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ADAPTIVE BACKSTEPPING CONTROL OF UNCERTAIN LORENZ SYSTEM

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In this paper, we consider the problem of controlling chaos in the well-known Lorenz system. Firstly we show that the Lorenz system can be transformed into a kind of nonlinear system in the so-called general strict-feedback form. Then, adaptive backstepping design is used to control the Lorenz system with three key parameters unknown. By exploiting the property of the system, the resulting controller is singularity free, and the closed-loop system is stable globally. Simulation results are conducted to show the effectiveness of the approach.

1. Introduction

In recent years, there has been considerable interest in the control of chaos in nonlinear dynamical systems, see [Chen & Dong, 1998; Fradkov & Pogromsky, 1998] for a survey of recent developments. In particular, the control of the well-known Lorenz system [Lorenz, 1963] has been studied extensively [Gallegos, 1994; Yu, 1996; Lenz & Obradovic, 1997; Zeng & Singh, 1997; Liao, 1998] (to name just a few). Gallegos [1994] considered the input-state feedback linearization for the control of the Lorenz system based on nonlinear differential geometric control theory [Isidori, 1989]. It has been noticed, however, that the complete input-state linearization leads to controller singularity when applied to the Lorenz system [Lenz & Obradovic, 1997]. Yu [1996] elegantly proposed a variable structure control strategy to stabilize Lorenz chaos. To prevent an exploding control action due to singular control law, lower and upper bounds for the control action have been introduced. As a consequence, the resulting controller is only locally stabilizing [Yu, 1996].

To overcome the singularity problem in controlling the Lorenz system, more recently, Lenz and Obradovic [1997] and Zeng and Singh [1997] have obtained some interesting singularity free results on the control of the Lorenz system. In [Lenz & Obradovic, 1997], a global approach for the control of the Lorenz system with partial linearization was presented. The stabilities of the resulting system were guaranteed by sequentially proving the stability of each individual state. Zeng and Singh [1997] proposed an excellent adaptive controller based on Lyapunov stability theorem for set-point control of uncertain Lorenz system with three parameters unknown. Both approaches avoid the singularity problem based on the analysis of the specific structure of the Lorenz system, and in both cases the control action is introduced into the second equation of the Lorenz system.

In the past decade, backstepping [Kanellakopoulos *et al.*, 1991] has become one of the most popular design methods for adaptive nonlinear control because it can guarantee global stabilities, tracking and transient performance for the broad class of strict-feedback system [Krstic *et al.*, 1995, and the references therein]. In [Ge *et al.*, 2000; Ge & Wang, 2000], it has been shown that many well-known chaotic systems as paradigms in the

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research of chaos, including Duffing oscillator, van der Pol oscillator, Rössler system and several types of Chua's circuits, can be transformed into a class of nonlinear systems in the "strict-feedback" form, and adaptive backstepping has been employed and extended to the control of these chaotic systems. However, the Lorenz system, as indicated in [Ge *et al.*, 2000], cannot be controlled directly using backstepping method for its singularity problem. As a design tool, the backstepping method is less restrictive than feedback linearization. In some situations it can overcome singularities such as the lack of controllability [Krstic *et al.*, 1995]. In this letter, firstly we show that the Lorenz system in the research of chaos can be transformed into a kind of nonlinear system in the so-called general strict-feedback form. Secondly, by exploiting the specific property of the Lorenz system, a global approach for adaptive control of the uncertain Lorenz system, with three key parameters unknown, is presented. In this approach, the control action u is introduced into the third equation of the Lorenz system. By employing adaptive backstepping design procedure, the resulting controller is singularity free, and the closed-loop system is stable globally. Simulation results are included to show the effectiveness of the approach.

2. The Lorenz System in General Strict-Feedback Form

The Lorenz system is an approximation of a partial differential equation for fluid convection, where a flat fluid layer is heated from below and cooled from above. It has become one of paradigms in the research of chaos, and is described by

$$\begin{aligned} \dot{x}_1 &= p_1x_2 - p_1x_1 \\ \dot{x}_2 &= -x_1x_3 + p_2x_1 - x_2 \\ \dot{x}_3 &= x_1x_2 - p_3x_3 + u \end{aligned} \tag{1}$$

where x_1, x_2 and x_3 are the states, p_1, p_2 and p_3 are positive constant parameters, and a controller $u(\cdot)$ is fed into the third equation to form the controlled Lorenz system.

