

Tip Tracking Control of a Single-Link Flexible Robot: A Backstepping Approach

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Abstract. In this paper, tip tracking control is investigated for a single-link flexible robot. The flexible beam is first lumped to a spring-mass system, to which the so called *backstepping* approach is applicable, then two robust controllers and an adaptive controller are developed in the presence of system disturbances/uncertainties. The controllers are rigorously proven to be able to achieve stable tip position and velocity tracking control in the sense of *Global Uniform Ultimate Boundedness* (GUUB). Numerical simulation results are provided which show that the proposed controllers are effective.

Keywords: tracking control, flexible robot, backstepping

1. Introduction

Tip tracking control of robots with flexible links is difficult because of the nonminimum phase property of such systems. Normally the tip tracking problem of flexible robots is solved by transferring it into two sub-problems, i.e., (i) tracking of the joint motion (when the output is selected to be the joint instead of the tip, the system is of minimum phase), and (ii) suppression of the elastic vibrations of the flexible links. Such a description of tracking problem directly leads to the application of the Singular Perturbation (SP) method, which has been developed in [1] [2]. In SP method, the original system is two-time-scaled and divided into a slow sub-system (related to joint motion) and a fast sub-system (corresponding to elastic vibrations). Two sub-controllers are designed accordingly, in which the slow control is designed to make the joint angle track a pre-defined trajectory and the fast control to stabilize the elastic vibrations. SP approach is also combined with a feedback linearization procedure in [3] to give a smaller perturbation parameter and thus improve the control result.

Some tracking control approaches which have been used for rigid-link robots were also investigated for the flexible-link case, e.g., the model reference adaptive control [4] and the inverse dynamics (computed torque) method [5] [6]. It should be noted that the methods

are actually not directly applicable to flexible robots because (i) the number of inputs is less than the number of degrees of freedom of the system, and (ii) the system is of nonminimum phase. In [4], an assumption was made that the changing of the system matrices was much slower than the updating speed of the adaptive algorithm and thus some terms were simply neglected in the derivation to solve the tracking problem. As for the inverse dynamics method, a flexible robot system from joint input to tip output is theoretically not stably invertible because it is of nonminimum phase. The problem has been discussed in detail in both time domain [5] and frequency domain [6]. Inverse dynamics method seems to be able to result in better tip tracking performance over other techniques. However, the successful application of this method heavily depends on highly accurate models and efficient computational algorithms/powerful computing facilities.

In engineering practice, it is desirable to have a simple dynamic model which leads to easy controller design, implementation and tuning, if only the system performance is satisfactory and certain degree of robustness can be guaranteed. In this paper, we first simplify a single-link flexible robot system by lumping it into a spring-mass system. The lumped model will be shown to possess a form to which the backstepping approach [8] is applicable. The backstepping approach has been proved to be a powerful method in dealing with nonlinear uncertain systems. Using this method, many robust and adaptive controllers have been reported in the literature to solve the control problem of a large class of systems with high nonlinearity and various kinds of uncertainties, e.g., [20]–[22] and so on. Due to the fact that the dynamics of robots possess serious nonlinearity, and usually there exist parameter uncertainties in practical applications, the backstepping approach has also been investigated in control of robots. A very good survey for backstepping control of flexible joint robots can be found in [11]. In this paper, by lumping and simplification of the dynamics, it will be shown that the backstepping method is also applicable to flexible link robots.

In order to improve robustness of the controlled system, the probable disturbances and uncertainties and the neglected part in the lumping procedure are also included in the lumped model as some bounded terms. Then, based on the model, three backstepping controllers are developed to directly control the tip position and its velocity to track some pre-defined trajectories. It will be shown that the three controllers (two of them are of robust type and another is an adaptive controller) can achieve stable tracking control in the sense of Global Uniform Ultimate Boundedness (GUUB) [12]. Numerical simulations are provided to verify the effectiveness of the presented controllers.

The rest of the paper is organized as follows: the simplified dynamic model is presented in Section 2, design of the backstepping controllers is discussed in Section 3, and followed by simulation tests in Section 4 and conclusion remarks in Section 5.

2. Simplified Dynamic Model

This section considers lumping a single-link flexible robot into a simple spring-mass system. Lumping is a commonly used method which can reasonably simplify the dynamic model of flexible robots. For example, in [13], an adaptive tip position controller was presented for a multi-link flexible robot based on a lumped model, in which each flexible link is lumped into two rigid pieces connected by a fictitious joint. While in [7], the flexible link of the

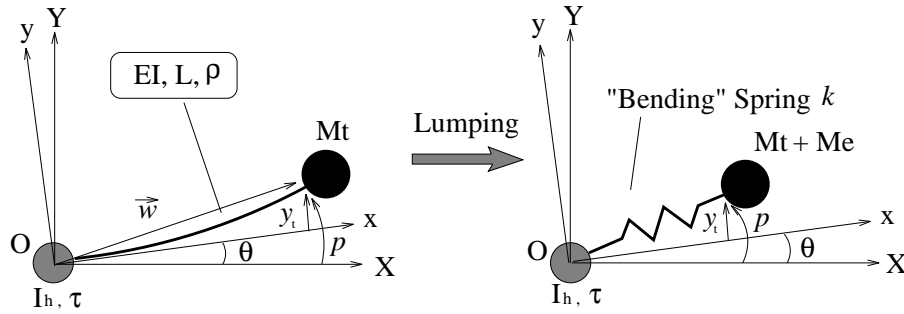


Figure 1. Lumping of a single-link flexible beam.

robot is lumped into a cascade of spring-mass units based on modal analysis of a clamped-free vibrating beam. Each spring-mass unit provides a DOF. The springs, representing the flexibility of the original link, are assumed to be weightless and linear. Simultaneously, the concentrated masses are used to represent the original distributed mass of the link. The equivalent spring constants are determined such that the strain energy of a clamped-free vibrating beam (suppose there is no rigid motion) is equal to the elastic potential energy stored in the springs; and the equivalent concentrated masses are such that the kinetic energy of the vibrating beam is equal to that of the masses. The most dominant flexible mode is considered in [7], which implies that the lumped model is able to describe the main dynamic behaviour of the original system. The lumping process which we shall introduce here is a simplified case of that in [7], for the flexible link will be lumped into a *single* spring-mass unit. As mentioned above, the method is based on the modal analysis of a clamped-free vibrating beam (no rigid motion), and can be briefly described as (i) lumping the distributed mass of the flexible beam to a point mass located at its tip, and (ii) representing the flexibility of the beam by a weightless linear “bending” spring.

On the other hand, as stated in [8], backstepping approach can be used to handle a nonlinear system in the absence of matching conditions. However, the system is required to have the following “strict feedback form”:

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1, x_2) \\
 \dot{x}_2 &= x_3 + f_2(x_1, x_2) \\
 &\vdots \\
 \dot{x}_n &= u + f_n(x_1, x_2, \dots, x_n)
 \end{aligned}$$

In general, the models of flexible manipulators obtained from the traditional Assumed Modes Method or Finite Element Method do not possess such a form. In order to use the backstepping method, we shall, under the assumption of small deflection, use the aforementioned lumping method to simplified the single-link flexible robot system shown in Fig. 1.

