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Exponential stabilization of non-holonomic systems: an ENI approach

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In this paper, a new canonical form, called extended non-holonomic integrators (ENI), is firstly introduced for non-holonomic systems. Next, a recursive design technique is presented to exponentially stabilize ENI systems. The relationships between the convergence rates of the states and the controller parameters are explicitly given, in particular, the convergence rate of individual state can be specified explicitly. Then, it is shown that non-holonomic systems in ENI form, chained and power forms are equivalent, thus can be dealt with in a unified framework. Finally, a car-like mobile robot is used to demonstrate the effectiveness of the proposed control.

1. Introduction

In recent years, much attention has been devoted to the problem of controlling non-holonomic systems. Many mechanical systems (such as wheeled mobile robots, tractor–trailer systems, free-floating space robots, underwater vehicles, etc.) are subjected to non-holonomic velocity constraints, and these constraints can be modelled as driftless non-holonomic control systems (Kolmanovsky and McClamroch 1995). Because non-holonomic control systems do not satisfy the Brockett's necessary condition (Brockett 1983), they cannot be asymptotically stabilized to their equilibrium points by any continuous pure state feedback (Bloch *et al*. 1992). This makes the stabilization problem of non-holonomic systems one of the most challenging topics in control theory and applications.

The commonly used approach for controller design of non-holonomic systems is to convert, with appropriate state and input transformations, the original systems into some canonical forms for which controller design can be carried out more easily. This idea has motivated a thorough and systematic investigation into the problem of non-holonomic system control. The chained system (Murray and Sastry 1991) and power system (M'Closkey and Murray 1992) are two of the most important canonical forms of non-holonomic control systems. Using the special algebra structures of chained and power systems, various efficient feedback control laws, such as time-varying feedback, discontinuous feedback, hybrid feedback control and intelligent control strategies, have been proposed to stabilize non-holonomic systems (see Kolmanovsky and McClamroch 1995 and references therein).

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Among these control strategies, the simplest and efficient one may be the exponential stabilization law derived with discontinuous state transformation in Astolfi (1996). Recently, some elegant recursive algorithms to design exponentially convergent controllers for chained and power systems have been presented using backstepping (Tayebi *et al*. 1997) and invariant manifold methods (Luo and Tsiotras 1997, 1998). These algorithms reduce the exponential stabilization design of n-dimensional chained or power systems to that of three-dimensional ones, and make controller design for these non-holonomic systems more easily.

In this paper, a new canonical form, called extended non-holonomic integrator (ENI), is firstly presented for non-holonomic systems. An iterative order reduction procedure is then established, and a new recursive design algorithm is given to construct state feedback stabilization laws with specified exponential decay rates. The main contributions of the paper lie in four aspects:

(1) the introduction of a new canonical form, ENI, of non-holonomic systems;
(2) the formulation of a new family of exponential controllers for ENI;
(3) the identification of the relationship between the convergence rates of the states and the controller parameters, in fact, the convergence rate of individual state can be specified; and
(4) the establishment of the equivalences among the canonical forms: ENI, chained and power forms, thus they can be dealt with in a unified framework.

The proposed controllers possess a simpler structure and the stability proof of the closed-loop system is more straightforward, since there is no need to construct control Lyapunov functions or invariant manifolds. The 'burden' due to coordinate transformation can be reduced if a symbolic computation software can be written to do the iterative transformations systematically.

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This paper is organized as fellows. In § 2, a canonical form, the extended non-holonomic integrators (ENI), is introduced, and a recursive algorithm is presented to reduce the system order. This order reduction algorithm is utilized to design a class of exponentially convergent controllers for ENI in § 3. Section 4 establishes the equivalence relationships among ENI, chained form and power form. A simulation study for the kinematic model of a car-like mobile robot is reported in § 5. The last section presents concluding remarks.

2. ENI and recursive order reduction

2.1. *Extended non-holonomic integrators*

Consider the non-holonomic system described by

$$\left.\begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \\ \dot{y}_3 &= k_2 y_2 u_1 - k_1 y_1 u_2 \\ \dot{y}_j &= y_{j-1} u_1, \quad j = 4, \ldots, n \end{aligned}\right\} \tag{1}$$

where y_i, $i = 1, \ldots, n$, and u_i, $i = 1, 2$ are the states and inputs of the system, respectively. Since the well-known Brockett's non-holonomic integrator described by

$$\left.\begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \\ \dot{y}_3 &= y_2 u_1 - y_1 u_2 \end{aligned}\right\} \tag{2}$$

is a special case of form (1) by chosen $n = 3$ and $k_1 = k_2 = 1$, the new canonical form (1) is referred to as *the extended non-holonomic integrators (ENI)* in this paper.

2.2. *Recursive order reduction*

Letting $y_j = y_{1,j} (j = 1, \ldots, n)$ and $u_i = u_{1,i} (i = 1, 2)$, the extended non-holonomic integrators (1) can be rewritten as

$$\left.\begin{aligned} \dot{y}_{1,1} &= u_{1,1} \\ \dot{y}_{1,2} &= u_{1,2} \\ \dot{y}_{1,3} &= k_2 y_{1,2} u_{1,1} - k_1 y_{1,1} u_{1,2} \\ \dot{y}_{1,j} &= y_{1,j-1} u_{1,1}, \quad j = 4, \ldots, n \end{aligned}\right\} \tag{3}$$

It is referred to as *the first generation of extended non-holonomic integrators*, i.e. the original ENI system (1). In the following development, let k_i, $i = 3, \ldots, n$, denote the constant design parameters which can be determined easily in the controller presented later.

