Switched Controllability via Bumpless Transfer Input and Constrained Switching

S. S. Ge and Zhendong Sun

***Abstract*—In this note, we investigate the controllability of linear hybrid systems via bumpless transfer input and constrained switching. By bumpless transfer input, we mean that the control input signals are as close as possible at switching times. By constrained switching, we mean that the switching index sequence is cyclic and the switching time sequence is possibly with pre-assigned duration intervals. While the problem is well motivated in several practical situations, it is also theoretically interesting. A complete criterion for controllability is presented, and a computational procedure is developed for finding a switching signal and a control input to achieve the controllability.**

***Index Terms*—Bumpless transfer input, controllability, linear hybrid systems, switching signal.**

I. Introduction

A hybrid system is a dynamic system that consists of continuous dynamics, discrete dynamics and interactions between them. The evolution of continuous state depends on the discrete state, and vice versa. Such interactions may lead to complicated situations that are very difficult to handle. Since 1990s, the study on hybrid systems has attracted increasingly more attention because of their practical importance and theoretical challenge. The reader is referred to [3], [11], [9], [13], and [14] for recent development.

A practical motivation for studying hybrid dynamical systems stems from the fact that the hybrid control scheme provides an effective approach for controlling complex dynamical systems. Indeed, while linear design techniques are widely used in control system synthesis, as far as complex dynamics are concerned, in practice a linear controller is only valid around a specific operating region. It is thus a common practice to design more than one linear controller, each at a proper operating region, and a switching mechanism that coordinates the switching among them. A typical example is the hybrid control approach for hard disk drives, where two different controllers—one for the setting and the other for the tracking—were designed [10]. While effective in achieving multiple control objectives, the main drawback of the hybrid control scheme is the transient generated by switching, which may cause transient resonance effects that are damageable and even dangerous in many situations. The suppression of these transient efforts between controllers is referred to as "bumpless transfer" control, a terminology originally stemmed from the anti-windup literature [1], [7]. In this work, we are to design the control input and the switching law such that, at the time of switching, the pre-active and post-active controllers produce control input signals which are as close as possible so as to reduce the magnitude of the discontinuity that occurs during transfer [2].

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Another motivation of the work is that, in the hybrid control scheme, the switching is not arbitrarily designed in many practical situations. For example, the speed control of an automotive is achieved through the gear-box, which in fact is a switching device. To achieve a higher speed, we need to switch from a lower gear to a higher one, step by step, through the neutral gear. It should be stressed that no control force is connected to the neutral gear, while the duration for the occupation of the neutral gear is not arbitrarily small. Another example is that, in workshops, the order of the activated subsystems is pre-assigned rather than arbitrarily assigned. In this case, for instance, we must activate subsystem 1 first, then switch to subsystem 2, then subsystem 3, etc. These impose a restriction on the switching mechanism, which motivates an interesting and nontrivial theoretical problem for switched controllability of hybrid systems [12].

In the literature, much work has been done on the controllability of hybrid systems, mostly focused on unconstrained linear hybrid systems [16], [4], [5], [15], [19], [17], [8]. In particular, for the unconstrained systems, complete geometric and equivalent algebraic criteria have been presented in [15], [6], and [17]. Recently, Krastanov and Veliov [8] extended the controllability condition to the case where the control input is constrained in a cone. In this paper, we investigate the situation that more practical constraints are imposed to the switching device and the control input. The contribution of this work includes: 1) the proof that the controllable set with respect to the constrained dwell-time switching and continuous input is in fact the controllable subspace and 2) the synthesis of bumpless transfer control inputs that achieve controllability[18].

II. Problem Statement

A linear hybrid control system is given by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the continuous state, $\sigma(t) \in M \overset{\text{def}}{=} \{1, \ldots, m\}$ is the discrete state, also known as the switching signal, $u(t) \in \mathbf{R}^p$ is the control input, and $A_i \in \mathbf{R}^{n \times n}$ and $B_i \in \mathbf{R}^{n \times p}, i \in M$ are real constant matrices.

