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# SYNCHRONIZATION OF TWO UNCERTAIN CHAOTIC SYSTEMS VIA ADAPTIVE BACKSTEPPING

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In this letter, adaptive synchronization of two uncertain chaotic systems is presented using adaptive backstepping with tuning functions. The master system is any smooth, bounded, linear-in-the-parameters nonlinear chaotic system, while the slave system is a nonlinear chaotic system in the strict-feedback form. Both master and slave systems are with key parameters unknown. Global stability and asymptotic synchronization between the outputs of master and slave systems can be achieved. The proposed approach offers a systematic design procedure for adaptive synchronization of a large class of continuous-time chaotic systems in the chaos research literature. Simulation results are presented to show the effectiveness of the approach.

## 1. Introduction

Since the discovery of chaos synchronization [Pecora & Carroll, 1990], there have been tremendous interests in studying the synchronization of chaotic systems, see [Chen & Dong, 1998; Fradkov & Pogromsky, 1998] and the references therein for a survey of recent development. As chaotic signals could be used to transmit information from a master system to a slave system in a secure and robust manner, chaos synchronization has been intensively studied in communications research [Cuomo *et al.*, 1993; Dedieu *et al.*, 1993; Chua *et al.*, 1996; Dedieu & Ogorzalek, 1997; Kolumban *et al.*, 1997, 1998] (to name just a few).

Recently, specialists from nonlinear control theory turned their attention to the study of chaos synchronization and its potential applications in communications. Fradkov and Pogromsky [1996] presented a speed-gradient method for adaptive synchronization of chaotic systems. Nijmeijer and Mareels [1997] casted the problem of chaos synchronization as a special case of observer design. Suykens *et al.* [1997] proposed a robust nonlinear $H_\infty$ synchronization method for chaotic Lur's systems with applications to secure communications. Pogromsky [1998] considered the problem of controlled synchronization of nonlinear systems using a passivity-based design method. More recently, Fradkov *et al.* [1999] presented an adaptive observer-based synchronization scheme, where an adaptive observer for estimating the unknown parameters of the master system was designed, which corresponds to the parameter modulation for message transmission. Due to these developments, chaos synchronization as well as chaos communications have attracted revived interests in the nonlinear control community.

Over the past decade, backstepping [Kanellakopoulos *et al.*, 1991] has become one of the most popular design methods for adaptive nonlinear control because it can guarantee global stabilities, tracking, and transient performance for a broad class of strict-feedback systems ([Krstiċ *et al.*, 1995], and the references therein). In [Ge *et al.*, 2000],

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it has been shown that many well-known chaotic systems as paradigms in the research of chaos, including Duffing oscillator, van der Pol oscillator, Rössler system, and several types of Chua's circuits, can be transformed into a class of nonlinear systems in the so-called nonautonomous "strict-feedback" form, and the adaptive backstepping and tuning functions control schemes have been employed and extended to control these chaotic systems with key parameters unknown. Global stability and asymptotic tracking have been achieved. In particular, the output of the controlled chaotic system has been designed to asymptotically track any smooth and bounded reference signal generated from a known reference model which may be a chaotic system.

In this paper, we propose an approach for adaptive synchronization of two uncertain nonlinear (chaotic) systems, using adaptive backstepping with tuning functions [Krstiċ *et al.*, 1992, 1995]. In our approach, the master system is any smooth, bounded, linear-in-the-parameters nonlinear chaotic system, while the slave system is a nonlinear chaotic system in the strict-feedback form. All the key parameters of both master and slave systems are unknown. Global stability and asymptotic synchronization between the outputs of master and slave systems can be achieved. In particular, chaos synchronization can be realized when the master system is in chaotic states. With this approach, we can synchronize not only two chaotic systems of the same type with different system parameters, but also two completely different chaotic systems, e.g. the Rössler system and the Duffing oscillator, as will be demonstrated in the Simulation Section.

Compared with the observer-based synchronization schemes [Nijmeijer & Mareels, 1997; Fradkov *et al.*, 1999], one drawback of the presented approach is that at least two states of the master system are employed by the slave system due to its strict-feedback form. However, there are two advantages which make this approach attractive. Firstly, a systematic design procedure for adaptive synchronization is presented for a wide class of nonlinear systems with key parameters unknown, including most of the continuous-time chaotic and hyperchaotic systems in the literature. Secondly, the master system can be chosen as a nonlinear (chaotic) dynamical system of any order, which implies that much complicated high-order chaotic systems can be employed to improve the security in chaos communications.

The rest of the paper is organized as follows: The problem formulation is presented in Sec. 2. Adaptive backstepping with tuning functions is extended to the adaptive synchronization problem in Sec. 3. In Sec. 4, the Rössler system and the Duffing oscillator, both with key constant parameters unknown, are used as the master and the slave systems respectively, to show the effectiveness of the proposed approach. Section 5 contains the conclusions.

## 2. Problem Formulation

Consider the master system in the form of any smooth, bounded, linear-in-the-parameters nonlinear (chaotic) system as

$$\begin{aligned} \dot{x}_{di} &= f_{di}(x_d, t) + \theta^T F_{di}(x_d, t),\ 1 \le i \le m \\ y_d &= x_{d1} \end{aligned} \tag{1}$$

where $x_d = [x_{d1}, x_{d2}, \ldots, x_{dm}]^T \in R^m$ is the state vector; $y_d \in R$ is the output, $\theta = [\theta_1, \theta_2, \ldots, \theta_p]^T \in R^p$ is the vector of unknown constant parameters; $f_{di}(\cdot)$, $F_{di}(\cdot)$, $i = 1, 2, \ldots, m$ are known smooth nonlinear functions, with their $j$th derivatives $(j = 0, \ldots, m-i)$ uniformly bounded in $t$.

