



Stabilization of underactuated mechanical systems: a non-regular backstepping approach

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This paper presents a design framework for the stabilization of a class of underactuated mechanical systems. By utilizing non-regular static state feedbacks, these systems are transformed into a class of non-linear systems with chain structures. Then, controller design is presented by applying the backstepping design technique. The design procedure is applied to an underactuated robotic system and simulation tests are carried out for illustrating the effectiveness of the proposed approach.

1. Introduction

Underactuated mechanical systems are systems with fewer inputs than degrees of freedom. During the past few years, control system design of underactuated mechanical systems has attracted much attention in the literature, see, for example, Oriolo and Nakamura (1991), Seto and Baillieul (1994), De Luca (1998 a,b), Laiou and Astolfi (1999), Reyhanoglu *et al.* (1999), Su and Stepanenko(1999), and the references therein.

In Seto and Baillieul (1994), underactuated systems are classified into three types according to their control flow diagram representation, namely, the chain, tree and isolated-point structures. For a system with a chain structure, both the feedback linearization technique and the backstepping design procedure can be applied to design control laws. A system with a tree structure, however, does not share this advantage. The difficulty lies in the need to control certain variables in parallel, i.e. one input must control several degrees of freedom simultaneously. As for systems with isolated points, certain control goals are difficult to achieve because the control inputs have no influence on some variables at certain states. Control design for the last two classes of systems is presently not well understood.

In this paper, controller design is investigated for a class of underactuated mechanical systems with tree structures. The idea behind this approach is very simple. Through the introduction of appropriate non-regular static state feedback, the systems are transformed into systems with chain structures. Then, controller design can be easily carried out by applying the standard backstepping design procedure (Kanellakopoulos *et al.* 1991, Seto *et al.* 1994, Freeman and Kokotovic 1996, Sepulchre *et al.* 1997).

The paper is organized as follows. The problem formulation is given in §2. A sufficient condition is presented for transforming an underactuated mechanical

system into a system with a chain structure in §3. In §4, an underactuated robotic system is used for illustrating the design procedure and simulation verification. The last section presents a short concluding remark.

2. Problem formulation

Consider an underactuated mechanical system described by Reyhanoglu *et al.* (1999)

$$\left. \begin{aligned} M_1(q)\ddot{q}_1 + M_2(q)\ddot{q}_2 + N_1(q, \dot{q}) &= 0 \\ M_2^T(q)\ddot{q}_1 + M_3(q)\ddot{q}_2 + N_2(q, \dot{q}) &= \Lambda(q)\tau \end{aligned} \right\} \quad (1)$$

where $q = [q_1^T, q_2^T]^T$ is the vector of generalized coordinates, $q_1 \in \mathbb{R}^n$ represents the unactuated degrees of freedom, and $q_2 \in \mathbb{R}^m$ represents the actuated degrees of freedom

$$M(q) = \begin{bmatrix} M_1(q) & M_2(q) \\ M_2^T(q) & M_3(q) \end{bmatrix}$$

is the inertia matrix which is symmetric and positive definite for all q , $\Lambda(q)$ is invertible for all q , τ is the input generalized force produced by the m actuators, and $N_i(q, \dot{q}), i = 1, 2$ represent the Coriolis, centrifugal and perhaps gravitational and/or elastic force vectors. Throughout this paper, all functions are assumed to be smooth functions.

$$\begin{aligned} \text{Let } \bar{M}(q) &= M_3(q) - M_2^T(q)M_1^{-1}(q)M_2(q), \text{ and} \\ \tau &= \Lambda^{-1}(q)[N_2(q, \dot{q}) - M_2^T(q)M_1^{-1}(q)N_1(q, \dot{q}) + \bar{M}(q)u] \end{aligned} \quad (2)$$

where $u = [u_1, \dots, u_m]^T$ are new control inputs to be designed. Then system (1) can be rewritten in the form of partial feedback linearization (Spong 1996, Reyhanoglu *et al.* 1999)

$$\left. \begin{aligned} \ddot{q}_1 &= J(q)\ddot{q}_2 + R(q, \dot{q}) = J(q)u + R(q, \dot{q}) \\ \ddot{q}_2 &= u \end{aligned} \right\} \quad (3)$$

where

$$J(q) = -M_1^{-1}(q)M_2(q), \quad R(q, \dot{q}) = -M_1^{-1}(q)N_1(q, \dot{q})$$

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System (3) have a cascade form that captures the important feature of underactuated mechanical systems. The latter equation defines the decoupled and linearized dynamics of the m actuated degrees of freedom, while the former equation defines the coupled dynamics of the n unactuated degrees of freedom.

