



Feedback linearization and stabilization of second-order non-holonomic chained systems

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This paper presents a theoretical framework for non-regular feedback linearization and stabilization of second-order non-holonomic chained systems. By giving a new criterion for the problem of non-smooth non-regular feedback linearization, it is proved that second-order chained systems are non-regular static state feedback linearizable. A discontinuous control law is obtained based on linear system theory and the inversion technique. The design mechanism is generalized to higher-order non-holonomic chained systems. Simulation studies are carried out to show the effectiveness of the approach.

1. Introduction

The problem of controlling non-holonomic systems has attracted much attention in the last decade. These studies were primarily limited to first-order non-holonomic systems which undergo non-integrable kinematic constraints (Kolmanovsky and McClamroch 1995, Canudas de Wit *et al.* 1996, Luo and Tsiotras 2000). There are only a few publications which address the problem of controlling non-holonomic systems satisfying non-integrable acceleration or dynamic relationships. This class of systems is referred to as second-order or high-order non-holonomic systems (Oriolo and Nakamura 1991, Laiou and Astolfi 1999).

Second-order and high-order non-holonomic systems arise very often in the study of mechanical systems. Typical examples include redundant manipulators (De Luca *et al.* 1996, 1998) and underactuated systems (Oriola and Nakamura 1991, Egeland and Berglund 1994, Seto and Baillieul 1994, Spong 1995, Reyhanoglu *et al.* 1999). Although the dynamics are well understood and many techniques have been investigated for these systems, controller design of these systems remains a challenging problem. Attempts were made to stabilize second-order or high-order non-holonomic systems recently. Among them, the most notable contribution is due to Laiou and Astolfi (1999), who successfully generalize the idea proposed in Astolfi (1996) to a class of high-order non-holonomic systems with two inputs. In this paper, a class of discontinuous state feedback laws was presented to exponentially stabilize the systems. Some research works have also been presented for specific classes of systems in the literature. Controllability and stabilizability have been investigated for a special class of underactuated systems

in Reyhanoglu *et al.* (1999). In Su and Stepanenko (1999), variable structure control has been applied to the control of underactuated robots. Homogeneous design technique is exploited in M'Closkey and Morin (1998) to obtain time-varying homogeneous feedback for a class of non-linear systems with drifts which includes some underactuated mechanical systems. Based on a novel design technique proposed in Sordalen (1993), a time-varying controller was obtained for a second-order chained form in Egeland and Berglund (1994) to achieve asymptotic stability with exponential convergence.

It has been proven that several classes of second-order non-holonomic systems are transformable into the following chained form by smooth state and input transformations

$$\left. \begin{aligned} \ddot{y}_1 &= u_1 \\ \ddot{y}_2 &= u_2 \\ \ddot{y}_3 &= y_2 u_1 \end{aligned} \right\} \quad (1)$$

Typical examples include a three-link planar manipulator with the third joint unactuated (Arai *et al.* 1997), an (underactuated) underwater vehicle with non-holonomic acceleration constraints (Egeland and Berglund 1994), an underactuated surface vessel with two independent propellers (Reyhanoglu *et al.* 1999, Wichlund *et al.* 1995), and a kinematic redundant planar PPR manipulator with all joints passive and steered by the forces imposed at the end-effector (De Luca *et al.* 1996, 1998).

In this paper, the problem of feedback stabilization is addressed for second-order non-holonomic chained system (1) in the framework of non-regular feedback linearization. In §2, a new sufficient condition for non-smooth non-regular feedback linearization is presented, and the second-order non-holonomic chained systems are then proved to be non-regular feedback linearizable. In §3, a discontinuous feedback control law is designed based on linear system theory and the inversion technique. The design mechanism is extended to general high-order non-holonomic chained systems in §4. The

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effectiveness of the proposed control law is illustrated through simulation study in § 5. The last section presents concluding remarks.

2. Mathematical preliminaries

Feedback linearization is a standard technique for control of many non-linear systems, for example, robotic systems (Spong and Vidyasagar 1989, Ge *et al.* 1998), motor drives (Taylor 1994, Chiasson 1998) among others. For non-holonomic systems, however, it is known that they are not regular static state feedback linearizable. Accordingly, the classical approach of feedback linearization cannot be applied. To cope with this difficulty, more general classes of feedback should be introduced. One candidate is dynamic feedback, but it means that the original systems must be augmented to include additional dynamics. Another possible choice is non-regular static state feedback, which does not introduce additional dynamics. It happens that the latter is applicable to the second-order non-holonomic chained systems as illustrated below.

Consider the multi-input affine non-linear system given by

$$\dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x) = f(x) + G(x)u \quad (2)$$

where $x \in \mathbb{R}^n$ are the states, $u \in \mathbb{R}^m$ are the inputs, entries of $f(x)$ and $G(x)$ are analytic functions of x , and $\text{rank } G(x) = m, \forall x \in \mathbb{R}^n$.

Non-linear control system (2) is said to be (non-smooth) non-regular (static state) feedback linearizable, if there exist a discontinuous state transformation

$$z = T(x), \quad z \in \mathbb{R}^n \quad (3)$$

and a non-regular state feedback

$$u(t) = \alpha(x) + \beta(x)v(t), \quad v \in \mathbb{R}^{m_0}, m_0 \leq m \quad (4)$$

such that the transformed system with state z and input v reads as a controllable linear system.

Note that state transformation (3) is not necessarily of diffeomorphism as required in Definition 2.1 of Sun and Xia (1997). However, its inverse transformation should be well defined everywhere except for on a lower-dimensional submanifold of \mathbb{R}^n . The term discontinuous is used to denote functions which are unbounded and hence undefined on a certain set (Astolfi 1996).

Remark 1: The problem of linearizing a non-linear system via non-regular state feedback arises naturally and was proposed as an open problem in Charlet *et al.* (1989). Though the concept is rather simple, it is very difficult to judge whether a system is non-regular feedback linearizable. Even if the answer is affirmative, it is also very hard to find a linearizing feedback and the

corresponding state coordinate transformation. Until now, only some preliminary results were reported in Sun and Xia (1997) and no extensive studies have been found in the literature.

The following lemma presents a new criterion for non-regular static state feedback linearizability.