The slave system is in the form of strict-feedback nonlinear (chaotic) system as

$$\begin{aligned} \dot{x}_i &= g_i(\bar{x}_i, t)x_{i+1} + \eta^T F_i(\bar{x}_i, t) \\ &\quad + f_i(\bar{x}_i, t), \quad 1 \le i \le n-1 \\ \dot{x}_n &= g_n(\bar{x}_n, t)u + \eta^T F_n(\bar{x}_n, t) \\ &\quad + f_n(\bar{x}_n, t), \quad n \le m \\ y &= x_1 \end{aligned} \tag{2}$$

where $\bar{x}_i = [x_1, x_2, \ldots, x_i]^T \in R^i (i = 1, \ldots, n)$, $y \in R$ and $u \in R$ are the states, output and control action, respectively; $\eta = [\eta_1, \eta_2, \ldots, \eta_q]^T \in R^q$ is the vector of unknown constant parameters; $g_i(\cdot) \neq 0$, $F_i(\cdot)$, $f_i(\cdot)$, $i = 1, \ldots, n$ are known, smooth nonlinear functions, with their $j$th derivatives $(j = 0, \ldots, n-i)$ uniformly bounded in $t$.

The problem is to design an adaptive synchronization algorithm

$$\begin{aligned} u &= U(x, x_d, \hat{\theta}, \hat{\eta}, t) \\ \dot{\hat{\eta}} &= H_\eta(x, x_d, \hat{\theta}, \hat{\eta}, t) \\ \dot{\hat{\theta}} &= H_\theta(x, x_d, \hat{\theta}, \hat{\eta}, t) \end{aligned} \tag{3}$$

where $\hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p] \in R^p$ and $\hat{\eta} = [\hat{\eta}_1, \hat{\eta}_2, \ldots, \hat{\eta}_q]^T \in R^q$ are parameter estimates of the unknown parameters $\theta$ and $\eta$, respectively, to guarantee global stability and force the output $y(t)$ of the slave system (2) to asymptotically synchronize with the output $y_d(t)$ of the master system (1), i.e. to achieve

$$y_(t) - y_d(t) \to 0, \quad \text{as} \quad t \to \infty\,, \tag{4}$$

while guarantees global boundedness of all the signals in the closed-loop system.

## 3. Adaptive Synchronization via Backstepping

In order to design an adaptive synchronization algorithm (3) to achieve objective (4), adaptive backstepping with tuning functions is employed. The global stability of the closed-loop system and the asymptotic synchronization of the outputs of master (1) and slave (2) systems are summarized in Theorem 1.

**Theorem 1.** *Consider the master system (1) and the slave system (2) both with key parameters $\theta$ and $\eta$ unknown. Consider the coordinate transformation*:

$$\begin{cases} z_1 = x_1 - x_{d1} \\ z_{i+1} = x_{i+1} - x_{d(i+1)} - \alpha_i,\ 1 \le i \le n-1 \\ z = [z_1,\, z_2, \ldots,\, z_n]^T \end{cases} \tag{5}$$

*with*

$$\begin{cases} \alpha_1 = \dfrac{1}{g_1}(-c_1 z_1 - \hat{\eta}^T F_{1\eta} + \hat{\theta}^T F_{1\theta} - f_{1s}) \\ \alpha_i = \dfrac{1}{g_i}\Bigg(-c_i z_i - g_{i-1} z_{i-1} - \hat{\eta}^T F_{i\eta} + \hat{\theta}^T F_{i\theta} - f_{is} \\ \qquad + \displaystyle\sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\eta}} \Gamma_\eta F_{i\eta} - \sum_{k=1}^{i-2} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma_\theta F_{i\theta}\Bigg),\ 2 \le i \le n-1 \end{cases} \tag{6}$$

*where*

$$\begin{cases} F_{1\eta} = F_1,\ F_{1\theta} = F_{d1} \\ f_{1s} = f_1 - f_{d1} + g_1 x_{d2} \\ F_{i\eta} = F_i - \displaystyle\sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} F_k \\ F_{i\theta} = F_{di} + \displaystyle\sum_{k=1}^{m} \frac{\partial \alpha_{i-1}}{\partial x_{dk}} F_{dk} \\ f_{is} = f_i - f_{di} + g_i x_{d(i+1)} - \displaystyle\sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}(g_k x_{k+1} + f_k) \\ \qquad - \dfrac{\partial \alpha_{i-1}}{\partial \hat{\eta}} \pi_i - \dfrac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \tau_i - \displaystyle\sum_{k=1}^{m} \frac{\partial \alpha_{i-1}}{\partial x_{dk}} f_{dk} - \frac{\partial \alpha_{i-1}}{\partial t},\ 2 \le i \le n-1 \\ f_{ns} = f_n - f_{dn} - \displaystyle\sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}(g_k x_{k+1} + f_k) \\ \qquad - \dfrac{\partial \alpha_{n-1}}{\partial \hat{\eta}} \pi_n - \dfrac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \tau_n - \displaystyle\sum_{k=1}^{m} \frac{\partial \alpha_{n-1}}{\partial x_{dk}} f_{dk} - \frac{\partial \alpha_{n-1}}{\partial t} \end{cases} \tag{7}$$

*Consider the adaptation laws*

$$\begin{cases} \dot{\hat{\eta}} = \pi_n = \tau_{n-1} + \Gamma_\eta F_{n\eta} z_n \\ \dot{\hat{\theta}} = \tau_n = \tau_{n-1} - \Gamma_\theta F_{n\theta} z_n \end{cases} \tag{8}$$

