International Journal of Bifurcation and Chaos, Vol. 10, No. 5 (2000) 1149–1156

ADAPTIVE BACKSTEPPING CONTROL OF A CLASS OF CHAOTIC SYSTEMS

S. S. GE*, C. WANG and T. H. LEE
Department of Electrical Engineering,
National University of Singapore, Singapore 117576

Received March 26, 1999; Revised September 2, 1999

This paper is concerned with the control of a class of chaotic systems using adaptive backstepping, which is a systematic design approach for constructing both feedback control laws and associated Lyapunov functions. Firstly, we show that many chaotic systems as paradigms in the research of chaos can be transformed into a class of nonlinear systems in the so-called nonautonomous "strict-feedback" form. Secondly, an adaptive backstepping control scheme is extended to the nonautonomous "strict-feedback" system, and it is shown that the output of the nonautonomous system can asymptotically track the output of any known, bounded and smooth nonlinear reference model. Finally, the Duffing oscillator with key constant parameters unknown, is used as an example to illustrate the feasibility of the proposed control scheme. Simulation studies are conducted to show the effectiveness of the proposed method.

1. Introduction

Recent years have seen much progress in the study of controlling chaotic systems ([Chen & Dong, 1998] and the references therein). In particular, many adaptive control schemes have been successfully applied to the control and synchronization of chaotic systems [Wu *et al.*, 1996; Dong *et al.*, 1997; Fradkov & Pogromosky, 1996; Yang *et al.*, 1998]. As a general tool, Lyapunov stability theory has also been used for adaptive control of chaos. Nijmeijer and Berghuis [1995] suggested a Lyapunov control method for the control of the chaotic Duffing oscillator, in which the controller is an adaptive, PD-like, observer-combined controller. Bernardo [1996a, 1996b] proposed a complete adaptive control method based on the rigorous Lyapunov argument, along with the differential inclusion principle and observer design method, for both chaos control and synchronization. Dong *et al.* [1997] developed an adaptive feedback controller based on rigorous Lyapunov argument for an uncertain chaotic Duffing oscillator, in which the three key system parameters are essentially unknown. All these methods are based on rigorous Lyapunov stability theorem and Lyapunov function methods. But the construction of the Lyapunov functions remains to be a difficult task.

Adaptive control is one of the main approaches in control engineering that deal with uncertain systems. Over the last few years, adaptive control of nonlinear systems has emerged as an exciting research area, which has witnessed rapid and impressive developments leading to global stability and tracking results for a large class of nonlinear "strict-feedback" systems [Krstič *et al.*, 1995]. These results are all Lyapunov-based, i.e. the design procedure achieves the desired objectives by constructing a suitable Lyapunov function and rendering its derivative nonpositive. These results are based on several design tools such as *adaptive backstepping* [Kanellakopoulos *et al.*, 1991] and *tuning*

*Corresponding author.
E-mail: elegesz@nus.edu.sg

functions [Krstič *et al.*, 1992], which are used as building blocks for the construction of systematic design procedures.

The present paper is to discuss the control of chaotic systems using *adaptive backstepping* method. Our objectives are twofold: the first one is to show that many chaotic systems as paradigms in the research of chaos can be transformed into a class of nonlinear systems in the so-called nonautonomous "strict-feedback" form, as detailed in Sec. 2; and the second is to extend the adaptive backstepping design method to the nonautonomous "strict-feedback" system, and to show that adaptive backstepping design method may be naturally applied to control a class of chaotic systems, as detailed in Sec. 3. Finally, the Duffing oscillator with key constant parameters unknown is used as an example in simulation studies to show the effectiveness of the proposed method.

2. Chaotic Systems in Strict-Feedback Form

In the study of nonlinear dynamics, such as bifurcation and chaos, and their control, several low-dimensional systems are frequently used as benchmark examples for verification and validation of a proposed theory, method and algorithm. These examples include Duffing oscillator [Duffing, 1918], van der Pol oscillator [van der Pol, 1927], Rössler system [Rössler, 1976], and Chua's circuit [Chua *et al.*, 1986]. It is interesting to note that all these chaotic systems mentioned above can be rewritten into the nonautonomous "strict-feedback" form as follows

$$\begin{aligned}
\dot{x}_1 &= g_1(x_1, t)x_2 + \theta^T F_1(x_1, t) + f_1(x_1, t)\\
\dot{x}_2 &= g_2(x_1, x_2, t)x_3 + \theta^T F_2(x_1, x_2, t)\\
&\quad + f_2(x_1, x_2, t)\\
&\vdots\\
\dot{x}_{n-1} &= g_{n-1}(x_1, \dots, x_{n-1}, t)x_{n-1}\\
&\quad + \theta^T F_{n-1}(x_1, \dots, x_{n-1}, t)\\
&\quad + f_{n-1}(x_1, \dots, x_{n-1}, t)\\
\dot{x}_n &= g_n(x, t)u + \theta^T F_n(x, t) + f_n(x, t)\\
y &= x_1
\end{aligned} \tag{1}$$

