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Adaptive NN control for a class of discrete-time non-linear systems

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In this paper, adaptive neural network (NN) control is investigated for a class of single-input single-output (SISO) discrete-time unknown non-linear systems with general relative degree in the presence of bounded disturbances. Firstly, the systems are transformed into a causal state space description, adaptive state feedback NN control is presented based on Lyapunov's stability theory. Then, by converting the systems into a causal input–output representation, adaptive output feedback NN control is given. Finally, adaptive NN observer design and observer-based adaptive control are presented under the assumption of persistent excitation (PE). All the control schemes avoid the so-called controller singularity problem in adaptive control. By suitably choosing the design parameters, the closed-loop systems are proven to be semi-globally uniformly ultimately bounded (SGUUB). Simulation studies show the effectiveness of the newly proposed schemes.

1. Introduction

In recent years, control system design for systems with complex non-linear dynamics has attracted an ever increasing attention in the control community. Many remarkable results have been obtained owing to the advances in geometric non-linear control theory, especially feedback linearization techniques (Marino and Tomei 1995), Lyapunov design (Slotine and Li 1991) and NN control (Ge *et al.* 1998). Both state feedback and output feedback linearization methods have been studied in the literature. A general form of control structure of adaptive feedback linearization is $u = \hat{N}(x)/\hat{D}(x)$, where $\hat{D}(x)$ must be bounded away from zero to avoid the possible controller singularity problem (Yesidirek and Lewis 1995). The approach is only applicable to the class of systems whose dynamics are linear-in-the-parameters and satisfy the so-called matching conditions. The matching condition was relaxed to the extended matching condition in Kanellakopoulos *et al.* (1991 b) and Campion and Bastin (1990), and the extended matching barrier was broken in Kanellakopoulos *et al.* (1991 a) by using adaptive backstepping (Kanellakopoulos 1991, Krstic *et al.* 1995, Ge *et al.* 2002).

The design methodologies for both continuous-time systems and discrete-time systems are very different. Similar formulations in continuous-time and discrete-time domains may describe two totally different systems. Many properties in continuous-time domain may disappear in discrete-time domain, and vice versa. The same concepts in continuous-time and discrete-time domains may have different meanings. For example, the *relative degrees* defined for continuous-time and discrete-time systems have totally different physical explanations (Cabrera and Narendra 1999). As a consequence, results obtained in continuous-time domain may not be obtainable at all in discrete-time domain. Therefore, it is necessary to investigate them separately. Adaptive control of discrete-time systems using input–output data is of great interest owing to the wide application of digital computers. In the study of non-linear discrete-time control, one of the most popular representations is the NARMAX (non-linear auto regressive moving average with exogenous inputs) model (Leontaritis and Billings 1985). As only input and output sequences appear in the NARMAX model, it is straightforward to use approximation based methods to construct the 'inverse' of the system to emulate the desired control input, which can then drive the system output to the desired trajectory.

For non-linear systems in discrete-time, linear control theory is of little use. As such, adaptive robust control schemes have been investigated by many researchers (Baras and Patel 1998, Zhang *et al.* 2000, Zhao and Kanellakopoulos 2002). In Baras and Patel (1998), robust control was given for a class of 'set-valued' discrete-time dynamical systems subject to persistent bounded noises. In Zhang *et al.* (2000), by using the backstepping procedures with parameter projection update laws, robust adaptive control was designed for systems with the *a priori* range of unknown time-varying parameters. In Zhao and Kanellakopoulos (2002), a systematic design method was given for global stabilization and tracking of discrete-time output feedback non-linear systems with unknown parameters. Most of the schemes require the structures of the system models to be known. In this research, approximation based control is presented for unknown non-linear systems because of the difficulty in obtaining realistic models for complex systems in practice. Compared with robust adaptive control schemes, the universal approximate ability of a neural network makes it one of the best and attractive

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alternatives in discrete-time control though using neural networks may raise the complexity of computing. This is especially true when the model of the system is unknown or impossible to obtain.

In the past several years, active research has been carried out in neural network control. It has been proven that artificial neural networks can approximate a wide range of non-linear functions to any desired degree of accuracy under certain conditions. Recently, several excellent NN control approaches have been proposed based on Lyapunov's stability theory, such as Lewis *et al.* (1995, 1996), Yesidirek and Lewis (1995), Polycarpou (1996), Ge *et al.* (1999, 2002), Seshagiri and Khalil (1999), among others. One main advantage of these schemes is that the adaptive laws are derived based on Lyapunov synthesis and therefore guarantee the stability of continuous-time systems without the requirement for off-line training. All the works mentioned above deal with *static neural networks* (feed-forward networks). Recently, the study of *dynamic neural networks* (recurrent networks) in identification and control also attracted much attention in the control community. The most successful one may be the work of Poznyak *et al.* (1999), in which an on-line learning dynamic NN identifier was combined with an optimal controller to solve the trajectory tracking problem for non-linear continuous-time systems.

There are fewer analytical mathematical tools in the discrete-time domain than in the continuous-time domain. Thus, an easy problem in continuous-time may become difficult in discrete-time. This may be one of the reasons that NN control for non-linear systems in discrete-time is less studied than that for non-linear systems in continuous-time in the literature. Nevertheless, owing to its universal function approximation ability, NN control is drawing more attention to discrete-time output feedback control (Chen and Khalil 1995, Jagannathan and Lewis 1996, Adetona *et al.* 2000, Dabroom and Khalil 2002) among others. In Chen and Khalil (1995), multi-layer NNs were used to control a class of discrete-time non-linear systems with general relative degree through back propagation. To provide a good starting point for on-line adaptive control, off-line training was needed. The convergence results are local with respect to the initial parameters. In Jagannathan and Lewis (1996), NN control was studied for a class of discrete-time non-linear systems with relative degree 1. The controller singularity problem was excellently solved but not avoided completely. The existence of reversibility or invertibility of non-linear discrete-time systems studied in Fliess (1992) inspires us to search for stable adaptive NN control for discrete-time non-linear systems. In this paper, both state and output feedback adaptive NN controllers are presented for a class of unknown SISO discrete-time non-linear systems with general relative degree under external disturbances using high-order neural networks (HONNs). The controllers not only guarantee that the overall system is semi-globally uniformly ultimately bounded without the requirement for persistent excitation, but also avoid the controller singularity problem completely as in Ge *et al.* (2002).

When the states are not available for feedback control, observers are frequently used to reconstruct the states. Non-linear discrete-time observers have been discussed in the literature (e.g. Moraal and Grizzle 1995, Dabroom and Khalil 2001). In Dabroom and Khalil (2001), a high gain observer was firstly designed in continuous time domain, then it was discretized and implemented for state estimation in a sampled-data control problem. In Moraal and Grizzle (1995), novel observers (different from Luenberger ones) for non-linear discrete-time systems were constructed by solving sets of non-linear equations simultaneously. Research on control system design based on this discrete-time observer is still yet to be investigated. As an additional contribution in this paper, an adaptive NN observer is firstly constructed which guarantees the estimated states of the unknown system converge to a small neighbourhood of the system states under persistent exciting (PE) condition (Sadegh 1993), then observer-based adaptive NN control is presented for completeness.

The paper is organized as follows. Section 2 presents the dynamics of the systems and the preliminaries. In § 3, adaptive NN control is proposed when full states are measurable. In § 4, output feedback adaptive NN control is presented. In § 5, novel NN observer design and observer-based control are presented under the assumption of PE. Finally, simulation results are given in § 6 to show the effectiveness of the schemes.

2. System description and preliminaries

2.1. *Input–output model description*

Consider the SISO non-linear discrete-time system in the NARMAX form

$$\begin{aligned} y_{k+1} = & f_o(y_k, \ldots, y_{k-n+1}, u_{k-\tau}, \ldots, u_{k-\tau-m+1}) \\ & + g_o(y_k, \ldots, y_{k-n+1}, u_{k-\tau}, \ldots, u_{k-\tau-m+1}) u_{k-\tau+1} \\ & + d_k \end{aligned} \tag{1}$$

where $m \le n$, $y \in R$ is the measured output, $u \in R$ is the input, d_k denotes the external disturbance bounded by a known constant $d_0 > 0$, i.e. $|d_k| \le d_0$, τ is the system delay which satisfies $1 \le \tau \le n$ (τ is also called the relative degree of the system), and $f_o(*)$ and $g_o(*)$ are unknown smooth non-linear functions.

The relative degree for discrete-time systems was well defined and explained in Cabrera and Narendra (1999), which is much different from the counterpart in continuous-time. For a system of relative degree 1, the controller design procedure is causal and simple, because the control input u_k appears explicitly on the right side of equation (1). However, for a system with relative degree larger than 1, we cannot obtain control u_k directly from equation (1) because u_k does not appear in the equation explicitly.

Moving forward $(\tau - 1)$ steps, equation (1) becomes

$$\begin{aligned} y(k+\tau) = &f_o(y_{k+\tau-1}, \dots, y_{k+\tau-n}, u_{k-1}, \dots, u_{k-m}) \\ &+ g_o(y_{k+\tau-1}, \dots, y_{k+\tau-n}, u_{k-1}, \dots, u_{k-m})u_k \\ &+ d_{k+\tau-1} \end{aligned} \tag{2}$$

where u_k appears explicitly. But we still have difficulty in designing u_k because of the existence of future outputs $y_{k+1}, \dots, y_{k+\tau-1}$, which are unavailable for feedback. By carefully examining equations (1) and (2), it can be seen that control input u_k only affects future output $y_{k+\tau}$ and those beyond, and has no influence on the intermediate future outputs $y_{k+1}, \dots, y_{k+\tau-1}$. When the external disturbance $d_{k+\tau-1}$ is ignored, there will be no non-causal problem to predict the future outputs on the right-hand side of equation (2) if f_o and g_o are known.

Definition 1: The future output of a discrete-time control system is said semi-determined future output (SDFO) at time instant k, if it can be determined based on the available system information up to time instant k and controls up to time instant $k-1$ under the assumption that the dynamics of the plant and the disturbance are known.

For example, y_{k+1} is a semi-determined future output because it can be predicted from equation (2) if f_o, g_o and d_k are known. Actually, $y_{k+1}, \dots, y_{k+\tau-1}$ are all SDFOs. Further explanation can be found in § 4.

2.2. *State space model description*

For convenience of analysis, define system states $x(k) = [x_1(k), \dots, x_n(k)]^{\mathrm{T}}$ as

$$x_i(k) = y(k - n - 1 + \tau + i) \qquad i = 1, 2, \dots, n$$

and rewrite (2) in state space description

$$\left.\begin{aligned} x_i(k+1) &= x_{i+1}(k) \qquad i = 1, 2, \dots, n-1 \\ x_n(k+1) &= f(x(k), \upsilon_{k-1}(k)) + g(x(k), \upsilon_{k-1}(k))u_k \\ &\quad + d_{k+\tau-1} \\ y_k &= x_{n-\tau+1}(k) \end{aligned}\right\} \tag{3}$$

where $x(k) \in R^n$, and

$$\upsilon_{k-1}(k) = [u_{k-1}, \dots, u_{k-m}]^{\mathrm{T}} \in R^m$$

$$f(x(k), \upsilon_{k-1}(k)) = f_o(y_{k+\tau-1}, \dots, y_{k+\tau-n}, u_{k-1}, \dots, u_{k-m})$$

$$g(x(k), \upsilon_{k-1}(k)) = g_o(y_{k+\tau-1}, \dots, y_{k+\tau-n}, u_{k-1}, \dots, u_{k-m})$$

For clarity, define

$$f(k) = f(x(k), \upsilon_{k-1}(k))$$
$$g(k) = g(x(k), \upsilon_{k-1}(k))$$

which are functions of states $x(k)$ and all past control inputs from u_{k-1} up to the most delayed input u_{k-m} on the right-hand side of (2).

Assumption 1: *The sign of* $g(k)$ *is known and there exist two constants* $g_0, g_1 > 0$ *such that* $g_0 \le |g(k)| \le g_1$, $\forall x(k) \in \Omega \subset R^n$ *and* $\forall u_k \in \Omega_u \subset R$.

Without losing generality, we shall assume that $g(k)$ is positive in the following discussion.

The control objective is to drive the system output y_k to follow a desired trajectory $y_d(k) \in \Omega_y \subset R$. Define vector $x_d(k)$ as the desired system states

$$x_d(k) = [y_d(k+\tau-n), \dots, y_d(k), \dots, y_d(k+\tau-1)]^{\mathrm{T}}$$

Assumption 2: $y_d(k)$ *is smooth and known, and* $x_d \in \Omega_d$ *with* Ω_d *being a connected small subset of* Ω.

Define error vector $e(k)$ as

$$e(k) = x(k) - x_d(k) = [e_1(k), e_2(k), \dots, e_n(k)]^{\mathrm{T}}$$

Then the error equation of $e(k)$ can be written as

$$\left.\begin{aligned} e_i(k+1) &= e_{i+1}(k), \qquad i = 1, 2, \dots, n-1 \\ e_n(k+1) &= f(k) + g(k)u_k + d_{k+\tau-1} - y_d(k+\tau) \end{aligned}\right\} \tag{4}$$

Define the one-step open-loop tracking error $e_f(k)$ and one-step open-loop speed error $v_e(k)$ as

$$e_f(k) = e_n(k+1)\Big|_{u_k=0} \tag{5}$$

$$v_e(k) = e_f(k) - e_n(k) \tag{6}$$

Assumption 3: $(1/g_0)|e_f(k)| \in \Omega_f \subset \Omega_u$, $(1/g_0)|v_e(k)| \in \Omega_f$, $\forall x(k) \in \Omega$ *and* $\forall u_k \in \Omega_u$.

Assumption 3 is reasonable for many physical systems because the states of a physical system cannot change too fast within a small time interval in open-loop due to the 'inertia' of the systems.

Since Assumptions 2 and 3 are only valid on the compact sets Ω and Ω_u, it is necessary to guarantee that the system's states and the control signal remain in Ω and Ω_u $\forall k \ge 0$ respectively. We will design an adaptive control u_k for system (3) which makes system output y_k follow the desired trajectory $y_d(k)$, and simultaneously guarantees $x(k) \in \Omega, u_k \in \Omega_u$ $\forall k > 0$ for $x(0) \in \Omega$.

Definition 2: The solution of (4) is semi-globally uniformly ultimately bounded (SGUUB), if for any Ω, a compact subset of R^n and all $x(k_0) = x_0 \in \Omega$, there exist an $\epsilon > 0$, and a number $N(\epsilon, x_0)$ such that $\|e(k)\| < \epsilon$ for all $k \geq k_0 + N$.

Definition 3: The sequence $S(k)$ is said to be persistently exciting (PE) if there is a constant $\bar{\lambda} > 0$ and integer $L > 0$ such that

$$\lambda_{\min}\left[\sum_{k=k_0}^{k_0+L-1} S(k)S^{\mathrm{T}}(k)\right] \geq \bar{\lambda}, \qquad \forall k_0 \geq 0 \tag{7}$$

where $\lambda_{\min}(M)$ denotes the smallest eigenvalue of M (Sadegh 1993).