Under the assumption that p_1, p_2 and p_3 are unknown, the controlled Lorenz system (1) can be easily transformed into the general strict-feedback form with $n = 3$

$$\begin{aligned} \dot{x}_i &= b_ig_i(\overline{x}_i)x_{i+1} + \theta^TF_i(\overline{x}_i) + f_i(\overline{x}_i)\,, \\ &\qquad i = 1, \ldots, n-1 \\ \dot{x}_n &= b_ng_n(\overline{x}_n)u + \theta^TF_n(\overline{x}_n) + f_n(\overline{x}_n) \\ y &= x_1 \end{aligned} \tag{2}$$

where $\overline{x}_i = [x_1, x_2, \ldots, x_i]^T \in R^i$, $i = 1, \ldots, n$, $u \in R$, and $y \in R$ are the states, input and output, respectively; $b = [b_1, b_2, \ldots, b_n]^T \in R^n$ and $\theta \in R^p$ are the vectors of unknown constant parameters of interest; $g_i(\cdot)$, $F_i(\cdot)$, $f_i(\cdot)$, $i = 1, \ldots, n$ are known, smooth nonlinear functions.

Backstepping design procedure cannot be directly applied to the general strict-feedback system (2) when $g_i(\cdot) = 0$. However, for some systems it can overcome the singularity problems (caused by $g_i(\cdot) = 0$) by exploiting the structure properties of the controlled system. For the controlled Lorenz system in the forms of (1), where $g_2(\cdot) = x_1$, which may take the value of zero, our objective is to design an adaptive state-feedback controller for uncertain Lorenz system (1) that guarantees global stability and regulates the state $x_1(t)$ of system (1) to the set-point $x_1^e = 0$.

3. Adaptive Backstepping Control Design

For the uncertain Lorenz system with three key parameters unknown, we develop a design procedure via backstepping which can overcome the singularity caused by $-x_1x_3$. The backstepping design procedure contains three steps. At Step i, an intermediate virtual control function α_i shall be developed using an appropriate Lyapunov function V_i. Let us first consider the first equation in (1).

Step 1. Define $z_1 = x_1$. Its derivative is given by

$$\begin{aligned} \dot{z}_1 = \dot{x}_1 &= p_1x_2 - p_1x_1 \\ &= p_1z_2 + p_1\alpha_1 - p_1x_1 \end{aligned} \tag{3}$$

where $z_2 = x_2 - \alpha_1$, α_1 is an artificial control to be defined later.

Using α_1 as a control to stabilize the z_1-subsystem defined by (3), we choose the following Lyapunov function candidate

$$V_1 = \frac{1}{2}z_1^2\,. \tag{4}$$

Its derivative is given by

$$\begin{aligned} \dot{V}_1 = z_1\dot{z}_1 &= z_1(p_1x_2 - p_1x_1) \\ &= p_1z_1z_2 + z_1p_1(\alpha_1 - x_1)\,. \end{aligned} \tag{5}$$

By noticing that $p_1 > 0$ and $x_1 = z_1$, we choose

$$\alpha_1 = -c_0 z_1 \,, \quad c_0 > -1 \tag{6}$$

which leads to

$$\dot{V}_1 = -c_1 z_1^2 + p_1 z_1 z_2 \,, \quad c_1 = c_0 p_1 + p_1 > 0 \,. \tag{7}$$

The second term $p_1 z_1 z_2$ will be canceled in the next step. The closed-loop form of (3) with (6) is

$$\dot{z}_1 = -c_1 z_1 + p_1 z_2 \,. \tag{8}$$

Remark 1. In the first step of controller design for α_1, $V_1 = (1/2)z_1^2$ rather than $V_1 = (1/2)z_1^2 + (1/2\gamma)(\hat{p}_1 - p_1)^2$ was chosen as the Lyapunov function candidate. This is for the reason that $p_1 > 0$ is a stabilizing element as shown in Eq. (5), though its value is unknown. The unknown parameter p_1 will be canceled in the second step. The choice of V_1 is different from that of traditional backstepping design for parametric strict-feedback system, where the unknown parameter in the first equation has to be coped with in the first step.

Step 2. In this step, we will deal with the singularity problem caused by $-x_1 x_3$ in the second equation of (1). The derivative of z_2 is expressed as

$$\begin{aligned}\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 \\ &= -x_1 x_3 - x_2 + p_2 x_1 + c_0(p_1 x_2 - p_1 x_1) \\ &= -x_1 z_3 - x_1 \alpha_2 + \hat{p}_2 x_1 - c_0 \hat{p}_1 x_1 - (1 - c_0 \hat{p}_1)\alpha_1 \\ &\quad - (1 - c_0 p_1) z_2 - (\hat{p}_2 - p_2) x_1 + c_0(\hat{p}_1 - p_1) x_1 \\ &\quad - c_0(\hat{p}_1 - p_1)\alpha_1\end{aligned} \tag{9}$$

where $z_3 = x_3 - \alpha_2$, α_2 is the virtual control to be defined.