Lemma 1: For a two-input affine non-linear system

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2, \quad x \in \mathbb{R}^n \quad (5)$$

suppose there exist a natural number $\mu \geq 1$ and a set of integer $1 \leq \kappa_1 < \kappa_2 < \dots < \kappa_\mu \leq n - 1$, such that the nested distributions defined by

$$\left. \begin{aligned} \Delta_0 &= \text{span} \{g_2\} \\ \Delta_i &= \Delta_{i-1} + \text{ad}_{g_1} \Delta_{i-1}, \quad i = \kappa_1, \dots, \kappa_{\mu-1} \\ \Delta_{\kappa_\mu} &= \Delta_{\kappa_\mu-1} + \text{span} \{g_1\} \\ \Delta_i &= \Delta_{i-1} + \text{ad}_f \Delta_{i-1}, \quad i \geq 1, i \neq \kappa_1, \dots, \kappa_\mu \end{aligned} \right\} \quad (6)$$

satisfy

- (i) Δ_i is involutive and of constant rank for $0 \leq i \leq n - 1$;
- (ii) $\text{rank } \Delta_{n-1} = n$;
- (iii) $[f, \Delta_{\kappa_j-1}] \subseteq \Delta_{\kappa_j-1}$ for $j = 1, \dots, \mu$; and
- (iv) $[g_1, \Delta_i] \subseteq \Delta_{i+1}$ for $0 \leq i \leq n - 3$

then system (5) is non-regular feedback linearizable.

Proof: By the Frobenius Theorem, there exist real-valued functions $\phi(x)$ and $h(x): \mathbb{R}^n \rightarrow \mathbb{R}$, such that

$$\left. \begin{aligned} d\phi \perp \Delta_{\kappa_\mu-2}, \quad d\phi \not\perp \Delta_{\kappa_\mu-1}, \\ dh \perp \Delta_{n-2}, \quad \text{and} \quad L_{\text{ad}_f^{n-\kappa_\mu-1} g_1} h = 1 \end{aligned} \right\} \quad (7)$$

Let

$$u_1 = \phi(x), \quad f_0(x) = f(x) + \phi(x)g_1(x)$$

System (5) can be rewritten as

$$\dot{x} = f_0(x) + u_2 g_2(x) \quad (8)$$

It follows from (6) that, if $L_f^i g_1 \in \Delta_j$ for some i and j , then $L_f^{i+1} g_2 \in \Delta_j$. This condition is repeatedly used in the following derivation.

Compute the quantities

$$L_{g_2} h = 0, \quad L_{f_0} h = L_f h + \phi L_{g_1} h = L_f h$$

The derivative of h along system (8) is given by

$$\frac{dh}{dt} = L_{f_0} h + (L_{g_2} h)u_2 = L_f h \quad (9)$$

Its higher derivatives are given by

$$\frac{d^2h}{dt^2} = L_{f_0}^2 h + (L_{g_2} L_{f_0} h) u_2 = L_f^2 h$$

⋮

$$\frac{d^{n-\kappa_\mu-1}h}{dt^{n-\kappa_\mu-1}} = L_{f_0}^{n-\kappa_\mu-1} h + (L_{g_2} L_{f_0}^{n-\kappa_\mu-1} h) u_2 = L_f^{n-\kappa_\mu-2} h$$

$$\begin{aligned} \frac{d^{n-\kappa_\mu}h}{dt^{n-\kappa_\mu}} &= L_{f_0}^{n-\kappa_\mu} h + (L_{g_2} L_{f_0}^{n-\kappa_\mu-1} h) u_2 \\ &= L_f^{n-\kappa_\mu} h + \phi L_{ad_f^{n-\kappa_\mu-1} g_1} h = L_f^{n-\kappa_\mu} h + \phi \end{aligned}$$

From assumptions (iii), (iv) and conditions (7), it follows that

$$\begin{aligned} L_{g_2} L_f^i h &= 0, \\ i &= 0, 1, \dots \end{aligned}$$

$$\begin{aligned} L_{g_2} L_f^i \phi &= 0, \\ i &= 0, 1, \dots \end{aligned}$$

$$\begin{aligned} L_{g_2} L_f^j L_{g_1} L_f^{\kappa_l - \kappa_{l-1} - 1} \dots L_{g_1} L_f^{\kappa_{\mu+1} - \kappa_\mu - 1} h &= 0, \\ l &= \mu, \dots, 2, j = 0, 1, \dots, \xi_{l-1} \end{aligned}$$

$$\begin{aligned} L_{g_2} L_f^j L_{g_1} L_f^{\kappa_l - \kappa_{l-1} - 1} \dots L_{g_1} L_f^{\kappa_\mu - \kappa_{\mu-1} - 1} L_{g_1} \phi &= 0, \\ l &= \mu, \dots, 2, j = 0, 1, \dots, \xi_{l-1} \end{aligned}$$

$$L_{g_2} L_f^{\kappa_1-1} L_{g_1} L_f^{\kappa_2 - \kappa_1 - 1} \dots L_{g_1} L_f^{\kappa_\mu - \kappa_{\mu-1} - 1} L_{g_1} \phi \neq 0$$

where $\kappa_0 = 1$, $\kappa_{\mu+1} = n + 1$, $\xi_l = \kappa_l - \kappa_{l-1} - 1$, $l = 1, \dots, \mu$.

Keeping these in mind, we can further compute higher order derivatives of h along system (8) as

$$\frac{d^{n-\kappa_\mu+1}h}{dt^{n-\kappa_\mu+1}} = L_{f_0}^{n-\kappa_\mu+1} h + (L_{g_2} L_{f_0}^{n-\kappa_\mu} h) u_2 = L_{f_0} L_f^{n-\kappa_\mu} h + L_{f_0} \phi$$

⋮

$$\frac{d^{n-2}h}{dt^{n-2}} = L_{f_0}^{n-2} h + (L_{g_2} L_{f_0}^{n-3} h) u_2 = L_{f_0}^{\kappa_\mu} L_f^{n-\kappa_\mu} h + L_{f_0}^{\kappa_\mu} \phi$$

$$\begin{aligned} \frac{d^{n-1}h}{dt^{n-1}} &= L_{f_0}^{n-1} h + (L_{g_2} L_{f_0}^{n-2} h) u_2 = L_{f_0}^{\kappa_{\mu+1}} L_f^{n-\kappa_\mu} h + L_{f_0}^{\kappa_{\mu+1}} \phi \\ &\quad + (L_{g_2} L_f^{\kappa_1-1} L_{g_1} L_f^{\kappa_2 - \kappa_1 - 1} \dots L_{g_1} L_f^{\kappa_\mu - \kappa_{\mu-1} - 1} L_{g_1} \phi) u_2 \end{aligned}$$