By neglecting the second equation of the first generation of extended non-holonomic integrators (3), we obtain a $(n-1)$-dimensional subsystem

$$\left.\begin{aligned} \dot{y}_{1,1} &= u_{1,1} \\ \dot{y}_{1,3} &= k_2 y_{1,2} u_{1,1} - k_1 y_{1,1} u_{1,2} \\ \dot{y}_{1,j} &= y_{1,j-1} u_{1,1}, \quad j = 4, \ldots, n \end{aligned}\right\} \tag{4}$$

With state and input transformations given by

$$\left.\begin{aligned} y_{2,1} &= y_{1,1} \\ y_{2,2} &= y_{1,3} \\ y_{2,j} &= [(j-2)k_1 + k_3] y_{1,j+1} - k_1 y_{1,1} y_{1,j}, \\ & \qquad j = 3, \ldots, n-1 \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} u_{2,1} &= u_{1,1} \\ u_{2,2} &= k_2 y_{1,2} u_{1,1} - k_1 y_{1,1} u_{1,2} \end{aligned}\right\} \tag{6}$$

subsystem (4) can be converted to the following $(n-1)$-dimensional extended non-holonomic integrators

$$\left.\begin{aligned} \dot{y}_{2,1} &= u_{1,1} = u_{2,1} \\ \dot{y}_{2,2} &= k_2 y_{1,2} u_{1,1} - k_1 y_{1,1} u_{1,2} = u_{2,2} \\ \dot{y}_{2,3} &= (k_1 + k_3) y_{1,3} u_{1,1} - k_1 (y_{1,3} u_{1,1} + y_{1,1} u_{2,2}) \\ &= k_3 y_{2,2} u_{2,1} - k_1 y_{2,1} u_{2,2} \\ \dot{y}_{2,j} &= [(j-2)k_1 + k_3] y_{1,j} u_{1,1} \\ &\quad - k_1 (y_{1,j} u_{1,1} + y_{1,1} y_{1,j-1} u_{1,1}) \\ &= y_{2,j-1} u_{2,1}, \quad j = 4, \ldots, n-1 \end{aligned}\right\} \tag{7}$$

This $(n-1)$-dimensional extended non-holonomic integrators is said to be *the second generation of extended non-holonomic integrators*. For clarity, let the ith $(1 \le i \le n-3)$ generation of extended non-holonomic integrators be

$$\left.\begin{aligned} \dot{y}_{i,1} &= u_{i,1} \\ \dot{y}_{i,2} &= u_{i,2} \\ \dot{y}_{i,3} &= k_{i+1} y_{i,2} u_{i,1} - k_1 y_{i,1} u_{i,2} \\ \dot{y}_{i,j} &= y_{i,j-1} u_{i,1}, \quad j = 4, \cdots, n-i+1 \end{aligned}\right\} \tag{8}$$

Considering state and input transformations

$$\left.\begin{aligned} y_{i+1,1} &= y_{i,1} \\ y_{i+1,2} &= y_{i,3} \\ y_{i+1,j} &= [(j-2)k_1 + k_{i+2}] y_{i,j+1} - k_1 y_{i,1} y_{i,j}, \\ & \qquad j = 3, \cdots, n-i \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} u_{i+1,1} &= u_{i,1} \\ u_{i+1,2} &= k_{i+1} y_{i,2} u_{i,1} - k_1 y_{i,1} u_{i,2} \end{aligned}\right\} \tag{10}$$

for the subsystem obtained by neglecting the second equation of (8), we have the $(i+1)$th generation of extended non-holonomic integrators of dimension $n-i$ as

$$\left.\begin{aligned} \dot{y}_{i+1,1} &= u_{i,1} = u_{i+1,1} \\ \dot{y}_{i+1,2} &= k_{i+1} y_{i,2} u_{i,1} - k_1 y_{i,1} u_{i,2} = u_{i+1,2} \\ \dot{y}_{i+1,3} &= (k_1 + k_{i+2}) y_{i,3} u_{i,1} - k_1 (y_{i,3} u_{i,1} + y_{i,1} u_{i+1,2}) \\ &= k_{i+2} y_{i+1,2} u_{i+1,1} - k_1 y_{i+1,1} u_{i+1,2} \\ \dot{y}_{i+1,j} &= [(j-2)k_1 + k_{i+2}] y_{i,j} u_{i,1} \\ &\quad - k_1 (y_{i,j} u_{i,1} + y_{i,1} y_{i,j-1} u_{i,1}) \\ &= y_{i+1,j-1} u_{i+1,1}, \quad j = 4, \ldots, n-i \end{aligned}\right\} \tag{11}$$

The same process can be repeated till the $(n-2)$th generation of extended non-holonomic integrators of dimension 3 is obtained as

$$\left.\begin{aligned}\dot{y}_{n-2,1} &= u_{n-2,1}\\ \dot{y}_{n-2,2} &= u_{n-2,2}\\ \dot{y}_{n-2,3} &= k_{n-1}y_{n-2,2}u_{n-2,1} - k_1 y_{n-2,1}u_{n-2,2}\end{aligned}\right\} \quad (12)$$

3. Controller design for ENI

Before presenting the main results, the following lemmas are necessary.