As no rank condition is imposed on matrices B_k's, the subsystems (A_i, B_i)'s are possibly forced-free.

A function pair $(x(\cdot), \sigma(\cdot))$ defined on a time interval $[t_0, t_1)$ with $t_1 > t_0$, where $x: [t_0, t_1) \mapsto \mathbf{R}^n$ is absolutely continuous, and $\sigma: [t_0, t_1) \mapsto M$ is piecewise constant and continuous from the right, is said to be a solution (or a state trajectory) of the linear hybrid system, if (1) holds for almost all $t \in [t_0, t_1)$. The term "for almost all $t \in [t_0, t_1)$" means that "for all $t \in [t_0, t_1)$ except for a set of Lebesgue measure zero".

For a switching signal, any jump instant is said to be a switching time. Accordingly, at a switching time t, we have $\lim_{s \uparrow t} \sigma(s) \neq \sigma(t)$. The ordered sequence of the switching times $t_0, t_1, t_2, \ldots$ (t_0 is the initial time) is said to be the switching time sequence of the switching signal, while the sequence $t_1 - t_0, t_2 - t_1, \cdots$ is said to be the switching duration sequence of the switching signal. Similarly, the ordered discrete state sequence at the switching times $\sigma(t_0), \sigma(t_1), \sigma(t_2), \ldots$ is said to be the switching index sequence of the switching signal. It is clear that a switching signal is uniquely determined by its switching time/index sequence or index/duration sequence, and vice versa.

Let $\phi(t; t_0, x_0, u, \sigma)$ denote the continuous state trajectory at time t of hybrid system (1) starting from $x(t_0) = x_0$ with input u and switching signal σ. It follows from the linearity that

$$\phi(t; t_0, x_0, u, \sigma) = \Phi(t, t_0, \sigma)x_0 + \int_{t_0}^{t} \Phi(t, \tau, \sigma)u(\tau)d\tau$$

where $\Phi(t, t_0, \sigma)$ is the state transition matrix satisfying

$$\frac{d\Phi(t,t_0,\sigma)}{dt} = A_{\sigma(t)}\Phi(t,t_0,\sigma), \Phi(t_0,t_0,\sigma) = I_n.$$

Suppose that $t_0, \ldots, t_j$ is the switching time sequence in $[t_0, t)$. Then, it is clear that

$$\Phi(t,t_0,\sigma) = e^{A_{\sigma(t_j)}(t-t_j)} e^{A_{\sigma(t_{j-1})}(t_j-t_{j-1})} \cdots e^{A_{\sigma(t_0)}(t_1-t_0)}.$$

An index sequence $\Lambda = (k_1, k_2, \ldots)$ is said to be spanning (w.r.t. the set M) if $k_j \in M$ for any j and each element in M appears at least once in the sequence. The sequence is said to be cyclic, if there is a natural number μ such that

$$k_{\mu+j} = k_j, \forall j = 1, 2, \ldots.$$

An interval sequence $\Delta = (\Delta_1, \Delta_2, \ldots)$ where $\Delta_j = [h_0^j, h_f^j)$ is said to be dwell if $0 \leq h_0^j < h_f^j$ for any j.

Definition 2.1: Suppose that the index sequence $\Lambda = (k_1, k_2, \ldots)$ is spanning and cyclic, and the interval sequence $\Delta = (\Delta_1, \Delta_2, \ldots)$ is dwell. The allowed set of switching signals w.r.t. Λ and Δ, denoted $\mathcal{S}_\Lambda^\Delta$, is the set of switching signals with switching index sequence Λ and Δ-constrained switching duration sequence $h_1, h_2, \ldots$, that is

$$h_j \in \Delta_j, \forall j = 1, 2, \ldots.$$

In what follows, we fix a spanning and cyclic index sequence Λ and a dwell interval sequence Δ. The set $\mathcal{S}_\Lambda^\Delta$ will be denoted by $\mathcal{S}_c$ in short.