Define new coordinates z and new input v respectively as

$$\begin{aligned} z &= [h, L_f h, \dots, L_f^{n-\kappa_\mu} h, L_{f_0} L_f^{n-\kappa_\mu} h \\ &\quad + L_{f_0} \phi, \dots, L_{f_0}^{\kappa_{\mu+1}} L_f^{n-\kappa_\mu} h + L_{f_0}^{\kappa_{\mu+1}} \phi]^T \end{aligned} \quad (10)$$

$$\begin{aligned} v &= L_{f_0}^{\kappa_{\mu+1}} L_f^{n-\kappa_\mu} h + L_{f_0}^{\kappa_{\mu+1}} \phi \\ &\quad + (L_{g_2} L_f^{\kappa_1-1} L_{g_1} L_f^{\kappa_2 - \kappa_1 - 1} \dots L_{g_1} L_f^{\kappa_\mu - \kappa_{\mu-1} - 1} L_{g_1} \phi) u_2 \end{aligned} \quad (11)$$

The state space description of system (8) in the z coordinates is then given by

$$\dot{z} = [z_2, z_3, \dots, z_n, v]^T \quad (12)$$

which is exactly the single-input Brunovsky canonical system.

The above analysis shows that, under the state feedback

$$\left. \begin{aligned} u_1 &= \phi(x) \\ u_2 &= (L_{g_2} L_f^{\kappa_1-1} L_{g_1} L_f^{\kappa_2 - \kappa_1 - 1} \dots L_{g_1} L_f^{\kappa_\mu - \kappa_{\mu-1} - 1} L_{g_1} \phi)^{-1} \\ &\quad \times (v - L_{f_0}^{\kappa_{\mu+1}} L_f^{n-\kappa_\mu} h + L_{f_0}^{\kappa_{\mu+1}} \phi) \end{aligned} \right\} \quad (13)$$

and the coordinate transformation (10), system (5) changes into the single-input controllable system (12). Because the input channel u_1 is pure state feedback, the overall input transformation (13) is non-regular. \square

Remark 2: Note that the above analysis is essentially non-smooth analysis in that the functions involved are not limited to continuous functions. State transformation (10) is not necessarily of diffeomorphism, whereas input transformations (11) and (13) may not be well defined on a lower-dimensional submanifold of \mathfrak{R}^n .

As a direct application of Lemma 1, we claim that the high-order non-linear system

$$\left. \begin{aligned} y_1^{(r_1)} &= \xi_1(x^1, u_1) \\ y_2^{(r_2)} &= \xi_2(x, u_2) \\ y_i^{(r_i)} &= \xi_i(x^1, y_{i-1}, x^i, \dots, x^l, u_1), i = 3, \dots, l \end{aligned} \right\} \quad (14)$$

is non-regular feedback linearizable, where $l \geq 3$, $r_i \geq 1$, $x^i = [y_i, \dots, y_i^{(r_i-1)}]^T$, $i = 1, \dots, l$, $x = [x^1, \dots, x^l]^T$, and ξ_i , $i = 1, \dots, l$ are analytic functions vanishing at the origin with

$$\frac{\partial \xi_1}{\partial u_1} \neq 0, \quad \frac{\partial \xi_2}{\partial u_2} \neq 0, \quad \frac{\partial^2 \xi_i}{\partial y_{i-1} \partial u_1} \neq 0, \quad i = 3, \dots, l$$

The linearizing output $h(x)$ and the real-valued function $\phi(x)$ in the proof of Lemma 1 could be explicitly constructed, say

$$\left. \begin{aligned} h(x) &= x_1 \\ \phi(x) &= \phi_1(y_1, \dots, y_1^{(r_1)}, y_l) \quad \text{with } \frac{\partial \phi_1}{\partial y_l} \neq 0 \end{aligned} \right\} \quad (15)$$

Note that the above model includes the high order single chained systems

$$\left. \begin{aligned} y_1^{(r_1)} &= u_1 \\ y_2^{(r_2)} &= u_2 \\ y_3^{(r_3)} &= y_2 u_1 \\ &\vdots \\ y_l^{(r_l)} &= y_{l-1} u_1 \end{aligned} \right\}$$

as special cases. In general, system (14) and the generalized chained system (1) in Laiou and Astolfi (1999) are not special cases of each other. For example, the simple system

$$\left. \begin{aligned} \ddot{y}_1 &= u_1 \\ \ddot{y}_2 &= u_2 \\ \ddot{y}_3 &= y_2 u_1 + y_3^2 u_1 \end{aligned} \right\}$$

is in form (14) but is not in form (1) of Laiou and Astolfi (1999).

Remark 3: As the restriction imposed on $\phi(x)$ by (15) is not very restrictive, we can fully explore the freedom in controller design in practice. Different choices of $\phi(x)$ will result in different linearizing feedback transformations.

3. Controller design

For second-order non-holonomic chained system (1), a stabilizing strategy in the framework of non-regular feedback linearization is proposed in this section. The resulting control laws render the system exponentially convergent to the origin.