Lemma 1 (Wang *et al.* 1999)**:** *Consider linear system*

$$\dot{x}(t) = -\beta x(t) + g(t)$$

where $t \geq 0, x \in R$ *and* $\beta > 0$. *If* $g(t)$ *converges exponentially to zero with rate* γ *(i.e. there exist constants* g_0 *and* γ *such that* $|g(t)| \leq g_0\, \mathrm{e}^{-\gamma t}, \forall t \geq 0$*) and* $\gamma > \beta$, *then* $x(t)$ *tends to zero exponentially with rate* β *.*

Lemma 2: *If the parameters in transformations* (9) *and* (10) *are chosen such that*

$$k_1 > 0; \quad k_2 > 0; \quad k_s > k_{s-1} + k_1, \quad s = 3, \ldots, n \qquad (13)$$

and the $(n-2)$th *generation of ENI satisfies that the initial state* $y_{n-2,1}(0) \neq 0$ *and the control inputs*

$$\left.\begin{aligned}u_{n-2,1} &= -k_1 y_{n-2,1}\\ u_{n-2,2} &= -k_{n-1}y_{n-2,2} + \frac{k_n y_{n-1,2}}{k_1 y_{n-2,1}}\end{aligned}\right\} \quad (14)$$

where $y_{n-1,2} \triangleq y_{n-2,3}$, *then, for any* $i \in \{1, 2, \ldots, n-2\}$, *the states and inputs of the* ith *generation of ENI have the following properties:*

(1) *There exist constants* a_1 *and* $a_{i,j}$ $(j = 2, \ldots, n-i+1)$ *such that*

$$\left.\begin{aligned}&|y_{i,1}(t)| = a_1\, \mathrm{e}^{-k_1 t}\\ &|y_{i,2}(t)| \leq a_{i,2}\, \mathrm{e}^{-k_{i+1} t}\\ &|y_{i,j}(t)| \leq a_{i,j}\, \mathrm{e}^{-[k_{i+2}+(j-3)k_1]t}, \quad j = 3, \ldots, n-i+1\end{aligned}\right\} \quad (15)$$

(2) *Control input* $u_{i,2}$ *can be expressed as*

$$u_{i,2} = -\sum_{j=i}^{n-1} \frac{k_{j+1} y_{j,2}}{(-k_1 y_{1,1})^{j-i}} \qquad (16)$$

(3) *There exist constants* b_1 *and* $b_{i,2}$ *such that*

$$|u_{i,1}(t)| = b_1 e^{-k_1 t}, \quad |u_{i,2}(t)| \leq b_{i,2} e^{-k_{i+1} t}$$

Proof: First, it is easy to check that if k_i $(i = 1, \ldots, n)$ are chosen according to (13), all state transformations (9) for deducing every new generation of ENI are non-singular. Substituting control (14) into the $(n-2)$th generation of extended non-holonomic integrators (12) gives

$$\dot{y}_{n-2,1} = -k_1 y_{n-2,1} \qquad (17)$$

$$\dot{y}_{n-2,2} = -k_{n-1}y_{n-2,2} + \frac{k_n y_{n-2,3}}{k_1 y_{n-2,1}} \qquad (18)$$

$$\dot{y}_{n-2,3} = -k_n y_{n-2,3} \qquad (19)$$

From (17) and the definitions of $y_{i,1}$ $(i = n-2, \ldots, 1)$, we know that, if $y_{n-2,1}(0) \neq 0$

$$\begin{aligned}y_{1,1}(t) = y_{2,1}(t) = \cdots &= y_{n-2,1}(t)\\ &= y_{n-2,1}(0)\, \mathrm{e}^{-k_1 t} \neq 0, \quad \forall t \in [0, \infty) \qquad (20)\end{aligned}$$

Thus, input transformations (13) for deriving every new generation of ENI are also non-singular, and the first expression in (15) holds for all $i \in \{1, \ldots, n-2\}$ by choosing $a_1 = |y_{n-2,1}(0)|$. In addition, from (14), (20) and the definition of $u_{i,1}$, we see that

$$|u_{i,1}(t)| = k_1 |y_{n-2,1}(0)|\, \mathrm{e}^{-k_1 t} \triangleq b_1\, \mathrm{e}^{-k_1 t}$$

In the following, we will prove the remaining results in Lemma 2 by induction. From (19), we obtain

$$y_{n-2,3}(t) = y_{n-2,3}(0)\, \mathrm{e}^{-k_n t} \qquad (21)$$

$$|y_{n-2,3}(t)| \leq |y_{n-2,3}(0)|\, \mathrm{e}^{-k_n t} \triangleq a_{n-2,3}\, \mathrm{e}^{-k_n t} \qquad (22)$$

Substituting (20) and (21) into (18) gives

$$\dot{y}_{n-2,2} = -k_{n-1}y_{n-2,2} + \frac{k_n y_{n-2,3}(0)}{k_1 y_{n-2,1}(0)}\, \mathrm{e}^{-(k_n - k_1)t}$$

Since $k_n - k_1 > k_{n-1}$, it follows, from Lemma 1, that there exist a constant $a_{n-2,2}$ such that

$$|y_{n-2,2}(t)| \leq a_{n-2,2}\, \mathrm{e}^{-k_{n-1} t} \qquad (23)$$

From (20)–(22), (14) and considering that $k_n - k_1 > k_{n-1}$, we have

$$\begin{aligned}|u_{n-2,2}| &\leq k_{n-1}a_{n-2,2}\, \mathrm{e}^{-k_{n-1}t} + \frac{k_n a_{n-2,3}}{k_1 a_1}\, \mathrm{e}^{-(k_n - k_1)t}\\ &\leq \left(k_{n-1}a_{n-2,2} + \frac{k_n a_{n-2,3}}{k_1 a_1}\right) \mathrm{e}^{-k_{n-1}t}\end{aligned}$$

Equations (22), (23), (14) and the above expression imply that Lemma 2 holds for $i = n-2$.