In the figure, the left part represents the original flexible manipulator, which is constrained at the rotor of a motor and moves in the horizontal plane, while the right part is the lumped spring-mass system. XOY is the fixed base frame and xOy is the local reference frame rotating with the hub. System parameters and variables are defined as: L , the length of the flexible beam; EI , the uniform flexural rigidity of the beam; M_t , the concentrated mass tip payload; M_e , the equivalent lumped mass of the beam located at its tip; k , the equivalent bending spring constant; ρ , the uniform weight per unit length of the beam; I_h , the hub inertia; τ , the control torque; θ , the rotation angle of the hub; y_t , tip deflection measured from the undeformed beam; p , arc approximation of tip position, $p = L\theta + y_t$; and \vec{w} , tip position vector with respect to the base frame XOY .

Firstly, the distributed mass of the flexible link is lumped to an equivalent point mass M_e at the tip. In doing this, we shall assume that the flexible beam undergoes no rigid motion, i.e., the motor is locked such that $\dot{\theta} \equiv 0$, which implies that the flexible modes obtained here are the constrained assumed modes [15]. This assumption eliminates the effect of I_h in the lumping procedure, but the advantage is that the mode shape functions of the vibrating beam and subsequently the equivalent mass M_e and spring constant k can be analytically determined. If rigid motion is permitted, the flexible modes (known as the unconstrained assumed modes [15] in the literature) can be identified experimentally by inputting sinusoidal voltages with different frequencies to the motor [16]. However, the analytical form of the mode shape functions and hence M_e and k are difficult to obtain.

Consider only the most dominant flexible mode, the deflection of the beam can be approximately given by:

$$y(x, t) = F(x)q(t) \quad (2.1)$$

where $q(t)$ is the most dominant mode (with lowest vibration frequency), and $F(x)$ is the corresponding clamped-free mode shape function given by [9]

$$F(x) = \frac{1}{A} \left[\cosh\left(\frac{\beta}{L}x\right) - \cos\left(\frac{\beta}{L}x\right) - \gamma \left(\sinh\left(\frac{\beta}{L}x\right) - \sin\left(\frac{\beta}{L}x\right) \right) \right] \quad (2.2)$$

$$0 \leq x \leq L$$

where A is a constant, and

$$\gamma = \frac{\cosh(\beta) + \cos(\beta)}{\sinh(\beta) + \sin(\beta)}$$

with β is the minimum positive solution of the following equation

$$1 + \cosh(\beta) \cos(\beta) + \frac{M_t \beta}{\rho L} [\sinh(\beta) \cos(\beta) - \cosh(\beta) \sin(\beta)] = 0$$

When there is no rigid motion, equating the kinetic energy of the lumped mass M_e and that of the original flexible link, i.e.,

$$\frac{1}{2} M_e \dot{y}_t^2 = \frac{\rho}{2} \int_0^L \dot{y}^2(x, t) dx \quad (2.3)$$

and substituting (2.1) into (2.3), we arrive at

$$M_e = \frac{\rho \int_0^L F^2(x) dx}{F^2(L)}$$

Here, we should mention that many approaches have been introduced in [17] for lumping distributed mass. We have selected the kinetic-energy method, i.e., as stated above, letting the kinetic energy of the equivalent lumped mass equal that of the vibrating flexible link. The reason we choose this method is that if the true mode shape function under certain boundary conditions (clamped-free boundary conditions are considered here) is used, the kinetic-energy method will yield the true value of the lumped mass, i.e., the true solution under certain boundary conditions. This implies that the true value of lumped mass of, for example, a pinned-free beam can be obtained if the true pinned-free mode shape function is used. In the case when we do not know the exact mode shape function (such as the case of a beam with nonuniform and unknown flexural rigidity), some approximated mode shape functions may be constructed from experiments to calculate the equivalent mass, however, they may not provide the true value. The selection of different types of boundary conditions depends on the configuration of the practical flexible robot system [18], which subsequently leads to the different mode shape function used in the lumping.

The next is to deal with the flexibility of the beam. In [13], this is done by introducing a fictitious joint between the two rigid pieces of each link. Here, the flexibility of the beam is represented by a linear “bending” spring. Such a spring is actually very similar to the flexible beam in Fig. 1, except that its elastic potential energy is assumed to be $ky_t^2/2$ with k being the equivalent spring constant. In [7], the idea of linear “angular” spring, instead of the “bending” spring is used. An “angular” spring is such that its elastic potential energy is in proportion to the square of the tip rotation of the beam (which is defined by the tangent to the beam at its tip), instead of y_t^2 . We introduce the bending spring by observing that y_t is comparatively easy to obtain in practice.

By equating the potential energy of the “bending” spring and that of the original flexible beam, we have

$$\begin{aligned} \frac{1}{2}EI \int_0^L [y''(x, t)]^2 dx &= \frac{1}{2}ky_t^2 \\ \Rightarrow \frac{1}{2}EIq^2(t) \int_0^L [F''(x)]^2 dx &= \frac{1}{2}kq^2(t)F^2(L) \\ \Rightarrow k &= EI \frac{\int_0^L [F''(x)]^2 dx}{F^2(L)} \end{aligned}$$

Now, we have obtained the equivalent mass M_e and spring constant k , through which the original system reduces to a simple spring-mass system. In the lumping procedure above, we consider the most dominant flexible mode. Since it has been shown in [19] that the natural frequencies calculated by the constrained assumed modes are very close to the experimental values, we can expect that the resulting lumped model is able to describe the main dynamic behaviour of the original system.

In the following, we shall derive the dynamic model of the lumped system shown by the right part in Fig. 1.

The vector of tip position \vec{w} can be given by:

$$\vec{w} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} L \\ y_t \end{bmatrix}$$

Note that the “bending” spring is assumed to be weightless, the total kinetic energy of the system is given by:

$$E_k = \frac{1}{2} I_h \dot{\theta}^2 + \frac{1}{2} m \dot{w}^T \dot{w} \quad (2.4)$$

where $m = M_t + M_e$. The potential energy of the “bending” spring, as stated above, is given by:

$$E_p = \frac{1}{2} k y_t^2 \quad (2.5)$$

Using the well-known Lagrange Equations, we arrive at

$$\begin{cases} m(\ddot{y}_t + L\ddot{\theta}) - m y_t \dot{\theta}^2 = -k y_t \\ (I_h + m y_t^2 + m L^2)\ddot{\theta} + 2m y_t \dot{y}_t \dot{\theta} + m L \ddot{y}_t = \tau \end{cases} \quad (2.6)$$

Substituting the first equation in (2.6) into the second yields

$$\begin{cases} m(\ddot{y}_t + L\ddot{\theta}) - m y_t \dot{\theta}^2 = -k y_t \\ (I_h + m y_t^2)\ddot{\theta} - k L y_t + m y_t (L \dot{\theta}^2 + 2\dot{\theta} \dot{y}_t) = \tau \end{cases} \quad (2.7)$$

Further, to improve robustness of the controllers that we shall design based on the lumped system, the effects of system uncertainties and disturbances, and the part which may have been neglected in the lumping procedure are also included as two bounded terms D_1 and D_2 . Thus, we obtain

$$\begin{cases} m(\ddot{y}_t + L\ddot{\theta}) - m y_t \dot{\theta}^2 + D_1 = -k y_t \\ (I_h + m y_t^2)\ddot{\theta} - k L y_t + m y_t (L \dot{\theta}^2 + 2\dot{\theta} \dot{y}_t) + D_2 = \tau \end{cases} \quad (2.8)$$

through which backstepping controller design will be carried out in the next section.