where $x = [x_1, x_2, \dots, x_n]^T \in R^n$, $u \in R$, and $y \in R$ are the states, input and output, respectively; $\theta \in R^p$ is the vector of unknown constant parameters of interest; $g_i(\cdot) \neq 0$, $F_i(\cdot)$, $f_i(\cdot)$, $i = 1, \dots, n-1$ are known, smooth nonlinear functions with their jth derivatives ($j = 0, \dots, n-i$) uniformly bounded in t, $g_n(\cdot) \neq 0$, $F_n(\cdot)$, $f_n(\cdot)$ are known continuous nonlinear functions which are uniformly bounded in t.

In a more general case, the terms $g_i(x_1, \dots, x_i, t)x_{i+1}$, $i = 1, \dots, n-1$, and $g_n(x, t)u$ are multiplied by unknown constant parameters $\gamma_i \neq 0$, $i = 1, \dots, n$. Thus, system (1) is changed to

$$\begin{aligned}
\dot{x}_1 &= \gamma_1 g_1(x_1, t)x_2 + \theta^T F_1(x_1, t) + f_1(x_1, t)\\
\dot{x}_2 &= \gamma_2 g_2(x_1, x_2, t)x_3 + \theta^T F_2(x_1, x_2, t)\\
&\quad + f_2(x_1, x_2, t)\\
&\vdots\\
\dot{x}_{n-1} &= \gamma_{n-1} g_{n-1}(x_1, \dots, x_{n-1}, t)x_{n-1}\\
&\quad + \theta^T F_{n-1}(x_1, \dots, x_{n-1}, t)\\
&\quad + f_{n-1}(x_1, \dots, x_{n-1}, t)\\
\dot{x}_n &= \gamma_n g_n(x, t)u + \theta^T F_n(x, t) + f_n(x, t)\\
y &= x_1
\end{aligned} \tag{2}$$

where, in addition to the unknown parameter vector θ, the constant coefficients γ_i are also unknown. We assume that the signs of these parameters γ_i are known.

For example, the controlled Duffing oscillator is described by

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= u - p_1 x_2 - p_2 x_1 - p_3 x_1^3 + p_4 \cos \omega t
\end{aligned} \tag{3}$$

where ω is a constant frequency parameter, p_1, p_2, p_3 and p_4 are constant parameters. In the literature of chaos research, it is assumed that ω is known, while $\theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T = [p_1, p_2, p_3, p_4]^T$ are unknown. Accordingly, the controlled Duffing oscillator can be rewritten into the second-order nonautonomous "strict-feedback" form (1) with

$$\begin{gathered}
g_1(x_1, t) = 1\,, \quad g_2(x_1, x_2, t) = 1\,,\\
f_1(x_1, t) = 0\,, \quad f_2(x_1, x_2, t) = 0\\
F_1(x_1, t) = \begin{bmatrix} 0\\ 0\\ 0\\ 0 \end{bmatrix}, \; F_2(x_1, x_2, t) = \begin{bmatrix} -x_2\\ -x_1\\ -x_1^3\\ \cos \omega t \end{bmatrix},\\
\theta = \begin{bmatrix} \theta_1\\ \theta_2\\ \theta_3\\ \theta_4 \end{bmatrix}
\end{gathered} \tag{4}$$

The Bonhoeffer–van der Pol oscillator (BVP) [Fitzhugh, 1961], a generalization of the van der Pol oscillator, is described by

$$\begin{aligned}\dot{x}_1 &= x_1 - \frac{1}{3}x_1^3 - x_2 + p_1 + p_2\cos\omega t\\ \dot{x}_2 &= p_3(x_1 + p_4 - p_5x_2)\end{aligned} \tag{5}$$

where ω is a constant frequency parameter, p_1, p_2, p_3, p_4 and p_5 are constant parameters. We can see that it can be directly written into the second-order nonautonomous "strict-feedback" form (1). But for the Rössler system [Rössler, 1976]

$$\begin{aligned}\dot{z}_1 &= -z_2 - z_3\\ \dot{z}_2 &= z_1 + \theta_1 z_2\\ \dot{z}_3 &= \theta_2 + z_3(z_1 - \theta_3)\end{aligned} \tag{6}$$

and the famous Chua's circuit [Chua *et al.*, 1986] (in dimensionless form)

$$\begin{aligned}\dot{z}_1 &= p_1(z_2 - z_1 - f(z_1))\\ \dot{z}_2 &= z_1 - z_2 + z_3\\ \dot{z}_3 &= -p_2z_2\end{aligned} \tag{7}$$