Next, let $\mathcal{U}_c$ be the set of continuous inputs. Note that the input function under a switching signal is in fact a concatenation from the inputs of different subsystems between the switching times. As a result, input continuity means that the input is not only continuous over the non-switched period, but also continuous at the switching time instances. Hence, the control inputs in $\mathcal{U}_c$ are "bumpless transfer" at the switching times. In contrast, let $\mathcal{U}_{pc}$ be the set of piecewise continuous inputs.

Definition 2.2: Suppose that $\mathcal{S}$ is a set of switching signals and $\mathcal{U}$ is a set of input functions. The controllable set of system (1) under $\mathcal{S}$ and $\mathcal{U}$, denoted $C(\mathcal{U}, \mathcal{S})$, is the set

$$\{x \in \mathbf{R}^n : \exists T \geq 0, u \in \mathcal{U}, \sigma \in \mathcal{S} s.t. \phi(T; 0, x, u, \sigma) = 0\}.$$

Problem Statement. The objective of this work is to address the following problems.

P_1: Characterize the controllable set $C(\mathcal{U}_c, \mathcal{S}_c)$.

P_2: Given an initial state x_0, and a target state x_f in $C(\mathcal{U}_c, \mathcal{S}_c)$, find a time $T > 0$, an allowed switching signal, and a continuous input that steer the system from $x(0) = x_0$ to $x(T) = x_f$.

III. Supporting Lemmas

Let $\mathcal{V}$ be the minimum subspace of $\mathbf{R}^n$ which is invariant under all $A_i, i \in M$ and containing all the image spaces of $B_i, i \in M$. It has been well established that $\mathcal{V}$ is the controllable subspace of the linear hybrid system with unconstrained switching signals and control input [15]. The subspace can be explicitly expressed by

$$\mathcal{V} = \sum_{j=0,\ldots,n-1}^{i_0,\ldots,i_j \in M} A_{i_j} \cdots A_{i_1} \mathrm{Im} B_{i_0}$$

where $\mathrm{Im} B$ is the image space of matrix B.

Suppose that N is a natural number, $(i_1, \ldots, i_N)$ is a sequence of indices, and $(h_1, \ldots, h_N)$ is a sequence of positive real numbers. Define the subspace

$$\begin{aligned}\mathcal{W}(i_1,\ldots,i_N; h_1,\ldots,h_N) \\ &= e^{A_{i_N}h_N} \cdots e^{A_{i_2}h_2} < A_{i_1}|B_{i_1} > + \cdots \\ &\quad + e^{A_{i_N}h_N} < A_{i_{N-1}}|B_{i_{N-1}} > \\ &\quad + < A_{i_N}|B_{i_N} >\end{aligned}$$

where $< A|B > \stackrel{\mathrm{def}}{=} \mathrm{Im}[B, AB, \ldots, A^{n-1}B]$ is the controllable subspace of pair (A, B).

Lemma 3.1: $C(\mathcal{U}_{pc}, \mathcal{S}_c) = \mathcal{V}$.

Proof: According to [15] [Proof of Theorem 1], there is a natural number l, such that for almost all positive real sequence $(s_1, s_2, \ldots, s_l)$, we have

$$\mathcal{W}(k_1,\ldots,k_l; s_1,\ldots,s_l) = \mathcal{V}.$$

This means that, there is a Δ-constrained switching duration sequence $h_1, \ldots, h_l$, such that

$$\mathcal{W}(k_1,\ldots,k_l; h_1,\ldots,h_l) = \mathcal{V}. \quad (2)$$

Let $t_0 = 0$ and $t_j = t_{j-1} + h_j$ for $j = 1, \ldots, l$. By the switching control design procedure in [15, Sect. 4.2], for any $x \in \mathcal{V}$, there is a control input $u \in \mathcal{U}_{pc}$ that is continuous over any subinterval $[t_{j-1}, t_j)$ for $j = 1, \ldots, l$, such that