In general, the cascade system (3) is highly coupled and with a tree structure as defined in Seto and Baillieul (1994). Accordingly, one control input must control several degrees of freedom simultaneously, which makes the control and stabilization of all the degrees of freedom very difficult.

A second-order system with a chain structure is described by Seto and Baillieul (1994)

$$\left. \begin{aligned} \ddot{q}_i &= \phi_i(q_1, \dots, q_{i+1}, \dot{q}_1, \dots, \dot{q}_i), \quad i = 1, \dots, n-1 \\ \ddot{q}_n &= \phi_n(q, \dot{q}) + \gamma(q, \dot{q})v \end{aligned} \right\} (4)$$

where $v \in \mathfrak{R}$ is the input, $\gamma(q, \dot{q}) \neq 0$, and $\partial\phi_i/\partial q_{i+1} \neq 0, \forall q, \dot{q}$ for $i = 1, \dots, n$.

For a system with a chain structure, the backstepping design technique is applicable to obtain a control law which will exponentially stabilize the system (Kanellakopoulos *et al.* 1991, Seto *et al.* 1994).

It is readily seen that, if system (3) can be transformed into system (4) via an appropriate state feedback, then a stabilizing controller could be designed for (3) following the well-established backstepping design procedure. This simple observation motivates the use of non-regular static state feedbacks as triangulating transformations.

The problem studied in this paper can be formulated as follows.

Non-regular static state feedback triangulation (NSSFT): Given a multi-input underactuated mechanical system (3), find, if possible, a non-regular static state feedback (Sun and Xia 1997)

$$u = \alpha(q, \dot{q}) + \beta(q, \dot{q})v, \quad v \in \mathfrak{R} \quad (5)$$

such that the transformed system is with a chain structure as in (4).

Remark 1: Note that the gain matrix $\beta(q, \dot{q})$ in (5) is non-square (and hence singular) for the multi-actuator case (i.e. $m > 1$). The idea of using non-regular state feedbacks in control system design could be traced back to Morgan (1964) and Heymann (1968) for linear systems and later Tsinias and Kalouptsidis (1981, 1987) and Huijberts (1992) for non-linear systems.

Remark 2: The above problem shares some similarities with the problem of non-regular state feedback linearization (Sun and Xia 1997). However, feedback triangulation is more attractive because it is more closely tied to the physical description of the systems

and does not require any coordinate transformation. This approach can avoid some undesired properties of feedback linearization such as cancelation of beneficial non-linearities. It can also enhance robustness and softness through appropriate backstepping design of Lyapunov functions (Freeman and Kokotovic 1996).

Remark 3: The approach of NSSFT is mainly applicable to a class of multi-input underactuated systems with tree structures. Possible practical examples include robots with elastic joints (Tomei 1991, De Luca 1998 b) and robot with mixed rigid/elastic joints (De Luca 1998 a). This approach, however, is not directly applicable to underactuated systems which undergo non-integrable (non-holonomic) constraints (Bloch *et al.* 1992, Reyhanoglu *et al.* 1999). As a consequence of Brockett's theorem (Brockett 1983), a non-holonomic system is not triangulable via any smooth static state feedback.

3. Main results

In this section, we identify a class of underactuated mechanical systems which are non-regular static state feedback triangulable.

Consider an underactuated mechanical system given by

$$\ddot{x} = f(x, \dot{x}) + H(x, \dot{x})y + G(x, \dot{x})u \quad (6)$$

$$\ddot{y} = u \quad (7)$$

where $x \in \mathfrak{R}^n, y \in \mathfrak{R}^m$ are the generalized coordinates, $u \in \mathfrak{R}^m$ are the inputs, $f(x, \dot{x}) = [f_1, \dots, f_n]^T$ is a smooth vector field, $H(x, \dot{x})$ and $G(x, \dot{x})$ are $n \times m$ matrices of real-valued functions. We assume $m \geq n$, that is, the number of actuated joints is not less than the number of unactuated joints.