where

$$f(z_1) = p_4z_1 + \frac{1}{2}(p_3 - p_4)(|z_1+1| - |z_1-1|) \tag{8}$$

it is clear that they are not in the "strict-feedback" form (1) or (2). However, they can be rendered into the desired "strict-feedback" form after some simple state transformations. Let

$$x_1 = z_2\,, \quad x_2 = z_1\,, \quad x_3 = z_3 \tag{9}$$

then (6) can be transformed into

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta_1x_1\\ \dot{x}_2 &= -x_3 - x_1\\ \dot{x}_3 &= u + \theta_2 + x_3(x_2 - \theta_3)\end{aligned} \tag{10}$$

where a controller $u(\cdot)$ is assumed to be fed into the third equation in (10). In this way the controlled Rössler system (10) can be easily written into the "strict-feedback" form (1). Similarly, for the Chua's circuit (7), let

$$x_1 = z_3\,, \quad x_2 = z_2\,, \quad x_3 = z_1 \tag{11}$$

and $\gamma_1 = p_2$, $\theta_1 = p_1$, $\theta_2 = p_1(1+p_4)$ and $\theta_3 = (1/2)p_1(p_3 - p_4)$, then (7) can be transformed into

$$\begin{aligned}\dot{x}_1 &= -\gamma_1x_2\\ \dot{x}_2 &= x_1 - x_2 + x_3\\ \dot{x}_3 &= u + \theta_1x_2 - \theta_2x_3\\ &\quad - \theta_3(|x_3+1| - |x_3-1|)\end{aligned} \tag{12}$$

where a controller $u(\cdot)$ is assumed to be fed into the third equation in (12). In this way the controlled Chua's circuit (12) can be easily written into the "strict-feedback" form (2).

However, it is with certainty that not all the chaotic systems can be transformed into the nonautonomous "strict-feedback" form (1) and (2), e.g. the famous Lorenz system [Lorenz, 1963], which has rich complex dynamics, including chaotic behavior, is described mathematically by

$$\begin{aligned}\dot{x}_1 &= \theta_1x_2 - \theta_1x_1\\ \dot{x}_2 &= -x_1x_3 + \theta_2x_1 - x_2\\ \dot{x}_3 &= x_1x_2 - \theta_3x_3\end{aligned} \tag{13}$$

where x_1, x_2 and x_3 are the states, θ_1, θ_2 and θ_3 are constant parameters.

3. Adaptive Backstepping Control

In this section we extend the adaptive backstepping method [Kanellakopoulos *et al.*, 1991] to the nonautonomous strict-feedback system in the form (1). For the chaotic system in the form (1), consider a known, bounded and smooth reference model as follows

$$\begin{aligned}\dot{x}_{ri} &= f_{ri}(x_r, t), \quad 1 \le i \le m\,, \quad n \le m\\ y_r &= x_{r1}\end{aligned} \tag{14}$$

where $x_r = [x_{r1}, x_{r2}, \ldots, x_{rm}]^T \in R^m$ and $y_r \in R$ are the states and output respectively; $f_{ri}(\cdot)$, $i = 1, 2, \ldots, m$ are known smooth nonlinear functions with their jth derivatives uniformly bounded in t.

Our objective is to design an adaptive state-feedback controller for system (1) that guarantees global stability and forces the output $y = x_1(t)$ of system (1) to asymptotically track the output $y_r = x_{r1}(t)$ of the reference model, i.e.

$$|y(t) - y_r(t)| \to 0\,, \quad \text{as } t \to \infty\,. \tag{15}$$

The backstepping design procedure contains n steps. At Step i, an intermediate control function α_i shall be developed using an appropriate Lyapunov function V_i. Let us first consider the equation in (1) when $i = 1$.

Step 1. Define the first error variable

$$z_1 = x_1 - x_{r1}\,. \tag{16}$$

Its derivative is

$$\begin{aligned}\dot{z}_1 &= \dot{x}_1 - \dot{x}_{r1}\\ &= g_1(x_2 - x_{r2}) + \theta^T F_1 + g_1 x_{r2} + f_1 - f_{r1}\,. \end{aligned}\tag{17}$$

In (17), we take $x_2 - x_{r2}$ as a "virtual control" and design for it a *stabilizing function* α_1. The difference between the actual value of $x_2 - x_{r2}$ and its "desired" value α_1 is defined to be the second error variable

$$z_2 = x_2 - x_{r2} - \alpha_1\,. \tag{18}$$

Thus, (17) can be written as

$$\dot{z}_1 = g_1 z_2 + g_1\alpha_1 + \theta^T F_{1s} + f_{1s} \tag{19}$$

where, for uniformity with subsequent steps and to simplify the notation, we have let

$$\begin{aligned}F_{1s} &= F_1\\ f_{1s} &= f_1 - f_{r1} + g_1 x_{r2}\end{aligned}$$

Since θ is unknown, the stabilizing function α_1 employs a parameter estimate $\hat{\theta}_{1st}$ as

$$\alpha_1 = \frac{1}{g_1}(-c_1 z_1 - \hat{\theta}_{1\mathrm{st}}^T F_{1s} - f_{1s}) \tag{20}$$

where $c_1 > 0$ is a positive design constant.