$$\int_{t_0}^{t_l} \Phi(t_l, \tau, \sigma) u(\tau) d\tau = x \quad (3)$$

where σ is the switching signal with index sequence $k_1, \ldots, k_l$ and duration sequence $h_1, \ldots, h_l$. Let $\mathcal{S}_0$ be the set of (unconstrained) switching signals. It is clear that

$$\begin{aligned}C(\mathcal{U}_{pc}, \mathcal{S}_c) \subseteq C(\mathcal{U}_{pc}, \mathcal{S}_0) = \mathcal{V} \\ = \mathcal{W}(k_1,\ldots,k_l; h_1,\ldots,h_l) \subseteq C(\mathcal{U}_{pc}, \mathcal{S}_c)\end{aligned}$$

which leads directly to the conclusion. $\diamond$

1) Lemma 3.2: Suppose that $M : [t_1, t_2] \mapsto \mathbf{R}^{n \times p}$ is a matrix function with continuous entries, and $z_0, z_f \in \mathbf{R}^p$ are any two given vectors. Then, there is a continuous vector function $z : [t_1, t_2] \mapsto \mathbf{R}^p$, such that $z(t_1) = z_0, z(t_2) = z_f$, and

$$\int_{t_1}^{t_2} M(\tau) z(\tau) d\tau = 0 \quad (4).$$

The proof is simple and is implied in the following steps, which provide a computational procedure for calculating the vector function z.

Computational Procedure for Calculating $z(t)$.

i) Denote the jth column of M by M_j for $j = 1, \ldots, p$, and let $\mathcal{M}_j$ be the subspace spanned by the set of vectors $\{M_j(t) : t \in [t_1, t_2]\}$.

ii) Select $\tau_1^j, \cdots, \tau_{r_j}^j$ in (t_1, t_2) such that $M_j(\tau_1^j), \cdots, M_j(\tau_{r_j}^j)$ form a basis of $\mathcal{M}_j$.

iii) For $j = 1, \ldots, p$, define scalar functions

$$\xi_k^j(t) = \begin{cases} \cos\delta_j(t - \tau_k^j), & t \in \left[\tau_k^j - \frac{\pi}{2\delta_j}, \tau_k^j + \frac{\pi}{2\delta_j}\right] \\ 0, & \text{otherwise} \end{cases} \quad k = 1, \ldots, r_j$$

where δ_j is chosen such that $\int_{t_1}^{t_2} M_j(\tau)\xi_k^j(\tau) d\tau, k = 1, \ldots, r_j$ are independent, and $t_1 < \tau_k^j - (\pi)/(2\delta_j) < \tau_k^j + (\pi)/(2\delta_j) < t_2$ for all $k = 1, \ldots, r_j$.

iv) Let ω_i^j denote the ith entry of column $M_j(\cdot)$. Solve the compatible linear equations on α_k^j's for $j = 1, \ldots, p$

$$\begin{bmatrix} \int_{t_1}^{t_2} \omega_1^j(\tau)\xi_1^j(\tau)d\tau & \cdots & \int_{t_1}^{t_2} \omega_1^j(\tau)\xi_{r_j}^j(\tau)d\tau \\ & \ddots & \\ \int_{t_1}^{t_2} \omega_n^j(\tau)\xi_1^j(\tau)d\tau & \cdots & \int_{t_1}^{t_2} \omega_n^j(\tau)\xi_{r_j}^j(\tau)d\tau \end{bmatrix} \times \begin{bmatrix} \alpha_1^j \\ \vdots \\ \alpha_{r_j}^j \end{bmatrix} = \begin{bmatrix} \beta_1^j \\ \vdots \\ \beta_n^j \end{bmatrix} \tag{5}$$