Using α_2 as a control to stabilize the (z_1, z_2)-subsystem defined by (8) and (9), we choose the following Lyapunov function candidate

$$V_2 = V_1 + \frac{1}{2} z_2^2 + \frac{1}{2\gamma}(\hat{p}_1 - p_1)^2 + \frac{1}{2\gamma}(\hat{p}_2 - p_2)^2 \,. \tag{10}$$

The derivative of V_2 is

$$\begin{aligned}\dot{V}_2 &= -c_1 z_1^2 + p_1 z_1 z_2 + z_2(\dot{x}_2 - \dot{\alpha}_1) \\ &\quad + (\hat{p}_1 - p_1)\gamma^{-1}\dot{\hat{p}}_1 + (\hat{p}_2 - p_2)\gamma^{-1}\dot{\hat{p}}_2 \\ &= -c_1 z_1^2 - x_1 z_2 z_3 + z_2[\hat{p}_1 z_1 - x_1 \alpha_2 + \hat{p}_2 x_1 \\ &\quad - c_0 \hat{p}_1 x_1 - (1 - c_0 \hat{p}_1)\alpha_1 - (1 - c_0 p_1) z_2] \\ &\quad + (\hat{p}_1 - p_1)\gamma^{-1}[\dot{\hat{p}}_1 + \gamma(c_0 x_1 - c_0 \alpha_1 - z_1) z_2] \\ &\quad + (\hat{p}_2 - p_2)\gamma^{-1}(\dot{\hat{p}}_2 - \gamma x_1 z_2)\end{aligned} \tag{11}$$

The $(\hat{p}_1 - p_1)$ and $(\hat{p}_2 - p_2)$-terms are now eliminated with the update laws

$$\begin{aligned}\dot{\hat{p}}_1 &= -\gamma(c_0 x_1 - c_0 \alpha_1 - z_1) z_2 \\ \dot{\hat{p}}_2 &= \gamma x_1 z_2 \,.\end{aligned}$$

Substituting $\alpha_1 = -c_0 z_1$ and $z_1 = x_1$ into Eq. (11), we have

$$\begin{aligned}\dot{V}_2 &= -c_1 z_1^2 - x_1 z_2 z_3 + z_2[x_1(\hat{p}_1 + \hat{p}_2 + c_0(1 - \hat{p}_1 \\ &\quad - c_0 \hat{p}_1)) - x_1 \alpha_2 - (1 - c_0 p_1) z_2] \,.\end{aligned} \tag{12}$$

Remark 2. The possibility of $x_1 = 0$ makes α_2 impossible to cancel the term $(1 - c_0 p_1) z_2$ in (12). However, by choosing c_0 such that $1 - c_0 p_1 > 0$, i.e. $-(1 - c_0 p_1) z_2$ stabilizing, there is no need to cancel this term for its good characteristics. Thus, the controller singularity problem is avoided.

Accordingly, Eq. (12) can be rewritten as

$$\begin{aligned}\dot{V}_2 &= -c_1 z_1^2 - x_1 z_2 z_3 + z_2 x_1[\hat{p}_1 + \hat{p}_2 - c_0 \hat{p}_1 \\ &\quad + (1 - c_0 \hat{p}_1)c_0 - \alpha_2] - (1 - c_0 p_1) z_2^2 \,.\end{aligned} \tag{13}$$

Choosing

$$\alpha_2 = \hat{p}_1 + \hat{p}_2 - c_0 \hat{p}_1 + (1 - c_0 \hat{p}_1)c_0 \tag{14}$$

which yields

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 - x_1 z_2 z_3 \,, \quad c_2 = (1 - c_0 p_1) \,. \tag{15}$$

Since $p_1 > 0$, it is sufficient to choose $-1 < c_0 \le 0 < 1/p_1$, such that $c_1 = c_0 p_1 + p_1 > 0$ and $c_2 = (1 - c_0 p_1) > 0$. The third term $-x_1 z_2 z_3$ will be canceled in the final step. The closed-loop form of (9) with (14) is

$$\begin{aligned}\dot{z}_2 &= -x_1 z_3 - \hat{p}_1 x_1 - (1 - c_0 p_1) z_2 - (\hat{p}_2 - p_2) x_1 \\ &\quad + c_0(\hat{p}_1 - p_1) x_1 - c_0(\hat{p}_1 - p_1)\alpha_1 \,.\end{aligned} \tag{16}$$