*where*

$$\begin{cases} \pi_i = \pi_{i-1} + \Gamma_\eta F_{i\eta} z_i,\ \pi_1 = \Gamma_\eta F_{1\eta} z_1, \\ \tau_i = \tau_{i-1} - \Gamma_\theta F_{i\theta} z_i,\ \tau_1 = -\Gamma_\theta F_{1\theta} z_1, \\ \quad 2 \le i \le n-1 \end{cases} \tag{9}$$

*with* $\Gamma_\eta = \Gamma_\eta^T > 0$ *and* $\Gamma_\theta = \Gamma_\theta^T > 0$. *Then the control law*

$$u = \frac{1}{g_n}\left(-c_n z_n - g_{n-1} z_{n-1} - \hat{\eta}^T F_{n\eta} + \hat{\theta}^T F_{n\theta} - f_{ns} + \sum_{k=1}^{n-2} z_{k+1}\frac{\partial \alpha_k}{\partial \hat{\eta}}\Gamma_\eta F_{n\eta} - \sum_{k=1}^{n-2} z_{k+1}\frac{\partial \alpha_k}{\partial \hat{\theta}}\Gamma_\theta F_{n\theta}\right) \tag{10}$$

*guarantees* (*i*) *global boundedness of all the signals in the closed-loop system, including the states of the slave system* $x = [x_1, \ldots, x_n]^T$, *the control* $u$ *and parameter estimates* $\hat{\theta}$ *and* $\hat{\eta}$, *and* (*ii*) $\lim_{t\to\infty} z(t) = 0$, *which means that asymptotic synchronization is achieved*

$$\lim_{t\to\infty}[y(t) - y_d(t)] = 0 \tag{11}$$

*Proof.* The backstepping design procedure is recursive. At the $i$th step, the $i$th-order subsystem is stabilized with respect to a Lyapunov function $V_i$ by the design of a stabilizing function $\alpha_i$, and tuning functions $\pi_i$ and $\tau_i$. The control law $u$ and the update laws $\dot{\hat{\eta}}$ and $\dot{\hat{\theta}}$ are given in the last step.

***Step 1.*** The derivative of $z_1 = x_1 - x_{d1}$ is given by

$$\begin{aligned} \dot{z}_1 &= g_1 x_2 + \eta^T F_1 + f_1 - f_{d1} - \theta^T F_{d1} \\ &= g_1 z_2 + g_1\alpha_1 + \hat{\eta}^T F_{1\eta} - \hat{\theta}^T F_{1\theta} + f_{1s} \\ &\quad - (\hat{\eta} - \eta)^T F_{1\eta} + (\hat{\theta} - \theta)^T F_{1\theta} \end{aligned} \tag{12}$$

where $z_2$, $F_{1s}$ and $f_{1s}$ are defined in (5) and (7), respectively; virtual control $\alpha_1$ (6) is used to stabilize (12) with respect to the Lyapunov function candidate

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}(\hat{\eta} - \eta)^T\Gamma_\eta^{-1}(\hat{\eta} - \eta) + \frac{1}{2}(\hat{\theta} - \theta)^T\Gamma_\theta^{-1}(\hat{\theta} - \theta) \tag{13}$$

The derivative of $V_1$ is

$$\begin{aligned} \dot{V}_1 &= z_1\dot{z}_1 + (\hat{\eta} - \eta)^T\Gamma_\eta^{-1}\dot{\hat{\eta}} + (\hat{\theta} - \theta)^T\Gamma_\theta^{-1}\dot{\hat{\theta}} \\ &= g_1 z_1 z_2 + z_1(g_1\alpha_1 + \hat{\eta}^T F_{1\eta} - \hat{\theta}^T F_{1\theta} + f_{1s}) \\ &\quad + (\hat{\eta} - \eta)^T\Gamma_\eta^{-1}(\dot{\hat{\eta}} - \Gamma_\eta F_{1\eta} z_1) \\ &\quad + (\hat{\theta} - \theta)^T\Gamma_\theta^{-1}(\dot{\hat{\theta}} + \Gamma_\theta F_{1\theta} z_1) \end{aligned} \tag{14}$$

Define the tuning functions $\pi_1$ and $\tau_1$ for $\hat{\eta}$ and $\hat{\theta}$ as in (9). Note that the terms with $(\hat{\eta} - \eta)$ and $(\hat{\theta} - \theta)$ would have been eliminated if we had chosen the following update laws $\dot{\hat{\eta}} = \pi_1$ and $\dot{\hat{\theta}} = \tau_1$. Since this is not the last design step, we postpone the choice of update laws and tolerate the presence of $(\hat{\eta} - \eta)$ and $(\hat{\theta} - \theta)$ in $\dot{V}_1$. Choose $\alpha_1$ as in (6) such that the bracketed term multiplying $z_1$ in Eq. (14) be equal to $-c_1 z_1^2$, then $\dot{V}_1$ becomes

$$\dot{V}_1 = -c_1 z_1^2 + g_1 z_1 z_2 + (\hat{\eta} - \eta)^T\Gamma_\eta^{-1}(\dot{\hat{\eta}} - \pi_1) + (\hat{\theta} - \theta)\Gamma_\theta^{-1}(\dot{\hat{\theta}} - \tau_1) \tag{15}$$