Lemma 1: *Consider the linear time varying discrete-time system*

$$x(k+1) = A(k)x(k) + u_k, \qquad y_k = x(k) \tag{8}$$

Let $\Phi(k_1, k_0)$ be the state transition matrix corresponding to $A(k)$ for system (8), *i.e.* $\Phi(k_1, k_0) = \prod_{k=k_0}^{k_1-1} A(k)$. *If $\|A(k)\|_2$ is strictly less than 1 (implies $\|\Phi(k_1, k_0)\|_2 < 1$), then $x(k+1)$ is bounded for bounded control input $u(k)$.*

2.3. *High-order neural network and function approximation*

For function approximation, different neural networks can be used. They include high-order neural networks (HONN) and radial basis function (RBF) neural networks. In this paper, for closed-loop stability, we need the basis function of the neural networks to be less than 1. Therefore, let us consider the following high-order neural networks which satisfies the above condition (Kosmatopoulos *et al.* 1995)

$$\phi(W, z) = W^{\mathrm{T}}S(z), \qquad W \text{ and } S(z) \in R^l$$

$$S(z) = [s_1(z), s_2(z), \ldots, s_l(z)]^{\mathrm{T}} \tag{9}$$

$$s_i(z) = \prod_{j \in I_i} [s(z_j)]^{d_j(i)}, \qquad i = 1, 2, \ldots, l \tag{10}$$

where $z = [z_1, z_2, \ldots, z_q]^{\mathrm{T}} \in \Omega_z \subset R^q$; positive integer l denotes the NN node number; $\{I_1, I_2, \ldots, I_l\}$ is a collection of l not-ordered subsets of $\{1, 2, \ldots, q\}$ and $d_j(i)$ are non-negative integers; W is an adjustable synaptic weight vector; $s(z_j)$ is chosen as a hyperbolic tangent function

$$s(z_j) = \frac{e^{z_j} - e^{-z_j}}{e^{z_j} + e^{-z_j}}$$

It has been shown in Kosmatopoulos *et al.* (1995) that neural network $W^{\mathrm{T}}S(z)$ satisfies the conditions of the Stone–Weierstrass theorem and can approximate any continuous function to any desired accuracy over a compact set. Because the non-linearity $u^*(z)$ in desired feedback control input (14) is a smooth function on Ω_z, for an arbitrary constant $\mu_0 > 0$, there exist an integer l^* and an ideal constant weight vector W^*, such that for all $l \geq l^*$

$$u^*(z) = W^{*\mathrm{T}}S(z) + \mu_z, \qquad \forall z \in \Omega_z \tag{11}$$

where μ_z is called the NN approximation error satisfying $|\mu_z| \leq \mu_0$ and

$$W^* := \arg\min_{W \in R^l}\{\sup_{z \in \Omega_z} |W^{\mathrm{T}}S(z) - u^*(z)|\}$$

The NN approximation error is a critical quantity, representing the minimum possible deviation of the ideal approximator $W^{*\mathrm{T}}S(z)$ from the unknown ideal control $u^*(z(k))$. The NN approximation error can be reduced by increasing the number of the adjustable weights. Universal approximation results for neural networks (Gupta and Rao 1994) indicate that, if NN node number l is sufficiently large, then $|\mu_z|$ can be made arbitrarily small on a compact region.

3. State feedback adaptive control

3.1. *Desired state feedback control*

Assuming that states $x(k)$ are measurable, non-linear functions $f(k)$ and $g(k)$ are known exactly, and there is no disturbance in the system, i.e. $d_k = 0$, we present a desired control, u_k^*, such that the output y_k follows the desired trajectory $y_d(k)$ asymptotically.

The following lemma establishes the existence of the ideal control, u_k^*, which brings the output of the system to the desired trajectory.

Lemma 2: *Consider system* (3) *with Assumptions 1–3 satisfied and $d_k = 0$. If the desired control input is chosen as*

$$u_k^* = -\frac{1}{g(k)}\left[f(k) - y_d(k+\tau) - k_v\frac{g(k)}{g_1}e_n(k)\right] \tag{12}$$

where $0 \leq k_v < 1$ is a design parameter, then $\lim_{k\to\infty}\|e(k)\| = 0$.

Proof: From equation (4) and the definition of $e_f(k)$ in (5), we obtain

$$e_f(k) = f(k) - y_d(k+\tau)$$

Accordingly, we have

$$\begin{aligned}|u_k^*| &= \frac{1}{g(k)}\left|f(k) - y_d(k+\tau) - k_v\frac{g(k)}{g_1}e_n(k)\right| \\ &= \frac{1}{g(k)}\left|e_f(k) - k_v\frac{g(k)}{g_1}e_n(k)\right| \\ &= \frac{1}{g(k)}\left|\left[e_f(k) - k_v\frac{g(k)}{g_1}e_n(k)\right]\mathrm{sgn}(e_n(k))\right|\end{aligned}$$

Based on Assumption 1 and the condition that $0 \le k_v < 1$, it is obvious that

$$0 \le k_v \frac{g(k)}{g_1} < 1$$

To prove $u_k^* \in \Omega_f$, let us consider the following three cases:

(i) If $\mathrm{sgn}(e_f(k)) = \mathrm{sgn}(e_n(k))$ and $|e_f(k)| \ge k_v(g(k)/g_1)|e_n(k)|$, then

$$\begin{aligned}|u_k^*| &= \frac{1}{g(k)}\left||e_f(k)| - k_v\frac{g(k)}{g_1}|e_n(k)|\right| \\ &\le \frac{1}{g(k)}|e_f(k)|\end{aligned}$$

From Assumption 3, we conclude that $u_k^* \in \Omega_f$ for this case.

(ii) If $\mathrm{sgn}(e_f(k)) = \mathrm{sgn}(e_n(k))$ and $|e_f(k)| < k_v(g(k)/g_1)|e_n(k)|$, then

$$\begin{aligned}|u_k^*| &= \frac{1}{g(k)}\left||e_f(k)| - k_v\frac{g(k)}{g_1}|e_n(k)|\right| \\ &= \frac{1}{g(k)}\left|k_v\frac{g(k)}{g_1}|e_n(k)| - |e_f(k)|\right| \\ &< \frac{1}{g(k)}\left||e_n(k)| - |e_f(k)|\right| \\ &\le \frac{1}{g(k)}|e_n(k) - e_f(k)| \\ &= \frac{1}{g(k)}|v_e(k)|\end{aligned}$$

From Assumption 3, we conclude that $u_k^* \in \Omega_f$ for this case.

(iii) If $\mathrm{sgn}(e_f(k)) = -\mathrm{sgn}(e_n(k))$, then

$$\begin{aligned}|u_k^*| &= \frac{1}{g(k)}\left|-|e_f(k)| - k_v\frac{g(k)}{g_1}|e_n(k)|\right| \\ &= \frac{1}{g(k)}\left||e_f(k)| + k_v\frac{g(k)}{g_1}|e_n(k)|\right| \\ &< \frac{1}{g(k)}||e_f(k)| + |e_n(k)|| \\ &= \frac{1}{g(k)}\left|-\mathrm{sgn}(e_n(k))e_f(k) + \mathrm{sgn}(e_n(k))e_n(k)\right| \\ &= \frac{1}{g(k)}|e_n(k) - e_f(k)| \\ &= \frac{1}{g(k)}|v_e(k)|\end{aligned}$$

From Assumption 3, we conclude that $u_k^* \in \Omega_f$ for this case.

In summary, we conclude that $u_k^* \in \Omega_f$.

Substituting the desired controller $u_k = u_k^*$ into (4) and noting that $d_k = 0$, we have

$$e_n(k+1) = k_v\frac{g(k)}{g_1}e_n(k) \tag{13}$$

Since $|k_v(g(k)/g_1)|$ is always less than 1, then

$$|e_n(k+1)| < |e_n(k)|$$

It is obvious that $\lim_{k\to\infty}\|e_n(k)\| = 0$, which leads to $\lim_{k\to\infty}\|e(k)\| = 0$, because

$$e_1(k+n-1) = e_2(k+n-2) = \cdots = e_n(k)$$

This means that the system output y_k will follow the desired trajectory $y_d(k)$ asymptotically. Under Assumption 2, $x(k)$ will remain in Ω for all $k > 0$ if $x(0) \in \Omega$ and will be attracted into Ω_d asymptotically if $x(0)$ is outside Ω_d. Due to $\Omega_f \subset \Omega_u$, u_k^* will remain in Ω_u for all $k > 0$. □

Noting expression (12), the desired control input u_k^* is a function of $x(k)$, $\upsilon_{k-1}(k)$, $y_d(k+\tau)$ and $e_n(k)$, it can be expressed as

$$\left.\begin{aligned}&u_k^* = u^*(z(k)),\\ &z = [x^{\mathrm{T}}(k), y_d(k+\tau), \upsilon_{k-1}^{\mathrm{T}}(k), k_v e_n(k)]^{\mathrm{T}} \in \Omega_z \subset R^{n+m+2}\end{aligned}\right\} \tag{14}$$

where the compact set Ω_z is defined as

$$\begin{aligned}\Omega_z = \Big\{&(x, y_d, \upsilon_{k-1}, k_v e_n)\Big|\upsilon_{k-1}(k) = [u_{k-1}\ldots u_{k-m}]^{\mathrm{T}},\\ &e_n(k) = x_n(k) - y_d(k+\tau-1), x \in \Omega,\\ &y_d \in \Omega_y, u \in \Omega_u\Big\}\end{aligned}$$

If non-linear functions $f(k)$ and $g(k)$ are unknown, $u^*(z(k))$ is unavailable. In the following discussion, neural networks will be used to approximate the unknown function $u^*(z(k))$.

Remark 1: When $k_v = 0$ in (12), the controller is a deadbeat controller because $e_n(k) = 0$ is achieved after one step.

3.2. *Adaptive state feedback control*

In this section, state feedback NN adaptive control is investigated. For convenience of analysis, define the variables

$$\begin{aligned}&e_n(k) = x_n(k) - y_d(k+\tau-1)\\ &z(k) = [x_1(k), \ldots, x_{n-\tau+1}(k), x_{n-\tau+2}(k), \ldots, x_n(k),\\ &\qquad y_d(k+\tau), \upsilon_{k-1}^{\mathrm{T}}(k), k_v e_n(k)]^{\mathrm{T}}\end{aligned} \tag{15}$$

The adaptive NN controller is given by

$$u_k = \hat{W}^{\mathrm{T}} S(z(k)) \tag{16}$$

$$\hat{W}(k+1) = \hat{W}(k) - \Gamma\left[S(z(k))e_n(k+1) + \sigma\hat{W}(k)\right] \tag{17}$$

where adaptation diagonal gain matrix $\Gamma = \Gamma^{\mathrm{T}} > 0$ and constant $\sigma > 0$. In adaptive law (17), σ-modification (Ioannou and Sun 1995) is used to eliminate the requirement of PE condition for the boundedness of NN weights.

Substituting controller (16) into (4), the error equation (4) can be re-written as

$$\begin{aligned} e_n(k+1) &= f(k) - y_d(k+\tau) \\ &\quad + g(k)\hat{W}^{\mathrm{T}}(k)S(z(k)) + d_{k+\tau-1} \end{aligned} \tag{18}$$

Adding and subtracting $g(k)u^*(z(k))$ on the right-hand side of (18) and noting (11), we have

$$\begin{aligned} e_n(k+1) &= f(k) - y_d(k+\tau) + g(k)[\hat{W}^{\mathrm{T}}(k)S(z(k)) \\ &\quad - W^{*\mathrm{T}}S(z(k)) - \mu_z] + g(k)u^*(z(k)) + d_{k+\tau-1} \end{aligned} \tag{19}$$

Substituting (12) into (19) leads to

$$\begin{aligned} e_n(k+1) &= g(k)\left[\hat{W}^{\mathrm{T}}(k)S(z(k)) - W^{*\mathrm{T}}(k)S(z(k)) \right. \\ &\quad \left. - \mu_z + \frac{k_v}{g_1}e_n(k)\right] + d_{k+\tau-1} \\ &= g(k)\left[\tilde{W}^{\mathrm{T}}(k)S(z(k)) - \mu_z + \frac{k_v}{g_1}e_n(k)\right] + d_{k+\tau-1} \end{aligned} \tag{20}$$

where $\tilde{W}(k) = \hat{W}(k) - W^*$.

Since NN approximation (11), and Assumptions 1–3 are only valid on the compact sets Ω and Ω_u, it is necessary to guarantee that the system's states remain in Ω and the control signal remains in Ω_u for all time.

Due to $u_k^* \in \Omega_f \subset \Omega_u$, there must exist two non-zero compact sets $\Omega_w \subset R^l$ and $\Omega_s \subset R^l$ such that for all $\tilde{W}(k) \in \Omega_w$ and $S(k) \in \Omega_s$ guarantee $u_k \in \Omega_u$.

In the following theorem, we show that for appropriate initial conditions $x(0)$, $\tilde{W}(0)$, and suitably chosen design parameters, adaptive controller (16) and adaptive law (17) guarantees $x(k) \in \Omega, u_k \in \Omega_u \ \forall k \geq 0$.

Theorem 1: *Consider the closed-loop system consisting of system* (3), *controller* (16) *and adaptation law* (17). *There exist compact sets* $\Omega_0 \subset \Omega$, $\Omega_{w_0} \subset \Omega_w$ *and positive constants* $\alpha_1, k_v^*, l^*, \gamma^*$ *and* σ^* *satisfying*

$$\left.\begin{aligned} &\alpha_1 < 1, \quad k_v^* = \frac{2-\alpha_1}{3}, \\ &\gamma^* = \frac{2-\alpha_1-3k_v^*}{2+g_1 l^*} \quad \textit{and} \quad \sigma^* = \frac{1}{\gamma^* + g_1\gamma^* l^*} \end{aligned}\right\} \tag{21}$$

such that if

(i) *Assumptions 1–3 are satisfied, the initial condition* $x(0) \in \Omega_0$, $\tilde{W}(0) \in \Omega_{w_0}$, *and*

(ii) *the design parameters are suitably chosen such that* $k_v < k_v^*, l > l^*, \sigma < \sigma^*$, *and* $\gamma < \gamma^*$, *with* γ *being the largest eigenvalue of diagonal gain matrix* Γ,

then, the closed-loop system is SGUUB. The tracking error can be made arbitrarily small by increasing the approximation accuracy of the neural networks.

Proof: See Appendix A.1. □

4. Direct output feedback NN control

Considering the good estimation property of NN, it may be feasible to search for a direct output feedback NN controller which is easier to implement than the state feedback NN controller.