Substituting (18) into (19), we obtain

$$\dot{z}_1 = -c_1 z_1 + g_1 z_2 + (\theta - \hat{\theta}_{1\mathrm{st}})^T F_{1s}\,. \tag{21}$$

To design the update law for the parameter estimate $\hat{\theta}_{1\mathrm{st}}$, we form the partial Lyapunov function

$$V_1(z_1, \hat{\theta}_{1\mathrm{st}}) = \frac{1}{2}z_1^2 + \frac{1}{2}(\theta - \hat{\theta}_{1\mathrm{st}})^T\Gamma^{-1}(\theta - \hat{\theta}_{1\mathrm{st}}) \tag{22}$$

where $\Gamma = \Gamma^T > 0$ is the adaptive gain matrix. The derivative of V_1 along the solution of (21) is

$$\begin{aligned}\dot{V}_1 &= z_1\dot{z}_1 - (\theta - \hat{\theta}_{1\mathrm{st}})^T\Gamma^{-1}\dot{\hat{\theta}}_{1\mathrm{st}}\\ &= g_1 z_1 z_2 - c_1 z_1^2\\ &\quad + (\theta - \hat{\theta}_{1\mathrm{st}})^T(F_{1s}z_1 - \Gamma^{-1}\dot{\hat{\theta}}_{1\mathrm{st}})\end{aligned} \tag{23}$$

To cancel the last term in the above derivative, we choose the update law

$$\dot{\hat{\theta}}_{1\mathrm{st}} = \Gamma F_{1s} z_1 \tag{24}$$

which yields

$$\dot{V}_1 = g_1 z_1 z_2 - c_1 z_1^2\,. \tag{25}$$

For global stability, the coupling term $g_1 z_1 z_2$ will be canceled at the next step.

Step i $(2 \le i \le n-1)$. After some simple algebraic manipulations, the derivative of $z_i = x_i - x_{ri} - \alpha_{i-1}$ can be expressed as

$$\begin{aligned}\dot{z}_i &= \dot{x}_i - \dot{x}_{ri} - \dot{\alpha}_{i-1}\\ &= g_i(x_{i+1} - x_{r(i+1)}) + \theta^T F_{is} + f_{is}\end{aligned} \tag{26}$$

where

$$F_{is} = F_i - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}F_k$$

$$\begin{aligned}f_{is} = f_i - f_{ri} + g_i x_{r(i+1)} &- \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}(f_k + g_k x_{k+1})\\ &- \sum_{k=1}^{m}\frac{\partial\alpha_{i-1}}{\partial x_{rk}}f_{rk}\\ &- \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{kth}}\dot{\hat{\theta}}_{kth} - \frac{\partial\alpha_{i-1}}{\partial t}\end{aligned}$$

The difference $x_{i+1} - x_{r(i+1)}$ in (26) is now again viewed as the "virtual control". Accordingly, the new error variable is defined as

$$z_{i+1} = x_{i+1} - x_{r(i+1)} - \alpha_i \tag{27}$$

The stabilizing function α_i and update law for $\hat{\theta}_{i\mathrm{th}}$ are now designed to render nonpositive the derivative of the following Lyapunov function

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2}(\theta - \hat{\theta}_{i\mathrm{th}})^T\Gamma^{-1}(\theta - \hat{\theta}_{i\mathrm{th}}) \tag{28}$$

whose derivative, using (26) and (27), is

$$\begin{aligned}\dot{V}_i &= \dot{V}_{i-1} + z_i\dot{z}_i - (\theta - \hat{\theta}_{i\mathrm{th}})^T\Gamma^{-1}\dot{\hat{\theta}}_{i\mathrm{th}}\\ &= -\sum_{k=1}^{i-1}c_k z_k^2 + z_i[g_{i-1}z_{i-1} + g_i z_{i+1} + g_i\alpha_i\\ &\quad + \theta^T F_{is} + f_{is}] - (\theta - \hat{\theta}_{i\mathrm{th}})^T\Gamma^{-1}\dot{\hat{\theta}}_{i\mathrm{th}}\end{aligned} \tag{29}$$

The choice for α_i is then given as

$$\alpha_i = \frac{1}{g_i}(-c_i z_i - g_{i-1} z_{i-1} - \hat{\theta}^T_{\text{ith}} F_{is} - f_{is})\,. \quad (30)$$