with

$$\begin{aligned} \beta_1^j &= -\frac{z_f(j)}{t_2 - t_1}\int_{t_1}^{t_2} \omega_1^j(\tau)(\tau - t_1)d\tau \\ &\quad - \frac{z_0(j)}{t_2 - t_1}\int_{t_1}^{t_2} \omega_1^j(\tau)(t_2 - \tau)d\tau \\ &\vdots \\ \beta_n^j &= -\frac{z_f(j)}{t_2 - t_1}\int_{t_1}^{t_2} \omega_n^j(\tau)(\tau - t_1)d\tau \\ &\quad - \frac{z_0(j)}{t_2 - t_1}\int_{t_1}^{t_2} \omega_n^j(\tau)(t_2 - \tau)d\tau. \end{aligned}$$

v) Define the functions

$$\begin{aligned} z_j(t) &= \frac{t - t_1}{t_2 - t_1} z_f(j) \\ &\quad + \frac{t_2 - t}{t_2 - t_1} z_0(j) + \sum_{k=1}^{r_j} \alpha_k^j \xi_k^j(t), j = 1, \ldots, p. \end{aligned}$$

Denote $z(t) = [z_1(t), \ldots, z_p(t)]^T$. Then, $z(t)$ satisfies (4).

IV. Main Results

The following theorem solves Problem P_1.

Theorem 4.1: The controllable set via bumpless transfer input and constrained switching, $C(\mathcal{U}_c, \mathcal{S}_c)$, is exactly the subspace $\mathcal{V}$.

Proof: First, it follows from Lemma 3.1 that $C(\mathcal{U}_{pc}, \mathcal{S}_c) = \mathcal{V}$, that is, the controllable set with constrained switching and piecewise continuous input is the controllable subspace. Define recursively $t_0 = 0$ and $t_j = t_{j-1} + h_j$ for $j = 1, \ldots, l$, where h_j is as in (2). It is clear that $(t_0, t_1, \ldots, t_{l-1})$ is the switching time sequence of σ over interval $[t_0, t_l)$. Let x be an arbitrarily given but fixed state in $\mathcal{V}$. It follows from Lemma 3.1 that there exists an input $u_p^x \in \mathcal{U}_{pc}$ satisfying

$$\int_{t_0}^{t_l} \Phi(t_l, \tau, \sigma)u_p^x(\tau)d\tau = -\Phi(t_l, t_0, \sigma)x. \tag{6}$$

Second, pick a $j \in \{1, 2, \ldots, l-1\}$. Let $\varpi_j = u_p^x(t_j-) - u_p^x(t_j+)$ where $u(t-)$ and $u(t+)$ are the limits from the left and the right, respectively. By Lemma 3.2, there is a continuous function $z_j: [t_j, t_{j+1}] \mapsto \mathbf{R}^p$ with $z_j(t_j) = \varpi_j$ and $z_j(t_{j+1}) = 0$, such that

$$\int_{t_j}^{t_{j+1}} e^{A_{\sigma(t_j)}(t_{j+1}-\tau)} B_{\sigma(t_j)} z_j(\tau)d\tau = 0. \tag{7}$$

Next, define a piecewise continuous function $u_c^x: [t_0, t_l] \mapsto \mathbf{R}^p$ by

$$u_c^x(t) = \begin{cases} 0 & t \in [t_0, t_1) \\ z_1(t) & t \in [t_1, t_2) \\ \vdots & \\ z_{l-1}(t) & t \in [t_{l-1}, t_l]. \end{cases}$$