Assumption 1: *There exists a positive definite matrix $P(x, \dot{x}) \in \mathfrak{R}^{n \times n}$ such that*

$$H(x, \dot{x}) = P(x, \dot{x})H_0(x, \dot{x}), \quad G(x, \dot{x}) = P(x, \dot{x})G_0(x, \dot{x})$$

where matrix $H_0(x, \dot{x})$ has an upper triangular structure

$$H_0(x, \dot{x}) = \begin{bmatrix} h_{1,1}(x, \dot{x}) & h_{1,2}(x, \dot{x}) & \cdots & h_{1,m}(x, \dot{x}) & \cdots & h_{1,n}(x, \dot{x}) \\ 0 & h_{2,2}(x, \dot{x}) & \cdots & h_{2,m}(x, \dot{x}) & \cdots & h_{2,n}(x, \dot{x}) \\ & & \ddots & & & \\ 0 & 0 & \cdots & h_{n,m}(x, \dot{x}) & \cdots & h_{n,n}(x, \dot{x}) \end{bmatrix} \quad (8)$$

$$h_{i,i}(x, \dot{x}) \neq 0, \quad i = 1, \dots, n$$

and matrix $G_0(x, \dot{x})$ has a strictly upper triangular structure (see top of next page).

Theorem 1: *System (6)–(7) satisfying Assumption 1 is non-regular static state feedback triangulable.*

$$G_0(x, \dot{x}) = \begin{bmatrix} 0 & g_{1,2}(x, \dot{x}) & \cdots & g_{1,n}(x, \dot{x}) & g_{1,n+1}(x, \dot{x}) & \cdots & g_{1,m}(x, \dot{x}) \\ 0 & 0 & \cdots & g_{2,n}(x, \dot{x}) & g_{2,n+1}(x, \dot{x}) & \cdots & g_{2,m}(x, \dot{x}) \\ & & \ddots & & & & \\ 0 & 0 & \cdots & 0 & g_{n,n+1}(x, \dot{x}) & \cdots & g_{n,m}(x, \dot{x}) \end{bmatrix} \quad (9)$$

$$g_{i,i+1}(x, \dot{x}) \neq 0, \quad i = 1, \dots, n-1$$

Proof: First, if $m > n$, then let

$$\left. \begin{aligned} u_{n+1} &= \alpha_{n+1} = y_{n+2} + \psi_{n+1}(y_{n+1}) \\ &\vdots \\ u_{m-1} &= \alpha_{n+2} = y_m + \psi_{m-1}(y_{n+1}, \dots, y_{m-1}) \\ u_m &= \alpha_m = x_1 + \psi_m(y_{n+1}, \dots, y_{m-1}, y_m) \end{aligned} \right\} \quad (10)$$

where $\psi_i, i = n+1, \dots, m$ are any smooth functions of appropriate arguments.

Rewrite equations in (6) as

$$P^{-1}(\bar{x})(\ddot{x} - f(\bar{x})) = H_0(\bar{x})y + G_0(\bar{x})u \quad (11)$$

where $\bar{x} = [x^T, \dot{x}^T]^T$.

Let $d_i(\bar{x}) = [d_{i,1}(\bar{x}), \dots, d_{i,n}(\bar{x})]$ be the i th row of matrix $P^{-1}(\bar{x})$. By virtue of Assumption 1 and equations in (10), the last equation of (11) is given by

$$d_n(\bar{x})(\ddot{x} - f(\bar{x})) = \sum_{j=n}^m h_{n,j}(\bar{x})y_j + \sum_{j=n+1}^m g_{n,j}(\bar{x})\alpha_j \quad (12)$$

from which we have

$$\ddot{x}_n = f_n + d_{n,n}^{-1} \left(\sum_{j=n}^m h_{n,j}y_j + \sum_{j=n+1}^m g_{n,j}\alpha_j - \sum_{j=1}^{n-1} d_{n,j}(\ddot{x}_j - f_j) \right) \quad (13)$$

where the arguments of the related functions are omitted for clarity. Note that $d_{n,n}(\bar{x}) \neq 0$ from the positive definiteness of $P(\bar{x})$.

Next, let us design inputs $u_i, i = 2, \dots, n$, such that the x -dynamics have a chain structure, i.e.

$$\left. \begin{aligned} \ddot{x}_1 &= x_2 + \varphi_1(\varsigma, x_1) \\ &\vdots \\ \ddot{x}_{n-1} &= x_n + \varphi_{n-1}(\varsigma, x_1, \dots, x_{n-1}) \end{aligned} \right\} \quad (14)$$

where $\varsigma = [y_{n+1}, \dot{y}_{n+1}, \dots, y_m, \dot{y}_m]^T$ and $\varphi_i, i = 1, \dots, n-1$ are arbitrary given smooth functions.

