

A global Implicit Function Theorem without initial point and its applications to control of non-affine systems of high dimensions [☆]

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Abstract

Control system design for non-affine systems is a difficult problem because of the lack of mathematical tools. The key to the problem is solving for an implicit function but the known results for implicit functions are not applicable for higher dimensional systems except for single-input and single-output systems. In this paper, a new version of a global implicit function theorem in higher dimension is presented and proved. This result can be applied to show the controllability of a class of non-affine multi-input and multi-output (MIMO) system so that approximation based control system design can be applied with ease.

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1. Introduction

The implicit function problem is to solve for the implicitly defined functions $u = g(x)$ from a functional equation $f(x, u) = 0$. The basic and well-known version of the Implicit Function Theorem is as follows.

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Lemma 1 (Implicit Function Theorem). *Assume that $f: \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^m$, $V \times U \subset \mathbf{R}^n \times \mathbf{R}^m$ is a neighborhood of $(x_0, u_0) \in \mathbf{R}^n \times \mathbf{R}^m$, and that f is continuous on $V \times U$ and is continuously differentiable in the second variable $u \in U$ for each fixed $x \in V$. If*

$$f(x_0, u_0) = 0, \quad \det \frac{\partial}{\partial u} f(x_0, u_0) \neq 0, \quad (1)$$

then there exist a neighborhood $V_0 \subset V$ of x_0 and a unique continuous mapping $g: V_0 \rightarrow \mathbf{R}^m$ such that $f(x, g(x)) = 0$ and $g(x_0) = u_0$. Moreover, if additionally f is continuously differentiable, then the obtained g is also continuously differentiable.

The Implicit Function Theorem and its variations are very important tools in the study of nonlinear control, mechanics and other engineering applications (see, for example, [3,6–9,16]). Along the path of research in nonlinear sciences, many updated requirements and concepts such as nonsmoothness, measurability, analyticity, convexity, Jacobian vanishing, degeneracy and approximation are raised, and in fact, many of these are practical problems. Those requirements motivated many generalizations of the Implicit Function Theorem in various spaces and manifolds [1,4,5,17,20]. The book [18] by S.G. Krantz and H.R. Parks gives an extensive survey on progress of implicit function theorems.

In control system design, most systems considered are affine in the control u for technical reasons, and, as said in [10], no effective method for controlling non-affine systems exists in the literature at the present stage, especially for multi-input and multi-output systems. A single-input single-output (SISO) non-affine nonlinear system was discussed in [8], where the input u and output y are both one-dimensional, using the basic result Implicit Function Theorem (Lemma 1) which only defines the function (mapping) g locally and requires assuming the existence of the point (x_0, u_0) which satisfies $f(x_0, u_0) = 0$. The existence of a continuous ideal control input u^* requires a (semi-)global version of the Implicit Function Theorem. The needed global result was actually given later in [10] without the stringent condition $f(x_0, u_0) = 0$ and a control system was constructed using neural networks although they can be constructed using any other approximation tools as stated in [9].

Global existence for the implicit function problem is also interesting and many versions of global implicit function theorem have been given. In 1981, I.W. Sandberg [22] discussed the problem on convex open subset V and open subset U of normed linear spaces and provided a necessary and sufficient condition for existence and uniqueness of $g: V \rightarrow U$ which satisfies $f(x, g(x)) = 0$. This condition requires that

- (S1) for some $x_0 \in V$ there is exactly one $u_0 \in U$ such that $f(x_0, u_0) = 0$,
- (S2) f is locally solvable for u , and
- (S3) for each $S \in \mathcal{A}$ there is a $T \in \mathcal{B}$, where \mathcal{A} and \mathcal{B} are special families of compact subsets of V and U respectively, such that $f(x, u) = 0$ with $(x, u) \in S \times U$ implies $x \in T$.