It follows from (7) that

$$\phi(t_l; t_0, 0, u_c^x, \sigma) = 0. \tag{8}$$

Finally, let $u^x = u_p^x + u_c^x$. It can be seen that $u^x: [t_0, t_l] \mapsto \mathbf{R}^p$ is continuous at the switching times and hence is continuous over the interval $[t_0, t_l]$. Moreover, combining (6) and (8) gives

$$\begin{aligned} \phi(t_l; t_0, x, u^x, \sigma) &\\ &= \phi(t_l; t_0, 0, u^x, \sigma) + \phi(t_l; t_0, x, 0, \sigma) \\ &= \phi(t_l; t_0, 0, u_p^x, \sigma) + \phi(t_l; t_0, 0, u_c^x, \sigma) \\ &\quad + \Phi(t_l, t_0, \sigma)x \\ &= -\Phi(t_l, t_0, \sigma)x + \Phi(t_l, t_0, \sigma)x = 0. \end{aligned} \tag{9}$$

As x is arbitrary given in $C(\mathcal{U}_{pc}, \mathcal{S}_c)$, we have

$$C(\mathcal{U}_c, \mathcal{S}_c) = C(\mathcal{U}_{pc}, \mathcal{S}_c) = \mathcal{V}. \diamondsuit$$

Remark 4.1: Note that $\mathcal{V}$ is the controllable subspace of the linear hybrid system where both the control input and switching signal are unconstrained design variables. Theorem 4.1 shows that any state in the controllable subspace can be steered to the origin by means of bumpless input and constrained switching.

Remark 4.2: From the proof it can be seen that, the controllability input u^x is the summation of the piecewise continuous nominal input u_p^x and the compensating input u_c^x. Lemma 3.2 assures that the input can take any pre-assigned value at the switching time instants.

From the proof, we can outline a design procedure for calculating an allowed switching signal σ and a bumpless transfer input u^x that steers any initial state $x_0 \in \mathcal{V}$ to a target state $x_f \in \mathcal{V}$.

Step 1. Find a natural number l, and a Δ-constrained duration sequence $h_1, \ldots, h_l$, such that

$$\dim \mathcal{W}(k_1, \ldots, k_l; h_1, \ldots, h_l) = \dim \mathcal{V}. \tag{10}$$

Let $t_0 = 0$ and $t_i = \sum_{j=1}^{i} h_j$ for $j = 1, \ldots, l$. Let σ be the switching signal with switching index sequence $(k_1, \ldots, k_l)$ and switching time sequence $(t_0, \ldots, t_{l-1})$.

Step 2. Find a piecewise continuous function $u_p^{x_0}: [t_0, t_l] \mapsto \mathbf{R}^p$ that is continuous over any sub-interval $[t_{j-1}, t_j)$ for $j = 1, \ldots, l$, such that

$$\int_0^{t_l} \Phi(t, \tau, \sigma)u_p^{x_0}(\tau)d\tau = x_f - \Phi(t_l, t_0, \sigma)x_0.$$

Step 3. Find continuous functions $z_j: [t_j, t_{j+1}] \mapsto \mathbf{R}^p, j = 1, \ldots, l-1$ with $z_j(t_j) = u_p^{x_0}(t_j-) - u_p^{x_0}(t_j+), z_j(t_{j+1}) = 0$, and $\int_{t_j}^{t_{j+1}} e^{A_{\sigma(t_j)}(t_{j+1}-\tau)} B_{\sigma(t_j)} z_j(\tau)d\tau = 0$. Let

$$u_c^{x_0}(t) = \begin{cases} 0 & t \in [t_0, t_1) \\ z_1(t) & t \in [t_1, t_2) \\ \vdots & \\ z_{l-1}(t) & t \in [t_{l-1}, t_l]. \end{cases}$$

Conclusion: The bumpless transfer input $u_c^{x_0} + u_p^{x_0}$ and the (allowed) switching signal σ steer the hybrid system from $x(0) = x_0$ to $x(t_l) = x_f$. Hence, Problem P_2 is solved.

Remark 4.3: For Step 1, it was proved in [15] that for sufficiently large l, relation (10) holds true for almost all sequence $(h_1, \ldots, h_l)$. For Step 2, a computational procedure was presented in [15] to calculate u_p^x, while z_j in Step 3 can be computed following Theorem 4.1 and Lemma 3.2.