In [10] [11], backstepping approach was used to develop controllers for Rigid Link Flexible Joint (RLFJ) robot. It may appear that the model given in (2.8) is quite similar to their RLFJ robot model. However, there actually exist significant differences, i.e., (i) the coefficient of $\ddot{\theta}$ in the second equation in (2.8) contains a term of y_t^2 , while the coefficient of \ddot{q}_m (q_m is the actuator displacement) in the RLFJ model is constant, and (ii) the left hand side of the first equation in (2.8) is dependent on $\dot{\theta}^2$, while no such very nonlinear term of q_m exists in the first equation of the RLFJ model. Here, we should note that θ in (2.8) corresponds to q_m in the RLFJ model. Further, model (2.8) itself also exhibits more serious nonlinearity than the RLFJ model does.

3. Design of Backstepping Controllers

The backstepping approach in [10] [11] is of a “two-step” type as it takes two steps of backstepping integral procedure to explicitly arrive at the physical controller. Actually, the two-step backstepping procedure can be applied to our case as well. However, we shall present in this paper a “one-step” backstepping approach to design controllers for the single-link flexible robot. Since our approach needs only one step of backstepping procedure, the corresponding derivation and the resulting controllers are also simplified.

The controllers can achieve stable tip position and tip velocity tracking in the sense of GUUB under some types of acceleration feedback. It has been shown in [10] [11] that the acceleration feedback is not necessary in RLFJ robot case. However, the acceleration feedback, though quite undesirable, cannot be totally removed in our case. This is mainly due to the existence of the very nonlinear term $y_t \dot{\theta}^2$ in the first equation in (2.8). As a matter of fact, if we similarly carry out a two-step backstepping procedure to model (2.8), there also requires the measurement of $\dot{\theta}$, which is generated from the term $y_t \dot{\theta}^2$.

The control objective is to control the combined output, i.e. the approximated tip position $p = L\theta + y_t$, to track a pre-defined desired trajectory p_d . The tracking error is denoted by

$$e = p - p_d \quad (3.1)$$

and a filtered tracking error to be considered is defined as

$$r = \dot{e} + \lambda e \quad (3.2)$$

where $\lambda > 0$. Before further derivations, the following assumptions are made:

1. D_1 and D_2 are bounded by d_1 and d_2 respectively, i.e.,

$$\|D_1\| \leq d_1, \quad \|D_2\| \leq d_2$$

where d_1 and d_2 are supposed to be known positive scalars.

2. p_d is continuous and time-differentiable up to the 3rd order.

From (3.1), (3.2) and model (2.8), we have

$$\begin{aligned} m\dot{r} &= m(\ddot{y}_t + L\ddot{\theta}) - m\ddot{p}_d + m\lambda\dot{e} \\ &= [-ky_t + my_t\dot{\theta}^2 - m\ddot{p}_d + m\lambda\dot{y}_t - m\lambda\dot{p}_d] + m\lambda L\dot{\theta} - D_1 \\ &= \Delta_1 - m\lambda Lu_r + m\lambda L\eta - D_1 \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} \eta &= u_r + \dot{\theta} \\ \Delta_1 &= -ky_t + my_t\dot{\theta}^2 - m\ddot{p}_d + m\lambda\dot{y}_t - m\lambda\dot{p}_d \end{aligned} \quad (3.4)$$

and u_r is the embedded controller to be designed later. Using the second equation in (2.8), we can write the equation of $\dot{\eta}$ as

$$\begin{aligned} (I_h + my_t^2)\dot{\eta} &= [(I_h + my_t^2)\dot{u}_r + kLy_t - my_t(L\dot{\theta}^2 + 2\dot{\theta}\dot{y}_t) + m\lambda Lr + \eta my_t\dot{y}_t] \\ &\quad - m\lambda Lr - \eta my_t\dot{y}_t - D_2 + \tau \\ &= \Delta_2 - m\lambda Lr - \eta my_t\dot{y}_t - D_2 + \tau \end{aligned} \quad (3.5)$$

where

$$\Delta_2 = (I_h + my_t^2)\dot{u}_r + kLy_t - my_t(L\dot{\theta}^2 + 2\dot{\theta}\dot{y}_t) + m\lambda Lr + \eta my_t \dot{y}_t \quad (3.6)$$

Now, we can stop the backstepping procedure because the physical control input τ appears in (3.5) explicitly. The complete system for controller design now can be rewritten as

$$m\dot{r} = \Delta_1 - m\lambda Lu_r + m\lambda L\eta - D_1 \quad (3.7)$$

$$(I_h + my_t^2)\dot{\eta} = \Delta_2 - m\lambda Lr - \eta my_t \dot{y}_t - D_2 + \tau \quad (3.8)$$

Based on equations (3.7) and (3.8), three different controllers are presented in the following subsections.

3.1. Robust Backstepping Controller Design

The first robust controller we would like to present is given by:

$$\begin{cases} u_r = \frac{1}{m\lambda L}(\Delta_1 + k_1 r + \frac{rd_1^2}{(\sqrt{r^2 + \sigma} - \sqrt{\sigma})d_1 + \epsilon_1}) \\ \tau = -k_2 \eta - \Delta_2 - \text{sgn}(\eta)d_2 \end{cases} \quad (3.9)$$

where $k_1, k_2, \sigma, \epsilon_1 > 0$ are selected by designers, and $\text{sgn}(\ast)$ is the normal sign function. The GUUB of the closed-loop system (2.8) with control (3.9) can be guaranteed by the following theorem.

THEOREM 1 *The closed-loop system (2.8) and (3.9) is Globally Uniformly Ultimately Bounded (GUUB) in the sense that the tracking errors are ultimately confined in a ball of limit size.*

Proof: Choosing a diagonal and positive definite matrix

$$P = \begin{bmatrix} m & \\ & I_h + my_t^2 \end{bmatrix}$$

and construct a positive definite function as

$$V = \frac{1}{2} X^T P X \quad (3.10)$$

with $X = [r \ \eta]^T$. From (3.7), (3.8) and (3.9), the time derivative of V is given by:

$$\begin{aligned} \dot{V} &= rm\dot{r} + \eta(I_h + my_t^2)\dot{\eta} + \eta^2 my_t \dot{y}_t \\ &= r \left[-k_1 r - \frac{rd_1^2}{(\sqrt{r^2 + \sigma} - \sqrt{\sigma})d_1 + \epsilon_1} - D_1 \right] + \eta \left[-k_2 \eta - \text{sgn}(\eta)d_2 - D_2 \right] \\ &= -k_1 r^2 - k_2 \eta^2 - r \left[D_1 + \frac{rd_1^2}{(\sqrt{r^2 + \sigma} - \sqrt{\sigma})d_1 + \epsilon_1} \right] - \eta \left[D_2 + \text{sgn}(\eta)d_2 \right] \end{aligned}$$

$$\begin{aligned}
&\leq -k_1 r^2 - k_2 \eta^2 - \frac{r D_1 (\sqrt{r^2 + \sigma} - \sqrt{\sigma}) d_1 + r D_1 \epsilon_1 + r^2 d_1^2}{(\sqrt{r^2 + \sigma} - \sqrt{\sigma}) d_1 + \epsilon_1} \\
&\leq -k_1 r^2 - k_2 \eta^2 + \frac{|r| d_1 \epsilon_1}{(\sqrt{r^2 + \sigma} - \sqrt{\sigma}) d_1 + \epsilon_1} \\
&< -k_1 r^2 - k_2 \eta^2 + \epsilon_1
\end{aligned} \tag{3.11}$$

As has been shown in [10], such a form of \dot{V} implies the vector X is Globally Uniformly Ultimately Bounded, i.e., $\|X\|$ is ultimately confined in a ball of size $\sqrt{\frac{\epsilon_1}{\zeta_1 \zeta_2}}$ when $t \rightarrow \infty$, in which $\zeta_1 = \min(m, I_h)$, and $\zeta_2 = \min(k_1, k_2)$. Therefore, $\|r\|$, and subsequently $\|e\|$ (tip position tracking) and $\|\dot{e}\|$ (tip velocity tracking) are all ultimately bounded by a ball of limited size. For a certain system, a smaller convergence ball can be obtained by selecting larger k_1, k_2 or smaller ϵ_1 . ■

To realize the above robust controller, we have to obtain the time derivative of the embedded controller \dot{u}_r , and subsequently $\dot{\Delta}_1$ and \dot{r} . From (3.2) and (3.4), it can be seen that acceleration-feedback of $\ddot{\theta}$ and \ddot{y}_t are needed.