In 1985 Ichiraku [13] proved a topological global implicit function theorem. Fix $\theta \in W$, W a metric space, and let $G_{f,\theta} := \{(x, u) \in X \times Y: f(x, u) = \theta\}$ for the continuous map $f: X \times Y \rightarrow W$, where X, Y are a simply connected metric space and a globally path-connected metric space, respectively. If

- (I1) for every $x \in X$ the map $u \mapsto f(x, u)$ is locally homeomorphism, and
- (I2) each sequence $(x_n, u_n) \in G_{f,\theta}$ with (x_n) convergent has a convergent subsequence,

then the projection $\text{pr}: G_{f,\theta} \rightarrow X$ is a homeomorphism and $G_{f,\theta}$ can be seen to be the graph of a continuous map $g: X \rightarrow Y$ such that $f(x, g(x)) = \theta$ for all $x \in X$. In 1991 Blot [2] introduced and studied the maximal (nonextendable) implicit functions in a purely topological setting. Let $f: X \times Y \rightarrow W$ be a continuous map on topological spaces. Fix $\theta \in W$. Under the condition of local solvability, Blot studied relations between solutions of the functional equation $f(x, g(x)) = \theta$ and the connected components of $f^{-1}(\theta)$.

These known results are not applicable to the problem of controlling non-affine systems. As in [8], one has no way to check the existence of the pair (x_0, u_0) in Sandberg’s condition (S1). Checking (S3) and (I2) is also difficult. These difficulties were overcome using a result of one-dimensional global implicit functions as given in [10] without assuming the existence of a point (x_0, u_0) which satisfies $f(x_0, u_0) = 0$.

Lemma 2 ([10, Lemma 1] and [9, Lemma 2.8]). *Assume that $f: \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$ is continuously differentiable and there exists a positive constant d such that $\|\frac{\partial}{\partial u} f(x, u)\| > d > 0$ for all $(x, u) \in \mathbf{R}^n \times \mathbf{R}$. Then there exists a unique continuous (smooth) function $g: \mathbf{R}^n \rightarrow \mathbf{R}$ such that $f(x, g(x)) = 0$.*

As we know, one major difficulty in controlling non-affine systems comes from the solvability of a global implicit function without assuming the existence of a point (x_0, u_0) which satisfies $f(x_0, u_0) = 0$. The above mentioned result (Lemma 2) can only be used to discuss SISO problems (i.e., one-dimensional problems). An interesting problem is to generalize Lemma 2 to higher dimensions. Unfortunately the method used in [10] for Lemma 2 cannot be generalized to higher dimensions. Actually, Lemma 2 was proved by finding intermediate value of a continuous function. One can compare values of one-dimensional maps via the monotonicity of functions, but this is not the case in high dimensional spaces.

In this paper we will realize the generalization. Using the fixed point theory, a global implicit function theorem is presented for multi-input and multi-output functions without assuming the existence of a point (x_0, u_0) which satisfies $f(x_0, u_0) = 0$. Some corollaries and remarks are given for $f: \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^k$ where $k \neq m$. This global result in high dimension can achieve the control of non-affine MIMO systems. We further apply this theorem to such a system, proving the existence of an ideal control input and giving conditions under which the non-affine MIMO system is controllable.

2. Global Implicit Function Theorem

Let $[\frac{\partial}{\partial u} f(x, u)]_{ij}$ denote the ij th entry of the Jacobian $\frac{\partial}{\partial u} f(x, u)$.

Theorem 1. *Assume that $f: \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^m$ is a continuous mapping and it is continuously differentiable in the second variable $u \in \mathbf{R}^m$. If either*

$$\left| \left[\frac{\partial}{\partial u} f(x, u) \right]_{ii} \right| - \sum_{j \neq i} \left| \left[\frac{\partial}{\partial u} f(x, u) \right]_{ij} \right| \geq d, \quad \forall (x, u) \in \mathbf{R}^n \times \mathbf{R}^m, \quad i = 1, \dots, m, \quad (2)$$

for a fixed constant $d > 0$, then there exists a unique mapping $g: \mathbf{R}^n \rightarrow \mathbf{R}^m$ such that $f(x, g(x)) = 0$. Moreover, this mapping g is continuous. Additionally, if f is continuously differentiable, then the obtained g is also continuously differentiable.

In comparison with Lemma 2, the difficulty in the proof of Theorem 1 is substantial. Lemma 2 can be proved by the Intermediate Theorem for continuous functions since it is considered on \mathbf{R} , but in \mathbf{R}^m there is not a so simple order between vectors as in \mathbf{R} .