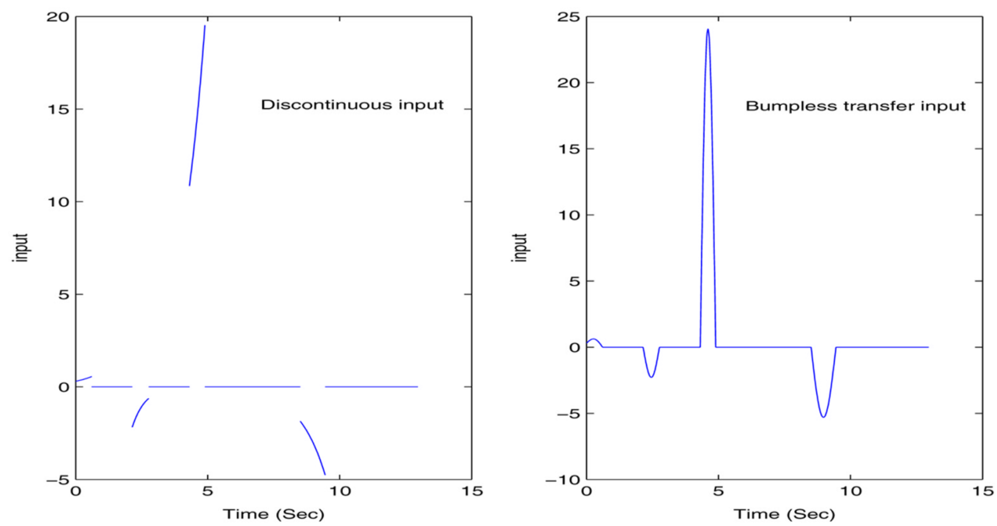

Fig. 1. Input trajectories.

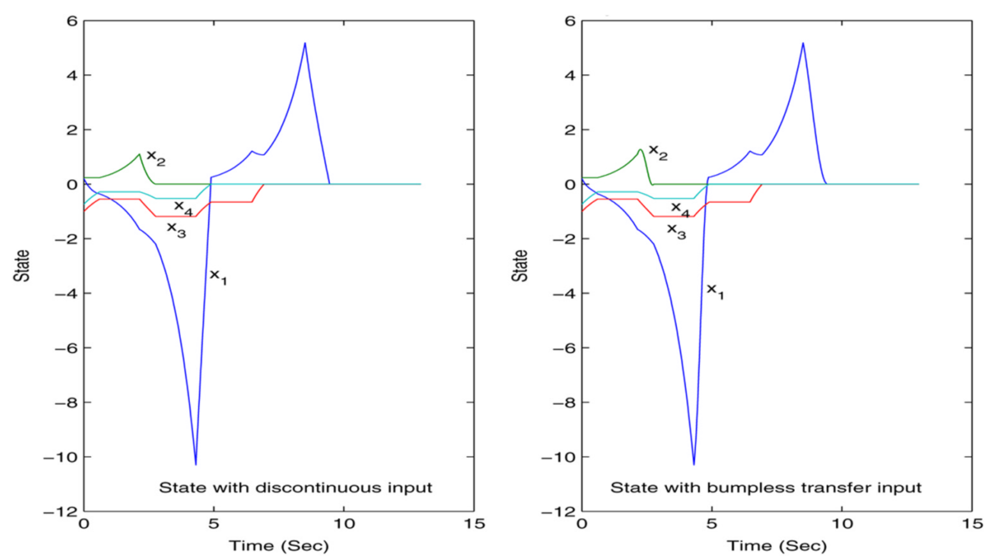

Fig. 2. State trajectories.

V. Illustrative Example

For the fourth-order linear hybrid system with three subsystems given by

$$A_1 = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & -3 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} \quad (11)$$

it is clear that the second subsystem is forced-free.