Substituting (30) into (26) and (29) results in

$$\dot{z}_i = -c_i z_i - g_{i-1} z_{i-1} + g_i z_{i+1} + (\theta - \hat{\theta}_{\text{ith}})^T F_{is} \quad (31)$$

and

$$\dot{V}_i = -\sum_{k=1}^{i-1} c_i z_i^2 + g_i z_{i+1} + (\theta - \hat{\theta}_{\text{ith}})^T (F_{is} z_i - \Gamma^{-1}\dot{\hat{\theta}}_{\text{ith}})\,. \quad (32)$$

The $(\theta - \hat{\theta}_{\text{ith}})$-term in (32) can be eliminated with the update law

$$\dot{\hat{\theta}}_{\text{ith}} = \Gamma F_{is} z_i \quad (33)$$

which yields

$$\dot{V}_i = g_i z_i z_{i+1} - \sum_{k=1}^{i-1} c_i z_i^2\,. \quad (34)$$

The coupling term $g_i z_i z_{i+1}$ can be eliminated next in the final step.

Step n. This is the final design step, since the actual control u appears in the derivative of z_n as given in

$$\begin{aligned} \dot{z}_n &= \dot{x}_n - \dot{x}_{rn} - \dot{\alpha}_{n-1} \\ &= g_n u + \theta^T F_{ns} + f_{ns} \end{aligned} \quad (35)$$

where

$$F_{ns} = F_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} F_k$$

$$f_{ns} = f_n - f_{rn} - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}(f_k + g_k x_{k+1})$$

$$- \sum_{k=1}^{m} \frac{\partial \alpha_{n-1}}{\partial x_{rk}} f_{rk} - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_{\text{kth}}} \dot{\hat{\theta}}_{kth} - \frac{\partial \alpha_{n-1}}{\partial t}$$

The control u and the update law for the nth estimate $\hat{\theta}_{\text{nth}}$ are designed to render nonpositive the derivative of the full Lyapunov function

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 + \frac{1}{2}(\theta - \hat{\theta}_{\text{nth}})^T \Gamma^{-1} (\theta - \hat{\theta}_{\text{nth}}) \quad (36)$$

whose derivative is

$$\begin{aligned} \dot{V}_n &= \dot{V}_n + z_n \dot{z}_n - (\theta - \hat{\theta}_{\text{nth}})^T \Gamma^{-1} \dot{\hat{\theta}}_{\text{nth}} \\ &= -\sum_{k=1}^{n-1} c_k z_k^2 + z_n(g_{n-1} z_{n-1} + g_n u \\ &\quad + \theta^T F_{ns} + f_{ns}) - (\theta - \hat{\theta}_{\text{nth}})^T \Gamma^{-1} \dot{\hat{\theta}}_{\text{nth}}\,. \end{aligned} \quad (37)$$

The choice of control u is given by

$$u = \frac{1}{g_n}(-c_n z_n - g_{n-1} z_{n-1} - \hat{\theta}^T_{\text{nth}} F_{ns} - f_{ns})\,. \quad (38)$$

Substituting (38) into (35) and (37) results in

$$\dot{z}_n = -c_n z_i - g_{n-1} z_{n-1} + (\theta - \hat{\theta}_{\text{nth}})^T F_{ns} \quad (39)$$

and

$$\dot{V}_n = -\sum_{k=1}^{n} c_k z_k^2 + (\theta - \hat{\theta}_{\text{nth}})^T (F_{ns} z_n - \Gamma^{-1} \dot{\hat{\theta}}_{\text{nth}})\,. \quad (40)$$

The $(\theta - \hat{\theta}_{\text{nth}})$-term is now eliminated with the update law

$$\dot{\hat{\theta}}_{\text{nth}} = \Gamma F_{ns} z_n \quad (41)$$

which yields

$$\dot{V}_n = -\sum_{i=1}^{n} c_i z_i^2 \quad (42)$$

Error subsystems (21), (31) and (39) form the complete error system

$$\begin{aligned} \dot{z}_1 &= -c_1 z_1 + g_1 z_2 + (\theta - \hat{\theta}_{\text{1st}})^T F_{1s} \\ \dot{z}_i &= -c_i z_i - g_{i-1} z_{i-1} + g_i z_{i+1} \\ &\quad + (\theta - \hat{\theta}_{\text{ith}})^T F_{is} \quad i = 2, \ldots, n-1 \\ \dot{z}_n &= -c_n z_i - g_{n-1} z_{n-1} + (\theta - \hat{\theta}_{\text{nth}})^T F_{ns} \end{aligned} \quad (43)$$

where

$$z_i = x_i - x_{ri} - \alpha_{i-1}\,,\; i = 1, \ldots, n\,,\; \alpha_0 = 0 \quad (44)$$

and the update laws for $\hat{\theta}_{\text{ith}}$ are

$$\dot{\hat{\theta}}_{\text{ith}} = \Gamma F_{is} z_i\,, \quad i = 1, \ldots, n\,. \quad (45)$$