For clarity, define

$$\begin{aligned} \boldsymbol{y}(k) &= [y_k, y_{k-1}, \ldots, y_{k-n+1}]^{\mathrm{T}} \\ \boldsymbol{u}_{k-1}(k) &= [u_{k-1}, \ldots, u_{k-\tau-m+1}]^{\mathrm{T}} \\ \boldsymbol{d}(k) &= [d_{k+\tau-1}, d_{k+\tau-2}, \ldots, d_k]^{\mathrm{T}} \\ \boldsymbol{z}(k) &= [\boldsymbol{y}^{\mathrm{T}}(k), \boldsymbol{u}_{k-1}^{\mathrm{T}}(k)]^{\mathrm{T}} \in \Omega_z \subset R^{n+m+\tau-1} \end{aligned}$$

From equations (1) and (3), we obtain

$$\begin{aligned} x_{n-\tau+2}(k) = y_{k+1} &= f_o(\boldsymbol{y}^{\mathrm{T}}(k), \upsilon_{k-1}^{\mathrm{T}}(k-\tau+1)) \\ &\quad + g_o(\boldsymbol{y}^{\mathrm{T}}(k), \upsilon_{k-1}^{\mathrm{T}}(k-\tau+1)) \\ &\quad u_{k-\tau+1} + d_k \end{aligned} \tag{22}$$

It means that $x_{n-\tau+2}(k)$ is a function of $\boldsymbol{y}(k)$, $u_{k-\tau+1}, \ldots, u_{k-\tau-m+1}$ and d_k. Though the right-hand side of (22) does not contain all the elements of $\boldsymbol{z}(k)$ and $\boldsymbol{d}(k)$, for uniformity of presentation, define

$$x_{n-\tau+2}(k) = y_{k+1} = F_1(\boldsymbol{z}(k), \boldsymbol{d}(k)) \tag{23}$$

Accordingly, we obtain the following equation from (1) and (3)

$$\begin{aligned} x_{n-\tau+3}(k) = y_{k+2} &= f_o(\boldsymbol{y}^{\mathrm{T}}(k+1), \upsilon_{k-1}^{\mathrm{T}}(k-\tau+2)) \\ &\quad + g_o(\boldsymbol{y}^{\mathrm{T}}(k+1), \upsilon_{k-1}^{\mathrm{T}}(k-\tau+2)) \\ &\quad \times u_{k-\tau+2} + d_{k+1} \end{aligned} \tag{24}$$

Substituting (23) into (24), $x_{n-\tau+3}(k)$ can be represented as a function of $\boldsymbol{y}(k), u_{k-\tau+2}, \ldots, u_{k-\tau-m+1}, d_k$ and d_{k+1}. Based on Definition 2, y_{k+2} is a SDFO. Define

$$x_{n-\tau+3}(k) = y_{k+2} = F_2(\boldsymbol{z}(k), \boldsymbol{d}(k)) \tag{25}$$

If we continue the iteration recursively, it is easy to prove that $x_n(k)$ is a function of $\boldsymbol{z}(k)$ and $\boldsymbol{d}(k)$ described as

$$x_n(k) = y_{k+\tau-1} = F_{\tau-1}(\boldsymbol{z}(k), \boldsymbol{d}(k)) \tag{26}$$

where $F_{\tau-1}(*)$ contains the full elements of $\boldsymbol{z}(k)$ and $\boldsymbol{d}(k)$. It is clear that $y_{k+\tau-1}$ is also a SDFO because all the elements of $\boldsymbol{z}(k)$ are available at time instant k. By the mean value theorem (Apostol 1974), equation (26) can be further written as

$$x_n(k) = F_{\tau-1}(\boldsymbol{z}(k), 0) + \bar{d}(k) \tag{27}$$

where

$$\bar{d}(k) = \left(\frac{\partial F_{\tau-1}}{\partial \boldsymbol{d}} \bigg|_{\boldsymbol{d}(k)=\xi(k)} \right)^{\mathrm{T}} \boldsymbol{d}(k), \qquad \xi(k) \in L(0, \boldsymbol{d}(k))$$

and $L(0, \boldsymbol{d}(k))$ is a line defined by

$$L(0, \boldsymbol{d}(k)) = \{\xi(k) \mid \xi(k) = \theta \boldsymbol{d}(k) \quad 0 < \theta < 1\}$$

We may assume that $\bar{d}(k)$ is bounded by a known constant $\bar{d}_0$ with $|\bar{d}(k)| < \bar{d}_0$ because $F_{\tau-1}(*)$ is a smooth function, $\boldsymbol{d}(k)$ is bounded and $\partial F/\partial d$ is evaluated on a finite line L.

Substituting the predicted future outputs from (23) to (26) for the predictable future outputs in equation (2), we obtain

$$y_{k+\tau} = F_\tau(\boldsymbol{z}(k), \boldsymbol{d}(k)) + G_\tau(\boldsymbol{z}(k), \boldsymbol{d}(k))u_k + d_{k+\tau-1}$$

where

$$\begin{aligned} &F_\tau(\boldsymbol{z}(k), \boldsymbol{d}(k)) \\ &\quad = f_0(F_{\tau-1}(*), \ldots, F_1(*), y_k, \ldots, y_{k+\tau-n}, u_{k-1}, \ldots, u_{k-m}) \\ &G_\tau(\boldsymbol{z}(k), \boldsymbol{d}(k)) \\ &\quad = g_0(F_{\tau-1}(*), \ldots, F_1(*), y_k, \ldots, y_{k+\tau-n}, u_{k-1}, \ldots, u_{k-m}) \end{aligned}$$

The above equation can be express as Taylor's formula by mean value theorem as

$$y_{k+\tau} = f_\tau(\boldsymbol{z}(k)) + \Delta_1(k) + [g_\tau(\boldsymbol{z}(k)) + \Delta_2(k)]u_k + d_{k+\tau-1}$$

where

$$f_\tau(\boldsymbol{z}(k)) = F_\tau(\boldsymbol{z}(k), 0), \qquad g_\tau(\boldsymbol{z}(k)) = G_\tau(\boldsymbol{z}(k), 0)$$

$$\Delta_1(k) = \left(\frac{\partial F_\tau}{\partial \boldsymbol{d}} \bigg|_{\boldsymbol{d}(k)=\xi_1(k)} \right)^{\mathrm{T}} \boldsymbol{d}(k), \qquad \xi_1(k) \in L(0, \boldsymbol{d}(k))$$

$$\Delta_2(k) = \left(\frac{\partial G_\tau}{\partial \boldsymbol{d}} \bigg|_{\boldsymbol{d}(k)=\xi_2(k)} \right)^{\mathrm{T}} \boldsymbol{d}(k), \qquad \xi_2(k) \in L(0, \boldsymbol{d}(k))$$

and $L(0, \boldsymbol{d}(k))$ is a line defined by

$$L(0, \boldsymbol{d}(k)) = \{\xi(k) \mid \xi(k) = \theta \boldsymbol{d}(k), \, 0 < \theta < 1\}$$

We can assume that $\Delta_1(k)$ and $\Delta_2(k)$ are bounded by known constants Δ_{10}, Δ_{20} with $|\Delta_1(k)| < \Delta_{10}$, $|\Delta_2(k)| < \Delta_{20}$ because $F_\tau(*)$ and $G_\tau(*)$ are smooth, and d_k is bounded.

Define the tracking error as $e_y(k) = y_k - y_d(k)$. The tracking error dynamics are

$$\begin{aligned} e_y(k+\tau) = &-y_d(k+\tau) + f_\tau(\boldsymbol{z}(k)) + \Delta_1(k) \\ &+ [g_\tau(\boldsymbol{z}(k)) + \Delta_2(k)]u_k + d_{k+\tau-1} \end{aligned} \tag{28}$$

Supposing that the non-linear functions $f_\tau(\boldsymbol{z}(k))$ and $g_\tau(\boldsymbol{z}(k))$ are known exactly, and there is no disturbance, i.e. $\boldsymbol{d}(k) = 0$, we present a desired deadbeat control, $\bar{u}_k^*$, such that the output y_k follows the desired trajectory $y_d(k)$ in deadbeat steps

$$\bar{u}_k^* = \frac{1}{g_\tau(\boldsymbol{z}(k))}(y_d(k+\tau) - f_\tau(\boldsymbol{z}(k))) \tag{29}$$

Substituting the desired control $\bar{u}_k^*$ into error equation (28), and noting that $\boldsymbol{d}(k) = 0$, we obtain

$$e_y(k+\tau) = 0$$

This means that after τ steps $e_y(k) = 0$, therefore $\bar{u}_k^*$ is a τ steps deadbeat control. Under Assumption 3, we can conclude that $\bar{u}_k^* \in \Omega_f$.

Accordingly, the desired control $\bar{u}_k^*$ can be expressed as

$$\left.\begin{aligned} &\bar{u}_k^* = \bar{u}^*(\bar{z}(k)) \\ &\bar{z}(k) = [\boldsymbol{z}^{\mathrm{T}}(k), y_d(k+\tau)]^{\mathrm{T}} \in \Omega_{\bar{z}} \subset R^{n+m+\tau} \end{aligned}\right\} \tag{30}$$

where the compact set $\Omega_{\bar{z}}$ is defined as

$$\begin{aligned} \Omega_{\bar{z}} = \{&(\boldsymbol{y}(k), \boldsymbol{u}_{k-1}(k), y_d(k)) | \boldsymbol{u}_{k-1}(k) \\ &= [u_{k-1}, \ldots, u_{k-\tau-m+1}]^{\mathrm{T}}; \\ &\boldsymbol{y}(k) = [y_k, \ldots, y_{k-n+1}]^{\mathrm{T}}, y \in \Omega_y, y_d \in \Omega_y, u \in \Omega_u\} \end{aligned}$$

As mentioned in §2.3, there exist an integer l_1^* and an ideal constant weight vector W_1^*, such that, for all $l_1 \geq l_1^*$

$$\bar{u}^*(\bar{z}) = W_1^{*\mathrm{T}} S(\bar{z}) + \mu_{\bar{z}}, \qquad \forall \bar{z} \in \Omega_{\bar{z}} \tag{31}$$

where $\mu_{\bar{z}}$ is the NN estimation error satisfying $|\mu_{\bar{z}}| < \mu_1$.

If we choose the control as

$$u_k = \hat{W}_1(k) S(\bar{z}(k)) \tag{32}$$

and the weight updating law as

$$\begin{aligned} \hat{W}_1(k+1) = &\hat{W}_1(k_1) - \Gamma_1[S(\bar{z}(k_1)) \\ &\times (y_{k+1} - y_d(k+1)) + \sigma_1 \hat{W}_1(k_1)] \end{aligned} \tag{33}$$

where adaptation diagonal gain matrix $\Gamma_1 = \Gamma_1^{\mathrm{T}} > 0$, $\sigma_1 > 0$ and $k_1 = k - \tau + 1$.

Since Ω_{yd} is a small subset of Ω_y, there must exist a large enough compact set $\Omega_e \subset R$, such that for any $e_y(k) \in \Omega_e$ guarantees that $y_k \in \Omega_y$.

Control input (32) can be rewritten as

$$u_k = (\tilde{W}_1(k) + W_1^*)S(\bar{z}(k)) = u_k^* + \tilde{W}_1(k)S(\bar{z}(k))$$

Since $u_k^* \in \Omega_f \subset \Omega_u$, there must exist a non-zero compact set $\Omega_{w_1} \subset R^{l_1}$ such that for any $\tilde{W}_1(k) \in \Omega_{w_1}$, $u_k \in \Omega_u$ is guaranteed.

In the following theorem, we show that for appropriate initial conditions and suitably chosen design parameters, adaptive controller (32) and adaptation law (33) guarantee $y_k \in \Omega_y$, $u_k \in \Omega_u$ $\forall k \geq 0$.

Theorem 2: *Consider the closed-loop system consisting of system (1), controller (32) and adaptation law (33). There exist compact sets $\Omega_{y0} \subset \Omega_y$, $\Omega_{w_{10}} \subset \Omega_{w_1}$ and positive constants $l_1^*, \bar{\gamma}_1^*$, and σ_1^* satisfying*

and
$$\left.\begin{aligned} \bar{\gamma}_1^* &= \frac{1}{1 + l_1^* + (g_1 + \Delta_{20})l_1^*} \\ \sigma_1^* &= \frac{1}{(1 + g_1 l_1^* + \Delta_{20} l_1^*)\bar{\gamma}_1^*} \end{aligned}\right\} \quad (34)$$

such that if

(i) *Assumptions 1–3 are being satisfied, the initial condition at time instant k_0 is*

$$\begin{aligned} y_{k_0-i} &\in \Omega_{y0}, \qquad i = 0, \ldots, n \\ u_{k_0-i} &\in \Omega_u, \qquad i = 1, \ldots, \tau + m \\ \tilde{W}_1(k_0 - i) &\in \Omega_{w_{10}}, \qquad i = 1, \ldots, \tau \end{aligned}$$

(ii) *the predictable future outputs at time instant k_0, $y_{k_0+1}, \ldots, y_{k_0+\tau-1}$ are all in compact set Ω_y, and*

(iii) *the design parameters are suitably chosen such that $l_1 > l_1^*, \sigma_1 < \sigma_1^*$ and $\bar{\gamma}_1 < \bar{\gamma}_1^*$ with $\bar{\gamma}_1$ being the largest eigenvalue of diagonal gain matrix Γ_1,*

then, the closed-loop system is SGUUB. The tracking error can be made arbitrarily small by increasing the approximation accuracy of the neural networks.

Proof: See Appendix A.2. □

5. HONN observer-based adaptive control

When states are not available for control, observer design and observer-based control are frequently being investigated as well. For completeness, they are also investigated here. Suppose that only output $y_k = x_{n-\tau+1}(k)$ is measurable and the rest of the $(\tau - 1)$ states are not available for feedback, we need to reconstruct the states $x_{n-\tau+2}(k), \ldots, x_n(k)$ in order to implement the state feedback adaptive NN controller presented in § 3.2.

Lemma 3: *For bounded system output y and control input u, consider the NN observer described by*

$$\left.\begin{aligned} \hat{x}_i(k) &= \hat{x}_{i+1}(k-1) \qquad i = 1, 2, \ldots, n-1 \\ \hat{x}_n(k) &= \hat{W}_o(k)S(\mathbf{z}(k)) \\ \hat{W}_o(k) &= \hat{W}_o(k_1) - \Gamma_o S(\mathbf{z}(k_1))(\hat{x}_{n-\tau+1}(k) - y_k) \\ k_1 &= k - \tau + 1 \end{aligned}\right\} \quad (35)$$

where $S(\mathbf{z}(k))$ defined by (9) is PE, $\mathbf{z}(k) \in \Omega_z \subset R^{n+m+\tau-1}$, and adaptive positive gain matrix $\Gamma_o = \Gamma_o^{\mathrm{T}} > 0$ is diagonal. There exist a positive number k^* and a small constant ϵ such that, for all $k > k^*$, the following equations hold

$$\hat{x}_i(k) - x_i(k) \leq \epsilon \qquad i = 1, 2, \ldots, n \quad (36)$$

Proof: According to the function approximation property of a high-order NN, for any arbitrary constant $\mu_{o0} > 0$, there exist an integer l_o^* and an ideal constant vector W_o^*, such that for all $l_o > l_o^*$, equation (27) can be expressed as

$$x_n(k) = W_o^{*\mathrm{T}} S(\mathbf{z}(k)) + \mu_z + \bar{d}(k), \qquad \forall \mathbf{z} \in \Omega_z$$

where l_o denotes the NN node numbers, and μ_z is called the NN approximation error satisfying $|\mu_z| \leq \mu_{o0}$. Equation (5) can then be rewritten as

$$\left.\begin{aligned} x_i(k) &= x_{i+1}(k-1) \qquad i = 1, 2, \ldots, n-1 \\ x_n(k) &= W_o^{*\mathrm{T}} S(\mathbf{z}(k)) + \mu_z + \bar{d}(k) \end{aligned}\right\} \quad (37)$$

Subtracting equation (37) from equation (35) and subtracting $W_o^*(k)$ on both sides of the last equation in (35), the state space error description of the observer is

$$\left.\begin{aligned} \tilde{x}_i(k) &= \tilde{x}_{i+1}(k-1) \qquad i = 1, 2, \ldots, n-1 \\ \tilde{x}_n(k) &= \tilde{W}_o^{\mathrm{T}}(k)S(\mathbf{z}(k)) - \mu_z - \bar{d}(k) \\ \tilde{W}_o(k) &= \tilde{W}_o(k_1) - \Gamma_o S(\mathbf{z}(k_1))\tilde{x}_{n-\tau+1}(k) \\ k_1 &= k - \tau + 1 \end{aligned}\right\} \quad (38)$$

where $\tilde{x}_i = \hat{x}_i - x_i$, $i = 1, 2, \ldots, n$ are the state estimation errors, and $\tilde{W}_o(k) = \hat{W}_o(k) - W_o^*$ is the NN weight estimation error.