Acceleration measurements are difficult to obtain if not impossible, especially for the \ddot{y}_t in this case. In the following, another robust controller is proposed without the need for feedback of \ddot{y}_t under the assumption that Δ_2 is bounded by a bounding function d_3

$$\|\Delta_2\| \leq d_3$$

Thus, we arrive at the second robust backstepping controller

$$\begin{cases} u_r = \frac{1}{m\lambda L} (\Delta_1 + k_1 r + \frac{r d_1^2}{(\sqrt{r^2 + \sigma} - \sqrt{\sigma}) d_1 + \epsilon_1}) \\ \tau = -k_2 \eta - \text{sgn}(\eta) d_3 - \text{sgn}(\eta) d_2 \end{cases} \tag{3.12}$$

It can be shown by the following theorem that the system (2.8) with control (3.12) is GUUB in the sense that the tracking errors are ultimately bounded by a ball of the same size as the bounding ball of the first controller.

THEOREM 2 *The closed-loop system (2.8) and (3.12) is Globally Uniformly Ultimately Bounded in the sense that the tracking errors are ultimately confined in a ball of limit size.*

Proof: Using the same Lyapunov candidate (3.10), its time derivative becomes

$$\begin{aligned}
\dot{V} &= r m \dot{r} + \eta (I_h + m y_t^2) \dot{\eta} + \eta^2 m y_t \dot{y}_t \\
&= -k_1 r^2 - k_2 \eta^2 - r \left[D_1 + \frac{r d_1^2}{(\sqrt{r^2 + \sigma} - \sqrt{\sigma}) d_1 + \epsilon_1} \right] \\
&\quad - \eta [D_2 + \text{sgn}(\eta) d_2] + \eta [\Delta_2 - \text{sgn}(\eta) d_3] \\
&\leq -k_1 r^2 - k_2 \eta^2 + \frac{|r| d_1 \epsilon_1}{(\sqrt{r^2 + \sigma} - \sqrt{\sigma}) d_1 + \epsilon_1} \\
&< -k_1 r^2 - k_2 \eta^2 + \epsilon_1
\end{aligned} \tag{3.13}$$

Similarly, X is GUUB and $\|X\|$ is ultimately bounded in a ball of size $\sqrt{\frac{\epsilon_1}{\zeta_1 \zeta_2}}$, which is of the same size as the ball obtained in the first robust controller for $\zeta_1 = \min(m, l_h)$, and $\zeta_2 = \min(k_1, k_2)$. Therefore, tip position and velocity tracking errors $\|e\|$ and $\|\dot{e}\|$ are ultimately confined in a ball of limit size, and larger k_1, k_2 or smaller ϵ_1 give a smaller convergence ball. ■

To realize the second robust controller, we only need to obtain the *bounding function* of Δ_2 , i.e. d_3 , instead of itself, and subsequently we need to calculate some bounding functions of $\dot{\Delta}_1$ and \dot{r} . A bounding function of \dot{r} can be easily obtained from (3.7) as follows:

$$\|\dot{r}\| \leq \frac{1}{m} \|\Delta_1 - m\lambda Lu_r + m\lambda L\eta\| + \frac{d_1}{m} \quad (3.14)$$

For the bounding function of $\dot{\Delta}_1$, in order to remove the acceleration feedback of \ddot{y} , we rewrite (3.4) as

$$\begin{aligned} \Delta_1 = & -ky_t + my_t\dot{\theta}^2 - m\ddot{p}_d \\ & + m\lambda(\dot{y}_t + L\dot{\theta}) - m\lambda\dot{p}_d - m\lambda L\dot{\theta} \end{aligned}$$

In combination of the first equation in (2.8), $\dot{\Delta}_1$ is given by

$$\begin{aligned} \dot{\Delta}_1 = & -k\dot{y}_t + m\dot{y}_t\dot{\theta}^2 + 2my_t\dot{\theta}\ddot{\theta} - mp_d^{(3)} \\ & + \lambda(my_t\dot{\theta}^2 - ky_t - D_1) - m\lambda\ddot{p}_d - m\lambda L\ddot{\theta} \end{aligned}$$

and a bounding function of $\dot{\Delta}_1$ can be given by

$$\|\dot{\Delta}_1\| \leq \|-k\dot{y}_t + m\dot{y}_t\dot{\theta}^2 + 2my_t\dot{\theta}\ddot{\theta} - mp_d^{(3)} + \lambda(my_t\dot{\theta}^2 - ky_t) - m\lambda\ddot{p}_d - m\lambda L\ddot{\theta}\| + \lambda d_1 \quad (3.15)$$

One can see that $\dot{\Delta}_1$ can be bounded by a bounding function independent of \ddot{y}_t , while the acceleration feedback of $\ddot{\theta}$ is still there. RLFJ robot was considered in [10] [11], in which the corresponding bounding functions were shown totally independent of acceleration or jerk terms. However, it is not difficult to check, that if the two-step backstepping procedure, which is similar to that in [10] [11] is applied here, we would have to calculate the time derivative of a bounding function of $y_t\dot{\theta}^2$. This implies that $\ddot{\theta}$ is still needed. Therefore, measurement of $\ddot{\theta}$ is inevitable for both one-step and two-step backstepping approaches. We reduce the backstepping procedure by one step to simplify the derivation and the resulting controllers, and also reduce the computational burden.

3.2. Adaptive Controller Design

In developing the above two robust controllers, the effect of system parameter uncertainties is considered to be included in the bounded terms D_1 and D_2 . It will be shown in the following that adaptive schemes can be introduced to handle the parameter uncertainties. In [11], an adaptive backstepping controller has been introduced for tracking control of

RLFJ robots, in which *no* system disturbances were taken into consideration. In this paper, an adaptive backstepping controller is developed in the presence of bounded terms D_1 and D_2 to improve robustness of system.