Suppose Lemma 2 is true for all $i \geq m$ $(1 < m \leq n-2)$, then for $i = m-1$, from (9), (10) and noting that $k_{m+2} > k_{m+1} + k_1$, we have

$$\begin{aligned}|y_{m-1,1}| &= |y_{m,1}| = a_1\, \mathrm{e}^{-k_1 t}\\ |y_{m-1,3}| &= |y_{m,2}| \leq a_{m,2}\, \mathrm{e}^{-k_{m+1}t} \triangleq a_{m-1,3}\, \mathrm{e}^{-k_{m+1}t}\\ |y_{m-1,4}| &= \frac{|y_{m,3} + k_1 y_{m-1,1} y_{m-1,3}|}{k_1 + k_{m+1}}\\ &\leq \frac{a_{m,3} + k_1 a_1 a_{m-1,3}}{k_1 + k_{m+1}}\, \mathrm{e}^{-(k_{m+1}+k_1)t}\\ &\triangleq a_{m-1,4}\, \mathrm{e}^{-(k_{m+1}+k_1)t}\end{aligned}$$

Following the same process, in the order of $j = 5, 6, \ldots, n - m + 2$, we can calculate

$$|y_{m-1,j}| = \frac{|y_{m,j-1} + k_1 y_{m-1,1} y_{m-1,j-1}|}{(j-3)k_1 + k_{m+1}}$$
$$\leq \frac{a_{m,j-1} + k_1 a_1 a_{m-1,j-1}}{(j-3)k_1 + k_{m+1}} \, e^{-[k_{m+1}+(j-3)k_1]t}$$
$$\triangleq a_{m-1,j} \, e^{-[k_{m+1}+(j-3)k_1]t}, \quad j = 5, \cdots, n - m + 2$$

$$u_{m-1,2} = \frac{1}{k_1 y_{m-1,1}} (k_m y_{m-1,2} u_{m-1,1} - u_{m,2})$$
$$= -\sum_{j=m-1}^{n-1} \frac{k_{j+1} y_{j,2}}{(-k_1 y_{1,1})^{j-(m-1)}} \tag{24}$$

Substituting them into the second expression of the $(m-1)$th generation of ENI yields

$$\dot{y}_{m-1,2} = u_{m-1,2} = -k_m y_{m-1,2} + g_{m-1}(t) \tag{25}$$

where

$$|g_{m-1}(t)| = \left| \sum_{j=m}^{n-1} \frac{k_{j+1} y_{j,2}}{(-k_1 y_{1,1})^{j-m+1}} \right|$$
$$\leq \sum_{j=m}^{n-2} \frac{k_{j+1} a_{j,2}}{(k_1 a_1)^{j-m+1}} \, e^{-[k_{j+1}-(j-m+1)k_1]t}$$
$$+ \frac{k_n a_{n-2,3}}{(k_1 a_1)^{n-m}} \, e^{-[k_n-(n-m)k_1]t}$$

From the choices for k_i $(i = 3, \ldots, n)$, it is easy to show that for $m \leq j \leq n-2$

$$k_{j+1} - (j - m + 1)k_1 \geq k_{m+1} - k_1,$$
$$k_n - (n - m)k_1 > k_{m+1} - k_1$$

Therefore, we have

$$|g_{m-1}(t)| \leq \left[\sum_{j=m}^{n-2} \frac{k_{j+1} a_{j,2}}{(k_1 a_1)^{j-m+1}} + \frac{k_n a_{n-2,3}}{(k_1 a_1)^{n-m}} \right] e^{-(k_{m+1}-k_1)t}$$
$$\triangleq d_{m-1} \, e^{-(k_{m+1}-k_1)t}$$

Since $k_{m+1} - k_1 > k_m$, from (25) and Lemma 1 we know that there exists a constant $a_{m-1,2}$ such that

$$|y_{m-1,2}(t)| \leq a_{m-1,2} \, e^{-k_m t} \tag{26}$$

Moreover, substituting (20), (22) and (26) into (24), we obtain

$$|u_{m-1,2}| \leq \sum_{j=m-1}^{n-2} \frac{k_{j+1} a_{j,2}}{(k_1 a_1)^{j-m+1}} \, e^{-[k_{j+1}-(j-m+1)k_1]t}$$
$$+ \frac{k_n a_{n-2,3}}{(k_1 a_1)^{n-m}} \, e^{-[k_n-(n-m)k_1]t}$$
$$\leq \left(\sum_{j=m-1}^{n-2} \frac{k_{j+1} a_{j,2}}{(k_1 a_1)^{j-m+1}} + \frac{k_n a_{n-2,3}}{(k_1 a_1)^{n-m}} \right) e^{-k_m t}$$
$$\triangleq b_{m-1,2} \, e^{-k_m t}$$

Equation (26), (24) and the above expression imply that the conclusion of Lemma 2 is true for $i = m - 1$, and the proof of Lemma 2 is completed. □

By letting $i = 1$ in Lemma 2, we immediately obtain the following results.

Theorem 1: *If ENI (1) satisfies $k_1 > 0$, $k_2 > 0$, $y_1(0) \neq 0$, and the control is chosen as*

$$\left.\begin{aligned} u_1 &= -k_1 y_1 \\ u_2 &= -\sum_{j=1}^{n-1} \frac{k_{j+1} y_{j,2}}{(-k_1 y_1)^{j-1}} \end{aligned}\right\} \tag{27}$$

where $k_i > k_{i-1} + k_1$ $(i = 3, \ldots, n)$, while $y_{j,2}$ $(j = 1, \ldots, n-2)$ and $y_{n-1,2} \triangleq y_{n-2,3}$ are determined recursively by (9), then

(1) *states y_1, y_2, y_i $(i = 3, \ldots, n)$ converge to the origin exponentially with rates k_1, k_2, $k_3 + (i-3)k_1$, respectively; and*
(2) *control inputs u_1 and u_2 are bounded along the closed-loop trajectories and tend to zero exponentially with rates k_1 and k_2, respectively.*

Remark 1: Theorem 1 provides a new family of control laws which yield exponential convergence of the state of ENI (1) to the origin. For the choice of the controller parameters, we need only the simple rules

$$k_1 > 0; \quad k_2 > 0; \quad k_i > k_{i-1} + k_1, \quad i = 3, 4, \ldots, n \tag{28}$$

this is the most simple way against the existing ones. The other variables in the approach are not design parameters and can be obtained by simple iterative algebraic manipulations.