Substituting (14) into (13) yields

$$\ddot{x}_n = l(\bar{x})y_n + \kappa(\bar{x}, \varsigma) \quad (15)$$

where $l(\bar{x}) = d_{n,n}^{-1}(\bar{x})h_{n,n}(\bar{x}) \neq 0$, and

$$\kappa(\bar{x}, \varsigma) = f_n + d_{n,n}^{-1} \left[\sum_{j=n+1}^m (h_{n,j}y_j + g_{n,j}\alpha_j) - \sum_{j=1}^{n-1} d_{n,j}(x_{j+1} + \varphi_j - f_j) \right]$$

The first $n-1$ equations of (11) are given by

$$\left. \begin{aligned} d_1(\bar{x})(\ddot{x} - f(\bar{x})) &= \sum_{j=1}^m h_{1,j}(\bar{x})y_j + \sum_{j=n+1}^m g_{1,j}(\bar{x})\alpha_j \\ &\quad + \sum_{j=2}^n g_{1,j}(\bar{x})u_j \\ &\vdots \\ d_{n-1}(\bar{x})(\ddot{x} - f(\bar{x})) &= \sum_{j=n-1}^m h_{n-1,j}(\bar{x})y_j \\ &\quad + \sum_{j=n+1}^n g_{n-1,j}(\bar{x})\alpha_j + g_{n-1,n}(\bar{x})u_n \end{aligned} \right\} \quad (16)$$

from which we can solve for $u_i, i = 2, \dots, n$ as

$$\left. \begin{aligned} u_n &= \alpha_n = c_n(\bar{x}, \varsigma, y_n) + e_n(\bar{x})y_{n-1} \\ u_{n-1} &= \alpha_{n-1} = c_{n-1}(\bar{x}, \varsigma, y_{n-1}, y_n) + e_{n-1}(\bar{x})y_{n-2} \\ &\vdots \\ u_2 &= \alpha_2 = c_2(\bar{x}, \varsigma, y_2, \dots, y_n) + e_2(\bar{x})y_1 \end{aligned} \right\} \quad (17)$$

where

$$c_n = g_{n-1,n}^{-1}(\bar{x}) \left[d_{n-1}(\bar{x})(\sigma - f(\bar{x})) - \sum_{j=n}^m h_{n-1,j}(\bar{x})y_j - \sum_{j=n+1}^m g_{n-1,j}(\bar{x})\alpha_j \right]$$

$$e_n = -g_{n-1,n}^{-1}(\bar{x})h_{n-1,n-1}(\bar{x})$$

\vdots

$$c_2 = g_{1,2}^{-1}(\bar{x}) \left[d_1(\bar{x})(\sigma - f(\bar{x})) - \sum_{j=2}^m h_{1,j}(\bar{x})y_j - \sum_{j=3}^m g_{1,j}(\bar{x})\alpha_j \right]$$

$$e_2 = -g_{1,2}^{-1}(\bar{x})h_{1,1}(\bar{x})$$

$$\sigma = [x_2 + \varphi_1(\varsigma, x_1), \dots, x_n + \varphi_{n-1}(\varsigma, x_1, \dots, x_{n-1}), l(\bar{x})y_n + \kappa(\bar{x}, \varsigma)]^T$$

Note that $e_2(\bar{x}), \dots, e_n(\bar{x})$ are non-zero real-valued functions.

Finally, let

$$u_1 = v \tag{18}$$

Gathering (10), (17) and (18), the non-regular static state feedback is given by

$$u = \begin{bmatrix} 0 \\ c_2(\bar{x}, \varsigma, y_2, \dots, y_n) + e_2(\bar{x})y_1 \\ \vdots \\ c_n(\bar{x}, \varsigma, y_n) + e_n(\bar{x})y_{n-1} \\ y_{n+2} + \psi_{n+1}(y_{n+1}) \\ \vdots \\ x_1 + \psi_m(y_{n+1}, \dots, y_{m-1}, y_m) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} v \tag{19}$$