To derive the adaptive controller, we first rewrite (3.7) and (3.8) into the following form

$$\begin{cases} m\dot{r} = \Delta_1 - m\lambda Lu_r + m\lambda L\eta - D_1 \\ (I_h + my_t^2)\dot{\eta} = \Delta_2 - \eta my_t \dot{y}_t - D_2 + \tau \end{cases} \quad (3.16)$$

in which the term $(-m\lambda Lr)$ in (3.8) disappears and subsequently Δ_2 is given by

$$\Delta_2 = (I_h + my_t^2)\dot{u}_r + kLy_t - my_t(L\dot{\theta}^2 + 2\dot{\theta}\dot{y}_t) + \eta my_t \dot{y}_t \quad (3.17)$$

Dividing by $(m\lambda L)$ on both sides of the first equation in (3.16), and noting that $\frac{\Delta_1}{m\lambda L}$ and Δ_2 can be written into the following linear-in-parameters forms:

$$\frac{\Delta_1}{m\lambda L} = \delta_1^T f_1, \quad \Delta_2 = \delta_2^T f_2$$

where δ_1, δ_2 are uncertain parameter vectors and f_1, f_2 are vectors of known functions:

$$\begin{aligned} \delta_1 &= \left[\frac{k}{m\lambda L} \quad \frac{1}{\lambda L} \right]^T \\ f_1 &= [-y_t \quad y_t \dot{\theta}^2 - \ddot{p}_d + \lambda \dot{y}_t - \lambda \dot{p}_d]^T \\ \delta_2 &= [I_h \quad m \quad kL \quad mL]^T \\ f_2 &= [\dot{u}_r \quad y_t^2 \dot{u}_r - 2y_t \dot{\theta} \dot{y}_t + \eta y_t \dot{y}_t \quad y_t \quad -y_t \dot{\theta}^2]^T \end{aligned}$$

System (3.16) can be further written as

$$\begin{cases} \frac{1}{\lambda L} \dot{r} = \delta_1^T f_1 + \eta - \frac{D_1}{m\lambda L} - u_r \\ (I_h + my_t^2)\dot{\eta} = \delta_2^T f_2 - \eta my_t \dot{y}_t - D_2 + \tau \end{cases} \quad (3.18)$$

Letting $\hat{\delta}_1$ and $\hat{\delta}_2$ denote the estimates of δ_1 and δ_2 , the error vectors of parameters are given by:

$$\tilde{\delta}_1 = \hat{\delta}_1 - \delta_1, \quad \tilde{\delta}_2 = \hat{\delta}_2 - \delta_2$$

We construct the adaptive controller as follows:

$$\begin{cases} u_r = \hat{\delta}_1^T f_1 + \frac{rv^2}{(\sqrt{r^2 + \sigma} - \sqrt{\sigma})v + \epsilon_1} - k_1 r \\ \tau = -\hat{\delta}_2^T f_2 - \text{sgn}(\eta)d_2 - k_2 \eta - r \end{cases} \quad (3.19)$$

where $\sigma, \epsilon_1, k_1, k_2 > 0$ are positive constants selected by designers, and $v = d_1/m_{min}\lambda L_{min}$, with m_{min} and L_{min} are the minimum tip mass and the minimum beam length, respectively, and the adaptive law is given by

$$\begin{cases} \dot{\hat{\delta}}_1 = r\Gamma_1 f_1 - \sigma_1 \Gamma_1 \hat{\delta}_1 \\ \dot{\hat{\delta}}_2 = \eta \Gamma_2 f_2 - \sigma_2 \Gamma_2 \hat{\delta}_2 \end{cases} \quad (3.20)$$

where $\sigma_1, \sigma_2 > 0$ are two positive scalars, $\Gamma_1 = \Gamma_1^T > 0$ and $\Gamma_2 = \Gamma_2^T > 0$ are two symmetric positive definite matrices with sizes of 2×2 and 4×4 , respectively. The adaptive laws (3.20) are the so-called σ -modification laws in the literature as introduced in [14]. Such adaptive laws, as has been analyzed in [14], are able to handle systems with disturbances. The stability of the closed-loop system (3.16) and (3.19) with the adaptive law (3.20) is stated in the following theorem.

THEOREM 3 *The system (3.16) with controller (3.19) and σ -modification adaptive law (3.20) is Globally Uniformly Ultimately Bounded in the sense that signals $X = [r \ \eta]^T$, $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are ultimately bounded by a limit compact region.*

Proof: Give a positive definite matrix P

$$P = \begin{bmatrix} 1/\lambda L & \\ & I_h + my_t^2 \end{bmatrix} > 0$$

and select Lyapunov candidate as:

$$V = \frac{1}{2} \tilde{\delta}_1^T \Gamma_1^{-1} \tilde{\delta}_1 + \frac{1}{2} \tilde{\delta}_2^T \Gamma_2^{-1} \tilde{\delta}_2 + \frac{1}{2} X^T P X$$

we have

$$\begin{aligned} \dot{V} &= \tilde{\delta}_1^T \Gamma_1^{-1} \dot{\tilde{\delta}}_1 + \tilde{\delta}_2^T \Gamma_2^{-1} \dot{\tilde{\delta}}_2 + r \frac{1}{\lambda L} \dot{r} + \eta (I_h + my_t^2) \dot{\eta} + \eta^2 my_t \dot{y}_t \\ &= \tilde{\delta}_1^T \Gamma_1^{-1} \dot{\tilde{\delta}}_1 + \tilde{\delta}_2^T \Gamma_2^{-1} \dot{\tilde{\delta}}_2 + r \left[\delta_1^T f_1 + \eta - \frac{D_1}{m\lambda L} - u_r \right] \\ &\quad + \eta \left[\delta_2^T f_2 - r - \eta my_t \dot{y}_t - D_2 + \tau + r \right] + \eta^2 my_t \dot{y}_t \\ &= \tilde{\delta}_1^T \Gamma_1^{-1} \dot{\tilde{\delta}}_1 + \tilde{\delta}_2^T \Gamma_2^{-1} \dot{\tilde{\delta}}_2 + r \left[\delta_1^T f_1 - \frac{D_1}{m\lambda L} - u_r \right] \\ &\quad + \eta \left[\delta_2^T f_2 - D_2 + \tau + r \right] \end{aligned} \quad (3.21)$$

Substituting (3.20) into (3.21) gives

$$\dot{V} = r \hat{\delta}_1^T f_1 + \eta \hat{\delta}_2^T f_2 - r \left[\frac{D_1}{m\lambda L} + u_r \right] + \eta [-D_2 + \tau + r] - \sigma_1 \tilde{\delta}_1^T \hat{\delta}_1 - \sigma_2 \tilde{\delta}_2^T \hat{\delta}_2$$

Using the controller (3.19), we have

$$\begin{aligned} \dot{V} &= -k_1 r^2 - k_2 \eta^2 - \sigma_1 \tilde{\delta}_1^T (\tilde{\delta}_1 + \delta_1) - \sigma_2 \tilde{\delta}_2^T (\tilde{\delta}_2 + \delta_2) + \epsilon_1 \\ &= -k_1 r^2 - k_2 \eta^2 - \sigma_1 \|\tilde{\delta}_1\|^2 - \sigma_2 \|\tilde{\delta}_2\|^2 - \sigma_1 \tilde{\delta}_1^T \delta_1 - \sigma_2 \tilde{\delta}_2^T \delta_2 + \epsilon_1 \\ &\leq -k_1 r^2 - k_2 \eta^2 + \sigma_1 (-\|\tilde{\delta}_1\|^2 + \|\delta_1\| \|\tilde{\delta}_1\|) \\ &\quad + \sigma_2 (-\|\tilde{\delta}_2\|^2 + \|\delta_2\| \|\tilde{\delta}_2\|) + \epsilon_1 \end{aligned} \quad (3.22)$$