Let $x = [x_1, \dots, x_6]^T = [y_1, \dot{y}_1, y_2, \dot{y}_2, y_3, \dot{y}_3]^T$. System (1) can be rewritten in the state space description

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u_2 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= x_3 u_1 \end{aligned} \right\} \quad (16)$$

Let function $\phi(x) = y_3^{1/3} = x_5^{1/3}$ which satisfies relationship (15). From Lemma 1, we have

$$u_1 = x_5^{1/3} \quad (17)$$

Let $h(x) = x_1$. From (10), we have the state and input transformations

$$z = \left[x_1, x_2, x_5^{1/3}, \frac{1}{3} \frac{x_6}{x_5^{2/3}}, \frac{1}{3} \frac{x_3}{x_5^{1/3}} - \frac{2}{9} \frac{x_6^2}{x_5^{5/3}}, \frac{1}{3} \frac{x_4}{x_5^{1/3}} - \frac{5}{9} \frac{x_3 x_6}{x_5^{4/3}} + \frac{10}{27} \frac{x_6^3}{x_5^{8/3}} \right]^T \quad (18)$$

$$v = -\frac{5}{9} \frac{x_3^2}{x_5} + \frac{50}{27} \frac{x_3 x_6^2}{x_5^3} - \frac{2}{3} \frac{x_4 x_6}{x_5^{4/3}} - \frac{80}{81} \frac{x_6^4}{x_5^{11/3}} + \frac{1}{3} \frac{u_2}{x_5^{1/3}} \quad (19)$$

which are discontinuous at the discontinuous surface so defined by

$$S_D = \{x: x_5 = 0\} = \{z: z_3 = 0\} \quad (20)$$

The inverses of the state and input transformations are

$$x = [z_1, z_2, 3z_3 z_5 + 6z_4^2, 3z_3 z_6 + 15z_4 z_5, z_3^3, 3z_3^2 z_4]^T \quad (21)$$

$$u = [z_3, 3z_3 v + 18z_4 z_6 + 45z_5^2]^T \quad (22)$$

which are well-define and smooth on \mathbb{R}^6 .

The corresponding transformed linear system is of sixth-order Brunovsky canonical form

$$\dot{z} = [z_2, z_3, z_4, z_5, z_6, v]^T \quad (23)$$

For linear system (23), controller design can be carried out easily. The desired closed-loop poles can be assigned in different ways. In fact it can be assigned to any arbitrary (symmetric) set. For completeness and simplicity of the paper, let us assign all the poles at $-\lambda$ with $\lambda > 0$. Accordingly, the controller is given by

$$v = -\sum_{i=1}^6 k_i z_i \quad (24)$$

where

$$\begin{aligned} k_1 &= \lambda^6, & k_2 &= 6\lambda^5, & k_3 &= 15\lambda^4, \\ k_4 &= 20\lambda^3, & k_5 &= 15\lambda^2, & k_6 &= 6\lambda \end{aligned}$$

Define an unbounded and open subset of \mathbb{R}^6

$$\begin{aligned} D = \{x \in \mathbb{R}^6: & x_5(3\lambda + x_6) > 0, 9\lambda^2 x_5^2 \\ & + 6\lambda x_5 x_6 + 3x_3 x_5^{4/3} - 2x_6^2 > 0, \\ & \lambda^3 x_5^2 + \lambda^2 x_5 x_6 + \lambda x_3 x_5^{4/3} - \frac{2}{3} \lambda x_6^2 + \frac{1}{2} x_4 x_5^{4/3} \\ & - \frac{5}{9} x_3 x_5^{1/3} x_6 + \frac{10}{27} x_6^3 / x_5 > 0, \\ & (\lambda^5 x_1 + 5\lambda^4 x_2) x_5^{5/3} + 10\lambda^3 x_5^2 + \frac{10}{3} \lambda^2 x_5 x_6 \\ & + (\frac{5}{3} \lambda x_3 + \frac{1}{3} x_4) x_5^{4/3} - \frac{10}{9} \lambda x_6^2 - \frac{5}{9} x_3 x_5^{1/3} x_6 + \frac{10}{27} x_6^3 / x_5 > 0, \\ & (\lambda^5 x_1 + 6\lambda^4 x_2) x_5^{5/3} + 14\lambda^3 x_5^2 + \frac{16}{3} \lambda^2 x_5 x_6 \\ & + (3\lambda x_3 + \frac{2}{3} x_4) x_5^{4/3} \\ & - 2\lambda x_6^2 - \frac{10}{9} x_3 x_5^{1/3} x_6 + \frac{20}{27} x_6^3 / x_5 < 0\} \end{aligned}$$

The closed-loop stability is summarized in Theorem 1.

Theorem 1: *If $x(0) \in D$, then the feedback control law*

$$\left. \begin{aligned} u_1 &= x_5^{1/3} \\ u_2 &= -k_5 x_3 - k_6 x_4 - 3k_1 x_1 x_5^{1/3} - 3k_2 x_2 x_5^{1/3} \\ &\quad - 3k_3 x_5^{2/3} - k_4 \frac{x_6}{x_5^{1/3}} + \frac{5}{3} \frac{x_3^2}{x_5^{2/3}} \\ &\quad + \frac{5}{3} k_6 \frac{x_3 x_6}{x_5} + 2 \frac{x_4 x_6}{x_5} + \frac{2}{3} k_5 \frac{x_6^2}{x_5^{4/3}} \\ &\quad - \frac{50}{9} \frac{x_3 x_6^2}{x_5^2} - \frac{10}{9} k_6 \frac{x_6^3}{x_5^{7/3}} + \frac{80}{27} \frac{x_6^4}{x_5^{10/3}} \end{aligned} \right\} \quad (25)$$

drives the second-order non-holonomic chained system (16) to the origin at an exponential rate with bounded input.

Proof: Due to the smooth nature of state transformation (21), exponential convergence of $z(t)$ implies exponential convergence of $x(t)$. The linearizing transformations (18) and (19) are well-defined if the trajectory of $z(t)$ do not cross the discontinuous surface (20). As a consequence, if $z_3(0)z_3(t) > 0$ for all $t > 0$, then $x(t)$ exponentially converge to the origin.