Remark 2: It should be noted that $y_1(0) \neq 0$ is not a very restrictive constraint on control (27), since it is always possible to apply an open-loop control driving the system away from $y_1 = 0$ in an arbitrary small period of time (Bloch and Drakunov 1994).

4. A united framework of canonical forms

4.1. *Equivalence of ENI, chained and power forms*

In this section, we prove that the non-holonomic systems in ENI, chained and power forms are all equivalent with appropriate state difformorphisms.

Consider the two-input non-holonomic systems in chained form described by

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_j &= x_{j-2}u_1, \quad j = 3, \dots, n \end{aligned}\right\} \tag{29}$$

and in power form represented by

$$\left.\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_j &= \frac{1}{(j-2)!}x_1^{j-2}u_2, \quad j = 3, \dots, n \end{aligned}\right\} \tag{30}$$

where $x_j (j = 1, \dots, n)$ are the states and u_1 and u_2 are the two inputs.

Proposition 1: *With non-singular state transformation*

$$\left.\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_j &= [(j-2)k_1 + k_2]x_j - k_1x_1x_{j-1}, \ k_1 > 0, \ k_2 > 0; \\ &\qquad j = 3, \dots, n \end{aligned}\right\} \tag{31}$$

chained system (29) *can be transformed to the extended non-holonomic integrators* (1).

Proof: Since $k_1 > 0, k_2 > 0$, it is easy to verify that state transformation (31) is non-singular. Differentiating (31) along system (29) gives

$$\begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \\ \dot{y}_3 &= (k_1 + k_2)x_2u_1 - k_1(x_2u_1 + x_1u_2) \\ &= k_2y_2u_1 - k_1y_1u_2 \\ \dot{y}_j &= [(j-2)k_1 + k_2]x_{j-1}u_1 - k_1(x_{j-1}u_1 + x_1x_{j-2}u_1) \\ &= y_{j-1}u_1, \quad j = 4, \dots, n \end{aligned}$$

□

Proposition 2: *Power system* (30) *can be converted to ENI* (1) *by the following transformation*

$$\left.\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_j &= \sum_{s=2}^{j}\frac{(-1)^s[(s-2)k_1 + k_2]}{(j-s)!}x_sx_1^{j-s}, \quad k_1 > 0, k_2 > 0; \\ &\qquad i = 3, \dots, n \end{aligned}\right\} \tag{32}$$

Proof: Differentiating (30) along the trajectories of system (32) gives

$$\left.\begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \\ \dot{y}_3 &= -(k_1 + k_2)x_1u_2 + k_2(x_1u_2 + x_2u_1) \\ &= k_2y_2u_1 - k_1y_1u_2 \\ \dot{y}_j &= \frac{(-1)^j[(j-2)k_1 + k_2]}{(j-2)!}x_1^{j-2}u_2 \\ &\quad + \sum_{s=3}^{j-1}\frac{(-1)^s[(s-2)k_1 + k_2]}{(j-s)!}\left[\frac{1}{(s-2)!}x_1^{s-2}u_2x_1^{j-s}\right. \\ &\quad \left. + (j-s)x_sx_1^{j-s-1}u_1\right] + \frac{k_2}{(j-2)!}[x_1^{j-2}u_2 \\ &\quad + (j-2)x_2x_1^{j-3}u_1] \\ &= \left\{\sum_{s=3}^{j}\frac{(-1)^sk_1}{(j-s)!(s-3)!}\right. \\ &\quad \left. + \sum_{s=2}^{j}\frac{(-1)^sk_2}{(j-s)!(s-2)!}\right\}x_1^{j-2}u_2 \\ &\quad + y_{j-1}u_1, \quad j = 4, \dots, n \end{aligned}\right\} \tag{33}$$

It is easy to show, see Appendix for detailed derivation, that for $j \geq 3$

$$\sum_{s=3}^{j}\frac{(-1)^s}{(j-s)!(s-3)!} = \sum_{s=2}^{j}\frac{(-1)^s}{(j-s)!(s-2)!} = 0 \tag{34}$$

Substituting the above expressions into (33), we have ENI (1). □

4.2. *Controller design for chained form*

Note that transformation (31) converts the chained system to the first generation of ENI, it is easy to prove the following theorem from Proposition 1.

Theorem 2: *If chained system* (29) *satisfies that* $x_1(0) \neq 0$, *and*

$$\left.\begin{aligned} u_1 &= -k_1x_1 \\ u_2 &= -\sum_{j=1}^{n-1}\frac{k_{j+1}y_{j,2}}{(-k_1x_1)^{j-1}} \end{aligned}\right\} \tag{35}$$

where $k_1 > 0, k_2 > 0, k_i > k_{i-1} + k_1 \ (i = 3, \dots, n)$; *while* $y_{j,2} \ (j = 1, \dots, n-2)$ *and* $y_{n-1,2} \triangleq y_{n-2,3}$ *are calculated recursively by* (31) *and* (9), *then*

(1) *the states* x_1, $x_i \ (i = 2, \dots, n)$ *converge to the origin exponentially with rates* k_1, $k_2 + (i-2)k_1$, *respectively; and*

(2) *the controls u_1 and u_2 are bounded and go to zero exponentially with rates k_1 and k_2, respectively.*

Proof: From Proposition 1, Theorem 1 and the definitions of $u_{1,1}, u_{1,2}$ and $y_{1,i} (i = 3, \ldots, n)$, when control (35) is applied to the chained system (29), we see that

(1) variables x_1, x_2 and $y_{1,i}$ $(i = 3, \ldots, n)$ decay exponentially to the origin with rates k_1, k_2, and $k_3 + (i-3)k_1$, respectively;
(2) controls u_1 and u_2 are bounded and tend exponentially to zero with rate k_1 and k_2, respectively.