Accordingly, the overall system of (6)–(7) and (19) is given by

$$\left. \begin{aligned} \ddot{y}_{n+1} &= y_{n+2} + \psi_{n+1}(y_{n+1}) \\ &\vdots \\ \ddot{y}_{m-1} &= y_m + \psi_{m-1}(y_{n+1}, \dots, y_{m-1}) \\ \ddot{y}_m &= x_1 + \psi_m(y_{n+1}, \dots, y_{m-1}, y_m) \\ \ddot{x}_1 &= x_2 + \varphi_1(\varsigma, x_1) \\ &\vdots \\ \ddot{x}_{n-1} &= x_n + \varphi_{n-1}(\varsigma, x_1, \dots, x_{n-1}) \\ \ddot{x}_n &= l(\bar{x})y_n + \kappa(\bar{x}, \varsigma) \\ \ddot{y}_n &= e_n(\bar{x})y_{n-1} + c_n(\bar{x}, \varsigma, y_n) \\ &\vdots \\ \ddot{y}_2 &= e_2(\bar{x})y_1 + c_2(\bar{x}, \varsigma, y_2, \dots, y_n) \\ \ddot{y}_1 &= v \end{aligned} \right\} \tag{20}$$

which has a chain structure. □

Remark 4: Note that there are some flexibilities in the triangulating control law (19) and the resulting system (20). In particular, the functions $\psi_i, i = n+1, \dots, m$ and $\varphi_i, i = 1, \dots, n-1$ are arbitrary smooth functions which can be chosen by the designer. These design flexibilities can be utilized to achieve certain control objectives other than stability.

Remark 5: It is readily seen that the upper triangular assumption of H_0 and G_0 in Assumption 1 can be replaced by the dual lower triangular assumption without violating the validity of Theorem 1.

Remark 6: If $m > n$, Assumption 1 can be relaxed to some extent. In fact, the upper triangular assumption (8) can be replaced by

$$H_0 = [0_k, \hat{H}_0] \quad 0 \leq k \leq m - n \quad \text{with } \hat{H}_0 \text{ in an upper triangular form}$$

where 0_k is $n \times k$ matrix whose entries are zeros. Similarly, the strict upper triangular assumption (9) can be replaced by

$$G_0 = [0_\mu, \hat{G}_0] \quad 0 \leq \mu \leq m - n \quad \text{with } \hat{G}_0 \text{ in a strict upper triangular form}$$

Remark 7: Theorem 1 can be readily extended to higher order systems

$$\left. \begin{aligned} x^{(k)} &= f(\bar{x}) + H(\bar{x})y + G(\bar{x})u \\ y^{(k)} &= u \end{aligned} \right\} \tag{21}$$

where $k > 2$, and

$$\bar{x} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(k-1)} \end{bmatrix}$$

4. Case Study

Consider a planar robot with two flexible revolute joints. Under the assumptions that the joint flexibilities are modelled as linear torsional springs and the rotors of the actuators are modelled as uniform bodies of revolution, the dynamics are given by Tomei (1991) and De Luca (1998 b)

$$\left. \begin{aligned} M_1(q_1)\ddot{q}_1 + M_2\ddot{q}_2 + C_1(q_1, \dot{q}_1)\dot{q}_1 \\ + K_e(q_1 - q_2) + \Gamma_1(q_1) + F_1(q_1, \dot{q}_1) = 0 \\ M_2^T\ddot{q}_1 + M_3\ddot{q}_2 + K_e(q_2 - q_1) + F_2(q_2, \dot{q}_2) = \tau \end{aligned} \right\} \tag{22}$$

where $q_1 = (q_{1,1}, q_{1,2})$ and $q_2 = (q_{2,1}, q_{2,2})$ represent the positions of the links and the actuators, respectively

$$M(q) = \begin{bmatrix} M_1(q_1) & M_2 \\ M_2^T & M_3 \end{bmatrix}$$

is the inertia matrix which is symmetric and positive definite, $C_1(q_1, \dot{q}_1)$ is the centripetal and Coriolis matrix, $K_e = \text{diag}[k_1, k_2]$ is the diagonal stiffness matrix whose entries are spring constants of the joints, $\Gamma_1(q_1)$ is the gravitational force vector, $F_1(q_1, \dot{q}_1)$ and $F_2(q_2, \dot{q}_2)$ represent the friction forces, and $\tau \in \mathbb{R}^n$ are the torques supplied by the motors.