For the closed-loop system of (23) and (24), it is straightforward to calculate that

$$z_3(t) = e^{-\lambda t} \sum_{i=1}^6 a_i t^{i-1} \quad (26)$$

where

$$\begin{aligned} a_1 &= z_3(0) \\ a_2 &= \lambda z_3(0) + z_4(0) \\ a_3 &= \frac{1}{2}(\lambda^2 z_3(0) + 2\lambda z_4(0) + z_5(0)) \\ a_4 &= \frac{1}{6}(\lambda^3 z_3(0) + 3\lambda^2 z_4(0) + 3\lambda z_5(0) + z_6(0)) \\ a_5 &= -\frac{1}{24}(\lambda^6 z_1(0) + 6\lambda^5 z_2(0) + 14\lambda^4 z_3(0) \\ &\quad + 16\lambda^3 z_4(0) + 9\lambda^2 z_5(0) + 2\lambda z_6(0)) \\ a_6 &= \frac{1}{120}(\lambda^7 z_1(0) + 5\lambda^6 z_2(0) + 10\lambda^5 z_3(0) \\ &\quad + 10\lambda^4 z_4(0) + 5\lambda^3 z_5(0) + \lambda^2 z_6(0)) \end{aligned}$$

Note that to guarantee $z_3(0)z_3(t) > 0, \forall t > 0$, a sufficient condition is $z_3(0)a_i > 0, i = 1, 2, \dots, 6$, which is equivalent to the assumption $x(0) \in D$. Accordingly, state $x(t)$ is bounded and converges to the origin at an exponential rate. By (22), the input $u(t)$ also exponentially converges to zero, and is bounded. \square

To make chained system (16) globally attractive, we only need to drive any initial configuration into set D by an appropriate control input. For completeness, a switching strategy is outlined below.

Firstly, if $x_3(0) = 0$, then apply $u_2(t) = k_1$ in duration $[0, t_1]$ to steer x_3 away from zero. Secondly, starting from t_1 , apply $u_1(t) = -k_2 \operatorname{sgn}(x_3 x_6)$ till $x_5(t_2) \neq 0$ and $x_6(t_2) = 0$. It can be verified that any

$$k_2 \neq \operatorname{sgn}(x_3 x_6) \frac{x_3(t_1)x_6^2(0)}{2x_5(0)}$$

with

$$t_2 = t_1 + \frac{|x_3(t_1)x_6(0)|}{k_2}$$

will do. Thirdly, apply the finite-time-stabilizing feedback controller (Bhat and Bernstein 1998)

$$u_2(t) = -k_3 [x_4^{1/3} + (x_3 + \frac{3}{5}x_4^{5/3})^{1/5}]$$

to achieve $x_3(t_3) = x_4(t_3) = 0$ in finite time $t_3 > t_2$. Fourthly, let

$$u_1(t) = -k_4 \operatorname{sgn}\left(x_1 - \frac{21}{\lambda^2} x_5^{1/3}\right) - k_5 \operatorname{sgn}\left(x_2 + \frac{6}{\lambda} x_5^{1/3}\right)$$

which steers the state trajectory into D in finite time t_4 . Finally, the feedback control law (25) steers chained system (16) approaching the origin at an exponentially convergent rate. All the k_i 's are positive numbers which can be assigned freely.

Denote the respective control laws as

$$\begin{aligned} u^1 &= \begin{bmatrix} 0 \\ k_1 \end{bmatrix} \\ u^2 &= \begin{bmatrix} -k_2 \operatorname{sgn}(x_3 x_6) \\ 0 \end{bmatrix} \\ u^3 &= \begin{bmatrix} 0 \\ -k_3 [x_4^{1/3} + (x_3 + \frac{3}{5}x_4^{5/3})^{1/5}] \end{bmatrix} \\ u^4 &= \begin{bmatrix} -k_4 \operatorname{sgn}\left(x_1 - \frac{6}{\lambda^2} x_5^{1/3}\right) - k_5 \operatorname{sgn}\left(x_2 + \frac{3}{\lambda} x_5^{1/3}\right) \\ 0 \end{bmatrix} \\ u^5 &= \begin{bmatrix} x_5^{1/3} \\ -k_5 x_3 - k_6 x_4 - 3k_1 x_1 x_5^{1/3} - 3k_2 x_2 x_5^{1/3} \\ -3k_3 x_5^{2/3} - k_4 \frac{x_6}{x_5^{1/3}} + \frac{5}{3} \frac{x_3^2}{x_5^{2/3}} \\ + \frac{5}{3} k_6 \frac{x_3 x_6}{x_5} + 2 \frac{x_4 x_6}{x_5} + \frac{2}{3} k_5 \frac{x_6^2}{x_5^{4/3}} \\ - \frac{50}{9} \frac{x_3 x_6^2}{x_5^2} - \frac{10}{9} k_6 \frac{x_6^3}{x_5^{7/3}} + \frac{80}{27} \frac{x_6^4}{x_5^{10/3}} \end{bmatrix} \end{aligned}$$

Define a logic-based switching controller

$$u = \begin{cases} u^5 & \text{if } x \in D \\ u^4 & \text{if } x \notin D \text{ and } x_3^2 + x_4^2 + x_6^2 = 0 \neq x_5 \\ u^3 & \text{if } x \notin D \text{ and } (x_3^2 + x_4^2)x_5 \neq 0 = x_6 \\ u^2 & \text{if } x \notin D \text{ and } x_3 \neq 0 \text{ and } (x_6 \neq 0 \text{ or } x_5 = 0) \\ u^1 & \text{else} \end{cases} \quad (27)$$

This controller steers system (16) globally approaching with an exponential rate of convergence. The total number of switching is less or equal to four.

Remark 4: Note that in the switching strategy (27), the numerically non-detectable criterion $x_3^2 + x_4^2 + x_6^2 = 0$ can be relaxed to $x_3^2 + x_4^2 + x_6^2 < \epsilon$ which are numerically detectable, where ϵ is an appropriate small positive constant. The resulting state configuration (before applying u^5) is

$$x^* = \left[\frac{21}{\lambda^2} x_5^{1/3}, \frac{-6}{\lambda} x_5^{1/3}, x_3, x_4, x_5, x_6 \right]$$

with $x_i, i = 3, 4, 6$ close to zero. Simple computation shows that, if

$$6\lambda \frac{|x_6|}{|x_5|} + 5\lambda^2 \frac{|x_3|}{x_5^{2/3}} + 2\lambda^3 \frac{|x_4|}{x_5^{2/3}} < 1$$

then $x^* \in D$. Accordingly, ϵ can be selected as

$$\epsilon = \frac{1}{6 \max \{ \lambda^{1/2}, \lambda^{3/2} \} \max \{ |x_5|^{1/2}, |x_5|^{1/3} \}}$$

which is not very restrictive.

Remark 5: As discussed in Remark 3, there might be many different choices for $\phi(x)$ satisfying relationship (15). Besides the choice made in the paper above, other candidates are also feasible. Indeed, it can be verified that any choice of the form $\phi(x) = x_5^{1/(2i+1)}$ with $i \geq 1$ will lead to (different) controllers for system (16). Generally speaking, the larger i is, the more computations are involved, and the slower the convergence of the resulting controller is.