Furthermore, we can prove by induction that x_i $(i = 3, \ldots, n)$ decrease exponentially to zero, each with corresponding rate $k_2 + (i-2)k_1$. As a matter of fact, when $i = 3$, from (31) we obtain

$$|x_3(t)| = \frac{|y_{1,3}(t) + k_1 x_1(t) x_2(t)|}{k_1 + k_2}$$

$$\leq \frac{a_{1,3} + k_1 a_1 a_{1,2}}{k_1 + k_2} \mathrm{e}^{-(k_2+k_1)t} \triangleq c_3 \mathrm{e}^{-(k_2+k_1)t}$$

Suppose that for $3 \leq i \leq n-1$, $x_i(t)$ converges to zero exponentially with rate $k_2 + (i-2)k_1$, i.e. there exist constants c_i $(i = 3, \ldots, n-1)$ such that

$$|x_i(t)| \leq c_i \mathrm{e}^{-[k_2+(i-2)k_1]t}, \quad i = 3, \ldots, n-1$$

Then, for $i = n$, we have

$$|x_n(t)| = \frac{|y_{1,n}(t) + k_1 x_1(t) x_{n-1}(t)|}{(n-2)k_1 + k_2}$$

$$\leq \frac{a_{1,n} + k_1 a_1 c_{n-1}}{(n-2)k_1 + k_2} \mathrm{e}^{-[k_2+(n-2)k_1]t} \triangleq c_n \mathrm{e}^{-[k_2+(n-2)k_1]t}$$

Thus, Theorem 2 is proved by induction. □

Remark 3: According to Astolfi (1996), control design for the chained system (29) is given by

$$\left.\begin{aligned} u_1 &= -k_1 x_1 \\ u_2 &= [\alpha_1, \alpha_2, \ldots, \alpha_{n-1}] \begin{bmatrix} x_2 \\ x_3/x_1 \\ \vdots \\ x_n/x_1^{n-2} \end{bmatrix} \end{aligned}\right\}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are controller parameters to be determined. It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_2 \\ x_3/x_1 \\ \vdots \\ x_n/x_1^{n-2} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & & \alpha_{n-1} \\ -k_1 & k_1 & & & \\ & \ddots & \ddots & & \\ & & & -k_1 & (n-2)k_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ x_3/x_1 \\ \vdots \\ x_n/x_1^{n-2} \end{bmatrix} \triangleq A \begin{bmatrix} x_2 \\ x_3/x_1 \\ \vdots \\ x_n/x_1^{n-2} \end{bmatrix} \tag{36}$$

Clearly, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ can be chosen such that all the eigenvalues of A have negative real parts. Let $\lambda_i(A)$, $i = 1, \ldots, n-1$ denote the eigenvalues of A and define

$$\underline{\lambda} = \min_{i \in \{1, \ldots, n-1\}} |\mathrm{Re}\ \lambda_i(A)|$$

Equation (36) implies that $x_2, x_3/x_1, \ldots, x_n/x_1^{n-2}$ converge to zero exponentially with rate $\underline{\lambda}$. Considering x_1 tends to zero exponentially with rate k_1, it follows that x_i, $i = 2, \ldots, n$, converge to zero exponentially with rate $\underline{\lambda} + (i-2)k_1$, respectively. From (31) and (9), it can be seen that the control u_2 in (35) of the ENI approach is also a linear combination of $x_2, x_3/x_1, \ldots, x_n/x_1^{n-2}$. In this aspect, both the controller in Astolfi (1996) and that presented here are the same. Apart from the similarity, there are also differences as listed below:

(1) In order to guarantee exponential convergence of the states, the controller parameters of the ENI approach can be easily obtained by (28). However, using the method in Astolfi (1996), one has to determine controller parameters $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ by assigning the eigenvalues of the $(n-1)$-order matrix A.
(2) When convergence rates of the states are specified, controller design based on ENI can be easily done by simply setting k_2, on the other hand, using the method in Astolfi (1996), one has to regulate all controller parameters $\alpha_1, \ldots, \alpha_{n-1}$ such that $\underline{\lambda}$ meets the requirement, which is not an easy task in general since the relationship between $\alpha_1, \ldots, \alpha_{n-1}$ and $\underline{\lambda}$ is very complex for systems with higher order.

The above facts demonstrate that this paper provides a simpler controller design method for exponential stabilization of chained systems.

4.3. *Controller design for power form*

Following the same procedure in the proof of Theorem 2, we have Theorem 3.

Theorem 3: *For power system* (30) *which satisfies that* $x_1(0) \neq 0$*, and*

$$\left.\begin{aligned} u_1 &= -k_1 x_1 \\ u_2 &= -\sum_{j=1}^{n-1} \frac{k_{j+1} y_{j,2}}{(-k_1 x_1)^{j-1}} \end{aligned}\right\} \quad (37)$$

where $k_1 > 0, k_2 > 0, k_i > k_{i-1} + k_1$ $(i = 3, \ldots, n)$*; while* $y_{j,2}$ $(j = 1, \ldots, n-2)$ *and* $y_{n-1,2} \triangleq y_{n-2,3}$ *are calculated recursively by* (32) *and* (9)*, then:*

(1) *the states* x_1, x_i $(i = 2, \ldots, n)$ *converge exponentially to the origin with rates* k_1, $k_2 + (i-2)k_1$*, respectively; and*

(2) *controls* u_1 *and* u_2 *are bounded and tend exponentially to zero with rates* k_1 *and* k_2*, respectively.*

5. Simulation study

The kinematic model of a car-like mobile robot with a motorized front wheel and two passive rear-wheels can be described as (Murray and Sastry 1993)

$$\left.\begin{aligned} \dot{x} &= v \cos\theta \\ \dot{y} &= v \sin\theta \\ \dot{\theta} &= \frac{v}{d} \tan\phi \\ \dot{\phi} &= \omega \end{aligned}\right\} \quad (38)$$

where (x, y) are the coordinates of point M located at mid-distance of the rear-wheels, θ is the orientation angle of the vehicle with respect to the x–axis, ϕ is the steering angle of the front wheel, v is the linear velocity of point M, ω is the steering velocity of the front wheel, and d denotes the distance between point M and the center of the front wheel.