***Step 2.*** The derivative of $z_2 = x_2 - x_{d2} - \alpha_1$ is expressed as

$$\begin{aligned} \dot{z}_2 &= g_2 z_3 + g_2\alpha_2 + g_2 x_{3r} + \eta^T F_2 + f_2 - f_{d2} \\ &\quad - \theta^T F_{d2} - \frac{\partial \alpha_1}{\partial x_1}(g_1 x_2 + \eta^T F_1 + f_1) - \frac{\partial \alpha_1}{\partial \hat{\eta}}\dot{\hat{\eta}} \\ &\quad - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \sum_{k=1}^{m}\frac{\partial \alpha_1}{\partial x_{dk}}(f_{dk} + \theta^T F_{dk}) - \frac{\partial \alpha_1}{\partial t} \\ &= g_2 z_3 + g_2\alpha_2 + \hat{\eta}^T F_{2\eta} - \hat{\theta}^T F_{2\theta} + f_{2s} \\ &\quad + \frac{\partial \alpha_1}{\partial \hat{\eta}}(\pi_2 - \dot{\hat{\eta}}) + \frac{\partial \alpha_1}{\partial \hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) - (\hat{\eta} - \eta)^T F_{2\eta} \\ &\quad + (\hat{\theta} - \theta)^T F_{2\theta} \end{aligned} \tag{16}$$

where $z_3$, $F_{2s}$ and $f_{2s}$ are defined in (5) and (7), respectively; virtual control $\alpha_2$ is used to stabilize the $(z_1, z_2)$-subsystem with respect to the

Lyapunov function candidate

$$V_2 = V_1 + \frac{1}{2}z_2^2 \tag{17}$$

The derivative of $V_2$ is

$$\begin{aligned}\dot{V}_2 = &-c_1 z_1^2 + g_2 z_2 z_3 + z_2(g_1 z_1 + g_2\alpha_2 + \hat{\eta}^T F_{2\eta} \\ &- \hat{\theta}^T F_{2\theta} + f_{2s}) + z_2\frac{\partial\alpha_1}{\partial\hat{\eta}}(\pi_2 - \dot{\hat{\eta}}) \\ &+ z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) + (\hat{\eta}-\eta)^T\Gamma_\eta^{-1} \\ &\times(\dot{\hat{\eta}} - \pi_1 - \Gamma_\eta F_{2\eta}z_2) + (\hat{\theta}-\theta)^T\Gamma_\theta^{-1} \\ &\times(\dot{\hat{\theta}} - \tau_1 + \Gamma_\theta F_{2\theta}z_2)\end{aligned} \tag{18}$$

Define tuning functions $\pi_2$ and $\tau_2$ for $\hat{\eta}$ and $\hat{\theta}$ as in (9). Choose $\alpha_2$ as in (6) such that the bracketed term multiplying $z_2$ in Eq. (18) be equal to $-c_2z_2^2$, then $\dot{V}_2$ becomes

$$\begin{aligned}\dot{V}_2 = &-c_1 z_1^2 - c_2 z_2^2 + g_2 z_2 z_3 + z_2\frac{\partial\alpha_1}{\partial\hat{\eta}}(\pi_2 - \dot{\hat{\eta}}) \\ &+ z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}(\tau_2 - \dot{\hat{\theta}}) + (\hat{\eta}-\eta)^T\Gamma_\eta^{-1}(\dot{\hat{\eta}} - \pi_2) \\ &+ (\hat{\theta}-\theta)^T\Gamma_\theta^{-1}(\dot{\hat{\theta}} - \tau_2)\end{aligned} \tag{19}$$

***Step i.*** $3 \le i \le n-1$. The derivative of $z_i = x_i - x_{di} - \alpha_{i-1}$ is expressed as

$$\begin{aligned}\dot{z}_i = &\, g_i z_{i+1} + g_i\alpha_i + g_i x_{d(i+1)} \\ &+ \eta^T F_i + f_i - f_{di} - \theta^T F_{di} \\ &- \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}(g_k x_{k+1} + \eta^T F_k + f_k) \\ &- \frac{\partial\alpha_{i-1}}{\partial\hat{\eta}}\dot{\hat{\eta}} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ &- \sum_{k=1}^{m}\frac{\partial\alpha_{i-1}}{\partial x_{dk}}(f_{dk} + \theta^T F_{dk}) - \frac{\partial\alpha_{i-1}}{\partial t} \\ = &\, g_i z_{i+1} + g_i\alpha_i + \hat{\eta}^T F_{i\eta} - \hat{\theta}^T F_{i\theta} + f_{is} \\ &+ \frac{\partial\alpha_{i-1}}{\partial\hat{\eta}}(\pi_i - \dot{\hat{\eta}}) + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}(\tau_i - \dot{\hat{\theta}}) \\ &- (\hat{\eta}-\eta)^T F_{i\eta} + (\hat{\theta}-\theta)^T F_{i\theta}\end{aligned} \tag{20}$$

where $z_{i+1}$, $F_{i\eta}$, $F_{i\theta}$ and $f_{is}$ are defined in (5) and (7), respectively; virtual control $\alpha_{i-1}$ is used to stabilize the $(z_1, \ldots, z_i)$-subsystem with respect to the Lyapunov function candidate