Matrix M_2 has a strictly upper triangular structure given by

$$M_2 = \begin{bmatrix} 0 & m_{1,2} \\ 0 & 0 \end{bmatrix} \quad (23)$$

When $M_2 = 0$, that is, $m_{1,2} = 0$, there is no inertial coupling between the links and the actuators, and model (22) is reduced to the well-known simplified model (Spong 1987). This simplified model is always regular static state feedback linearizable. To avoid this case, we assume that $M_2 \neq 0$. In this case, it can be verified that the system is neither feedback linearizable nor decouplable via any regular static state feedback.

Let

$$\begin{aligned} \tau = & -M_2^T M_1^{-1}(q_1)(C_1(q_1, \dot{q}_1)\dot{q}_1 + K_e(q_1 - q_2) \\ & + \Gamma_1(q_1) + F_1(q_1, \dot{q}_1)) + K_e(q_2 - q_1) \\ & + F_2(q, \dot{q}) + (M_3 - M_2^T M_1^{-1}(q_1)M_2)u \end{aligned} \quad (24)$$

System (22) can be rewritten as

$$\left. \begin{aligned} \ddot{q}_1 = & -M_1^{-1}(q_1)(C_1(q_1, \dot{q}_1)\dot{q}_1 + K_e(q_1 - q_2) \\ & + \Gamma_1(q_1) + F_1(q_1, \dot{q}_1)) - M_1^{-1}(q_1)M_2u \\ \ddot{q}_2 = & u \end{aligned} \right\} \quad (25)$$

which is in form (3).

It can be verified that Theorem 1 is applicable to system (25). Accordingly, it is non-regular static state feedback triangulable. Following the procedure proposed in the proof of Theorem 1, a triangulating feedback can be constructed as follows.

First, rewrite system (25) as

$$\left. \begin{aligned} \ddot{q}_1 = & J(q_1, \dot{q}_1) + M_1^{-1}(q_1)(K_e q_2 - M_2 u) \\ \ddot{q}_2 = & u \end{aligned} \right\} \quad (26)$$

where

$$\begin{aligned} J(q_1, \dot{q}_1) = & \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = -M_1^{-1}(q_1)(C_1(q_1, \dot{q}_1)\dot{q}_1 \\ & + K_e q_1 + \Gamma_1(q_1) + F_1(q_1, \dot{q}_1)) \end{aligned} \quad (27)$$

Next, let us design the second input channel u_2 such that q_1 can be linearized to be a single chain of integrator, i.e.

$$\ddot{q}_{1,1} = q_{1,2} \quad (28)$$

Denote

$$M_1^{-1}(q_1) = \begin{bmatrix} d_1(q_1) & d_2(q_1) \\ d_2(q_1) & d_3(q_1) \end{bmatrix}$$

Substituting (28) into (26) gives

$$J_1(q_1, \dot{q}_1) + [d_1(q_1), d_2(q_1)](K_e q_2 - M_2 u) = q_{1,2}$$

from which we can solve for u_2 as

$$\begin{aligned} u_2 = & \alpha_2(q, \dot{q}) \\ = & \frac{1}{d_1(q_1)m_{1,2}}(J_1(q_1, \dot{q}_1) - q_{1,2} \\ & + [d_1(q_1), d_2(q_1)]K_e q_2) \end{aligned} \quad (29)$$

Finally, let

$$u_1 = v \quad (30)$$

The overall system of (26), (29) and (30) is given by

$$\left. \begin{aligned} \ddot{q}_{1,1} = & q_{1,2} \\ \ddot{q}_{1,2} = & \rho_1(q_1, \dot{q}_1) + \eta_1(q_1)q_{2,2} \\ \ddot{q}_{2,2} = & \rho_2(q_1, \dot{q}_1, q_{2,2}) + \eta_2 q_{2,1} \\ \ddot{q}_{2,1} = & v \end{aligned} \right\} \quad (31)$$

where

$$\begin{aligned} \rho_1 = & J_2(q_1, \dot{q}_1) + \frac{d_2(q_1)}{d_1(q_1)}(q_{1,2} - J_1(q_1, \dot{q}_1)) \\ \eta_1 = & d_3(q_1) - \frac{d_2^2(q_1)}{d_1(q_1)} \\ \rho_2 = & \frac{1}{d_1(q_1)m_{1,2}}(J_1(q_1, \dot{q}_1) - q_{1,2} + d_2(q_1)k_2 q_{2,2}) \\ \eta_2 = & \frac{d_1(q_1)k_1}{d_1(q_1)m_{1,2}} \end{aligned}$$

It is readily seen that system (31) processes a chain structure.