Proof. We first discuss the case that $[\frac{\partial}{\partial u} f(x, u)]_{ii} > 0$ for all i . From (2), we have

$$\left[\frac{\partial}{\partial u} f(x, u) \right]_{ii} - \sum_{j \neq i} \left| \left[\frac{\partial}{\partial u} f(x, u) \right]_{ij} \right| \geq d, \quad \forall (x, u) \in \mathbf{R}^n \times \mathbf{R}^m, \quad i = 1, \dots, m. \tag{3}$$

For each $x_0 \in \mathbf{R}^n$, let $c := f(x_0, 0)$. If $c = 0$, then we define $g(x_0) = 0$. If $c \neq 0$, then we consider the closed ball $\bar{B}(O, |c|/d) := \{u \in \mathbf{R}^m : |u| \leq |c|/d\}$ in \mathbf{R}^m , where the norm $|\cdot|$ of vectors is defined by

$$|c| = \max\{|c_1|, \dots, |c_m|\}, \quad \text{where } c = [c_1, \dots, c_m]^T. \tag{4}$$

Define a mapping $T : \bar{B}(O, |c|/d) \rightarrow \mathbf{R}^m$ by

$$Tu = u - \frac{1}{\ell} f(x_0, u), \quad \forall u \in \mathbf{R}^m, \tag{5}$$

where $\ell > 0$ is a large constant. For $u \in \mathbf{R}^m$, by the Mean Value Theorem, we have

$$f(x_0, u) = f(x_0, 0) + \frac{\partial}{\partial u} f(x_0, \xi)u, \tag{6}$$

where $|\xi| \leq |u|$. Thus

$$\begin{aligned} |Tu| &= \left| u - \frac{1}{\ell} f(x_0, u) \right| = \left| u - \frac{1}{\ell} \left(c + \frac{\partial}{\partial u} f(x_0, \xi)u \right) \right| \\ &\leq \frac{1}{\ell} |c| + \left\| I - \frac{1}{\ell} \frac{\partial}{\partial u} f(x_0, \xi) \right\| |u|, \end{aligned} \tag{7}$$

where $I \in \mathbf{R}^{m \times m}$ is a unit matrix and $\|\cdot\|$ denotes the norm of matrices corresponding to the norm defined in (4). Since $\frac{\partial}{\partial u} f(x_0, u)$ is continuous, all entries are bounded by a constant $M > 0$, i.e.,

$$\left| \left[\frac{\partial}{\partial u} f(x_0, u) \right]_{ij} \right| \leq M, \quad i = 1, \dots, m,$$

for all $|u| \leq |c|/d$. Thus

$$1 - \frac{1}{\ell} \left[\frac{\partial}{\partial u} f(x_0, u) \right]_{ii} > 0, \quad i = 1, \dots, m,$$

for large $\ell > M$. Therefore, by (3), we have

$$\begin{aligned} &\left\| I - \frac{1}{\ell} \frac{\partial}{\partial u} f(x_0, u) \right\| \\ &= \max_{i=1, \dots, m} \left\{ \left| 1 - \frac{1}{\ell} \left[\frac{\partial}{\partial u} f(x_0, u) \right]_{ii} \right| + \frac{1}{\ell} \sum_{j \neq i} \left| \left[\frac{\partial}{\partial u} f(x_0, u) \right]_{ij} \right| \right\} \\ &= \max_{i=1, \dots, m} \left\{ 1 - \frac{1}{\ell} \left(\left[\frac{\partial}{\partial u} f(x_0, u) \right]_{ii} - \sum_{j \neq i} \left| \left[\frac{\partial}{\partial u} f(x_0, u) \right]_{ij} \right| \right) \right\} \\ &\leq 1 - \frac{d}{\ell} < 1 \end{aligned} \tag{8}$$

for all $|u| \leq |c|/d$. It follows from (7) that

$$|\mathcal{T}u| \leq \frac{1}{\ell}|c| + \left(1 - \frac{d}{\ell}\right)\frac{|c|}{d} = \frac{|c|}{d}, \quad \forall |u| \leq |c|/d,$$

which implies that \mathcal{T} maps $\bar{B}(O, |c|/d)$ into itself as we take a larger $\ell > M$.