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 \tag{21}$$

The derivative of $V_i$ is

$$\begin{aligned}\dot{V}_i = &-\sum_{k=1}^{i-1}c_k z_k^2 + g_i z_i z_{i+1} + z_i(g_{i-1}z_{i-1} + g_i\alpha_i \\ &+ \hat{\eta}^T F_{i\eta} - \hat{\theta}^T F_{i\theta} + f_{is}) + \sum_{k=1}^{i-2}\left(z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\eta}}\right) \\ &\times(\pi_{i-1} - \dot{\hat{\eta}}) + \sum_{k=1}^{i-2}\left(z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\right)(\tau_{i-1} - \dot{\hat{\theta}}) \\ &+ z_i\frac{\partial\alpha_{i-1}}{\partial\hat{\eta}}(\pi_i - \dot{\hat{\eta}}) + z_i\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}(\tau_i - \dot{\hat{\theta}}) \\ &+ (\hat{\eta}-\eta)^T\Gamma_\eta^{-1}(\dot{\hat{\eta}} - \pi_{i-1} - \Gamma_\eta F_{i\eta}z_i) \\ &+ (\hat{\theta}-\theta)^T\Gamma_\theta^{-1}(\dot{\hat{\theta}} - \tau_{i-1} + \Gamma_\theta F_{i\theta}z_i)\end{aligned} \tag{22}$$

Define tuning functions $\pi_i$ and $\tau_i$ for $\hat{\eta}$ and $\hat{\theta}$ as in (9). Note that

$$\begin{aligned}\pi_{i-1} - \dot{\hat{\eta}} &= \pi_i - \dot{\hat{\eta}} + \pi_{i-1} - \pi_i \\ &= \pi_i - \dot{\hat{\eta}} - \Gamma_\eta F_{i\eta}z_i \\ \tau_{i-1} - \dot{\hat{\theta}} &= \tau_i - \dot{\hat{\theta}} + \tau_{i-1} - \tau_i \\ &= \tau_i - \dot{\hat{\theta}} + \Gamma_\theta F_{i\theta}z_i\end{aligned} \tag{23}$$

By choosing $\alpha_i$ as in (6) such that the bracketed term multiplying $z_i$ in Eq. (22) be equal to $-c_iz_i^2$, $\dot{V}_i$ becomes

$$\begin{aligned}\dot{V}_i = &-\sum_{k=1}^{i}c_k z_k^2 + g_i z_i z_{i+1} + \sum_{k=1}^{i-1}\left(z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\eta}}\right)(\pi_i - \dot{\hat{\eta}}) \\ &+ \sum_{k=1}^{i-1}\left(z_{k+1}\frac{\partial\alpha_k}{\partial\hat{\theta}}\right)(\tau_i - \dot{\hat{\theta}}) + (\hat{\eta}-\eta)^T\Gamma_\eta^{-1}(\dot{\hat{\eta}} - \pi_i) \\ &+ (\hat{\theta}-\theta)^T\Gamma_\theta^{-1}(\dot{\hat{\theta}} - \tau_i)\end{aligned} \tag{24}$$

***Step n.*** Since this is our last step, the derivative of $z_n = x_n - x_{dn} - \alpha_{n-1}$ is expressed as

$$
\begin{aligned}
\dot{z}_n &= g_n u + \eta^T F_n + f_n - f_{dn} - \theta^T F_{dn} \\
&\quad - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k}(g_k x_{k+1} + \eta^T F_k + f_k) \\
&\quad - \frac{\partial \alpha_{n-1}}{\partial \hat{\eta}}\dot{\hat{\eta}} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \\
&\quad - \sum_{k=1}^{m} \frac{\partial \alpha_{n-1}}{\partial x_{dk}}(f_{dk} + \theta^T F_{dk}) - \frac{\partial \alpha_{n-1}}{\partial t} \\
&= g_n u + \hat{\eta}^T F_{n\eta} - \hat{\theta}^T F_{n\theta} + f_{ns} \\
&\quad + \frac{\partial \alpha_{n-1}}{\partial \hat{\eta}}(\pi_n - \dot{\hat{\eta}}) \\
&\quad + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}(\tau_n - \dot{\hat{\theta}}) - (\hat{\eta} - \eta)^T F_{n\eta} \\
&\quad + (\hat{\theta} - \theta)^T F_{n\theta}
\end{aligned}
\tag{25}
$$

where $F_{n\eta}$, $F_{n\theta}$ and $f_{ns}$ are defined in (7).

Physical control $u$ is to stabilize the $(z_1, \ldots, z_n)$-system with respect to the Lyapunov function candidate

$$
V_n = V_{n-1} + \frac{1}{2} z_n^2 . \tag{26}
$$

The derivative of $V_n$ is

$$
\begin{aligned}
\dot{V}_n &= -\sum_{k=1}^{n-1} c_k z_k^2 + z_n(g_{n-1} z_{n-1} + g_n u \\
&\quad + \hat{\eta}^T F_{n\eta} - \hat{\theta}^T F_{n\theta} + f_{ns}) \\
&\quad + \sum_{k=1}^{n-2} \left( z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\eta}} \right)(\pi_{n-1} - \dot{\hat{\eta}}) \\
&\quad + \sum_{k=1}^{n-2} \left( z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right)(\tau_{n-1} - \dot{\hat{\theta}}) \\
&\quad + z_n \frac{\partial \alpha_{n-1}}{\partial \hat{\eta}}(\pi_n - \dot{\hat{\eta}}) + z_n \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}(\tau_n - \dot{\hat{\theta}}) \\
&\quad + (\hat{\eta} - \eta)^T \Gamma_\eta^{-1}(\dot{\hat{\eta}} - \pi_{n-1} - \Gamma_\eta F_{n\eta} z_n) \\
&\quad + (\hat{\theta} - \theta)^T \Gamma_\theta^{-1}(\dot{\hat{\theta}} - \tau_{n-1} + \Gamma_\theta F_{n\theta} z_n)
\end{aligned}
\tag{27}
$$