Choose the following Lyapunov function candidate

$$V(k) = \sum_{i=1}^{n} \tilde{x}_i^2(k) + \sum_{j=1}^{\tau-1} \tilde{W}_o^{\mathrm{T}}(k+j)\Gamma_o^{-1}\tilde{W}_o(k+j) \quad (39)$$

The first difference of (39) along (38) is given by

$$\begin{aligned}\Delta V &= V(k) - V(k-1)\\ &= \tilde{x}_n^2(k) - \tilde{x}_1^2(k-1)\\ &\quad + \tilde{W}_o^{\mathrm{T}}(k+\tau-1)\Gamma_o^{-1}\tilde{W}_o(k+\tau-1)\\ &\quad - \tilde{W}_o^{\mathrm{T}}(k)\Gamma_o^{-1}\tilde{W}_o(k)\\ &= \tilde{x}_n^2(k) - \tilde{x}_1^2(k-1) - 2\tilde{W}_o^{\mathrm{T}}(k)S(z(k))\tilde{x}_n(k)\\ &\quad + S^{\mathrm{T}}(z(k))\Gamma_o S(z(k))\tilde{x}_n^2(k)\\ &= -\tilde{x}_n^2(k) - \tilde{x}_1^2(k-1) - 2(\mu_z + \bar{d}(k))\tilde{x}_n(k)\\ &\quad + S^{\mathrm{T}}(z(k))\Gamma_o S(z(k))\tilde{x}_n^2(k)\end{aligned}$$

Using the fact that (see Appendix A.4 for more details)

$$2(\mu_z + \bar{d}(k))\tilde{x}_n(k) \le \bar{\gamma}_o\tilde{x}_n^2(k) + \frac{1}{\bar{\gamma}_o}(\mu_z + \bar{d}(k))^2$$

$$S^{\mathrm{T}}(z(k))\Gamma_o S(z(k)) \le \bar{\gamma}_o l_o$$

where $\bar{\gamma}_o$ is the largest eigenvalue of Γ_o, we obtain

$$\Delta V \le -\rho_o\tilde{x}_n^2(k) - \tilde{x}_1^2(k-1) + \beta_o$$

where

$$\rho_o = 1 - \bar{\gamma}_o - l_o\bar{\gamma}_o \quad \text{and} \quad \beta_o = \frac{(\mu_{o0} + \bar{d}_0)^2}{\bar{\gamma}_o}$$

If we choose the design parameter as

$$\bar{\gamma}_o < \frac{1}{l_o + 1} \tag{40}$$

then $\Delta V \le 0$ once the estimation error $\tilde{x}_1(k-1)$ is larger than $\sqrt{\beta_o}$. This means that the estimation error $\tilde{x}_1(k)$ is bounded for all $k \ge 0$, and $\tilde{x}_1(k)$ will asymptotically converge into the compact set denoted by $\epsilon < \sqrt{\beta_o}$. Namely, there must exist a positive number k_o^*, for all $k > k_o^*$, such that $|\tilde{x}_1(k)| = |\hat{x}_1(k) - x_1(k)| \le \epsilon$. Since $\tilde{x}_1(k+n-1) = \tilde{x}_2(k+n-2) = \cdots = \tilde{x}_n(k)$, it is easy to show that $\tilde{x}_i(k), i = 1, 2, \ldots, n$ are bounded, and

$$|\tilde{x}_i(k)| = |\hat{x}_i(k) - x_i(k)| \le \epsilon,$$
$$i = 1, 2, \ldots, n \quad \text{for all } k > k_o^* + n - 1$$

Next, let us prove the boundedness of $\tilde{W}_o(k)$, or equivalently, $\hat{W}_o(k)$. From the state estimation error dynamic equation, we obtain

$$\begin{aligned}\tilde{x}_n^2(k) &= [\tilde{W}_o^{\mathrm{T}}(k)S(z(k)) - \mu_z - \bar{d}(k)]^{\mathrm{T}}\\ &\quad \times [\tilde{W}_o^{\mathrm{T}}(k)S(z(k)) - \mu_z - \bar{d}(k)]\\ &= \tilde{W}_o^{\mathrm{T}}(k)S(z(k))S^{\mathrm{T}}(z(k))\tilde{W}_o(k)\\ &\quad - 2(\mu_z + \bar{d}(k))\tilde{x}_n(k) - (\mu_z + \bar{d}(k))^2\end{aligned} \tag{41}$$

Since $\tilde{x}_n(k)$ and $\mu_z, \bar{d}(k)$ are all bounded, $\tilde{W}_o(k)$ is also bounded as long as $S(z(k))$ satisfies PE condition (7). □

For convenience, define the variables

$$\hat{e}_n(k) = \hat{x}_n(k) - y_d(k+\tau-1)$$

$$\hat{z}(k) = [x_1(k), \ldots, x_{n-\tau+1}(k), \hat{x}_{n-\tau+2}(k), \ldots, \hat{x}_n(k), y_d(k+\tau), \upsilon_{k-1}^{\mathrm{T}}(k), k_v\hat{e}_n(k)]^{\mathrm{T}} \tag{42}$$

The adaptive NN controller based on observer (35) is given by

$$u_k = \hat{W}^{\mathrm{T}}S(\hat{z}(k)) \tag{43}$$

$$\hat{W}(k+1) = \hat{W}(k) - \Gamma[S(\hat{z}(k))\hat{e}_n(k+1) + \sigma\hat{W}(k)] \tag{44}$$

where adaptive gain matrix $\Gamma = \Gamma^{\mathrm{T}} > 0$ and constant $\sigma > 0$. In adaptive law (44), σ-modification is used to eliminate the requirement of PE condition for the boundedness of $\|\hat{W}(k)\|$, while PE condition is needed in the NN observer (38) in order to guarantee the boundedness of $\|\hat{W}_o(k)\|$.

Substituting controller (43) into (4), the error equation (4) can be re-written as

$$\begin{aligned}e_n(k+1) &= f(k) - y_d(k+\tau)\\ &\quad + g(k)\hat{W}^{\mathrm{T}}(k)S(\hat{z}(k)) + d_{k+\tau-1}\end{aligned} \tag{45}$$

Adding and subtracting $g(k)u^*(z(k))$ on the right-hand side of (45) and noting (11), we have

$$\begin{aligned}e_n(k+1) &= f(k) - y_d(k+\tau) + g(k)[\hat{W}^{\mathrm{T}}(k)S(\hat{z}(k))\\ &\quad - W^{*\mathrm{T}}S(z(k)) - \mu_z]\\ &\quad + g(k)u^*(z(k)) + d_{k+\tau-1}\end{aligned} \tag{46}$$

Substituting (12) into (46) leads to

$$\begin{aligned}e_n(k+1) &= g(k)\left[\hat{W}^{\mathrm{T}}(k)S(\hat{z}(k)) - W^{*\mathrm{T}}(k)S(z(k)) - \mu_z + \frac{k_v}{g_1}e_n(k)\right] + d_{k+\tau-1}\\ &= g(k)\left[\tilde{W}^{\mathrm{T}}(k)S(\hat{z}(k)) + 2W^{*T}(k)S_t(k) - \mu_z + \frac{k_v}{g_1}e_n(k)\right] + d_{k+\tau-1}\end{aligned} \tag{47}$$

where $\tilde{W}(k) = \hat{W}(k) - W^*$ and

$$S_t(k) = \tfrac{1}{2}[S(\hat{z}(k)) - S(z(k))]$$

The basis functions of HONN have the following properties $\lambda_{\max}[S_t(k)S_t^{\mathrm{T}}(k)] < 1$, $S_t^{\mathrm{T}}(k)S_t(k) < l$.

Since NN approximation (11), and Assumptions 1–3 are only valid on the compact sets Ω and Ω_u, it is necessary to guarantee that the system's states remain in Ω and the control signal remains in Ω_u for all time.

Due to $u_k^* \in \Omega_f \subset \Omega_u$, there must exist two nonzero compact sets $\Omega_w \subset R^l$ and $\Omega_s \subset R^l$ such that for all $\tilde{W}(k) \in \Omega_w$ and $S_t(k) \in \Omega_s$ guarantee $u_k \in \Omega_u$.

In the following theorem, we show that for appropriate initial conditions $x(0)$, $\tilde{W}(0)$, and suitably chosen design parameters, adaptive controller (43) and adaptive law (44) guarantees $x(k) \in \Omega, u_k \in \Omega_u \, \forall k \geq 0$.

Theorem 3: *Consider the closed-loop system consisting of system* (3)*, observer* (35) *with its PE requirement satisfied, controller* (43) *and adaptation law* (44)*. There exist compact sets* $\Omega_0 \subset \Omega$, $\Omega_{w_0} \subset \Omega_w$ *and positive constants* $\alpha_1, \alpha_2, k_v^*, l^*, l_o^*, \gamma^*, \sigma^*$ *and* γ_o^* *satisfying*

$$\left.\begin{aligned} &\alpha_1 < 1 \qquad\qquad \alpha_2 < 1 \\ &\gamma^* = \frac{1-\alpha_1-3k_v^*}{4+2g_1 l^*} \quad \gamma_o^* = \frac{1-\alpha_2-k_v^*-2g_0\gamma^* l^* - \gamma^*}{4+l_o^*} \\ &k_v^* = \frac{1-\alpha_1}{3} \\ &\sigma^* = \frac{1}{\gamma^*+g_1\gamma^* l^* + g_0\gamma^* l^*} \end{aligned}\right\} \tag{48}$$

such that if

(i) *Assumptions 1–3 are satisfied, the initial condition* $x(0) \in \Omega_0$, $\tilde{W}(0) \in \Omega_{w_0}$, *and*

(ii) *the design parameters are suitably chosen such that* $k_v < k_v^*$, $l > l^*$, $l_o > l_o^*$, $\sigma < \sigma^*$, $\gamma < \gamma^*$, *and* $\gamma_o < \gamma_o^*$, *with* γ *being the largest eigenvalue of diagonal gain matrix* Γ, *and* γ_o *being the largest eigenvalue of* Γ_o,

then, the closed-loop system is SGUUB. The tracking error can be made arbitrarily small by increasing the approximation accuracy of the neural networks.

Proof: Proof can be found in Appendix A.3. □

Remark 2: From the parameter conditions in equations (21), (34) and (48), it is clear that the adaptation gains γ^* and $\bar{\gamma}_1^*$ decreases with an increase in the number of HONN nodes l^* and l_1^* respectively, therefore the learning rate must slow down for guaranteed performance. This theoretically explains the phenomenon of large NN requiring very slow learning rates which has often been encountered in NN control literature but not adequately explained.

6. Simulation study

To show the effectiveness of the proposed control schemes, simulation studies are presented in this section. Consider a non-linear discrete-time SISO plant described by

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= f(x(k), u_{k-1}) + g(x(k))u_k + d_{k+\tau-1} \\ y_k &= x_2(k) \end{aligned}$$

where

$$f(x(k), u_{k-1}) = \frac{x_1(k)x_2(k)(x_1(k)+2.5)+u_{k-1}}{1+x_1^2(k)+x_2^2(k)+x_3^2(k)}$$

$$g(x(k)) = \frac{2.5}{1+x_1^2(k)+x_2^2(k)+x_3^2(k)}$$

$$d_k = 0.1\cos(0.05k)\cos(x_1(k))$$

It can be checked that Assumptions 1 and 2 are satisfied. The tracking objective is to make the output y_k follow a desired reference signal

$$y_d(k) = \frac{1}{2}\sin\left(\frac{k\pi}{20}\right) + \frac{1}{2}\sin\left(\frac{k\pi}{10}\right)$$

For illustration, the robust model based control scheme presented in Adetona *et al.* (2000) is first studied. Then state feedback, output feedback and observer-based NN controls are investigated respectively under the assumption that there is no *a priori* knowledge of the system non-linearities.

Robust model based control: The input–output model of the plant can be expressed as

$$y(k+\tau) = f_c(y(k+1), y(k), y(k-1), u(k), u(k-1))$$

where $\tau = 2$, and $f_c(k) =: f_c(y(k+1), \; y(k), \; y(k-1), \; u(k), \; u(k-1))$. According to (Adetona *et al.* 2000), the robust model based controller is given as

$$u(k) = \begin{cases} \hat{u}(k), & \text{if } |\hat{u}(k) - u(k-1)| \leq \epsilon(k) \\ u(k-1) + \epsilon(k)\mathrm{sgn}[\hat{u}(k) - u(k-1)], & \text{if } |\hat{u}(k) - u(k-1)| > \epsilon(k) \end{cases} \tag{49}$$

where

$$\left.\begin{aligned} \hat{u}(k) &= \frac{y_d(k+\tau) - y(k+\tau-1) + \dfrac{\partial f_c(k-1)}{\partial u(k-1)} \cdot u(k-1)}{\dfrac{\partial f_c(k-1)}{\partial u(k-1)}} \\ \frac{\partial f_c(k-1)}{\partial u(k-1)} &= \frac{2.5}{1+x_1^2(k-1)+x_2^2(k-1)+x_3^2(k-1)} \end{aligned}\right\} \tag{50}$$

It is clear that in the robust model based control, the model of the system plays an important role. This is exactly the problem that we are trying to solve using NN as it is very hard or impossible to obtain an accurate model in practice. The simulation parameter is chosen as follows: $\epsilon(k) = 0.1$.

State feedback NN control: For state feedback control described by (16) and (17), the following parameters are chosen: the number of neurons $l = 30$, $\hat{W}(0) = 0$, $\Gamma = 0.04I$, $k_v = 0.1$ and $\sigma = 0.005$.

Direct NN output feedback control: For the direct NN output feedback control described by (32) and (33), the parameters are chosen as follows: the number of neurons $l_1 = 30$, $\hat{W}_1(0) = 0$, $\Gamma_1 = 0.05I$ and $\sigma_1 = 0.004$.

NN observer-based NN control: For the observer-based control described by (43) and (44), the parameters are chosen as: the number of neurons $l = 30$, $\hat{W}(0) = 0$, $\Gamma = 0.04I$, $k_v = 0.1$, while for the neural network observer described by (35), we choose the number of neurons $l_o = 23$ and $\Gamma_o = 0.1I$.

Figure 1 shows the tracking performances of the four controllers. From figure 1, we find that though all the controllers are effective, the performance of direct NN output feedback control is better than that of the observer-based NN control at the beginning. This is because that, at the beginning of control process, a large estimation error exists in observer estimation, which leads to the relatively larger tracking error at the beginning. Figure 2 shows the boundedness of the control signals of the four control schemes. In figure 3, we can see that the estimation error $\tilde{x}_3(k)$ converges to a small neighbourhood of 0. Other estimation errors are not plotted because of $\tilde{x}_3(k) = \tilde{x}_2(k+1) = \tilde{x}_1(k+2)$ by definition. Figure 4 substantiates the boundedness of the HONN weights of the three NN controllers and the NN observer.