Note that the maximum of $(-\|\tilde{\delta}_1\|^2 + \|\delta_1\| \|\tilde{\delta}_1\|)$ is $(\|\delta_1\|^2/4)$ when $\|\tilde{\delta}_1\| = \|\delta_1\|/2$, while

the maximum of $(-\|\tilde{\delta}_2\|^2 + \|\delta_2\|\|\tilde{\delta}_2\|)$ is $(\|\delta_2\|^2/4)$ when $\|\tilde{\delta}_2\| = \|\delta_2\|/2$. Therefore, we have the following three cases:

$$(I) \quad \begin{aligned} \dot{V} &\leq -k_1 r^2 - k_2 \eta^2 + \frac{1}{4}(\sigma_1 \|\delta_1\|^2 + \sigma_2 \|\delta_2\|^2) + \epsilon_1 \\ &\leq -\zeta_2 \|X\|^2 + \frac{1}{4}(\sigma_1 \|\delta_1\|^2 + \sigma_2 \|\delta_2\|^2) + \epsilon_1, \quad \text{where } \zeta_2 = \min(k_1, k_2) \\ &< 0 \quad \text{for all} \\ \|X\| &> \frac{1}{2} \sqrt{\frac{\sigma_1 \|\delta_1\|^2 + \sigma_2 \|\delta_2\|^2 + 4\epsilon_1}{\zeta_2}}; \end{aligned}$$

$$(II) \quad \begin{aligned} \dot{V} &\leq -\sigma_1 \|\tilde{\delta}_1\|^2 + \sigma_1 \|\delta_1\| \|\tilde{\delta}_1\| + \frac{\sigma_2 \|\delta_2\|^2}{4} + \epsilon_1 \\ &< 0 \quad \text{for all} \\ \|\tilde{\delta}_1\| &> \frac{\|\delta_1\|}{2} + \frac{1}{2\sigma_1} \sqrt{\sigma_1^2 \|\delta_1\|^2 + \sigma_1 \sigma_2 \|\delta_2\|^2 + 4\sigma_1 \epsilon_1}; \end{aligned}$$

$$(III) \quad \begin{aligned} \dot{V} &\leq -\sigma_2 \|\tilde{\delta}_2\|^2 + \sigma_2 \|\delta_2\| \|\tilde{\delta}_2\| + \frac{\sigma_1 \|\delta_1\|^2}{4} + \epsilon_1 \\ &< 0 \quad \text{for all} \\ \|\tilde{\delta}_2\| &> \frac{\|\delta_2\|}{2} + \frac{1}{2\sigma_2} \sqrt{\sigma_2^2 \|\delta_2\|^2 + \sigma_1 \sigma_2 \|\delta_1\|^2 + 4\sigma_2 \epsilon_1}. \end{aligned}$$

so, we can conclude that $\dot{V} < 0$ outside the following compact region Φ :

$$\Phi := \left\{ (X, \tilde{\delta}_1, \tilde{\delta}_2) \left| \begin{aligned} \|X\| &\leq \frac{1}{2} \sqrt{\frac{\sigma_1 \|\delta_1\|^2 + \sigma_2 \|\delta_2\|^2 + 4\epsilon_1}{\zeta_2}}, \\ \|\tilde{\delta}_1\| &\leq \frac{\|\delta_1\|}{2} + \frac{1}{2\sigma_1} \sqrt{\sigma_1^2 \|\delta_1\|^2 + \sigma_1 \sigma_2 \|\delta_2\|^2 + 4\sigma_1 \epsilon_1}, \\ \|\tilde{\delta}_2\| &\leq \frac{\|\delta_2\|}{2} + \frac{1}{2\sigma_2} \sqrt{\sigma_2^2 \|\delta_2\|^2 + \sigma_1 \sigma_2 \|\delta_1\|^2 + 4\sigma_2 \epsilon_1}. \end{aligned} \right. \right\}$$

Thus, with σ -modification adaptive law (3.20) and controller (3.19), the system (3.16) is stable in the sense of GUUB, i.e., signals X , $\tilde{\delta}_1$ and $\tilde{\delta}_2$ ultimately converge into Φ . ■

The above adaptive controller requires time derivative of the embedded controller \dot{u}_r , and subsequently \dot{f}_1 . This indicates that both $\dot{\theta}$ and \ddot{y}_t are all needed for feedback.

Remarks. The backstepping procedure in this paper is of *one* step (i.e., taking only one step of backstepping integral procedure to explicitly obtain physical input τ). *Two-step* backstepping controller design for RLFJ robot was discussed in [10] [11]. The reduction of steps in backstepping procedure not only simplifies the derivation but also results in controllers of simpler forms. If a controller is designed by a two-step backstepping procedure for the flexible link robot, i.e., begin the backstepping procedure with a variable containing θ , instead of the η which is a function of $\dot{\theta}$, one will get two embedded controllers and increase the computational burden. Another advantage of one-step backstepping procedure that can be seen from the derivation above is that the desired trajectory p_d is required to be bounded up to only the 3rd-order, while the 4th-order time derivative of p_d corresponding to the two-step backstepping approach is also required to be bounded.

4. Numerical Simulations

Let us first consider the generation of the desired tip trajectory p_d . Since p_d is required such that its time derivatives up to the 3rd order are bounded, it can be generated by the following linear system:

$$p_d = \frac{b}{(s+a)^3}$$

Obviously, a larger a gives a faster response and the final position is given by:

$$p_d(\infty) = \frac{b}{a^3}$$

In simulations, we set $a = 3$ and $b = a^3\pi/10$ such that the final position is $p_d = \pi/10$.

The system parameters are given by $L = 1.0$ m, $EI = 10.0$ Nm², $\rho = 0.1$ Kg/m, $I_h = 0.1$ Kg m² and $M_t = 1.0$ Kg. This leads to $M_e = 0.024$ Kg and $k = 30.0$ N/m. The plant is simulated by a four-mode assumed modes model. A 4th-order Runge-Kutta program with adaptive steps is used to numerically solve the differential equations.

It should be mentioned that it is difficult in practice to obtain acceleration feedback. In our case, one possible solution is to measure the tip acceleration, i.e. $\ddot{p} = L\ddot{\theta} + \ddot{y}_t$ by attaching an accelerometer to the tip of the beam, and measure $\ddot{\theta}$ by mounting another accelerometer on a rigid attachment at the joint. Moreover, a noise filter may be needed in practical realization because accelerometer feedback often contains much noise.

4.1. Robust Backstepping Controllers

For the two robust backstepping controllers given in (3.9) and (3.12), the control parameters are set as $k_1 = k_2 = 0.000001$, $\lambda = 1$. The bounds of uncertain terms D_1 and D_2 are simply set to be $d_1 = d_2 = 0$ for the sake of testing robustness of the controllers. Since $d_1 = 0$, we do not have to give values to σ and ϵ_1 because the last terms in u_r in (3.9) and (3.12) are always zero. Furthermore, in the second controller (3.12), d_3 (the bounding function of Δ_2) is calculated, according to (3.6), as follows

$$d_3 = (I_h + my_t^2) \frac{1}{m\lambda L} (|\dot{\Delta}_1| + k_1|\dot{r}|) + \|kLy_t - my_t(L\dot{\theta}^2 + 2\dot{\theta}\dot{y}_t) + m\lambda Lr + \eta my_t\dot{y}_t\|$$

where $|\dot{\Delta}_1|$ and $|\dot{r}|$ are given by (3.15) and (3.14).

The tip position and velocity tracking performances of the two robust backstepping controllers are given in Fig. 2 and Fig. 3, respectively. It can be seen that the tip position tracking performances of both controllers are quite satisfactory. It is also seen from Fig. 3 that the first controller (dashed line) exhibits visible vibrations (though still very small) in the end stage of the velocity tracking. The filtered tracking error r for the two controllers are plotted in Fig. 4. The r of the second controller is quite rough at the beginning, but gives better convergence. From the control torques shown in Fig. 5, it can be concluded that the non-smoothness of r of the second controller is caused by its non-smooth control effort, which is also much larger than the first control torque because the bounding function d_3 , instead of Δ_2 itself is used.