Remark 6: Note that feedback controllers for linear system (23) are not limited to form (24). Other choices could also be made, and accordingly, different controllers for chained system (15) could be obtained by following the same procedure discussed above.

Remark 7: In Laiou and Astolfi (1999), the system is divided into the ‘base’ subsystem and the ‘extended’ subsystem. The base subsystem is linear and time-invariant, accordingly, linear feedback control laws can stabilize this subsystem. The resulting extended subsystem is linear and time-varying, and a time-varying design technique is utilized to present stabilizing feedback laws for it. The proposed controllers yield ex-

ponential convergence of the state to the origin. Similar ‘base-extended’ decomposition technique has also been used in Egeland and Berglund (1994) to develop a time-varying control strategy for system (1) to guarantee asymptotic stability with exponential convergence. In this paper, a new approach is based on non-regular feedback linearization of the whole system, thus it involves neither system division nor time-varying design technique. As the approaches are essentially different, the resulted controllers differ from each other accordingly.

4. Stabilization of high-order general chained systems

The design mechanism presented in §3 can be extended to the more general class of non-holonomic systems (14). The design procedure is outlined as follows.

Let the first control be

$$u_1 = \phi(x) = \begin{cases} y_1^{1/n} & \text{if } n \text{ is odd} \\ y_1^{1/(n+1)} & \text{if } n \text{ is even} \end{cases} \quad (28)$$

Denote the first-order state-space description of system (14) as

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2, \quad x \in \mathbb{R}^n$$

Let $f_0(x) = f(x) + g_1(x)\phi(x)$ and $g_0(x) = g_2(x)$.

Define a real-valued function $h(x) = x_1$, and new coordinates z and input v as

$$z = T(x) = [h, L_{f_0}h, L_{f_0}^2h, \dots, L_{f_0}^{n-1}h]^T \\ v = L_{f_0}^n h + u_2 L_{g_0} L_{f_0}^{n-1} h$$

The explicit expressions for z and v can be calculated routinely, though they become more and more tedious as n increasing. Note that $T(x)$ is discontinuous on the surface $S_D^n = \{x \in \mathbb{R}^n : y_l = 0\}$.

It may be verified that the inverse of $z = T(x)$, denoted by $x = T^{-1}(z)$, is a vector function with polynomial components. Therefore, exponential convergence of $z(t)$ implies exponential convergence of $x(t) = T^{-1}(z(t))$.

Consider the Brunovsky canonical system (12). It is standard that we may find a linear feedback control

$$v = -Kv = -\sum_{i=1}^n k_i z_i$$

to exponentially stabilize this system. Let W denote the subset of \mathbb{R}^n such that every trajectory of the closed-loop system started from W will never cross the surface S_D^n .

Define $D_n = \{x \in \mathbb{R}^n : T(x) \in W\}$. For non-holonomic system (14) with control law

$$\left. \begin{aligned} u_1 &= \phi(x) \\ u_2 &= -(L_{g_0} L_{f_0}^{n-1} h)^{-1} (L_{f_0}^n h(x) + KT(x)) \end{aligned} \right\} \quad (29)$$

every trajectory of the closed-loop system started within D_n will exponentially approach to the origin.

To achieve global stabilization, before applying control law (29), a control strategy must be exploited to drive an arbitrary configuration outside D_n into D_n in finite time. This strategy may depend on the parameters $k_i, i = 1, \dots, n$ and could be determined accordingly.

5. Simulation studies

In this section, simulation studies are carried out to demonstrate the effectiveness and procedure of the proposed approach.

Consider a PPR robot moving on a horizontal plain. All the three joints are passive and the only available inputs are forces acting on the end-effector. Let m_i be the mass of the i th link, d_3 the distance between the centre of mass of the third link and the third joint axis, l_3 the length of the third link, and \mathcal{I}_3 the central moment of inertia of the third link.

The dynamic model of the robot is (De Luca *et al.* 1998)

$$M(q_3)\ddot{q} + H(q_3, \dot{q}_3) = J^T(q_3)F \quad (30)$$

where $q_i, i = 1, 2, 3$ are the generalized coordinates, $F = [F_x, F_y]^T$ are the Cartesian forces acting on the end-effector, and

$$M(q_3) = \begin{bmatrix} a_1 & 0 & a_4 \cos q_3 \\ 0 & a_2 & -a_4 \sin q_3 \\ a_4 \cos q_3 & -a_4 \sin q_3 & -a_3 \end{bmatrix}$$

$$H(q_3, \dot{q}_3) = -a_4 \dot{q}_3^2 \begin{bmatrix} \sin q_3 \\ \cos q_3 \\ 0 \end{bmatrix}$$

$$J(q_3) = \begin{bmatrix} 0 & 1 & -l_3 \sin q_3 \\ 1 & 0 & l_3 \cos q_3 \end{bmatrix}$$

with

$$\begin{aligned} a_1 &= m_1 + m_2 + m_3 \\ a_2 &= m_2 + m_3 \\ a_3 &= \mathcal{I}_3 + m_3 d_3^2 \\ a_4 &= m_3 d_3 \end{aligned}$$

Under the condition that $q_3 \neq (\pi/2) + k\pi, k = 0, \pm 1, \pm 2, \dots$, the dynamic equations can be transformed into the second-order Caplygin form (De Luca *et al.* 1998)

$$\left. \begin{aligned} \ddot{q}_2 &= v_1 \\ \ddot{q}_3 &= v_2 \\ \ddot{q}_1 &= a_5 \tan q_3 v_1 + a_6 \sec q_3 v_2 \end{aligned} \right\} \quad (31)$$

where

$$\begin{aligned} v_1 &= \frac{(a_4^2 - a_1 a_3) a_4 c_3 \dot{q}_3^2}{\rho_1} + \frac{a_4 a_1 l_3 s_3^2 + a_4^2 c_3^2 - a_1 a_3}{\rho_1} F_x \\ &\quad + \frac{(a_4 - a_1 l_3) a_4 s_3 c_3}{\rho_1} F_y \\ v_2 &= \frac{(a_2 - a_1) a_4^2 s_3 c_3 \dot{q}_3^2}{\rho_2} + \frac{(a_2 l_3 - a_4) a_1 s_3}{\rho_2} F_x \\ &\quad + \frac{(a_4 - a_1 l_3) a_2 c_3}{\rho_2} F_y \\ a_5 &= \frac{a_4 - a_2 l_3}{a_4 - a_1 l_3}, \quad a_6 = \frac{a_4 l_3 - a_3}{a_4 - a_1 l_3}, \\ s_3 &= \sin q_3, \quad c_3 = \cos q_3 \\ \rho_1 &= -a_1 a_2 a_3 + a_1 a_4^2 - a_1 a_4^2 c_3^2 + a_4^2 a_2 c_3^2, \\ \rho_2 &= -a_1 a_2 a_3 + a_1 a_4^2 s_3^2 + a_2 a_4^2 c_3^2 \end{aligned}$$