Step 3. The derivative of z_3 is expressed as

$$\begin{aligned}\dot{z}_3 &= \dot{x}_3 - \dot{\alpha}_2 \\ &= u + x_1 x_2 - p_3 x_3 - \frac{\partial \alpha_2}{\partial \hat{p}_1}\dot{\hat{p}}_1 - \frac{\partial \alpha_2}{\partial \hat{p}_2}\dot{\hat{p}}_2 \,.\end{aligned} \tag{17}$$

Using control u to stabilize the (z_1, z_2, z_3)-subsystem defined by (8), (16) and (17), we choose

$$V_3 = V_2 + \frac{1}{2} z_3^2 + \frac{1}{2\gamma}(\hat{p}_3 - p_3)^2 \,. \tag{18}$$

The derivative of V_3 is

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 + z_3\bigg(-x_1 z_2 + u + x_1 x_2 - \hat{p}_3 x_3 - \frac{\partial \alpha_2}{\partial \hat{p}_1}\dot{\hat{p}}_1 - \frac{\partial \alpha_2}{\partial \hat{p}_2}\dot{\hat{p}}_2\bigg) + (\hat{p}_3 - p_3)\gamma^{-1}(\dot{\hat{p}}_3 + \gamma x_3 z_3)\,. \quad (19)$$

The $(\hat{p}_3 - p_3)$-term is now eliminated with the update law

$$\dot{\hat{p}}_3 = -\gamma x_3 z_3\,. \quad (20)$$

For the bracketed term multiplying z_3 in Eq. (19) to be equal to $-c_3 z_3^2$, $c_3 > 0$, we choose

$$u = -c_3 z_3 + x_1 z_2 - x_1 x_2 + \hat{p}_3 x_3 + \frac{\partial \alpha_2}{\partial \hat{p}_1}\dot{\hat{p}}_1 + \frac{\partial \alpha_2}{\partial \hat{p}_2}\dot{\hat{p}}_2 \quad (21)$$

which yields

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2\,. \quad (22)$$

Since $\dot{V}_3$ is negative definite, it follows from LaSalle–Yoshizawa theorem [LaSalle, 1968; Yoshizawa, 1966] that in the (z_1, z_2, z_3) coordinates the equilibrium $(0, 0, 0)$ is global asymptotically stable, and $(\hat{p}_1(t), \hat{p}_2(t), \hat{p}_3(t))$ are global uniformly bounded. In view of $z_1 = x_1$, $z_2 = x_2 - \alpha_1$ and $\alpha_1 = -c_0 z_1$, this implies that x_1 and x_2 go to zeros asymptotically. From $z_3 = x_3 - \alpha_2$ and (14), one concludes that $x_3 \to \hat{p}_1 + \hat{p}_2 - c_0\hat{p}_1 + (1 - c_0\hat{p}_1)c_0$, as $t \to \infty$, i.e. $x_3(t)$ remains bounded. Using (21), one concludes that the control u is also bounded.

Simulation studies are conducted to demonstrate the effectiveness of this approach.

4. Simulation Results

Assuming that the controlled Lorenz system (1) is originally ($u = 0$) in the chaotic state with parameters $p_1 = 10$, $p_2 = 28$ and $p_3 = 8/3$, our objective is to force the state $x_1(t)$ of the controlled Lorenz system (1) to asymptotically regulate to the set-point $x_1^e = 0$.

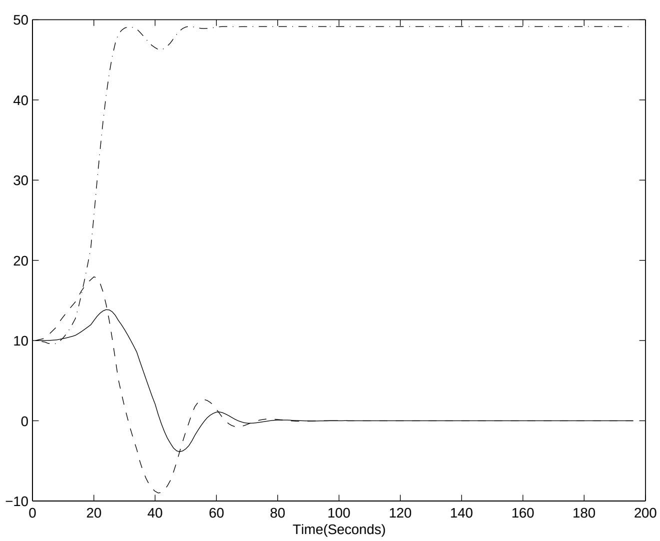

Fig. 1. System responses of controlled Lorenz system: $x_1(t)$ (solid line), $x_2(t)$ (dash line) and $x_3(t)$ (dashdot line).

