in this paper is very effective in handling disturbances and unknown parameters.

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