Substituting (29) and (30) into (24), we have the non-regular static state feedback

$$\begin{aligned} \tau = & -M_2^T M_1^{-1}(q_1)(C_1(q_1, \dot{q}_1)\dot{q}_1 + K_e(q_1 - q_2) \\ & + \Gamma_1(q_1) + F_1(q_1, \dot{q}_1)) + K_e(q_2 - q_1) + F_2(q, \dot{q}) \\ & + (M_3 - M_2^T M_1^{-1}(q_1)M_2) \begin{bmatrix} 0 \\ \alpha_2(q, \dot{q}) \end{bmatrix} + M_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \end{aligned} \quad (32)$$

which solves the problem of non-regular static state feedback triangulation for the flexible joint robots (22).

For system (31), the backstepping design technique is readily applied to obtain a stabilizing control law

$v = v(q, \dot{q})$. The corresponding torque input τ computed through equations (32) will steer the robot system (22) globally asymptotically stable. In addition, the rate of convergence can be designed through adjusting the gain constants of the backstepping design procedure.

For a simulation study, let us consider system (22) with

$$M_1(q_1) = \begin{bmatrix} p_1 + p_2 \cos^2 q_{1,2} & 0 \\ 0 & p_3 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0 & p_4 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} p_5 & 0 \\ 0 & p_6 \end{bmatrix}$$

$$C_1(q_1, \dot{q}_1) = \begin{bmatrix} -0.5p_2\dot{q}_{1,2} \sin 2q_{1,2} & -0.5p_2\dot{q}_{1,1} \sin 2q_{1,2} \\ 0.5p_2\dot{q}_{1,1} \sin 2q_{1,2} & 0 \end{bmatrix}$$

$$\Gamma_1(q_1) = \begin{bmatrix} 0 \\ p_7 \cos q_{1,2} \end{bmatrix}$$

$$F_1(q_1, \dot{q}_1) = F_2(q_2, \dot{q}_2) = 0$$

with $p_i, i = 1, \dots, 7$ are constant parameters.

A triangulating feedback can be computed from (32) as

$$\tau_1 = k_1 q_{2,1} - k_1 q_{1,1} + p_5 v$$

$$\tau_2 = -\frac{1}{p_4} (p_2 p_6 q_{1,2} \cos^2 q_{1,2} - p_2 p_6 \dot{q}_{1,1} \dot{q}_{1,2} \sin 2q_{1,2} \\ + k_2 p_4 q_{1,2} + p_6 p_1 q_{1,2} - k_2 p_4 q_{2,2} \\ + p_6 k_1 q_{1,1} - p_6 k_1 q_{2,1} - p_4^2 q_{1,2})$$

The transformed system is given by

$$\left. \begin{aligned} \ddot{q}_{1,1} &= q_{1,2} \\ \ddot{q}_{1,2} &= \frac{k_2}{p_3} q_{2,2} - \frac{1}{2p_3} (p_2 \dot{q}_{1,1}^2 \sin 2q_{1,2} \\ &\quad + 2k_2 q_{1,2} + 2p_7 \cos q_{1,2}) \\ \ddot{q}_{2,2} &= \frac{k_1}{p_4} q_{2,1} - \frac{1}{p_4} (-p_2 \dot{q}_{1,1} \dot{q}_{1,2} \sin 2q_{1,2} \\ &\quad + k_1 q_{1,1} + q_{1,2} p_1 + q_{1,2} p_2 \cos^2 q_{1,2}) \\ \ddot{q}_{2,1} &= v \end{aligned} \right\} \quad (33)$$

It can be verified that the only equilibrium point is $q_e = [0, 0, 0, p_7/k_2]^T$.

A stabilizing control law can be designed for (33) by utilizing the backstepping design procedure (Seto and Baillieul 1994, Theorem 4.1).

The values of the parameters $p_i, i = 1, \dots, 7$ and $k_i, i = 1, 2$ in this simulation are shown in table 1.