Furthermore, \mathcal{T} defined in (5) is clearly a continuous mapping. Applying (8) again, for arbitrary $u_1, u_2 \in \bar{B}(O, |c|/d)$, we obtain

$$\begin{aligned} |\mathcal{T}u_1 - \mathcal{T}u_2| &= \left| (u_1 - u_2) - \frac{1}{\ell} (f(x_0, u_1) - f(x_0, u_2)) \right| \\ &\leq \left\| I - \frac{1}{\ell} \frac{\partial}{\partial u} f(x_0, \eta) \right\| |u_1 - u_2| \\ &\leq \left(1 - \frac{d}{\ell}\right) |u_1 - u_2|, \end{aligned} \tag{9}$$

where $\eta \in \bar{B}(O, |c|/d)$ is given by the Mean Value Theorem. This means that \mathcal{T} is a contraction in $\bar{B}(O, |c|/d)$. By Banach’s fixed point theorem, there exists a unique $u_0 \in \bar{B}(O, |c|/d)$ such that $\mathcal{T}u_0 = u_0$, that is, $f(x_0, u_0) = 0$. Therefore, it is reasonable to define $g(x_0) = u_0$.

Following the above arguments, a correspondence $g: \mathbf{R}^n \rightarrow \mathbf{R}^m$ at each $x \in \mathbf{R}^n$ is defined uniquely and satisfies that $f(x, g(x)) = 0$. Moreover, by condition (2) and [19, Result 6.1.15], the Jacobian matrix $\frac{\partial}{\partial u} f(x, u)$ is invertible for all $(x, u) \in \mathbf{R}^n \times \mathbf{R}^m$ because it is strictly diagonally dominant uniformly with respect to $(x, u) \in \mathbf{R}^n \times \mathbf{R}^m$. Then the local result Lemma 1, the well-known Implicit Function Theorem, implies the continuity of g on \mathbf{R}^n . Continuous differentiability of g is also obtained similarly.

In the case that $[\frac{\partial}{\partial u} f(x, u)]_{ii} < 0$ for all i , we turn to consider $\hat{f}(x, u) = -f(x, u)$. Obviously, \hat{f} satisfies (3). As before, we can find a unique g such that $\hat{f}(x, g(x)) = 0$ for all $x \in \mathbf{R}^n$. Such a g is what we are looking for.

If $[\frac{\partial}{\partial u} f(x, u)]_{ii} > 0$ for some i and < 0 for other i ’s, with a permutation of components of f , then there is no loss of generality in assuming that $[\frac{\partial}{\partial u} f(x, u)]_{ii} > 0$ for $i = 1, \dots, k$ and < 0 for $i = k + 1, \dots, m$. Let

$$f(x, u) = (f_1(x, u), \dots, f_k(x, u), f_{k+1}(x, u), \dots, f_m(x, u))^T, \quad u = (u_1, \dots, u_m)^T,$$

where T means transpose of matrices and vectors. Define

$$\hat{f}(x, u) = (f_1(x, u), \dots, f_k(x, u), -f_{k+1}(x, u), \dots, -f_m(x, u))^T. \tag{10}$$

It follows from (2) that \hat{f} satisfies (3). As above we can find a unique $g: \mathbf{R}^n \rightarrow \mathbf{R}^m$ such that $\hat{f}(x, g(x)) = 0$. By the definition in (10), we see that $f(x, g(x)) = 0$. \square

3. Some generalization

We are also concerned with cases that the range \mathbf{R}^m of f in Theorem 1 is not the same as the domain of u . If the range of f is \mathbf{R}^1 , we have the following interesting and useful corollaries.

Corollary 1. Assume that $f: \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^1$ is a continuous mapping and it is continuously differentiable in the second variable $u \in \mathbf{R}^m$. If either

$$\left| \frac{\partial}{\partial u_i} f(x, u_1, \dots, u_m) \right| - \sum_{j \neq i} \left| \frac{\partial}{\partial u_j} f(x, u_1, \dots, u_m) \right| \geq d, \quad \forall (x, u) \in \mathbf{R}^n \times \mathbf{R}^m, \quad (11)$$

for some $i = 1, \dots, m$, where $d > 0$ is a fixed constant, then there exists a unique mapping $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$ such that $f(x, g(x)) = 0$. Moreover, g is continuous. Additionally, if f is continuously differentiable, then the obtained g is also continuously differentiable.