Remark 3: According to the simulation results under the current setting, it can be seen that the performance of the robust model based controller is better than that of proposed NN controllers though they all can control the system satisfactorily. This is understandable as the robust model based control required partial knowledge of the system, the proposed NN controllers know nothing about the systems and the initial NN weights are assumed to be zero. In fact, it has been proven that model based control is superior to non-model based control. However, if the model is not known, then the proposed controller is a good choice for practical engineers to try. However, it should be noted that the performance of the proposed output feedback adaptive NN controller can be improved by increasing the neural number used in HONN as clearly indicated in figure 5 using neural networks of different sizes with $l_1 = 6, 10, 15$ and 30. It is clear that the tracking error becomes smaller as the size of the neural network becomes larger. Of course, the improvement in tracking performance will be saturated when the size of the neural network becomes even larger. The current transient performance would be further improved if the weights of the

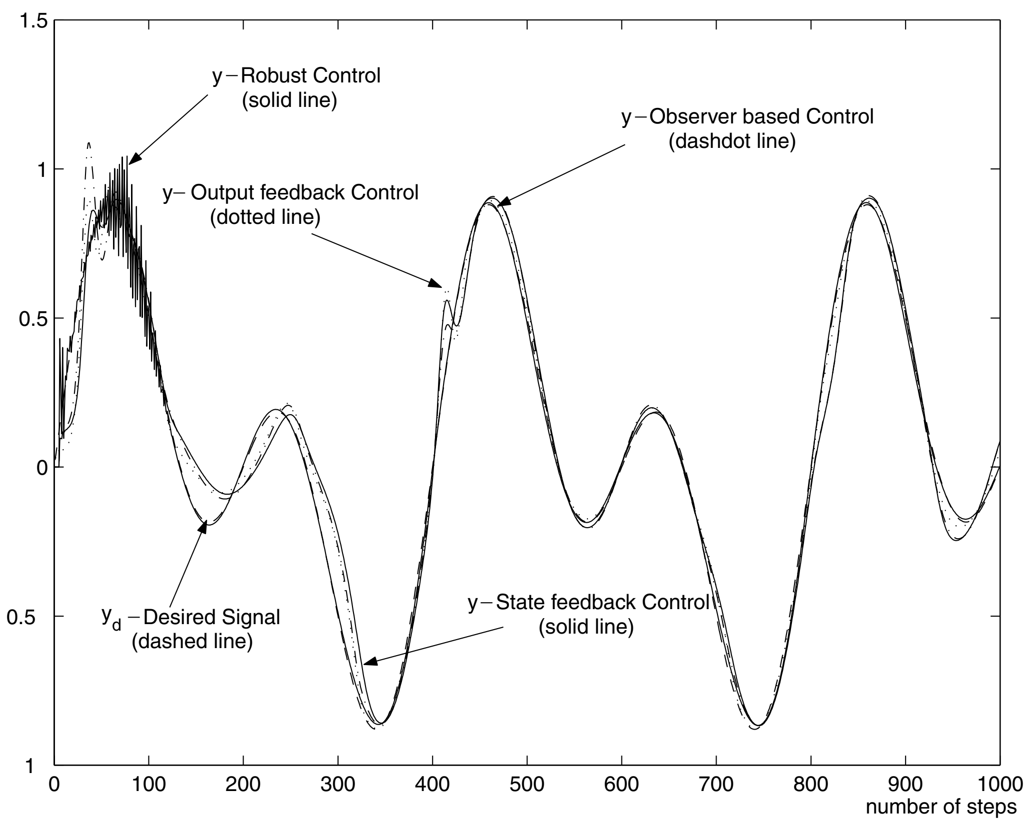

Figure 1. Tracking performance y_d, y.

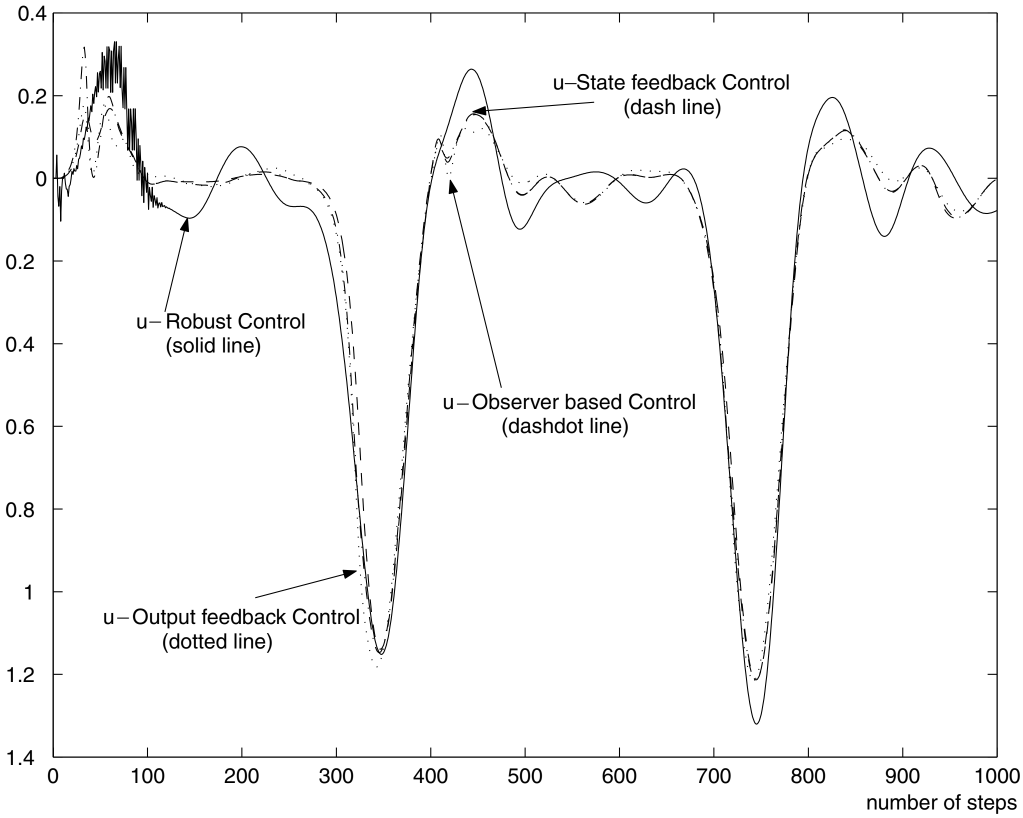

Figure 2. Control signal variations u.

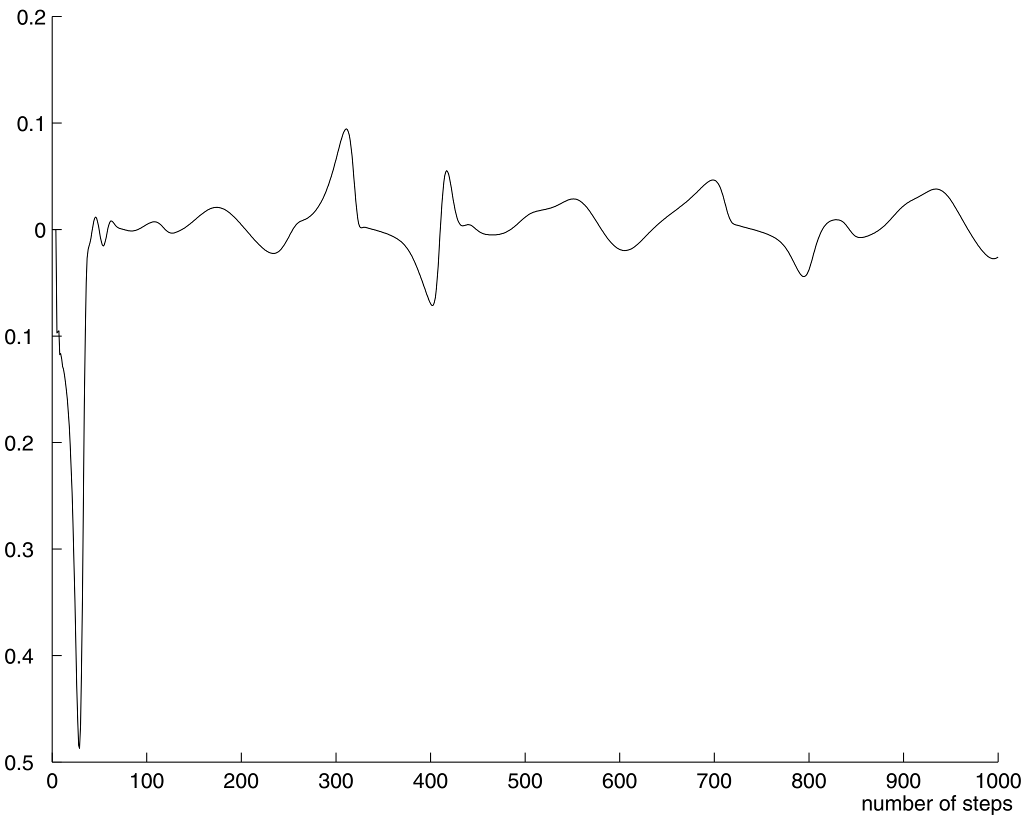

Figure 3. Variation of state estimation $\tilde{x}_3(k)$.

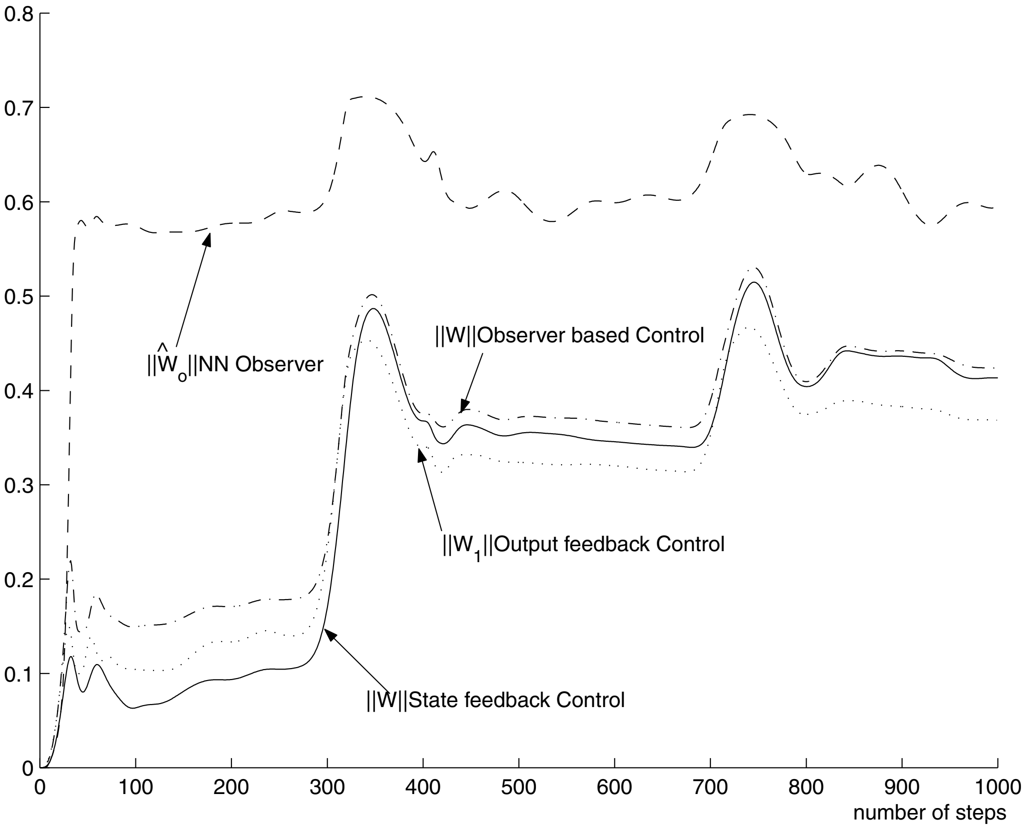

Figure 4. Variations of $\|\hat{W}\|$, $\|\hat{W}_1\|$ and $\|\hat{W}_o\|$.

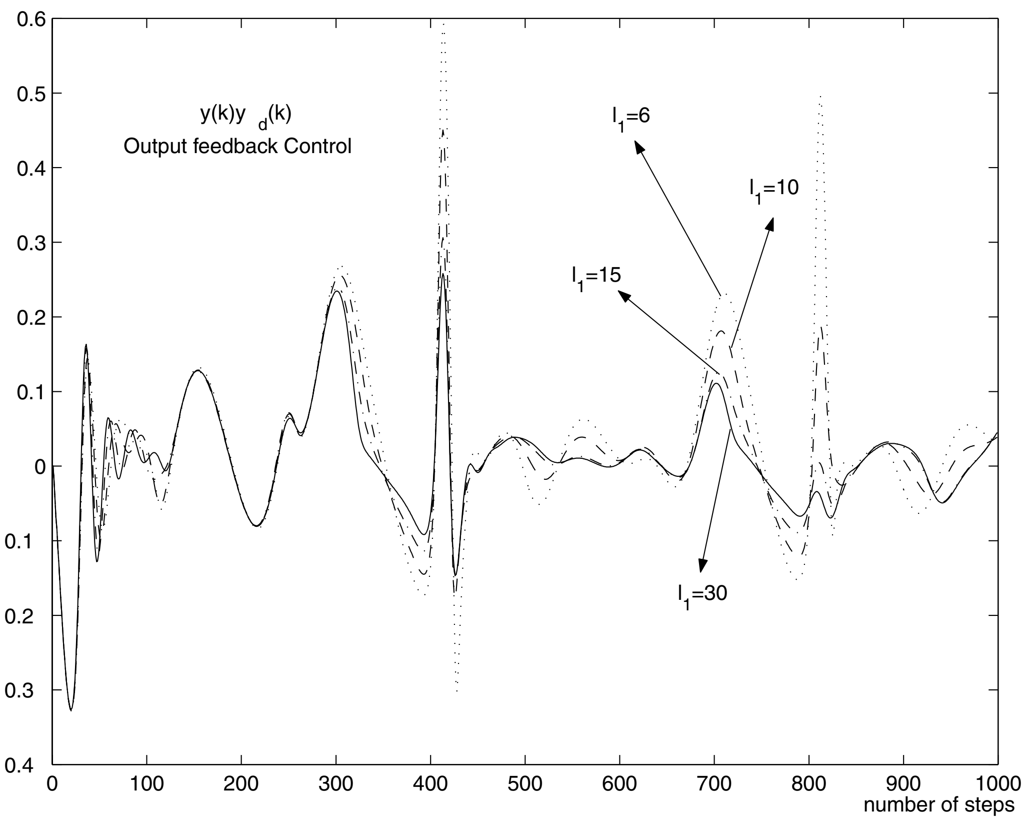

Figure 5. Tracking error variations using NN of different size.

neural networks are set to the values of last simulation run—a learning process in gaining knowledge about the system.

7. Conclusion

In this paper, adaptive neural network control has been investigated for a class of SISO discrete-time unknown non-linear systems with general relative degree in the presence of bounded disturbances. After the systems had been transformed into both causal state-space and input–output descriptions, state and output feedback adaptive NN controls were then presented based on Lyapunov stability analysis. Finally, adaptive NN observer design and observer-based adaptive control were investigated under the assumption of PE condition. All the control schemes avoid the so-called controller singularity problem in adaptive control. By suitably choosing the design parameters, the closed-loop systems are proven to be SGUUB. Simulation studies showed the effectiveness of the newly proposed schemes.