With the following local coordinate and input transformations defined over the subset $\Gamma = \{(x, y, \theta, \phi) \in R^4 | \theta \neq \pi/2 (\text{mod } \pi),\ \phi \neq \pi/2 (\text{mod } \pi)\}$

$$\left.\begin{aligned} y_1 &= x \\ y_2 &= \frac{\tan\phi}{d\cos^3\theta} \\ y_3 &= (k_1 + k_2)\tan\theta - k_1 \frac{x\tan\phi}{d\cos^3\theta} \\ y_4 &= (2k_1 + k_2)y - k_1 x \tan\theta \end{aligned}\right\},$$

$$\left.\begin{aligned} v &= \frac{u_1}{\cos\theta} \\ \omega &= -\frac{3\sin^2\phi\tan\theta}{d\cos\theta} u_1 + (d\cos^2\phi\cos^3\theta) u_2 \end{aligned}\right\} \quad (39)$$

system (38) can be transformed into a four-dimensional ENI

$$\left.\begin{aligned} \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \\ \dot{y}_3 &= k_2 y_2 u_1 - k_1 y_1 u_2 \\ \dot{y}_4 &= y_3 u_1 \end{aligned}\right\} \quad (40)$$

For system (40), Theorem 1 can be used to design an exponential stabilization law. Choosing $k_1 = 1.8$, $k_2 = 2$, $k_3 = 4$ and $k_4 = 6$ to meet the requirement of Theorem 1, a straightforward calculation from (9) gives

$$y_{2,1} = y_1, \quad y_{2,2} = y_{1,3} = y_3,$$
$$y_{3,2} \triangleq y_{2,3} = (k_1 + k_2)y_4 - k_1 y_1 y_3 = 5.8 y_4 - 1.8 y_1 y_3$$

According to (27), we obtain the exponential stabilization law

$$\left.\begin{aligned} u_1 &= -1.8 y_1 \\ u_2 &= -2y_2 + \frac{50 y_3}{9 y_1} - \frac{290 y_4}{27 y_1^2} \end{aligned}\right\} \quad (41)$$

Assume that system parameter $d = 1$ and initial states of the mobile robot are $[x(0),\ y(0),\ \theta(0),\ \phi(0)]^{\mathrm{T}} = [-2, -2, \pi/4, \pi/4]^{\mathrm{T}}$. From (39), we can calculate the corresponding initial states for the ENI (40) as $[y_1(0), y_2(0), y_3(0),\ y_4(0)]^{\mathrm{T}} = [-2, 2\sqrt{2},\ 3.8 + 7.2\sqrt{2},\ -7.6]^{\mathrm{T}}$. From Theorem 1, we conclude that when control (41) is applied to system (40):

(1) y_1, y_2, y_3 and y_4 converge exponentially to the origin with rates 1.8, 2, 3.8 and 5.8, respectively; and

(2) u_1 and u_2 are bounded and tend exponentially to zero with rate 1.8 and 2, respectively.

These theoretical conclusions are verified by simulation study as shown in figure 1 for the states and figure 2 for the control inputs, respectively.

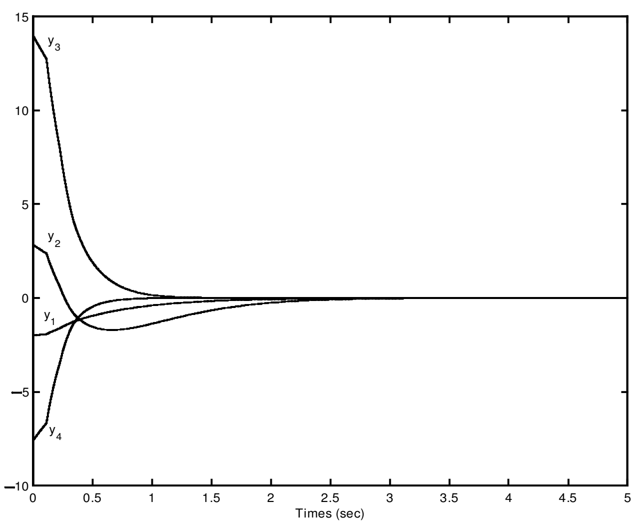

Figure 1. State trajectories of system (40).

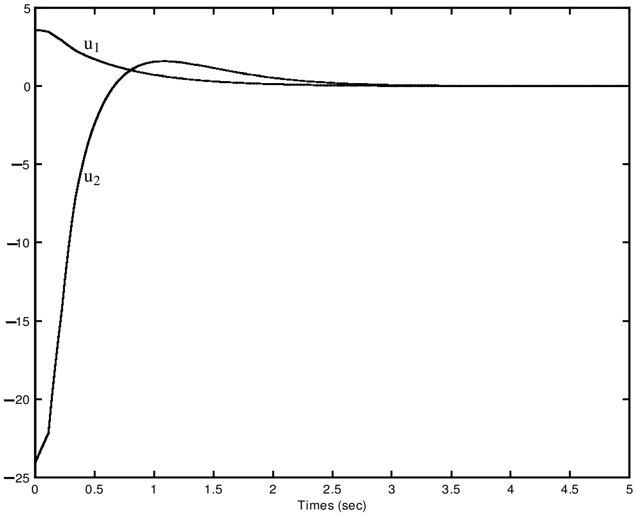

Figure 2. Control signals of system (40).