System (31) can be converted to form (1) by the following state and input transformations

$$\left. \begin{aligned} y_1 &= q_2 - \frac{a_6}{a_5} (\cos q_3 - 1) \\ y_2 &= a_5 \tan q_3 \\ y_3 &= q_1 - a_6 \sin q_3 \end{aligned} \right\} \quad (32)$$

$$\left. \begin{aligned} u_1 &= v_1 + \frac{a_6}{a_5} \sin q_3 v_2 + \frac{a_6}{a_5} \cos q_3 \dot{q}_3^2 \\ u_2 &= a_5 \sec^2 q_3 v_2 + 2a_5 \sec^2 q_3 \tan q_3 \dot{q}_3^2 \end{aligned} \right\} \quad (33)$$

In summary, the original PPR robot model (30) can be transformed into system (1) by state transformation (32) and input transformation

$$\left. \begin{aligned} u_1 &= \frac{(a_4^2 - a_1 a_3) a_7 c_3 \dot{q}_3^2}{a_5 \rho_2} + \frac{(a_4^2 c_3^2 - a_1 a_3) a_5 + a_1 a_9 s_3^2}{a_5 \rho_2} F_x \\ &\quad + \frac{s_3 c_3 a_7 a_4 - a_1 l_3}{a_5 \rho_2} F_y \\ u_2 &= \frac{\sec^2 q_3 a_5 a_4^2 c_3 s_3 (a_2 - a_1) \dot{q}_3^2}{\rho_2} + 2a_5 \sec^2 q_3 \tan q_3 \dot{q}_3^2 \\ &\quad + \frac{a_1 a_5 a_8 s_3}{\rho_2} F_x + \frac{a_2 c_3}{\rho_2} F_y \end{aligned} \right\}$$

where

$[m_1, m_2, m_3]$	$[0.5, 0.5, 1.0]$ kg
\mathcal{I}_3	1 kg m^2
$[d_3, l_3]$	$[1, 2]$ m

Table 1. Parameters used for simulation.

$$a_7 = a_2 a_6 + a_4 a_5$$

$$a_8 = \sec^2 q_3 (a_2 l_3 - a_4)$$

$$a_9 = (a_2 a_6 l_3 + a_4 a_5 l_3 - a_4 a_6)$$

The values of the parameters $m_i, i = 1, 2, 3$ and \mathcal{I}, l in this simulation are shown in table 1.

Suppose the joint configuration is initially at

$$q_0 = (0.93, 5.43, -0.40\pi)[\text{m}, \text{m}, \text{rad}]$$

$$\dot{q}_0 = (0, 2.50, 0)[\text{m/s}, \text{m/s}, \text{rad/s}]$$

The final desired state is the equilibrium

$$q_d = \dot{q}_d = [0, 0, 0]^T$$

The corresponding initial state for system (1) is computed to be

$$x_0 = [5.50, -2.50, -2.20, 0.00, 1.00, 0.00]^T \in D$$

Accordingly, pure state feedback controller (25) can be used. Let $\lambda = 1$. Figures 1 and 2 show the convergence of the generalized coordinates and velocities, while figure 3 demonstrates the boundedness of the applied forces. As shown in figure 1, the angle q_3 never crosses the singular points $\pm\pi/2$.

If the joint configuration is initially at

$$q_0 = (5, 0, \pi/4)[\text{m}, \text{m}, \text{rad}]$$

$$\dot{q}_0 = (0, 0, 0) [\text{m/s}, \text{m/s}, \text{rad/s}]$$

Then the corresponding initial state for system (1) is

$$x_0 = [0.029, 0, 0.714, 0, 4.949, 0]^T \notin D$$

Because the initial state is outside set D , appropriate control laws have to be conducted to drive the system into D in a finite time, then the discontinuous controller is applicable. Figures 4 and 5 shows the converging trajectories of the generalized coordinates and velocities, respectively. Figure 6 demonstrates the boundedness of the applied forces. Due to the switching of the control signals, the trajectories are not smooth as shown in figures 4 and 5.

6. Conclusion

In this paper, a new non-regular feedback linearization mechanism has been proposed to steer the second-order non-holonomic chained systems approaching the origin with an exponential rate of convergence. A suffi-

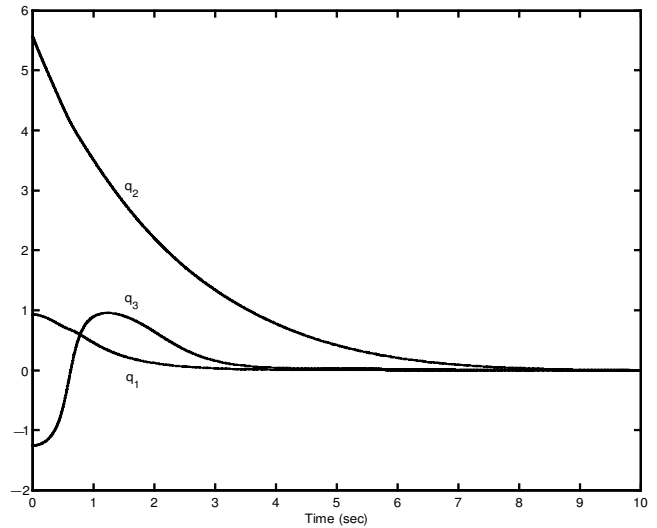


Figure 1. The generalized coordinates with continuous forces.