Appendix

A.1. *Proof of Theorem 1*

Under the new controller (16), the tracking error equation becomes (see equation (20))

$$e_n(k+1) = g(k)\left[\tilde{W}^{\mathrm{T}}(k)S(z(k)) - \mu_z + \frac{k_v}{g_1}e_n(k)\right] + d_{k+\tau-1}$$

or equivalently

$$\tilde{W}^{\mathrm{T}}(k)S(z(k)) = \frac{e_n(k+1)}{g(k)} - \frac{d_{k+\tau-1}}{g(k)} + \mu_z - \frac{k_v}{g_1}e_n(k)$$

Suppose that $x(k) \in \Omega$, $u(k-j) \in \Omega_u$, $j = 1, 2, \ldots, n-\tau\ \forall k \geq 0$, then NN approximation (11), Assumption 1–3 are valid.

Choose the following Lyapunov function candidate

$$V(k) = \frac{\alpha_1}{g_1}\sum_{i=1}^{n} e_i^2(k) + \frac{2k_v}{g_1}e_n^2(k) + \tilde{W}^{\mathrm{T}}(k)\Gamma^{-1}\tilde{W}(k) \tag{51}$$

where α_1 is a small positive constant which is less than 1. The first difference of (51) along (4), (16), (17) and (20) is given

$$\begin{aligned}
\Delta V(k) &= \frac{\alpha_1 + 2k_v}{g_1}e_n^2(k+1) - \frac{\alpha_1}{g_1}e_1^2(k) - \frac{2k_v}{g_1}e_n^2(k) \\
&\quad + \tilde{W}^{\mathrm{T}}(k+1)\Gamma^{-1}\tilde{W}(k+1) - \tilde{W}^{\mathrm{T}}(k)\Gamma^{-1}\tilde{W}(k) \\
&= \frac{\alpha_1 + 2k_v}{g_1}e_n^2(k+1) - \frac{\alpha_1}{g_1}e_1^2(k) - \frac{2k_v}{g_1}e_n^2(k) \\
&\quad - 2\tilde{W}^{\mathrm{T}}(k)S(z(k))e_n(k+1) - 2\sigma\tilde{W}^{\mathrm{T}}(k)\hat{W}(k) \\
&\quad + S^{\mathrm{T}}(z(k))\Gamma S(z(k))e_n^2(k+1) \\
&\quad + 2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(z(k))e_n(k+1) \\
&\quad + \sigma^2\hat{W}^{\mathrm{T}}(k)\Gamma\hat{W}(k) \\
&= \frac{\alpha_1 + 2k_v}{g_1}e_n^2(k+1) - \frac{\alpha_1}{g_1}e_1^2(k) - \frac{2k_v}{g_1}e_n^2(k) \\
&\quad - 2\left[\frac{e_n(k+1)}{g(k)} - \frac{d_{k+\tau-1}}{g(k)} + \mu_z - \frac{k_v}{g_1}e_n(k)\right] \\
&\quad \times e_n(k+1) - 2\sigma\tilde{W}^{\mathrm{T}}(k)\hat{W}(k) \\
&\quad + S^{\mathrm{T}}(z(k))\Gamma S(z(k))e_n^2(k+1) \\
&\quad + \sigma^2\hat{W}^{\mathrm{T}}(k)\Gamma\hat{W}(k) \\
&\quad + 2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(z(k))e_n(k+1)
\end{aligned}$$

Using the following facts (the explanation of these facts can be found or derived from Appendix A.4)

$$S^{\mathrm{T}}(z(k))S(z(k)) < l$$

$$S^{\mathrm{T}}(z(k))\Gamma S(z(k)) \leq \gamma l$$

$$2e_n(k)e_n(k+1) \leq e_n^2(k) + e_n^2(k+1)$$

$$2e_n(k)\tilde{x}_n(k+1) \leq e_n^2(k) + \tilde{x}_n^2(k+1)$$

$$2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(z(k))e_n(k+1) \leq \frac{\gamma e_n^2(k+1)}{g_1} + g_1\sigma^2\gamma l\|\hat{W}\|^2$$

$$2\left(\mu_z - \frac{d_{k+\tau-1}}{g(k)}\right)e_n(k+1) \leq \frac{\gamma e_n^2(k+1)}{g_1} + \frac{g_1}{\gamma}\left(\mu_z - \frac{d_{k+\tau-1}}{g(k)}\right)^2$$

$$2\tilde{W}^{\mathrm{T}}(k)\hat{W}(k) = \|\tilde{W}(k)\|^2 + \|\hat{W}(k)\|^2 - \|W^*\|^2$$

we obtain

$$\begin{aligned}
\Delta V &\leq -\frac{\rho_1}{g_1}e_n^2(k+1) - \frac{\alpha_1}{g_1}e_1^2(k) - \frac{k_v}{g_1}e_n^2(k) - \sigma\|\tilde{W}(k)\|^2 \\
&\quad - \sigma(1 - \sigma\gamma - g_1\sigma\gamma l)\|\hat{W}(k)\|^2 + \beta_1
\end{aligned} \tag{52}$$

where

$$\rho_1 = 2 - \alpha_1 - 3k_v - 2\gamma - \gamma l g_1$$

$$\beta_1 = \frac{g_1}{\gamma}\left(\mu_0 + \frac{d_0}{g_0}\right)^2 + \sigma\|W^*\|^2$$

If we choose the design parameters as

$$k_v < \frac{2-\alpha_1}{3}, \quad \gamma < \frac{2-\alpha_1-3k_v}{2+g_1 l} \quad \text{and} \quad \sigma < \frac{1}{\gamma + g_1 \gamma l} \tag{53}$$

then $\Delta V \le 0$ once the tracking error $e_1(k)$ is larger than ϵ_1, where

$$\epsilon_1 = \sqrt{\frac{g_1 \beta_1}{\alpha_1}}$$

This implies the boundedness of $V(k)$ for all $k \ge 0$, which leads to the boundedness of $e(k)$, because

$$V(k) = \sum_{j=0}^{k} \Delta V(j) + V(0) < \infty$$

Furthermore, the tracking error $e_1(k)$ will asymptotically converge to the compact sets denoted by ϵ_1.

Define a compact

$$\Omega_e := \{e(k) \,|\, |e_i(k)| \le \epsilon_1, i = 1, 2, \ldots, n\}$$

All tracking errors $e(k)$ will asymptotically converge to the compact set Ω_e, since

$$e_1(k+n-1) = e_2(k+n-2) = \cdots = e_n(k)$$

Due to the negativity of ΔV, we can conclude that $x(k+1) \in \Omega$ if $x(k) \in \Omega$, $u(k-j) \in \Omega_u$, $j = 1, 2, \ldots, n-\tau$ and Ω_e is small enough.

Now it still remains to show that the weight estimate $\hat{W}(k)$ is bounded, and $u_k \in \Omega_u$.

The dynamics of the NN weight updating (17) can be written as

$$\begin{aligned}
\tilde{W}(k+1) &= \tilde{W}(k) - \Gamma[S(z(k))e_n(k+1) \\
&\quad + \sigma(\tilde{W}(k) + W^*)] \\
&= \tilde{W}(k) - \Gamma\Big\{ S(z(k))g(k)[\tilde{W}^{\mathrm{T}}(k)S(z(k)) - \mu_z] \\
&\quad + S(z(k))\left[d_{k+\tau-1} + \frac{g(k)k_v}{g_1} e_n(k)\right] \\
&\quad + \sigma(\tilde{W}(k) + W^*)\Big\} \\
&= A(k)\tilde{W}(k) - \sigma \Gamma W^* - g(k)\Gamma S(z(k)) \\
&\quad \times \left[\frac{d_{k+\tau-1}}{g(k)} - \mu_z + \frac{k_v}{g_1} e_n(k)\right]
\end{aligned}$$

where

$$A(k) = I - \sigma\Gamma - g(k)\Gamma S(z(k))S^{\mathrm{T}}(z(k))$$

which satisfies $\|A(k)\| < 1$ (see Appendix A.5 for proof). By assumption, we know the boundedness of μ_z, $d_{k+\tau-1}$, $g(k)$ and W^*. We have already proved that $e_n(k)$ asymptotically converge to the small compact Ω_e. Applying Lemma 1, we know that $\tilde{W}(k)$ will asymptotically converge to a small compact set denoted by Ω_{we}, hence the boundedness of $\hat{W}(k)$ is assured without the requirement for PE condition.

Control input (16) can be rewritten as

$$\begin{aligned}
u_k &= \hat{W}^{\mathrm{T}}(k)S(z(k)) = (\tilde{W}^{\mathrm{T}}(k) + W^{*\mathrm{T}})S(z(k)) \\
&= u_k^* + \tilde{W}^{\mathrm{T}}(k)S(z(k))
\end{aligned}$$

Since $u_k^* \in \Omega_f \subset \Omega_u$, there must exist a nonzero compact set $\Omega_w \subset R^l$ such that $\tilde{W}(k) \in \Omega_w$ guarantees $u_k \in \Omega_u$.

Now we can conclude that $u_k \in \Omega_u$ if $x(k) \in \Omega$ and $u(k-j) \in \Omega_u, j = 1, 2, \ldots, n-\tau$, Ω_{we} is small enough.

Finally, if we initialize state $x(0) \in \Omega_0$, $\tilde{W}(0) \in \Omega_{w_0}$, and we choose suitable parameters l, k_v, γ and σ according to (53) to make Ω_e, Ω_{se} and Ω_{we} small enough, i.e. $\Omega_d \cap \Omega_e \subset \Omega$ and $\Omega_{we} \subset \Omega_w$, there exists a constant k^* such that all tracking errors asymptotically converge to Ω_e, and NN weight error asymptotically converges to Ω_{we} for all $k > k^*$, and all other signals in system are bounded. This implies that the closed-loop system is SGUUB. Then $x(k) \in \Omega$, $\hat{W}(k) \in L_\infty$ and $u_k \in \Omega_u$ will hold for all $k > 0$. □

A.2. *Proof of Theorem 2*

Substituting controller (32) into (28), the error equation (28) can be re-written as

$$\begin{aligned}
e_y(k+\tau) &= -y_d(k+\tau) + f_\tau(\mathbf{z}(k)) + \Delta_1(k) \\
&\quad + [g_\tau(\mathbf{z}(k)) + \Delta_2(k)]\hat{W}_1^{\mathrm{T}}(k)S(\bar{z}(k)) + d_{k+\tau-1}
\end{aligned} \tag{54}$$

Adding and subtracting $[g_\tau(\mathbf{z}(k)) + \Delta_2(k)]\bar{u}^*(\bar{z}(k))$ on the right side of (54) and noting (31), we have

$$\begin{aligned}
e_y(k+\tau) &= -y_d(k+\tau) + f_\tau(\mathbf{z}(k)) + \Delta_1(k) \\
&\quad + [g_\tau(\mathbf{z}(k)) + \Delta_2(k)]\bar{u}^*(\bar{z}(k)) \\
&\quad + [g_\tau(\mathbf{z}(k)) + \Delta_2(k)][\hat{W}_1^{\mathrm{T}}(k)S(\bar{z}(k)) \\
&\quad - W^{*\mathrm{T}}S(\bar{z}(k)) - \mu_z] + d_{k+\tau-1}
\end{aligned} \tag{55}$$

Substituting (29) into (55) leads to

$$e_y(k+\tau) = [g_\tau(\mathbf{z}(k)) + \Delta_2(k)]\tilde{W}_1^{\mathrm{T}}(k)S(\bar{z}(k)) + d_1(k) \tag{56}$$

where

$$\begin{aligned}
\tilde{W}_1(k) &= \hat{W}_1(k) - W_1^* \\
d_1(k) &= \Delta_1(k) + \frac{\Delta_2(k)}{g_\tau(\bar{z}(k))}(y_d(k+\tau) - f_\tau(\bar{z}(k))) \\
&\quad + d_{k+\tau-1} - [g_\tau(\mathbf{z}(k)) + \Delta_2(k)]\mu_z
\end{aligned}$$

Since $\Delta_1(k)$, $\Delta_2(k)$ and $d_{k+\tau-1}$ are generate by external disturbance, we can consider $d_1(k)$ as an external disturbance. Similarly, it is reasonable to consider that $d_1(k)$ is bounded by a small positive constant d_{10}, i.e. $|d_1(k)| < d_{10}$.

Since all the assumptions are only valid in compact sets Ω_y and Ω_u, we must prove that the system output and input will remain in these compact sets all the time. At time instant k, suppose that all past inputs are in Ω_u, current output and all past outputs are in Ω_y, the SDFOs, $y_{k+1}, \ldots, y_{k+\tau-1}$, are all in Ω_y, all past NN weight errors are in Ω_{w_1}, we will prove that all these conditions still hold after time instant k and the tracking error converges to a small neighborhood of zero.

Choose the following Lyapunov function candidate

$$V(k) = \frac{1}{g_1 + \Delta_{20}} \sum_{j=0}^{\tau-1} e_y^2(k+j) + \sum_{j=0}^{\tau-1} \tilde{W}_1^{\mathrm{T}}(k+j) \Gamma_1^{-1} \tilde{W}_1(k+j) \tag{57}$$

It should be noted that the Lyapunov function candidate (57) contains no future system information. From (33), we have

$$\tilde{W}_1(k+\tau) = \tilde{W}_1(k) - \Gamma_1[S(\bar{z}(k))e_y(k+\tau) + \sigma_1 \hat{W}_1(k)]$$

Because $e_y(k+\tau)$ is predetermined by the input and output sequences prior to time instant k, we can consider $\tilde{W}_1(k+\tau)$ contains no future system information. In the same way, all $W_1(k+j), j = 1, 2, \ldots, \tau-1$ can be proved that they contain no future system information. Therefore, the choice of Lyapunov function candidate is causal.