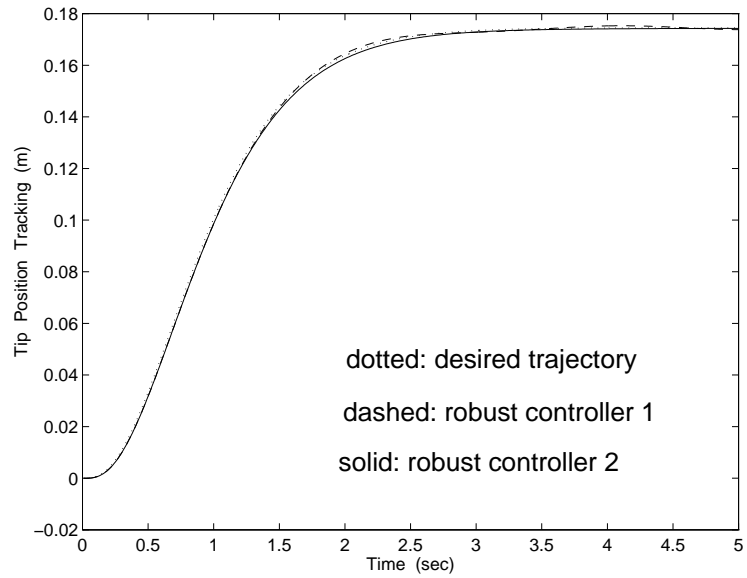


Figure 2. Tip position tracking performance of robust backstepping controllers.

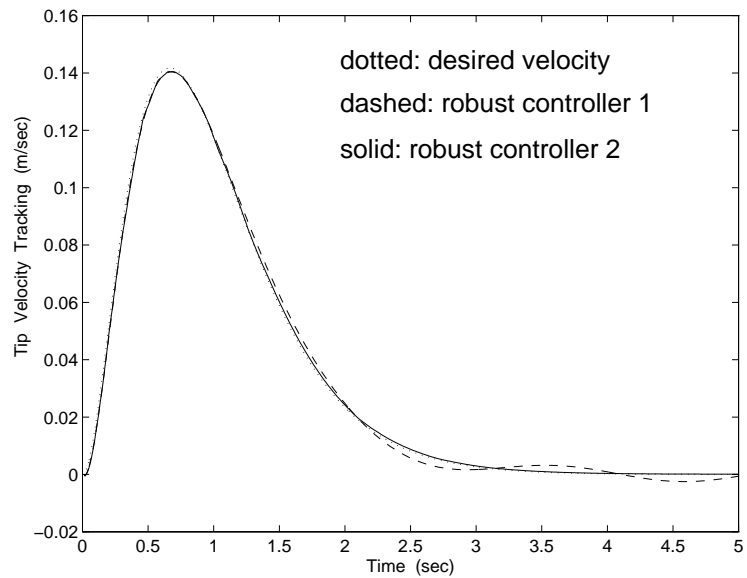


Figure 3. Tip velocity tracking performance of robust backstepping controllers.

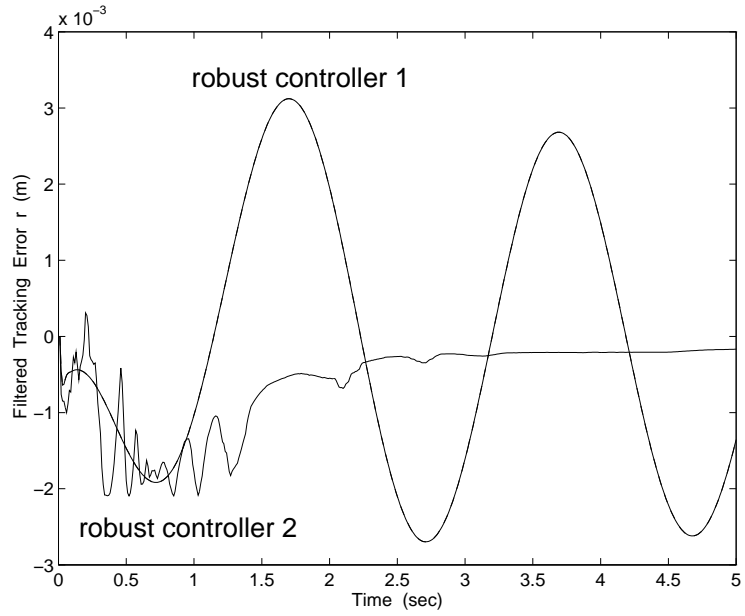


Figure 4. Filtered track error $r = \dot{e} + \lambda e$ of robust backstepping controllers.

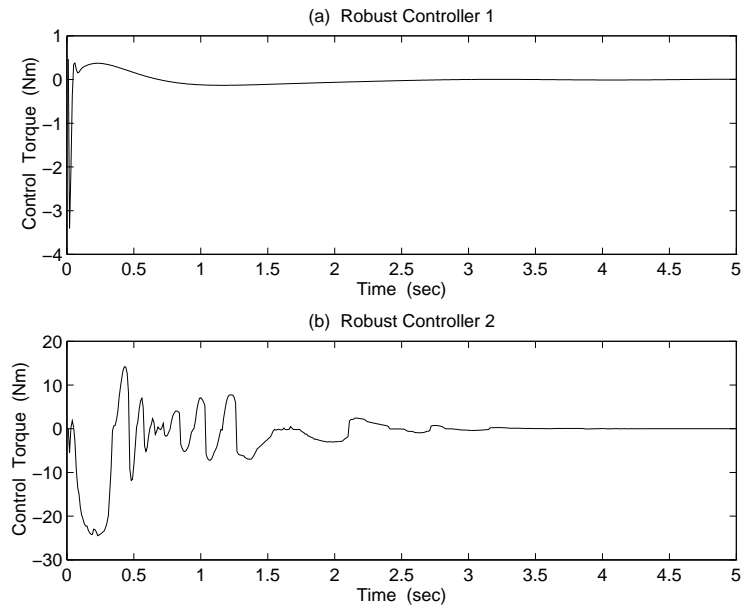


Figure 5. Control torque of robust backstepping controllers.

4.2. Adaptive Backstepping Controller

It is often the case that system parameters are not exactly known. Such parameter uncertainties can be handled by the adaptive controller presented above. In the following, we simulate the case by initially setting the parameter vectors δ_1 and δ_2 to be some estimated values. With the afore-given system parameters, the true parameter vectors are

$$\delta_1 = [29.31 \ 1]^T, \quad \delta_2 = [0.1 \ 1.0236 \ 30 \ 1.0236]^T$$

While the estimated initial vectors are selected to be

$$\delta_1 = [30 \ 0.8]^T, \quad \delta_2 = [0.08 \ 1.1 \ 29.5 \ 0.8]^T$$

which are different from the true values. With the similar reason to that in the robust control case, d_1 and d_2 are simply set to be zero. This implies that v , and subsequently the second term in the u_r in (3.19) are zero. The adaptive gain matrices are given as

$$\Gamma_1 = \text{diag}(0.01 \ 0.1), \quad \Gamma_2 = \text{diag}(0.1 \ 0.1 \ 0.01 \ 0.1)$$

In the σ -modification laws (3.20), $\sigma_1 = \sigma_2 = 2$. For comparison, we also simulate the normal adaptive laws without σ -modification by setting $\sigma_1 = \sigma_2 = 0$ (in this case, stability cannot be theoretically guaranteed because of the existence of system uncertainties).