To eliminate the terms with $(\hat{\eta} - \eta)$ and $(\hat{\theta} - \theta)$ in $\dot{V}_n$ from Eq. (27), we choose the parameter update laws for $\hat{\eta}$ and $\hat{\theta}$ as in (8). Noting that

$$
\begin{aligned}
\pi_{n-1} - \dot{\hat{\eta}} &= \pi_{n-1} - \pi_n = -\Gamma_\eta F_{n\eta} z_n \\
\tau_{n-1} - \dot{\hat{\theta}} &= \tau_{n-1} - \tau_n = \Gamma_\theta F_{n\theta} z_n
\end{aligned}
\tag{28}
$$

Equation (27) can be written as

$$
\begin{aligned}
\dot{V}_n = -\sum_{k=1}^{n-1} c_k z_k^2 + z_n \Bigg( & g_{n-1} z_{n-1} + g_n u + \hat{\eta}^T F_{n\eta} \\
& - \hat{\theta}^T F_{n\theta} + f_{ns} - \sum_{k=1}^{n-2} \left( z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\eta}} \right) \Gamma_\eta F_{n\eta} \\
& + \sum_{k=1}^{n-2} \left( z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \right) \Gamma_\theta F_{n\theta} \Bigg)
\end{aligned}
\tag{29}
$$

Finally, we choose the control $u$ as in (10) such that the bracketed term multiplying $z_n$ in Eq. (29) be equal to $-c_n z_n^2$, then $\dot{V}_n$ is rewritten as

$$
\dot{V}_n = -\sum_{k=1}^{n} c_k z_k^2 \tag{30}
$$

which proves that (i) equilibrium $z = [z_1, \ldots, z_n]^T = 0$ is globally uniformly stable, and (ii) $\hat{\eta}$ and $\hat{\theta}$ are bounded. Since $z_1 = x_1 - x_{d1}$ and $x_{d1}$ are bounded, we see that $x_1$ is also bounded. The boundedness of $x_i$, $i = 2, \ldots, n$ follows from the boundedness of $\alpha_{i-1}$ and $x_{di}$, and the fact that $x_i = z_i + x_{di} + \alpha_{i-1}$, $i = 2, \ldots, n$. Using (10), we conclude that the control $u$ is also bounded.

From the LaSalle–Yoshizawa theorem [Krstìc *et al.*, 1995], it further follows that, all the solutions of the $(z_1, \ldots, z_n)$-system converge to the manifold $z = 0$ as $t \to \infty$. From the definition $z_1 = x_1 - x_{d1} = y - y_d$, we conclude that $y(t) - y_d(t) \to 0$ as $t \to \infty$, which means that asymptotic synchronization is achieved.

Q.E.D.

*Remark 1.* It should be noted that when the proposed adaptive synchronization method is used for chaos secure communication, it is not the system parameters, but the known functions in the master system (1) and slave system (2) that act as the cryptographic key. Because the master system is in a very general form (1), a wide class of continuous-time chaotic and hyperchaotic systems can be designed as the transmitter. This implies that much complicated high-order chaotic systems can be employed to improve the security in chaos communications.

*Remark 2.* In the proposed approach, the master system (1) and slave system (2) only have parametric uncertainties that appear linearly with respect to the known nonlinear functions. For the case when both parametric uncertainty and unknown nonlinear functions are present in the systems, where these unknown nonlinear functions could be due to modeling errors, external disturbances, time variations in the system, robust adaptive control design can be used to guarantee robustness with respect to bounded uncertainties and exogenous disturbances (see e.g. [Ioannou & Sun, 1996] and the references therein). Ultimately uniform boundedness and generalized synchronization can be achieved. The results of robust adaptive control theory can be further employed and extended to the research on practical applications of chaos synchronization to communications.

## 4. Simulation Studies

In the simulation studies, we shall consider two cases: (i) two chaotic Rössler systems [Rössler, 1976] as master and slave systems respectively, with different system parameters; (ii) the chaotic Rössler system as the master system and the chaotic Duffing oscillator [Duffing, 1918] as the slave system. All the key system parameters are assumed to be unknown. Other chaotic systems such as the van der Pol oscillator, the Chua's circuits and the hyperchaotic Rössler system can all be taken as the master and the slave systems, and can be designed readily by the same design procedure.

Consider the Rössler system [Rössler, 1976] as the master system in both cases (i) and (ii) described as (after some simple state transformations)

$$\begin{aligned}\dot{x}_{d1} &= x_{d2} + \theta_1 x_{d1}\\ \dot{x}_{d2} &= -x_{d3} - x_{d1}\\ \dot{x}_{d3} &= \theta_2 + x_{d3}(x_{d2} - \theta_3)\end{aligned} \tag{31}$$

where $\theta_1$, $\theta_2$ and $\theta_3$ are constant system parameters. When $\theta_1 = 0.15$, $\theta_2 = 0.20$ and $\theta_3 = 10$, the Rössler system is in chaotic states.

For case (i), the slave system is designed as the same Rössler system (31) except that the system parameters are $\eta_1 = 0.20$, $\eta_2 = 0.20$ and $\eta_3 = 10$, and a control $u$ is fed into the third equation of the Rössler system.

In the following simulation, the design parameters of controller (10) and parameter update law (8) are chosen as $c_1 = 2$, $c_2 = 2$, $c_3 = 2$ and $\Gamma_\eta = \Gamma_\theta = \text{diag}\{0.001, 0.1, 0.1\}$. The initial conditions are chosen that $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 0$, $x_{d1}(0) = 5$, $x_{d2}(0) = 0.3$ and $x_{d3}(0) = 0.4$.