Theorem 1. *The closed-loop adaptive system consisting of the plant (1), the reference model (14), the controller (38) and the parameter update law (45) has a globally uniformly stable equilibrium at* $z = [z_1, z_2, \ldots, z_n]^T = 0$. *This guarantees the global boundedness of the states* $x = [x_1, x_2, \ldots, x_n]^T$, *the* *parameter estimates* $\hat{\theta}(t) = [\hat{\theta}_{1\text{st}}, \hat{\theta}_{2\text{nd}}, \ldots, \hat{\theta}_{n\text{th}}]^T$ *and the control action* u, *and* $\lim_{t\to\infty} z(t) = 0$, *i.e. subsequently*,

$$\lim_{t\to\infty} [y(t) - y_r(t)] = 0 \tag{46}$$

Proof. The error equations (43) correspond to the closed-loop adaptive system, which consists of the plant (1), the reference model (14), the controller (38) and the parameter update law (45). The derivative of the Lyapunov function (36) along the error equations (43) is (42), which proves that equilibrium $z = 0$ is globally uniformly stable.

Combining (36) with (42), we conclude that $\hat{\theta}_{1\text{st}}, \hat{\theta}_{2\text{nd}}, \ldots, \hat{\theta}_{n\text{th}}$ are bounded. Since $z_1 = x_1 - x_{r1}$ and x_{r1} is bounded, we see that x_1 is also bounded. The boundedness of x_i, $i = 2, \ldots, n$ follows from the boundedness of α_{i-1}, $i = 2, \ldots, n$ (defined in (30)) and x_{ri}, and the fact that $x_i = z_i + x_{ri} + \alpha_{i-1}$, $i = 2, \ldots, n$. Using (38), we conclude that the control u is also bounded.

From the LaSalle–Yoshizawa theorem [LaSalle, 1968; Yoshizawa, 1966], it further follows that, all the solutions of (43) converge to the manifold $z = 0$ as $t \to \infty$. From the definition in (17), we conclude that $|y(t) - y_r(t)| \to 0$ as $t \to \infty$. ■

Remark 3.1. The strict-feedback systems (1) and (2) are the cases with parametric uncertainties only. Nonlinearities of the system are assumed to be known and unknown parameters are assumed to appear linearly with respect to these known nonlinear functions. For the case when both parametric uncertainty and unknown nonlinear functions are present in the system, where these unknown nonlinear functions could be due to modeling errors, external disturbances, time variations in the system, robust adaptive control design can guarantee robustness with respect to bounded uncertainties and exogenous disturbances [Polycarpou & Ioannou, 1996; Yao & Tomizuka, 1997; Pan & Basar, 1998; Freeman *et al.*, 1998]. But in general they cannot achieve convergence of the tracking error to zero without high gain.

4. Simulation Study

To examine the effectiveness of the proposed design procedure, extensive computer simulations were carried out for the second-order Duffing oscillator. Other chaotic systems such as the van der Pol oscillator, the Rössler system, the Chua's circuit can be controlled readily by the same design procedure.

We assume that the controlled Duffing oscillator is originally ($u = 0$) in the chaotic state with parameters $\omega = 1.8$, $\theta = [0.4, -1.1, 1.0, 1.8]^T$. A periodic reference signal is generated from a different system — the BVP oscillator (5) with the system parameters $\omega = 1$, $p_1 = 0$, $p_2 = 0.74$, $p_3 = 0.1$, $p_4 = 0.7$ and $p_5 = 0.8$. The design parameters of controller (38) and parameter update law (45) are chosen as $c_1 = 1$, $c_2 = 1$, $\Gamma = \text{diag}\{1, 1, 1\}$. The initial conditions are chosen as $x_1(0) = 0$, $x_2(0) = 0$, $x_{r1}(0) = 0$ and $x_{r2}(0) = 0$.

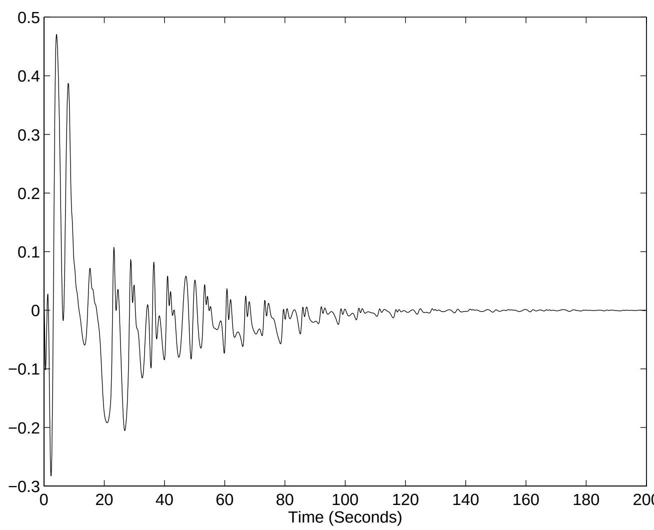

Fig. 1. Tracking error $x_1(t) - x_{r1}(t)$.