Suppose that the switching is constrained in the way that, the first subsystem is first activated, then the second, then the third, then the second again, and then back to the first. The switching index sequence is hence cyclic with the loop $1 \to 2 \to 3 \to 2 \to 1$. On the other hand, each duration on the second subsystem is between 1.0 s and 1.5 s. That is, the second subsystem admits a dwell time constraint.

To address the switched controllability problem with bumpless transfer input and constrained switching, we exploit the design procedure presented in the previous section.

First, simple calculation exhibits that the controllable subspace is the total state space, and hence the system is completely controllable.

Next, take a switching duration sequence

$$0.6084, 1.1490, 0.6210, 1.1974, 0.5874, 1.3360, 0.4648, \\ 1.3512, 0.9423, 1.2547, 0.4018, 1.2353$$

which generates an allowed switching signal in $\mathcal{S}_c$ with $l = 12$.

Then, take an initial state $x_0 = [0.2120, 0.2379, -1.0078, -0.7420]^T$. Applying the path planning design procedure (see [14, Sec. 4.3.3]), we can find a piecewise continuous nominal input that achieves controllability. The input trajectory is shown on the left of Fig. 1 and the corresponding state trajectory is presented on the left of Fig. 2. It is clear that the input is discontinuous at the switching times.

Finally, a compensating input is designed to compensate the discontinuity of the input function. The overall input and state trajectories are depicted on the right of Figs. 1 and 2, respectively. It can be seen that the input is continuous and hence bumpless transfer. The state is similar to that of the left side (in fact, as the compensator does not change the state at the switching times, the left-hand state and the right-hand state are the same at the switching times).

VI. Concluding Remarks

In this work, we formulated and solved the problem of switched controllability via bumpless transfer input and constrained switching. A computational procedure was presented to find the switching signal and the open-loop input. The numerical example verified the effectiveness of the proposed scheme.

It should be remarked that the switching and input strategies are open loop in this work, and hence the system performance might be sensitive to uncertainties and perturbations. To tackle this problem, a future work is looking for feedback strategies with event-driven switching and state/output feedback control input.

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Stability of Networked Control Systems Under a Multiple-Packet Transmission Policy

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***Abstract*—This paper is concerned with stability analysis of discrete-time networked control systems subject to packet loss under a multiple-packet transmission policy with the packet dropping probability of the communication channel bounded from above. Necessary and sufficient conditions for stability are obtained. In addition, the packet dropping margin as a measure of stability robustness of a system against packet loss is defined and its formula is derived. A design method is proposed for enhancing stability robustness subject to the constraint of a set of prescribed nominal closed-loop poles.**

***Index Terms*—Jump systems, networked control systems, packet loss, stability.**

I. Introduction

Networked control systems (NCSs), which send measurement and control signals via a real-time shared media network, reduce the cost of implementation, and offer flexibility in system design, installation and maintenance. This network is generally viewed as an unreliable communication channel, where several practical factors, such as limited bandwidth, packet dropping and packet delay, make the analysis and design of NCSs difficult. Stability analysis of NCSs subject to packet dropping has received much attention, see, e.g., [1]–[3]. Many control problems under network constraints have been considered as well, see e.g., [4]–[6].

In the design of NCSs, a multiple-packet transmission policy as opposed to a single-packet transmission policy may be required due to network or system requirements. For example, when a packet exceeding the limitation of the packet size of the communication channel is broken into multiple packets or in distributed control system, the each sensor or control of NCSs sends its own data via the individual packet, the multiple packet transmission policy has to be adopted [1], [7]. Therefore, it is of importance to fully understand and minimize the effect of the packet dropping on stability for the different transmission policies. Most of the previous work has only focused on a single packet transmission policy under which at any time instant, all the plant outputs are transmitted via only one data packet. For example, it was proven in [1] that a sufficiently fast sampling rate can guarantee stability of a continuous-time NCS, where the communication channel was treated as a switch that closes at a certain rate corresponding to a packet dropping probability (PDP). A stability analysis of a

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