The first difference of (57) along (56) and (33) is given

$$\begin{aligned}\Delta V = {} & \frac{1}{g_1 + \Delta_{20}} e_y^2(k+\tau) - \frac{1}{g_1 + \Delta_{20}} e_y^2(k) \\ & - 2\tilde{W}_1^{\mathrm{T}}(k) S(\bar{z}(k)) e_y(k+\tau) - 2\sigma \tilde{W}_1^{\mathrm{T}}(k) \hat{W}_1(k) \\ & + S^{\mathrm{T}}(\bar{z}(k)) \Gamma_1^{\mathrm{T}} S(\bar{z}(k)) e_y^2(k+\tau) \\ & + 2S^{\mathrm{T}}(\bar{z}(k)) \Gamma_1^{\mathrm{T}} \hat{W}_1(k) \sigma_1 e_y(k+\tau) \\ & + \sigma_1^2 \hat{W}_1^{\mathrm{T}}(k) \Gamma_1^{\mathrm{T}} \hat{W}_1(k)\end{aligned}$$

Noting equation (56), we obtain

$$\begin{aligned}\Delta V = {} & \frac{1}{g_1 + \Delta_{20}} e_y^2(k+\tau) - \frac{1}{g_1 + \Delta_{20}} e_y^2(k) \\ & - 2\frac{e_y^2(k+\tau)}{g_\tau + \Delta_2(k)} + 2\frac{d_1(k) e_y(k+\tau)}{g_\tau + \Delta_2(k)} \\ & - 2\sigma \tilde{W}_1^{\mathrm{T}}(k) \hat{W}_1(k) + S^{\mathrm{T}}(\bar{z}(k)) \Gamma_1^{\mathrm{T}} S(\bar{z}(k)) e_y^2(k+\tau) \\ & + 2S^{\mathrm{T}}(\bar{z}(k)) \Gamma_1^{\mathrm{T}} \hat{W}_1(k) \sigma_1 e_y(k+\tau) \\ & + \sigma_1^2 \hat{W}_1^{\mathrm{T}}(k) \Gamma_1^{\mathrm{T}} \hat{W}_1(k)\end{aligned}$$

Using the following facts (detailed explanations are in Appendix A.4)

$$-2\frac{e_y^2(k+\tau)}{g_\tau + \Delta_2(k)} \le -2\frac{e_y^2(k+\tau)}{g_1 + \Delta_{20}}$$

$$\begin{aligned}2\frac{d_1(k) e_y(k+\tau)}{g_\tau + \Delta_2(k)} & \le 2\frac{d_{10}|e_y(k+\tau)|}{g_0} \\ & \le \frac{\bar{\gamma}_1 l_1 e_y^2(k+\tau)}{g_1 + \Delta_{20}} + \frac{(g_1 + \Delta_{20}) d_{10}^2}{\bar{\gamma}_1 l_1 g_0^2}\end{aligned}$$

$$\begin{aligned}& - 2\sigma_1 \tilde{W}_1^{\mathrm{T}}(k) \hat{W}_1(k) \\ & \quad = -\sigma_1(\|\tilde{W}_1(k)\|^2 + \|\hat{W}_1(k)\|^2 - \|W_1^*\|^2)\end{aligned}$$

$$S^{\mathrm{T}}(\bar{z}(k)) S(\bar{z}(k)) < l_1$$

$$S^{\mathrm{T}}(\bar{z}(k)) \Gamma_1 S(\bar{z}(k)) \le \bar{\gamma}_1 l_1$$

$$\begin{aligned}& 2S^{\mathrm{T}}(\bar{z}(k)) \Gamma_1^{\mathrm{T}} \hat{W}_1(k) \sigma_1 e_y(k+\tau) \\ & \quad \le \frac{\bar{\gamma}_1 e_n^2(k+\tau)}{g_1 + \Delta_{20}} + (g_1 + \Delta_{20}) \sigma_1^2 \bar{\gamma}_1 l_1 \|\hat{W}_1(k)\|^2\end{aligned}$$

$$\sigma_1^2 \hat{W}_1^{\mathrm{T}}(k) \Gamma_1^{\mathrm{T}} \hat{W}_1(k) \le \sigma_1^2 \bar{\gamma}_1 \|\hat{W}_1(k)\|^2$$

we obtain

$$\begin{aligned}\Delta V \le {} & -\frac{\rho_1}{g_1 + \Delta_{20}} e_y^2(k+\tau) - \frac{1}{g_1 + \Delta_{20}} e_y^2(k) + \beta_1 \\ & - \sigma_1[1 - \sigma_1 \bar{\gamma}_1 - (g_1 + \Delta_{20}) \sigma_1 \bar{\gamma}_1 l_1] \|\hat{W}_1(k)\|^2\end{aligned}$$

where

$$\rho_1 = 1 - \bar{\gamma}_1 l_1 - (g_1 + \Delta_{20}) \bar{\gamma}_1 l_1 - \bar{\gamma}_1$$

and

$$\beta_1 = \frac{(g_1 + \Delta_{20}) d_{10}^2}{\bar{\gamma}_1 l_1 g_0^2} + \sigma_1 \|W_1^*\|^2$$

If we choose the design parameters as follows

$$\left.\begin{aligned}& \bar{\gamma}_1 < \frac{1}{1 + l_1 + (g_1 + \Delta_{20}) l_1} \\ \text{and} \quad & \\ & \sigma_1 < \frac{1}{(1 + g_1 l_1 + \Delta_{20} l_1) \bar{\gamma}_1}\end{aligned}\right\} \tag{58}$$

then $\Delta V \le 0$ once the tracking error $e_y(k)$ is larger than $\sqrt{(g_1 + \Delta_{20})\beta_1}$. This implies the boundedness of $V(k)$ for all $k \ge 0$, which leads to the boundedness of $e_y(k)$.

Furthermore, the tracking error $e_y(k)$ will asymptotically converge to the compact set denoted by $\epsilon \le \sqrt{(g_1 + \Delta_{20})\beta_1}$.

Due to negativeness of ΔV, we can conclude that $y_{k+1} \in \Omega_y$ if all past output $y_{k-j} \in \Omega_y, j = 0, \ldots, n-1,$

and input $u(k-j) \in \Omega_u, j = 1, 2, \ldots, \tau + m - 1$, and compact set ϵ small enough. The boundedness of $W_1(k)$ can be proved by following the same procedure in Appendix A.1.

We have proved that $y_{k+1} \in \Omega_y$, we can use the same techniques as in Appendix A.1 to show that the NN weight error asymptotically converges to small compact set $\Omega_{w_1 e}$, and the control signal $u_k \in \Omega_u$.

Finally, if we initialize state $y_0 \in \Omega_{y0}$, $\tilde{W}_1(0) \in \Omega_{w_1 0}$, and we choose suitable parameters $\bar{\gamma}_1$, σ_1 according to (58) to make ϵ small enough, there exists a constant k^* such that all tracking errors asymptotically converge to ϵ, and NN weight error asymptotically converges to $\Omega_{w_1 e}$ for all $k > k^*$. This implies that the closed-loop system is SGUUB. Then $y_k \in \Omega_y$, $\hat{W}(k) \in L_\infty$ and $u(k) \in \Omega_u$ will hold for all $k > 0$. □

A.3. *Proof of Theorem 3*

According to the definition of new variables (42), we note that

$$\hat{e}_n(k) = e_n(k) + \tilde{x}_n(k)$$

Under the new controller (43), the tracking error equation becomes (see equation (47))

$$e_n(k+1) = g(k)\left[\tilde{W}^{\mathrm{T}}(k)S(\hat{z}(k)) + 2W^{*\mathrm{T}}(k)S_t(k) - \mu_z + \frac{k_v}{g_1}e_n(k)\right] + d_{k+\tau-1}$$

or equivalently

$$\tilde{W}^{\mathrm{T}}(k)S(\hat{z}(k)) = \frac{e_n(k+1)}{g(k)} - \frac{d_{k+\tau-1}}{g(k)} - 2W^{*\mathrm{T}}(k)S_t(k) + \mu_z - \frac{k_v}{g_1}e_n(k)$$

Suppose that $x(k) \in \Omega$, $u(k-j) \in \Omega_u$, $j = 1, 2, \ldots, n-\tau$ $\forall k \geq 0$, then NN approximation (19), Assumptions 1–3 are valid.

Choose the following Lyapunov function candidate

$$V(k) = \frac{\alpha_1}{g_1}\sum_{i=1}^{n} e_i^2(k) + \frac{2k_v}{g_1}e_n^2(k) + \tilde{W}^{\mathrm{T}}(k)\Gamma^{-1}\tilde{W}(k) + \frac{\alpha_2}{g_0}\sum_{i=1}^{n}\tilde{x}_i^2(k) + \frac{1}{g_0}\sum_{j=1}^{\tau-1}\tilde{W}_o^{\mathrm{T}}(k+j)\Gamma_o^{-1}\tilde{W}_o(k+j) \tag{59}$$

where α_1 and α_2 are two small positive constants which are less than 1. The first difference of (59) along (4), (43), (44), (47) and (38) is given

$$\begin{aligned}
\Delta V(k) &= \frac{\alpha_1 + 2k_v}{g_1}e_n^2(k+1) - \frac{\alpha_1}{g_1}e_1^2(k) \\
&\quad - \frac{2k_v}{g_1}e_n^2(k) + \tilde{W}^{\mathrm{T}}(k+1)\Gamma^{-1}\tilde{W}(k+1) \\
&\quad - \tilde{W}^{\mathrm{T}}(k)\Gamma^{-1}\tilde{W}(k) + \frac{\alpha_2}{g_0}\tilde{x}_n^2(k+1) - \frac{\alpha_2}{g_0}\tilde{x}_1^2(k) \\
&\quad + \frac{1}{g_0}\tilde{W}_o^{\mathrm{T}}(k+\tau)\Gamma_o^{-1}\tilde{W}_o(k+\tau) \\
&\quad - \frac{1}{g_0}\tilde{W}_o^{\mathrm{T}}(k+1)\Gamma_o^{-1}\tilde{W}_o(k+1) \\
&= \frac{\alpha_1 + 2k_v}{g_1}e_n^2(k+1) - \frac{\alpha_1}{g_1}e_1^2(k) - \frac{2k_v}{g_1}e_n^2(k) \\
&\quad - 2\tilde{W}^{\mathrm{T}}(k)S(\hat{z}(k))\hat{e}_n(k+1) - 2\sigma\tilde{W}^{\mathrm{T}}(k)\hat{W}(k) \\
&\quad + S^{\mathrm{T}}(\hat{z}(k))\Gamma S(\hat{z}(k))\hat{e}_n^2(k+1) \\
&\quad + 2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(\hat{z}(k))\hat{e}_n(k+1) \\
&\quad + \sigma^2\hat{W}^{\mathrm{T}}(k)\Gamma\hat{W}(k) + \frac{\alpha_2}{g_0}\tilde{x}_n^2(k+1) - \frac{\alpha_2}{g_0}\tilde{x}_1^2(k) \\
&\quad - \frac{2}{g_0}\tilde{W}_o^{\mathrm{T}}(k+1)S(z(k+1))\tilde{x}_n(k+1) \\
&\quad + \frac{1}{g_0}S^{\mathrm{T}}(z(k+1))\Gamma_o S(z(k+1))\tilde{x}_n^2(k+1) \\
&= \frac{\alpha_1 + 2k_v}{g_1}e_n^2(k+1) - \frac{\alpha_1}{g_1}e_1^2(k) - \frac{2k_v}{g_1}e_n^2(k) \\
&\quad - 2\left[\frac{e_n(k+1)}{g(k)} - \frac{d_{k+\tau-1}}{g(k)} + \mu_z - \frac{k_v}{g_1}e_n(k)\right] \\
&\quad \times [e_n(k+1) + \tilde{x}_n(k+1)] \\
&\quad + 4W^{*\mathrm{T}}S_t(k)[e_n(k+1) + \tilde{x}_n(k+1)] \\
&\quad - 2\sigma\tilde{W}^{\mathrm{T}}(k)\hat{W}(k) \\
&\quad + S^{\mathrm{T}}(\hat{z}(k))\Gamma S(\hat{z}(k))[e_n(k+1) + \tilde{x}_n(k+1)]^2 \\
&\quad + \sigma^2\hat{W}^{\mathrm{T}}(k)\Gamma\hat{W}(k) \\
&\quad + 2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(\hat{z}(k))[e_n(k+1) + \tilde{x}_n(k+1)] \\
&\quad + \frac{\alpha_2}{g_0}\tilde{x}_n^2(k+1) - \frac{\alpha_2}{g_0}\tilde{x}_1^2(k) - \frac{2}{g_0}\tilde{x}_n^2(k+1) \\
&\quad - \frac{2}{g_0}[\mu_{z(k+1)} + \bar{d}(k+1)]\tilde{x}_n(k+1) \\
&\quad + \frac{1}{g_0}S^{\mathrm{T}}(z(k+1))\Gamma_o S(z(k+1))\tilde{x}_n^2(k+1)
\end{aligned}$$

Using the following facts (See Appendix A.4 for details)

$$S^{\mathrm{T}}(\hat{z}(k))S(\hat{z}(k)) < l$$

$$S^{\mathrm{T}}(z(k))S(z(k)) < l_o$$

$$S^{\mathrm{T}}(\hat{z}(k))\Gamma S(\hat{z}(k)) \leq \gamma l$$

$$S^{\mathrm{T}}(z(k))\Gamma_o S(z(k)) \le \gamma_o l_o$$

$$S^{\mathrm{T}}(z(k+1))\Gamma_o S(z(k+1)) \le \gamma_o l_o$$

$$|\lambda_{\max}[S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k))]| < l$$

$$|\lambda_{\max}[S_t(k)S_t^{\mathrm{T}}(k)]| < l$$

$$2e_n(k)e_n(k+1) \le e_n^2(k) + e_n^2(k+1)$$

$$2e_n(k)\tilde{x}_n(k+1) \le e_n^2(k) + \tilde{x}_n^2(k+1)$$

$$[e_n(k+1) + \tilde{x}_n(k+1)]^2 \le 2[e_n^2(k+1) + \tilde{x}_n^2(k+1)]$$

$$2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(\hat{z}(k))e_n(k+1) \le \frac{\gamma e_n^2(k+1)}{g_1} + g_1\sigma^2\gamma l\|\hat{W}\|^2$$

$$2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(\hat{z}(k))\tilde{x}_n(k+1) \le \frac{\gamma\tilde{x}_n^2(k+1)}{g_0} + g_0\sigma^2\gamma l\|\hat{W}\|^2$$

$$2\left(\mu_z - \frac{d_{k+\tau-1}}{g(k)}\right)e_n(k+1) \le \frac{\gamma e_n^2(k+1)}{g_1} + \frac{g_1}{\gamma}\left(\mu_z - \frac{d_{k+\tau-1}}{g(k)}\right)^2$$

$$2\left(\mu_z - \frac{d_{k+\tau-1}}{g(k)}\right)\tilde{x}_n(k+1) \le \frac{\gamma_o\tilde{x}_n^2(k+1)}{g_0} + \frac{g_0}{\gamma_o}\left(\mu_z - \frac{d_{k+\tau-1}}{g(k)}\right)^2$$

$$2[\mu_{z(k+1)} + \bar{d}(k+1)]\tilde{x}_n(k+1) \le \gamma_o\tilde{x}_n^2(k+1) + \frac{1}{\gamma_o}[\mu_{z(k+1)} + \bar{d}(k+1)]^2$$

$$2W^*S_t(k)e_n(k+1) \le \frac{\gamma e_n^2(k+1)}{g_1} + \frac{g_1}{\gamma}l\|W^*\|^2$$

$$2W^*S_t(k)\tilde{x}_n(k+1) \le \frac{\gamma_o\tilde{x}_n^2(k+1)}{g_0} + \frac{g_0}{\gamma_o}l\|W^*\|^2$$

$$2\tilde{W}^{\mathrm{T}}(k)\hat{W}(k) = \|\tilde{W}(k)\|^2 + \|\hat{W}(k)\|^2 - \|W^*\|^2$$

we obtain

$$\Delta V \le -\frac{\rho_1}{g_1}e_n^2(k+1) - \frac{\alpha_1}{g_1}e_1^2(k) - \frac{\rho_2}{g_0}\tilde{x}_n^2(k+1) - \frac{\alpha_2}{g_0}x_1^2(k) - \sigma\|\tilde{W}(k)\|^2 - \sigma(1 - \sigma\gamma - g_1\sigma\gamma l - g_0\sigma\gamma l)\|\hat{W}(k)\|^2 + \beta_1 + \beta_2 \quad (60)$$

where

$$\rho_1 = 1 - \alpha_1 - 3k_v - 4\gamma - 2\gamma l g_1$$

$$\rho_2 = 1 - 2\gamma l g_0 - k_v - \alpha_2 - \gamma - 4\gamma_0 - \gamma_0 l_0$$

$$\beta_1 = \frac{1}{g_0\gamma_0}(\mu_{o0} + \bar{d}_0)^2 + \left(\frac{g_1}{\gamma} + \frac{g_0}{\gamma_0}\right)\left(\mu_0 + \frac{\bar{d}_0}{g_0}\right)^2$$

$$\beta_2 = \left(\frac{2g_1 l}{\gamma} + \frac{2g_0 l}{\gamma_0} + \sigma\right)\|W^*\|^2$$

If we choose the design parameters as

$$\left.\begin{aligned} &k_v < \frac{1-\alpha_1}{3}, \quad \gamma < \frac{1-\alpha_1-3k_v}{4+2g_1 l} \\ &\gamma_o < \frac{1-\alpha_2-k_v-2g_0\gamma l-\gamma}{4+l_o} \quad \text{and} \\ &\sigma < \frac{1}{\gamma + g_1\gamma l + g_0\gamma l} \end{aligned}\right\} \quad (61)$$

then $\Delta V \le 0$ once the tracking error $e_1(k)$ is larger than ϵ_1, or the estimation error $\tilde{x}_1$ is larger than ϵ_2, where

$$\epsilon_1 = \sqrt{\frac{g_1(\beta_1+\beta_2)}{\alpha_1}} \quad \text{and} \quad \epsilon_2 = \sqrt{\frac{g_0(\beta_1+\beta_2)}{\alpha_2}}$$

This implies the boundedness of $V(k)$ for all $k \ge 0$, which leads to the boundedness of $e(k)$ and $\tilde{x}(k)$, because

$$V(k) = \sum_{j=0}^{k}\Delta V(j) + V(0) < \infty$$

Furthermore, the tracking error $e_1(k)$ and observer estimation error $\tilde{x}_1(k)$ will asymptotically converge to the compact sets denoted by ϵ_1 and ϵ_2 respectively.