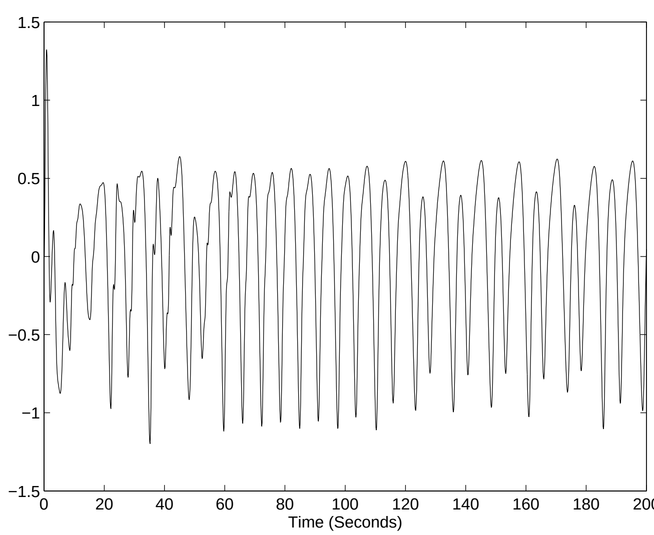

Fig. 2. Boundedness of system state $x_2(t)$.

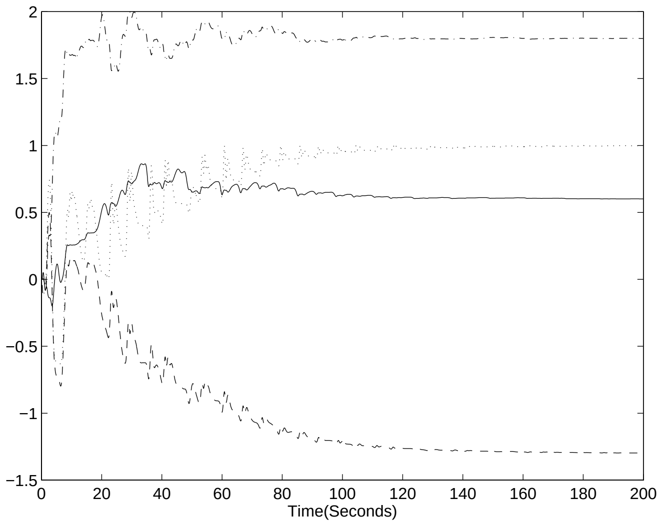

Fig. 3. Boundedness of the estimated parameters $\hat{\theta}_{2\mathrm{nd},1}$ (solid line), $\hat{\theta}_{2\mathrm{nd},2}$ (dashed line), $\hat{\theta}_{2\mathrm{nd},3}$ (dotted line), $\hat{\theta}_{2\mathrm{nd},4}$ (dashdot line) ($\hat{\theta}_{1\mathrm{st}} = [0, 0, 0, 0]^T$).

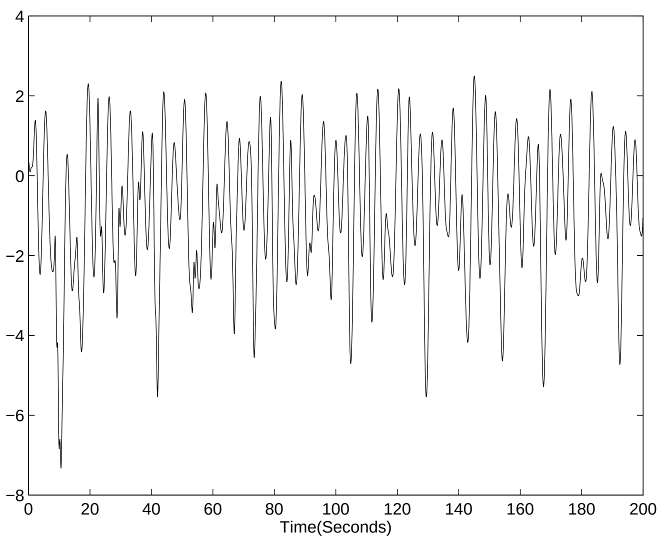

Fig. 4. Boundedness of control action u.

Numerical simulation results are shown in Figs. 1–4. As shown in Fig. 1, the output $y = x_1(t)$ of the controlled Duffing oscillator (3) asymptotically track the periodic reference signal $y_r = x_{r1}(t)$ of the BVP oscillator (5), while at the same time the state $x_2(t)$ of the controlled Duffing oscillator (3), and the parameter estimates $\hat{\theta}_{1\mathrm{st}}$ and $\hat{\theta}_{2\mathrm{nd}}$ and the control action u remain bounded as shown in Figs. 2–4 respectively.