Numerical simulation results are shown in Figs. 1–3. As shown in Fig. 1, the state $x_1(t)$ of the slave system asymptotically synchronizes with the state $x_{d1}(t)$ of the master system. It can be shown that at the same time the states $x_2(t)$ and $x_3(t)$ of the slave system, the parameter estimates $\hat{\eta}$ and $\hat{\theta}$ and the control $u$ remain bounded. The

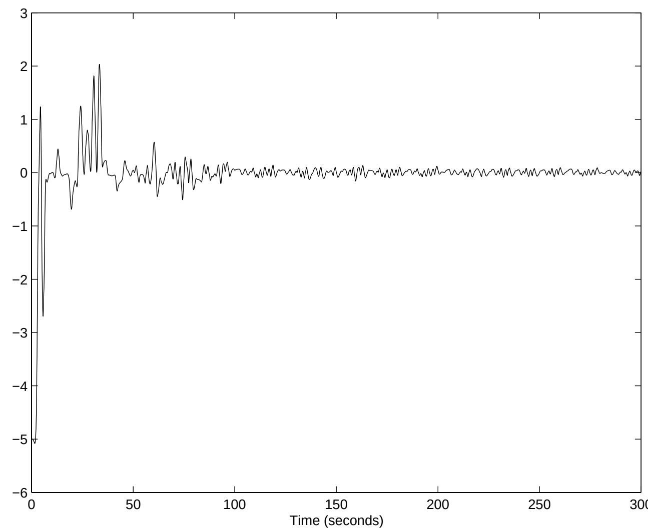


Fig. 1. Tracking error $x_1(t) - x_{d1}(t)$.

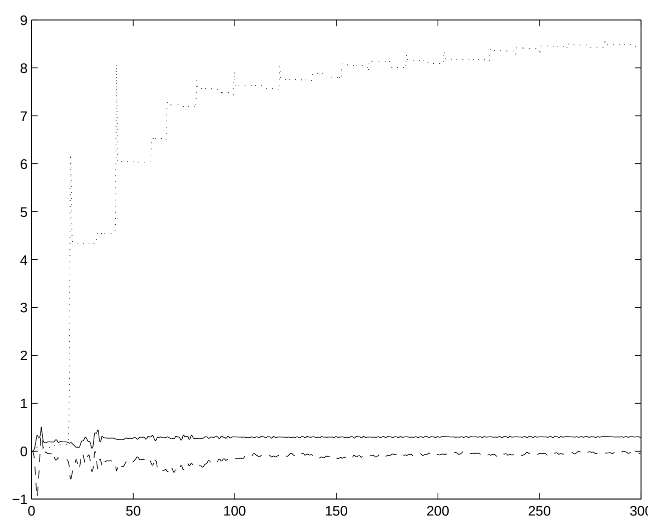


Fig. 2. Boundedness of the parameter estimates: $\hat{\theta}_1$ (solid line), $\hat{\theta}_2$ (dashdot line) and $\hat{\theta}_3$ (dash line).

boundedness of parameter estimates $\hat{\eta}$ and $\hat{\theta}$ are shown in Figs. 2 and 3, respectively.

For case (ii), the slave system is designed as the Duffing oscillator [Duffing, 1918]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u - \eta_1 x_2 - \eta_2 x_1 - \eta_3 x_1^3 + \eta_4 \cos \omega t \end{aligned} \tag{32}$$

where $x_1$ and $x_2$ are the states, $\omega$ is a constant frequency parameter, $\eta_1$, $\eta_2$, $\eta_3$ and $\eta_4$ are constant system parameters.

In the literature of chaos research, it is assumed that $\omega$ is known, while $\eta_1$, $\eta_2$, $\eta_3$ and $\eta_4$ are unknown. Assume that the Duffing oscillator (32) is originally ($u = 0$) in the chaotic states with parameters $\omega = 1.8$, $\eta_1 = 0.4$, $\eta_2 = -1.1$, $\eta_3 = 1.0$ and $\eta_4 = 1.8$.

In the following simulation, the design parameters of controller (10) and parameter update law (8) are chosen as $c_1 = 20$, $c_2 = 20$, $\Gamma_\eta = \text{diag}\{0.01, 0.001, 0.001, 0.1\}$ and $\Gamma_\theta = \text{diag}\{0.001, 0.01, 0.02\}$. The initial conditions are chosen that $x_1(0) = 0$, $x_2(0) = 0$, $x_{d1}(0) = -2$, $x_{d2}(0) = 0.3$ and $x_{d3}(0) = 0.4$.

Numerical simulation results are shown in Figs. 4–6. As shown in Fig. 4, the state $x_1(t)$ of the Duffing oscillator system (32) asymptotically

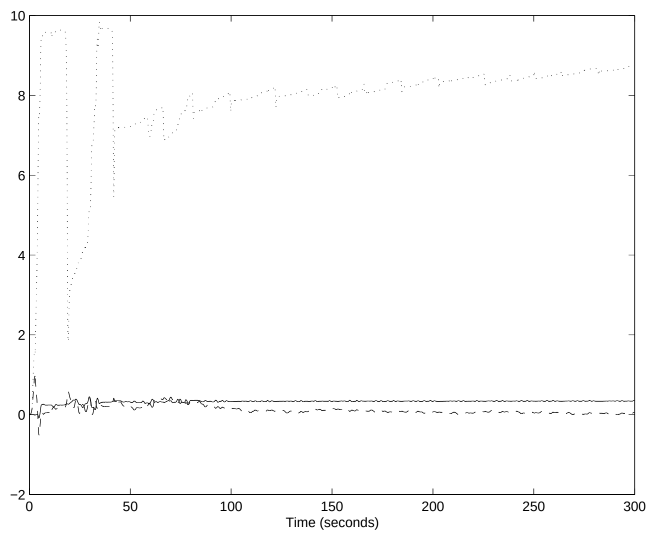


Fig. 3. Boundedness of the parameter estimates: $\hat{\eta}_1$ (solid line), $\hat{\eta}_2$ (dashdot line) and $\hat{\eta}_3$ (dash line).