Proof. There is no loss of generality in assuming in (11) that $i = 1$. Let

$$\tilde{f} := (f_1, \dots, f_m)^T, \quad (12)$$

where

$$f_1(x, z_1, z_2, z_3, \dots, z_m) := f(x, z_1, z_2, z_3, \dots, z_m),$$

$$f_2(x, z_1, z_2, z_3, \dots, z_m) := f(x, z_2, z_1, z_3, \dots, z_m),$$

...

$$f_m(x, z_1, z_2, z_3, \dots, z_m) := f(x, z_m, z_1, z_3, \dots, z_{m-1}),$$

that is, except for f_1 , all others are defined by the same function f with a permutation of variables. Clearly, $\tilde{f} : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^m$ and $\frac{\partial}{\partial z_1} f_2(x, z_1, z_2, \dots, z_m) = \frac{\partial}{\partial y_2} f(x, z_2, z_1, \dots, z_m)$, $\frac{\partial}{\partial z_2} f_2(x, z_1, z_2, \dots, z_m) = \frac{\partial}{\partial y_1} f(x, z_2, z_1, \dots, z_m)$. Similarly, we calculate the others, and, therefore, we obtain

$$\frac{\partial}{\partial z} \tilde{f}(x, z) = \begin{bmatrix} \frac{\partial}{\partial y_1} f(x, z) & \frac{\partial}{\partial y_2} f(x, z) & \cdots & \frac{\partial}{\partial y_m} f(x, z) \\ \frac{\partial}{\partial y_2} f(x, z) & \frac{\partial}{\partial y_1} f(x, z) & \cdots & \frac{\partial}{\partial y_m} f(x, z) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial y_2} f(x, z) & \frac{\partial}{\partial y_3} f(x, z) & \cdots & \frac{\partial}{\partial y_1} f(x, z) \end{bmatrix},$$

where $\frac{\partial}{\partial y_1} f(x, z)$ always appears on the diagonal. Therefore, condition (11) implies that

$$\left| \left[\frac{\partial}{\partial z} \tilde{f}(x, z) \right]_{ii} \right| - \sum_{j \neq i} \left| \left[\frac{\partial}{\partial z} \tilde{f}(x, z) \right]_{ij} \right| \geq d, \quad \forall (x, z) \in \mathbf{R}^n \times \mathbf{R}^m, \quad i = 1, \dots, m,$$

i.e., the corresponding condition in Theorem 1 is satisfied. It follows that there exists a unique mapping $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$ such that $\tilde{f}(x, g(x)) = 0$. Considering the first a component, we have $f(x, g(x)) = 0$. Continuity and continuous differentiability follow from Theorem 1. This completes the proof. \square

The idea in the proof of Corollary 1 also implies the following generalization to \mathbf{R}^k where $k \leq m$, but we have no need to repeat the same procedure for its proof.

Corollary 2. Assume that $f : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^k$ ($k \leq m$) is a continuous mapping and it is continuously differentiable in the second variable $u \in \mathbf{R}^m$. If either

$$\left| \frac{\partial}{\partial u_{i_s}} f(x, u_1, \dots, u_m) \right| - \sum_{j \neq i_s} \left| \frac{\partial}{\partial u_j} f(x, u_1, \dots, u_m) \right| \geq d, \quad s = 1, \dots, k, \quad (13)$$

for all $(x, u) \in \mathbf{R}^n \times \mathbf{R}^m$, where $i_1, \dots, i_k \in \{1, \dots, m\}$ are distinct and $d > 0$ is a fixed constant, then there exists a unique mapping $g: \mathbf{R}^n \rightarrow \mathbf{R}^m$ such that $f(x, g(x)) = 0$. Moreover, g is continuous. Additionally, if f is continuously differentiable, then the obtained g is also continuously differentiable.