5. Conclusion

In this paper, firstly we showed that many chaotic systems as paradigms in the research of chaos can be transformed into a class of nonlinear systems in the so-called nonautonomous "strict-feedback" form. Then, an adaptive backstepping control scheme has been extended to the nonautonomous "strict-feedback" system, and it has been used to control the output of these chaotic systems to asymptotically track arbitrarily given reference signals generated from known, bounded and smooth nonlinear reference model. Strong properties of global stability and asymptotic tracking have been achieved in a finite number of steps. Finally, the Duffing oscillator with key constant parameters unknown has been used as an example to illustrate the feasibility of the proposed adaptive backstepping control scheme. The adaptive backstepping approach as well as the procedures of control law design and parameters estimate law design do not use specific features of chaos and can be applied to track both chaotic and periodic motions. Along with its advantages, the backstepping design procedure has certain drawbacks. One of them is that for high-order systems the nonlinear expression of the controller becomes increasingly complex.

References

Bernardo, M. [1996a] "An adaptive approach to the control and synchronization of chaotic systems," *Int. J. Bifurcation and Chaos* **6**(3), 557–568.

Bernardo, M. [1996b] "A purely adaptive controller to synchronize and control chaotic systems," *Phys. Lett.* **A214**, 139–144.

Chen, G. & Dong, X. [1998] *From Chaos To Order* (World Scientific, Singapore).

Chua, L. O. [1993] "Global unfolding of Chua's circuit," *IEICE Trans. Fundamentals* **E76-A**, 704–734.

Chua, L. O., Komuro, M. & Matsumoto, T. [1986] "The double scroll family parts I and II," *IEEE Trans. Circuits Syst. I* **33**(11), 1073–1118.

Dong, X., Chen, G. & Chen, L. [1997] "Adaptive control of the uncertain Duffing oscillator," *Int. J. Bifurcation and Chaos* **7**(7), 1651–1658.

Duffing, G. [1918] "Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung," *Vieweg*, Braunschweig.

Fitzhugh, R. [1961] "Impulses and physiological states in theoretical models of nerve membrane," *Biophys. J.* **1**, 445–466.

Fradkov, A. L. & Pogromsky, A. Yu. [1996] "Speed gradient control of chaotic systems," *IEEE Trans. Circuits Syst. I* **43**, 907–913.

Freeman, R. A., Krstic, M. & Kokotovic, P. V. [1998] "Robustness of adaptive nonlinear control to bounded uncertainties," *Automatica* **34**(10), 1227–1230.

Kanellakopoulos, I., Kokotovič, P. & Morse, A. [1991] "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Trans. Automat. Contr.* **36**, 1241–1253.

Krstič, M., Kanellakopoulos, I. & Kokotovič, P. [1992] "Adaptive nonlinear control without overparametrization," *Syst. Contr. Lett.* **19**, 177–185.

Krstič, M., Kanellakopoulos, I. & Kokotovič, P. [1995] *Nonlinear and Adaptive Control Design* (John Wiley, NY).

LaSalle, J. P. [1968] "Stability theory for ordinary differential equations," *J. Diff. Eqs.* **4**, 57–65.

Lorenz, E. N. [1963] "Deterministic non-periodic flow," *J. Atmos. Sci.* **20**, 130–141.

Nijmeijer, H. & Berghuis, H. [1995] "On Lyapunov control of the Duffing equation," *IEEE Trans. Circuits Syst. I* **42**, 473–477.

Pan, Z. & Basar, T. [1998] "Adaptive controller design for tracking and disturbance attenuation in parametric strict-feedback nonlinear systems," *IEEE Trans. Automat. Contr.* **43**(8), 1066–1083.

Polycarpou, M. M. & Ioannou, P. A. [1996] "Robust adaptive nonlinear control design," *Automatica* **32**(3), 423–427.

Rössler, O. E. [1976] "An equation for continuous chaos," *Phys. Lett.* **57A**, 397–398.

van der Pol, B. [1927] "Forced oscillations in a circuit with nonlinear resistance (reception with reactive triode)," *Philos. Mag.* **7**, 65–80.

Wu, C. W., Yang, T. & Chua, L. O. [1996] "On adaptive synchronization and control of nonlinear dynamical systems," *Int. J. Bifurcation and Chaos* **6**(3), 455–472.

Yang, T., Yang, Ch.-M. & Yang, L.-B. [1998] "Detailed study of adaptive control of chaotic systems with unknown parameters," *Dyn. Contr.* **8**(3), 255–267.

Yao, B. & Tomizuka, M. [1997] "Adaptive robust control of SISO nonlinear systems in a semi-strict feedback form," *Automatica* **33**(5), 893–900.

Yoshizawa, T. [1966] *Stability Theory by Lyapunov's Second Method* (The Mathematical Society of Japan).