The tip position and velocity tracking performances are shown in Fig. 6 and Fig. 7. The trajectories obtained with and without σ -modification adaptation are quite different in the end stage, for the latter exhibits much larger vibrations in both position and velocity tracking. It is clear that better performance is obtained by using the σ -modification adaptive laws. The corresponding control torques are given in Fig. 8. Finally, the adaptation of parameter vectors are shown in Fig. 9, in which the dotted lines give the true values, dashed lines correspond to the σ -modification adaptation, and solid lines are that of the non- σ -modification adaptive laws. It is interesting to note that the parameters do not explicitly converge to the true values with either of the two types of adaptive laws. The phenomenon is actually not surprising since GUUB (for the σ -modification adaptation) indicates only a weak version of stability, i.e., boundedness of signals. It is also seen that the parameters changed very little in the non- σ -modification case. This may be partially caused by the fact that the gain matrices Γ_1 and Γ_2 are very small. However, it was found in our simulations that some large Γ_1 and Γ_2 result in unexpected or even unstable control performance. The Γ_1 and Γ_2 , and all the other control parameters are just used to show the effectiveness of the presented backstepping approach, but not necessarily the optimal choice.

5. Conclusion

In this paper, a lumped dynamic model of a single-link flexible robot is obtained, which makes the so-called backstepping approach applicable in studying the tip tracking problem. Three one-step (see the Remarks in Section 3) backstepping controllers (two robust controllers and one adaptive controller) are developed to achieve stable tip position and

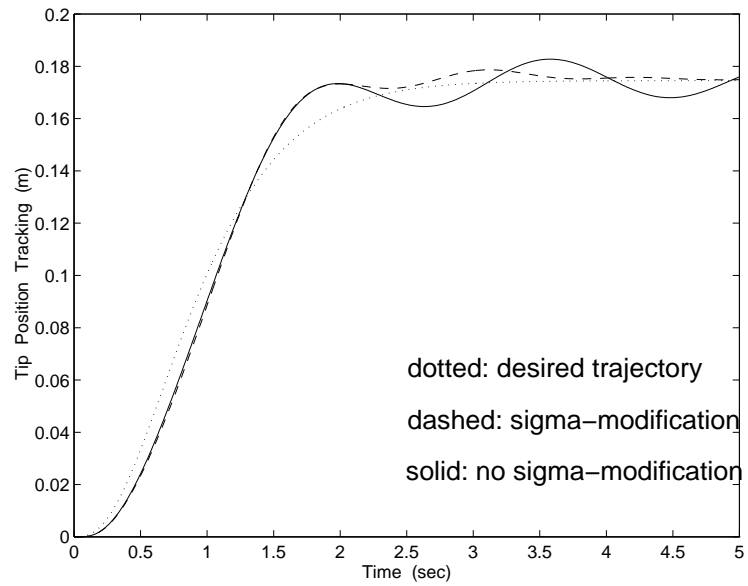


Figure 6. Tip position tracking performance of adaptive backstepping controller.

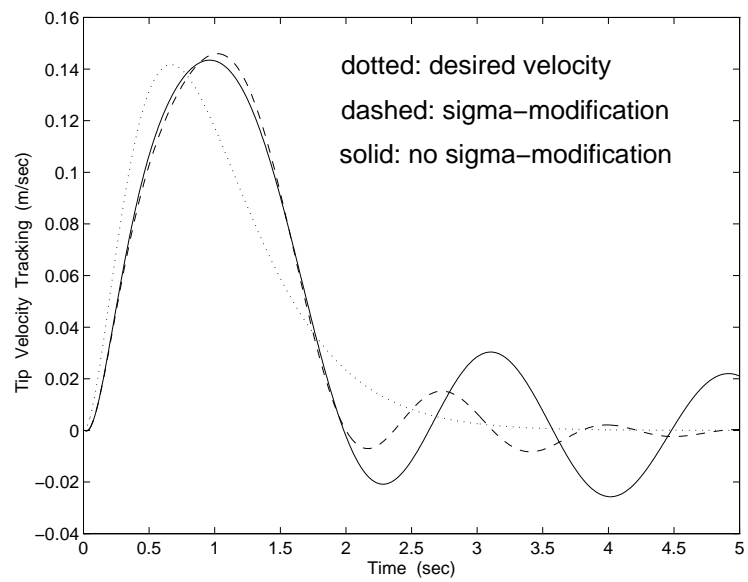


Figure 7. Tip velocity tracking performance of adaptive backstepping controller.

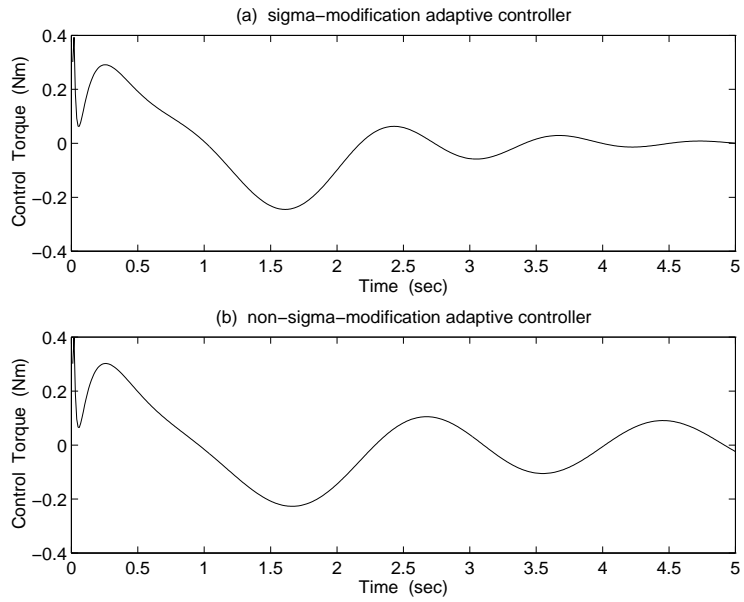


Figure 8. Control torque of adaptive backstepping controller.

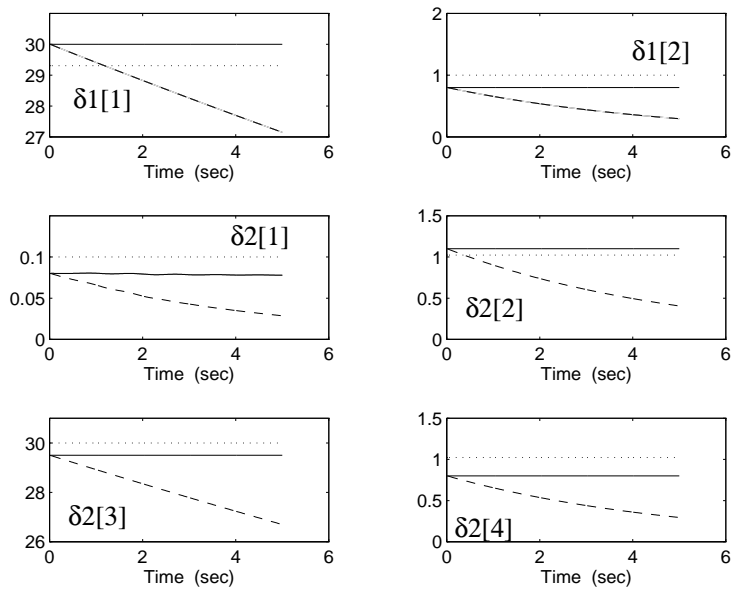


Figure 9. Parameters adaptation.

velocity tracking in the sense of GUUB. The controllers require acceleration feedback ($\ddot{\theta}$ and/or \ddot{y}_t), which cannot be removed as opposed to the case in the RLFJ robot case [10] [11]. Numerical simulations are provided and give reasonably satisfactory results.

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