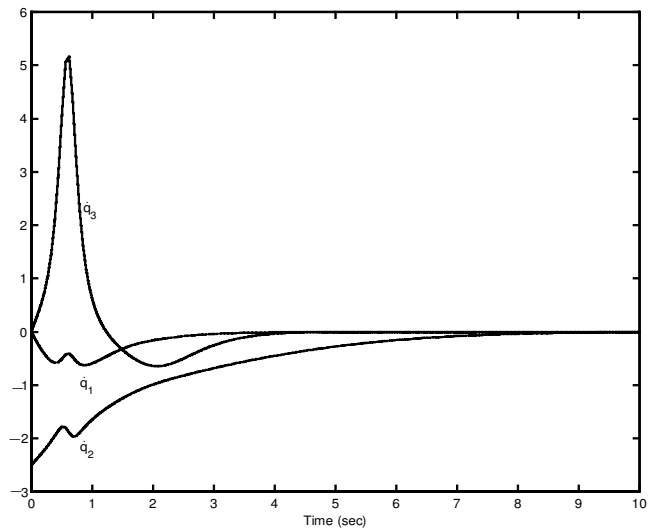


Figure 2. The generalized velocities with continuous forces.

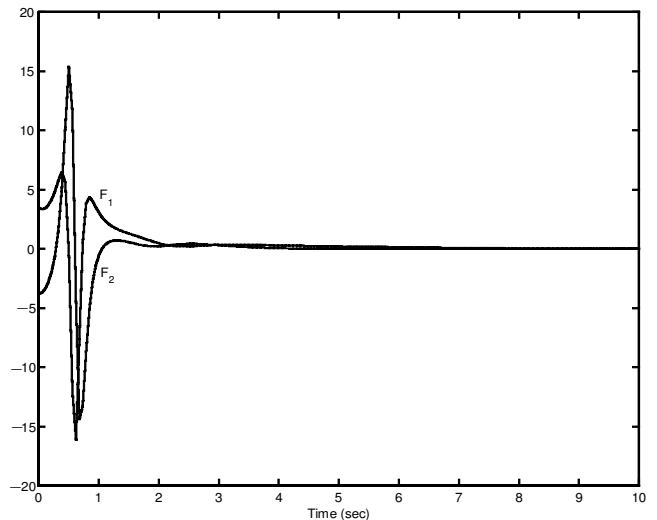


Figure 3. The applied forces.

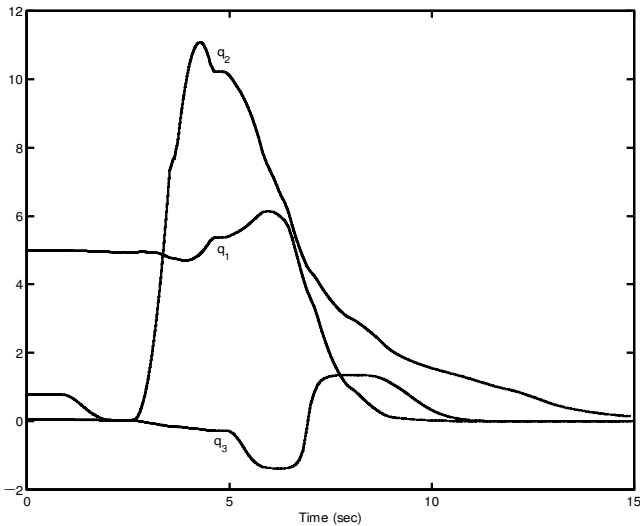


Figure 4. The generalized coordinates with discontinuous forces.

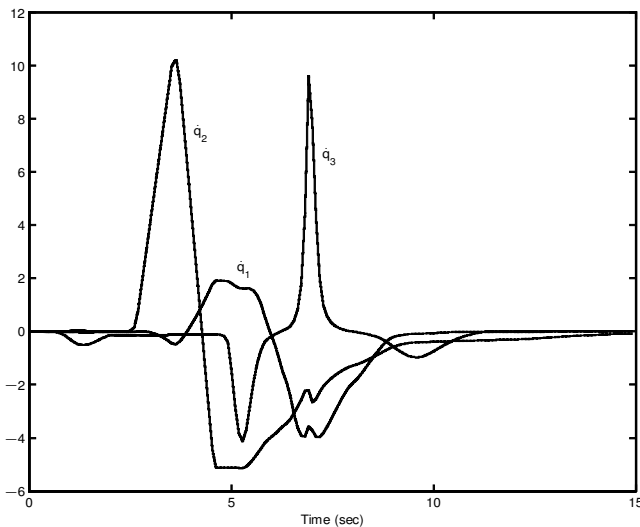


Figure 5. The generalized velocities with discontinuous forces.

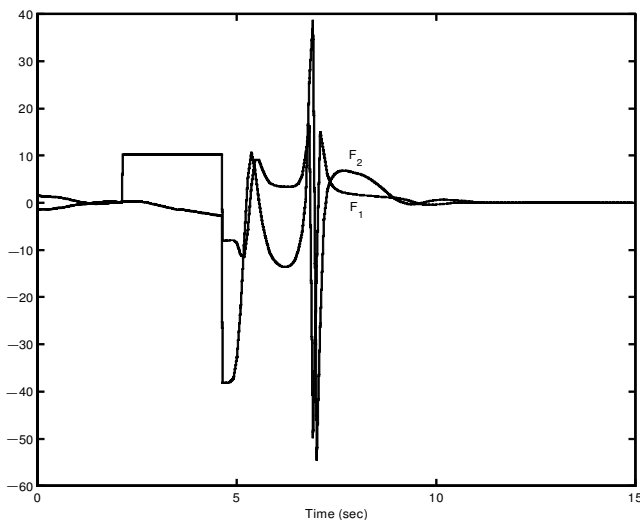


Figure 6. Trajectories of discontinuous forces.

cient condition for non-smooth non-regular feedback linearization was presented for affine non-linear systems with two inputs. It has been proven that the second-order non-holonomic chained systems are non-regular static state feedback linearizable. A discontinuous control law was designed based on linear system theory and the inversion technique. To globally stabilize the systems, a logic-based switching strategy among different control schemes was developed. A simulation study has also been provided to show the effectiveness of the control scheme.

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