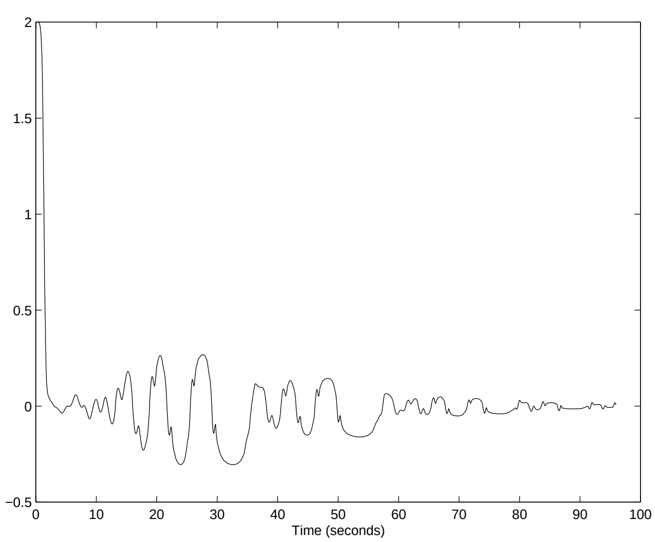


Fig. 4. Tracking error $x_1(t) - x_{d1}(t)$.

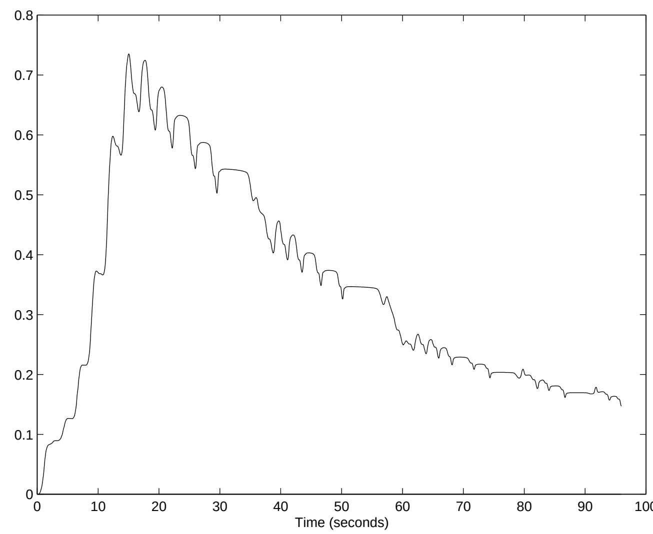


Fig. 5. Boundedness of the parameter estimates: $\hat{\theta}_1$ ($\hat{\theta}_2 = 0$, $\hat{\theta}_3 = 0$).

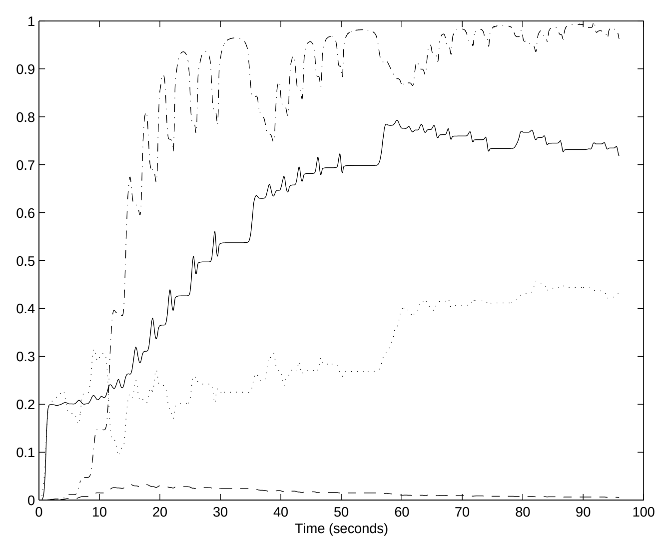


Fig. 6. Boundedness of the parameter estimates: $\hat{\eta}_1$ (solid line), $\hat{\eta}_2$ (dash line), $\hat{\eta}_3$ (dashdot line) and $\hat{\eta}_4$ (dotted line).

synchronizes with the state $x_{d1}(t)$ of the Rössler system (31). It can be shown that at the same time the state $x_2(t)$, the parameter estimates $\hat{\eta}$ and $\hat{\theta}$ and the control $u$ remain bounded. The boundedness of parameter estimates $\hat{\eta}$ and $\hat{\theta}$ are shown in Figs. 5 and 6, respectively.

## 5. Conclusion

An approach for adaptive synchronization of uncertain chaotic systems using backstepping with tuning functions method has been presented in this paper. Strong properties of global stability and asymptotic synchronization have been achieved in a finite number of steps. This approach can be used for the synchronization of two chaotic systems of the same type with different system parameters, as well as two completely different chaotic systems. The proposed approach offers a systematic design procedure for the adaptive synchronization of most of chaotic systems in the chaos research literature. These results demonstrate the fruitfulness of modern nonlinear and adaptive control theory for applications to chaos synchronization.

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