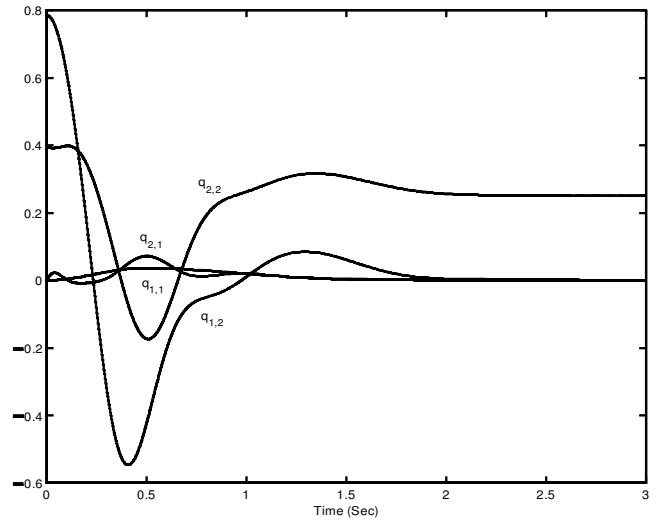


Figure 1. Trajectories of the link positions.

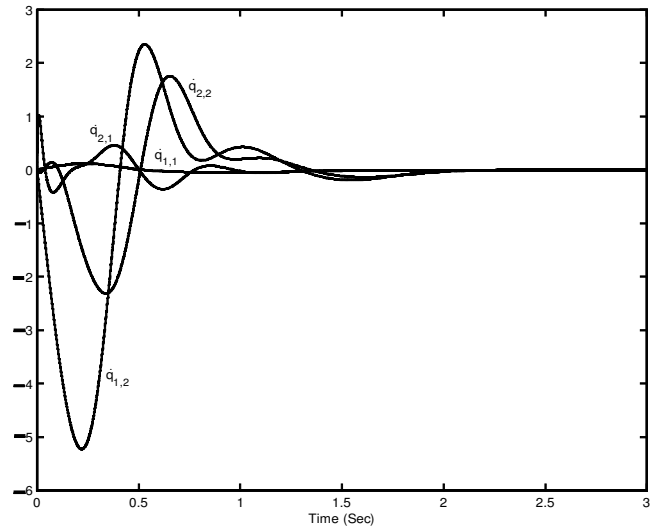


Figure 2. Trajectories of the link velocities.

Assume the manipulator is initially at rest with $q(0) = [0, \pi/4, 0, \pi/8]^T$ and $\dot{q}(0) = [0, 0, 0, 0]^T$. The trajectories of the link positions and link velocities are depicted in figures 1 and 2, respectively. It can be seen that the link positions converge to the equilibrium point at an exponential rate and with a satisfactory transient process. The applied torques are shown in figure 3. Note that steady state of τ_2 is equal to p_7 , which is exactly the effort needed to resist the gravity effect at the equilibrium point.

5. Conclusion

In this paper, a non-regular static state feedback triangulation approach has been proposed to design stabilizing controllers for a class of multi-input underactuated mechanical systems. The systems were transformed

$[p_1, p_2, p_3, p_4, p_5, p_6, p_7]$	$[1.5, 0.5, 2.0, 0.2, 1.0, 1.0, 50.0] \text{ kg m}^2$
$[k_1, k_2]$	$[100, 200] \text{ N m}^{-1}$

Table 1. Parameters used for simulation.

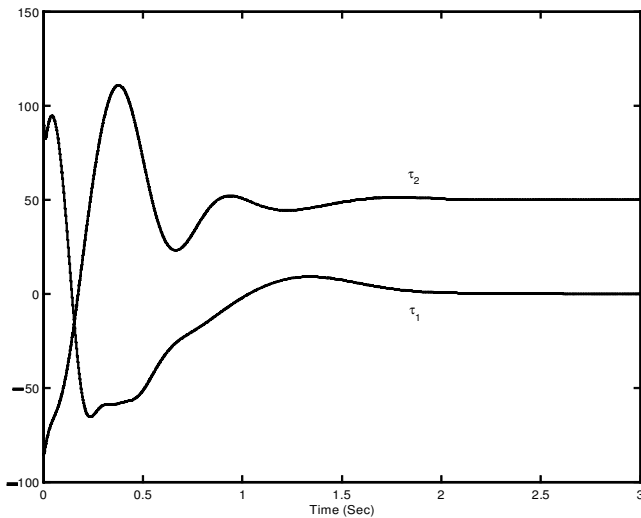


Figure 3. Trajectories of applied torques.

into a class of systems with chain structures via appropriate non-regular static state feedbacks, thus enable us to design a control law by applying the standard backstepping design procedure. A criterion for non-regular static state feedback triangulation was presented for a class of underactuated systems with tree structures. As an illustrating example, the design procedure has been applied to an underactuated robotic system and simulation tests have been carried out to show the effectiveness of the proposed approach.

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