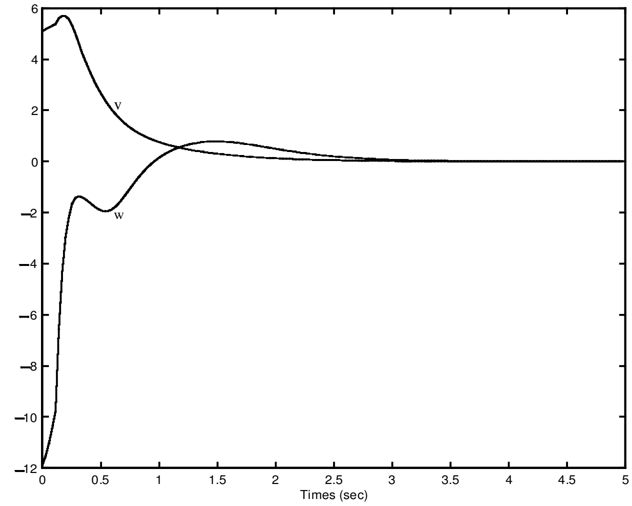

Figure 4. Control signals of mobile robot (38).

Using (39), we obtain the control inputs described with the original system variables as

$$\left.\begin{aligned} v &= -\frac{1.8x}{\cos\theta} \\ \omega &= \frac{5.4x\sin^2\phi\tan\theta}{\cos\theta} - \cos^2\phi\left[12\tan\phi - \frac{1}{x}\right. \\ &\quad \left.\times\left(40.44\tan\theta - \frac{60.14y}{x}\right)\cos^3\theta\right] \end{aligned}\right\} \tag{42}$$

Simulation studies have been carried out for mobile robot (38) with control (42) under the above initial conditions as well. Figure 3 shows the convergence of the states, and figure 4 demonstrates the boundedness and convergence of control inputs.

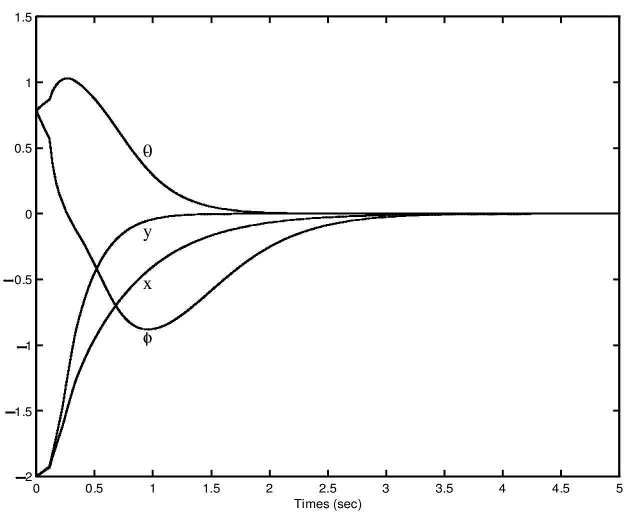

Figure 3. State trajectories of mobile robot (38).

6. Conclusions

In this paper, a new canonical form, called extended non-holonomic integrators (ENI), has been introduced for non-holonomic systems. For the introduced ENI, a simple recursive design method has been presented to exponentially stabilize ENI systems. The relationships between the convergence rates of the states and the controller parameters were explicitly given, in particular, the convergence rate of individual state can be specified explicitly. It has been proven that the non-holonomic systems in ENI, chained form and power form are equivalent to each other, thus they can be addressed in a united framework. A car-like mobile robot is used to demonstrate the effectiveness of the proposed control.

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Appendix: Proof of (34)

Let us first prove that

$$\sum_{t=0}^{k} \frac{(-1)^t k!}{(k-t)!t!} = 0, \quad \forall k \geq 1 \tag{43}$$

For $k = 1$, (43) is trivial. Suppose that, for $k = m$ (≥ 1)

$$\sum_{t=0}^{m} \frac{(-1)^t m!}{(m-t)!t!} = 0 \tag{44}$$

Then, for $k = m+1$, we have

$$\sum_{t=0}^{m+1} \frac{(-1)^t (m+1)!}{(m+1-t)!t!} = (-1)^{m+1} + \sum_{t=1}^{m} \frac{(-1)^t m!}{(m-t)!t!} + 1$$

$$= (-1)^{m+1} + \sum_{t=1}^{m} \frac{(-1)^t m!}{(m-t+1)!(t-1)!} + \sum_{t=1}^{m} \frac{(-1)^t m!}{(m-t)!t!} + 1$$

$$= -\sum_{s=0}^{m} \frac{(-1)^s m!}{(m-s)!s!} + \sum_{t=0}^{m} \frac{(-1)^t m!}{(m-t)!t!} = 0$$

by using (44). By induction, we know that (43) is true.

Using (43), it is easy to show that (34) is valid. By setting $s-3=t$, $j-3=k$, we know that

$$\sum_{s=3}^{j} \frac{(-1)^s}{(j-s)!(s-3)!} = \sum_{t=0}^{j-3} \frac{(-1)^{t+3}}{(j-3-t)!t!} = -\sum_{t=0}^{k} \frac{(-1)^t}{(k-t)!t!}$$

$$= \frac{-1}{k!} \sum_{t=0}^{k} \frac{(-1)^t k!}{(k-t)!t!} = 0$$

Similarly, by setting $s-2=t$, $j-2=k$, we have

$$\sum_{s=2}^{j} \frac{(-1)^s}{(j-s)!(s-2)!} = \sum_{t=0}^{j-2} \frac{(-1)^{t+2}}{(j-2-t)!t!} = \sum_{t=0}^{k} \frac{(-1)^t}{(k-t)!t!}$$

$$= \frac{1}{k!} \sum_{t=0}^{k} \frac{(-1)^t k!}{(k-t)!t!} = 0 \qquad \square$$

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