Define two compact sets

$$\Omega_e := \{e(k)|\ |e_i(k)| \le \epsilon_1, i = 1, 2, \ldots, n\}$$

$$\Omega_{\tilde{x}} := \{\tilde{x}(k)|\ |\tilde{x}_i(k)| \le \epsilon_2, i = 1, 2, \ldots, n\}$$

All tracking errors $e(k)$ will asymptotically converge to the compact set Ω_e, and all states estimation errors $\tilde{x}(k)$ will asymptotically converge to the compact set $\Omega_{\tilde{x}}$ since

$$e_1(k+n-1) = e_2(k+n-2) = \cdots = e_n(k)$$

$$\tilde{x}_1(k+n-1) = \tilde{x}_2(k+n-2) = \cdots = \tilde{x}_n(k)$$

Due to the negativity of ΔV, we can conclude that $x(k+1) \in \Omega$ if $x(k) \in \Omega, u(k-j) \in \Omega_u, j = 1, 2, \ldots, n-\tau$ and Ω_e is small enough.

The boundedness of $e(k)$ and $\tilde{x}(k)$ obviously leads to the boundedness of $\hat{e}(k)$, and $\hat{x}(k)$, but it still remains to show that the weight estimates $\hat{W}(k)$ and $\hat{W}_o(k)$ are bounded, and $u_k \in \Omega_u$.

The dynamics of the NN weight updating (44) can be written as

$$
\begin{aligned}
&\tilde{W}(k+1)\\
&\quad = \tilde{W}(k) - \Gamma[S(\hat{z}(k))\hat{e}_n(k+1) + \sigma(\tilde{W}(k) + W^*)]\\
&\quad = \tilde{W}(k) - \Gamma\Big\{ S(\hat{z}(k))g(k)\\
&\qquad \times \left[\tilde{W}^{\mathrm{T}}(k)S(\hat{z}(k)) + 2W^{*\mathrm{T}}S_t(k) - \mu_z\right] + S(\hat{z}(k))\\
&\qquad \times \left[d_{k+\tau-1} + \tilde{x}_n(k+1) + \frac{g(k)k_v}{g_1}e_n(k)\right]\\
&\qquad + \sigma(\tilde{W}(k) + W^*)\Big\}\\
&\quad = A(k)\tilde{W}(k) - \sigma\Gamma W^* - g(k)\Gamma S(\hat{z}(k))\\
&\qquad \times \left[\frac{d_{k+\tau-1}}{g(k)} - \mu_z + 2W^{*T}S_t(k)\right.\\
&\qquad \left. + \frac{\tilde{x}_n(k+1)}{g(k)} + \frac{k_v}{g_1}e_n(k)\right]
\end{aligned} \tag{62}
$$

where

$$A(k) = I - \sigma\Gamma - g(k)\Gamma S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k))$$

which satisfies $\|A(k)\| < 1$ (see Appendix A.5 for proof). By assumption, we know the boundedness of μ_z, $d_{k+\tau-1}$, $g(k)$, W^* and $S_t(k)$. We have already proved that $\tilde{x}_n(k+1)$ and $e_n(k)$ asymptotically converge to small compacts Ω_e and $\Omega_{\tilde{x}}$ respectively. Applying Lemma 1, we know that $\tilde{W}(k)$ will asymptotically converge to a small compact set denoted by Ω_{we}, hence the boundedness of $\hat{W}(k)$ is assured without the requirement for PE condition.

The boundedness of $\hat{W}_o$ has been proven in (41).

Control input (43) can be rewritten as

$$
\begin{aligned}
u_k &= \hat{W}^{\mathrm{T}}(k)S(\hat{z}(k)) = (\tilde{W}^{\mathrm{T}}(k) + W^{*\mathrm{T}})[S(z(k)) + 2S_t(k)]\\
&= u_k^* + \tilde{W}^{\mathrm{T}}(k)S(z(k)) + 2\tilde{W}^{\mathrm{T}}(k)S_t(k) + 2W^{*\mathrm{T}}S_t(k)
\end{aligned}
$$

Since $u_k^* \in \Omega_f \subset \Omega_u$, there must exist two nonzero compact sets $\Omega_w \subset R^l$ and $\Omega_s \subset R^l$ such that $\tilde{W}(k) \in \Omega_w$ and $S_t(k) \in \Omega_s$ guarantee $u_k \in \Omega_u$. Because $\tilde{x}(k)$ converge to the small compact set $\Omega_{\tilde{x}}$, $S_t(k)$ must converge to a small compact set Ω_{se}.

Now we can conclude that $u_k \in \Omega_u$ if $x(k) \in \Omega$, $u(k-j) \in \Omega_u$, $j = 1, 2, \ldots, n-\tau$, Ω_{we} and $\Omega_{\tilde{x}}$ are small enough.

Finally, if we initialize state $x(0) \in \Omega_0$, $\tilde{W}(0) \in \Omega_{w_0}$, and we choose suitable parameters l, l_o, k_v, γ, γ_o and σ according to (61) to make Ω_e, Ω_{se} and Ω_{we} small enough, i.e. $\Omega_d \cap \Omega_e \subset \Omega$, $\Omega_{we} \subset \Omega_w$ and $\Omega_{se} \subset \Omega_s$, there exists a constant k^* such that all tracking errors asymptotically converge to Ω_e, and NN weight error asymptotically converges to Ω_{we} for all $k > k^*$, and all other signals in system are bounded. This implies that the closed-loop system is SGUUB. Then $x(k) \in \Omega$, $\hat{W}(k) \in L_\infty$, $\hat{W}_o(k) \in L_\infty$ and $u_k \in \Omega_u$ will hold for all $k > 0$. □

A.4. *Explanation of facts used in the proof of system stability*

- $S^{\mathrm{T}}(\hat{z}(k))S(\hat{z}(k)) < l$

  Because the absolute value of each element in $S(\hat{z}(k))$ is less than 1, thus the inner product of $S(\hat{z}(k))$ must be less than its dimension.

- $S^{\mathrm{T}}(\hat{z}(k))\Gamma S(\hat{z}(k)) < \gamma l$

  Because Γ is a positive and symmetric matrix, it can be decomposed into $U\Lambda U^{\mathrm{T}}$ where Λ is a diagonal matrix with the eigenvalues of Γ as its diagonal elements, and U is a unitary orthogonal matrix. Accordingly, we have

$$
\begin{aligned}
S^{\mathrm{T}}(\hat{z}(k))\Gamma S(\hat{z}(k)) &= R^{\mathrm{T}}(k)\Lambda R(k)\\
&= \sum \gamma_i r_i(k) \le \gamma \sum r_i(k)\\
&= \gamma R^{\mathrm{T}}(k)R(k)\\
&= \gamma S^{\mathrm{T}}(\hat{z}(k))UU^{\mathrm{T}}S(\hat{z}(k))\\
&= \gamma S^{\mathrm{T}}(\hat{z}(k))S(\hat{z}(k)) < \gamma l
\end{aligned}
$$

  where

$$R(k) = U^{\mathrm{T}}S(\hat{z}(k))$$

- $|\lambda_{\max}[S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k))]| < l$

  It is clear that the inequality is true based on Gerschgorin Circle Theorem: Let $A \in L(C^n)$ and define the set of complex numbers

$$S = \bigcup_{i=1}^{n} \left\{ z \Big| |a_{ii} - z| \le \sum_{j=1, j\ne i}^{j=n} |a_{ij}| \right\}$$

  Then, every eigenvalue of A lies in S (Paretto and Niez 1986), i.e. the eigenvalue of matrix A is located within a circle with radius of $\sum_{j=1,j\ne i}^{j=n}|a_{ij}|$ and with its center at a_{ii}.

- $2e_n(k)e_n(k+1) \le e_n^2(k) + e_n^2(k+1)$

- $2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(\hat{z}(k))e_n(k+1) \le \dfrac{\gamma e_n^2(k+1)}{g_1} + g_1\sigma^2\gamma l\|\hat{W}\|^2$

Using the facts that $2ab \le a^2 + b^2$, we obtain the following inequality

- $2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(\hat{z}(k))e_n(k+1) \le \dfrac{\gamma e_n^2(k+1)}{g_1}$
  $+\dfrac{g_1}{\gamma}\sigma^2\hat{W}^{\mathrm{T}}(k)\Gamma S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k))\Gamma^{\mathrm{T}}\hat{W}(k)$

  Because $S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k)) = [S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k))]^{\mathrm{T}}$, $S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k))$ can be decomposed into $V \operatorname{diag} \lambda[S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k))]V^{\mathrm{T}}$ with V a unitary orthogonal matrix (Golub and Vanloan 1996). Using the fact that $|\lambda_{\max}[S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k))]| < l$, we obtain

$$\hat{W}(k)\Gamma S(\hat{z}(k))S^{\mathrm{T}}(\hat{z}(k))\Gamma^{\mathrm{T}}\hat{W}(k) < l\hat{W}(k)\Gamma VV^{\mathrm{T}}\Gamma^{\mathrm{T}}\hat{W}(k)$$
$$= l\hat{W}(k)\Gamma\Gamma^{\mathrm{T}}\hat{W}(k)$$
$$= l\hat{W}(k)U\Lambda^2U^{\mathrm{T}}\hat{W}(k)$$
$$\leq \gamma^2 l\|\hat{W}(k)\|$$

Thus, we have

$$2\sigma\hat{W}^{\mathrm{T}}(k)\Gamma S(\hat{z}(k))e_n(k+1) \leq \frac{\gamma e_n^2(k+1)}{g_1} + g_1\sigma^2\gamma l\|\hat{W}\|^2$$

- $2\tilde{W}^{\mathrm{T}}(k)\hat{W}(k) = \|\tilde{W}(k)\|^2 + \|\hat{W}(k)\|^2 - \|W^*\|^2$
  From the definition of $\tilde{W}(k)$, we obtain $W^* = \tilde{W}(k) - \hat{W}(k)$. Taking the norms, we obtain

$$\|W^*\|^2 = [W(k) - \hat{W}(k)]^{\mathrm{T}}[\tilde{W}(k) - \hat{W}(k)]$$
$$= \|\tilde{W}(k)\|^2 - 2\tilde{W}^{\mathrm{T}}(k)\hat{W}(k) + \|\hat{W}(k)\|^2$$

Therefore, $2\tilde{W}^{\mathrm{T}}(k)\hat{W}(k) = \|\tilde{W}(k)\|^2 + \|\hat{W}(k)\|^2 - \|W^*\|^2$.

A.5. *Proof of* $\|A(k)\| < 1$

For $\Gamma = \mathrm{diag}[\gamma_1, \gamma_2, \ldots, \gamma_l]$, $S(z(k)) = [s_1, s_2, \ldots, s_l]^{\mathrm{T}}$ and

$$A(k) = I - \sigma\Gamma - g(k)\Gamma S(z(k))S^{\mathrm{T}}(z(k))$$

we have

$$A(k) = I - \sigma\Gamma - g(k)\Gamma S(z(k))S^{\mathrm{T}}(z(k))$$
$$= \begin{bmatrix} 1-\sigma\gamma_1 - g(k)\gamma_1 s_1^2 & -g(k)\gamma_1 s_1 s_2 & \ldots & -g(k)\gamma_1 s_1 s_l \\ -g(k)\gamma_2 s_2 s_1 & 1-\sigma\gamma_2 - g(k)\gamma_2 s_2^2 & \ldots & -g(k)\gamma_2 s_2 s_l \\ \vdots & \vdots & \ddots & \vdots \\ -g(k)\gamma_l s_l s_1 & -g(k)\gamma_l s_l s_2 & \ldots & 1-\sigma\gamma_l - g(k)\gamma_l s_l^2 \end{bmatrix}$$

in which the iith diagonal elements are $1 - \sigma\gamma_i - g(k)\gamma_i s_i^2$, and the ijth $(i \neq j)$ off-diagonal elements are $-g(k)\gamma_i s_i s_j$. Let λ_i denote the ith $(1 \leq i \leq l)$ eigenvalue of $A(k)$. According to the Gerschgorin circle theorem (Paretto and Niez 1986), we obtain

$$\lambda_i \in \bigcup_{i=1}^{l}\left\{\lambda \,\middle|\, |\lambda - a_{ii}| \leq \sum_{j=1, j\neq i}^{j=l}|a_{ij}|\right\}$$

with a_{ij} denotes the ijth element of $A(k)$. As a consequence, we have

$$a_{qq} - \sum_{j\neq q}|a_{qj}| \leq \lambda_i \leq a_{qq} + \sum_{j\neq q}|a_{qj}|, \qquad 1 \leq i, j, q \leq l$$

that is

$$1 - \sigma\gamma_q - g(k)\gamma_q s_q^2 - \sum_{j\neq q}|g(k)\gamma_q s_q s_j|$$
$$\leq \lambda_i \leq 1 - \sigma\gamma_q - g(k)\gamma_q s_q^2 + \sum_{j\neq q}|g(k)\gamma_q s_q s_j|$$

Because each element of $S(z(k))$ satisfies $s_i < 1$ (if $z(k)$ is operating around 0, we have $s_i \ll 1$), the high order terms $s_i s_j$ $(i, j = 1, \ldots, l)$ must be even smaller. By neglecting the high order terms, we obtain $\lambda_p \simeq 1 - \sigma\gamma_q$. Noticing parameter conditions (53), we can obtain $0 < \sigma\gamma_i < 1$, and $0 < \lambda_p < 1$ which sequentially leads to $\|A(k)\| < 1$ and $\|\Phi(k_1, k_0)\| < 1$.

It should be noted that the adaptive gain matrix $\Gamma > 0$ is diagonal in this paper. For a general positive symmetric Γ, it can be diagonalized by unitary orthogonal matrix U as

$$A(k) = U^{\mathrm{T}}[I - \sigma\Lambda - g(k)\Lambda USS^{\mathrm{T}}U^{\mathrm{T}}]U$$

Define $B = I - \sigma\Lambda - g(k)\Lambda USS^{\mathrm{T}}U^{\mathrm{T}}$. If US has the same properties as $S(z(k))$, then following the same the procedure, we can conclude $\|A(k)\| < 1$. But this is not true in general. It only holds when vector $S(z(k))$ is within a pseudo unit sphere, i.e. $S^{\mathrm{T}}S \leq 1$. This assumption is too strong to be plausible, and further research is needed.

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