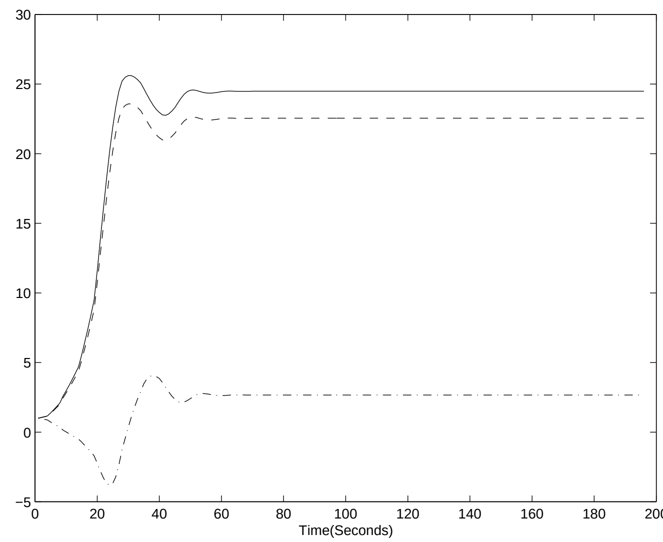

Fig. 2. System parameters estimates: $\hat{p}_1$ (solid line), $\hat{p}_2$ (dash line) and $\hat{p}_3$ (dashdot line).

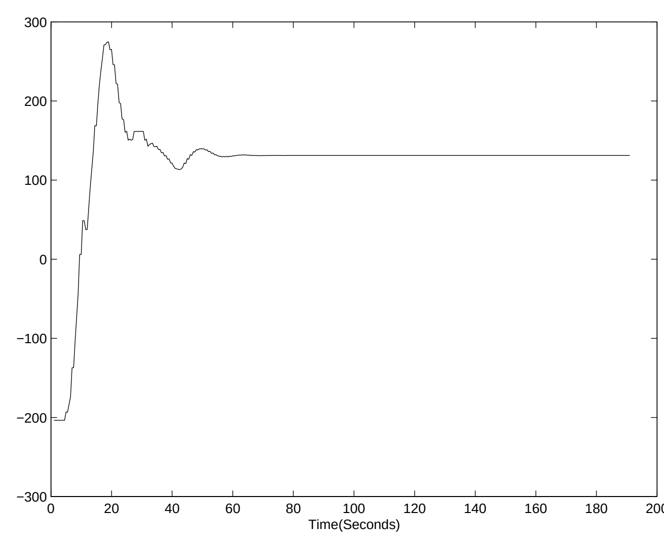

Fig. 3. Control action u.

In the following simulation, the design parameters of controller (19) and parameter update law (12) and (20) are chosen as $c_0 = -0.1$, $c_3 = 50$ and $\gamma = 1$. The initial conditions are chosen that $x_1(0) = 10$, $x_2(0) = 10$ and $x_3(0) = 10$.

Numerical simulation results are shown in Figs. 1–3. As shown in Fig. 1, the states $x_1(t)$ and $x_2(t)$ asymptotically regulate to $x_1 = 0$ and $x_2 = 0$ respectively, the state $x_3(t)$ remains bounded; while at the same time the parameter estimates $\hat{p}_1$, $\hat{p}_2$ and $\hat{p}_3$ and the control action u remain bounded as shown in Figs. 2 and 3 respectively.

5. Conclusion

In this letter, firstly we showed that the well-known Lorenz system as one of the paradigms in the research of chaos can be transformed into a kind of nonlinear system in the so-called general strict-feedback form. Then, by exploiting the specific property of the Lorenz system, an adaptive backstepping control scheme has been employed to control the state x_1 of uncertain Lorenz system to asymptotically regulate to the set-point $x_1^e = 0$. The strong property of global stability has been achieved in a finite number of steps. The proposed approach makes use of the flexibility of backstepping design method and does not lead to singular behavior with respect to the control action.

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