Note that the generalization to \mathbf{R}^k where $k > m$ is complicated. For simplicity, assume that $f: \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^k$ is a continuous mapping, continuously differentiable in the second variable $u \in \mathbf{R}^m$, such that

$$\left| \frac{\partial}{\partial u_i} f(x, u_1, \dots, u_m) \right| - \sum_{j \neq i} \left| \frac{\partial}{\partial u_j} f(x, u_1, \dots, u_m) \right| \geq d, \quad i = 1, \dots, m,$$

for all $(x, u) \in \mathbf{R}^n \times \mathbf{R}^m$, where $d > 0$ is a fixed constant. Let $f(x, u) = (f_1(x, u), \dots, f_k(x, u))^T$. Consider $\tilde{f} = (f_1, \dots, f_m)^T$. By Theorem 1, there exists a unique mapping $g: \mathbf{R}^n \rightarrow \mathbf{R}^m$ such that $\tilde{f}(x, g(x)) = 0$ and g is continuous. A problem is: Does the unique mapping g satisfy that $f_i(x, g(x)) = 0$ for all $i = m + 1, \dots, k$? An explicit counterexample against the case of $k > m$ is the mapping $f: \mathbf{R}^1 \times \mathbf{R}^1 \rightarrow \mathbf{R}^2$, defined by $f = (f_1, f_2)^T$ where

$$f_1(x, u) = u - x, \quad f_2(x, u) = x - u + 1.$$

Obviously, there is a unique function $g(x) = x$ such that $f_1(x, g(x)) = 0$. However, $f_2(x, g(x)) \equiv 1 \neq 0$.

4. Controllable MIMO non-affine system

Consider a multi-input multi-output (MIMO) nonlinear system

$$Y^{(n)} = F(Y, Y^{(1)}, Y^{(2)}, \dots, Y^{(n-1)}, U), \tag{14}$$

where $Y \in \mathbf{R}^m$ is the measured output, $U \in \mathbf{R}^m$ is the control input, $Y^{(i)}$ denotes the i th time derivative of the output Y ($i = 1, 2, \dots, n - 1$), and $F: \mathbf{R}^{m(n+1)} \rightarrow \mathbf{R}^m$ is an unknown nonlinear vector function. It should be noted that, unlike most recent results, the nonlinearity F is an implicit function with respect to U . The control objective can be described as: Given a desired output $Y_d(t)$, find a control U such that the output of the system tracks the desired trajectory with an acceptable accuracy, while all the states and the control remain bounded. This problem was discussed for $m = 1$ (that is, a SISO system) in [8,9], where the one-dimensional result Lemma 2 can be applied.

Let $X = [X_1, X_2, \dots, X_n] = [Y, Y^{(1)}, \dots, Y^{(n-1)}] \in \mathbf{R}^{m \times n}$ be the state matrix, where X_1, X_2, \dots, X_n and $Y, Y^{(1)}, \dots, Y^{(n-1)}$ are all m -dimensional vectors in column. We may represent system (14) in a state space model

$$\begin{cases} \dot{X}_j = X_{j+1}, & j = 1, \dots, n - 1, \\ \dot{X}_n = F(X, U), \\ Y = X_1. \end{cases} \tag{15}$$

The following assumptions are made for system (15):

- (A1) $F(X, U)$ is C^1 for $(X, U) \in \mathbf{R}^{m(n+1)}$.
- (A2) $|\left[\frac{\partial}{\partial U} F(X, U) \right]_{ii} - \sum_{j \neq i} \left| \left[\frac{\partial}{\partial U} F(X, U) \right]_{ij} \right| \geq d$ for all $(X, U) \in \mathbf{R}^{m(n+1)}$, $i = 1, \dots, m$, where $d > 0$ is a definite constant.
- (A3) The reference signals $Y_d(t), Y_d^{(1)}(t), Y_d^{(2)}(t), \dots, Y_d^{(n)}(t)$ are smooth and bounded.

Under assumptions (A1), (A2), system (15) includes the class of affine systems discussed in [14,15,23] and the class of non-affine SISO system in [10].

Define vectors X_d and Υ as

$$\begin{aligned} X_d &= [Y_d, Y_d^{(1)}, \dots, Y_d^{(n-1)}], \quad X_d \in \mathbf{R}^{m \times n}, \\ \Upsilon &= X - X_d \end{aligned} \tag{16}$$

and define a filtered tracking error as

$$E = \Upsilon[\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 1]^T, \quad E \in \mathbf{R}^m, \tag{17}$$

where the vector $\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_{n-1}]$ can be chosen appropriately so that $s^{n-1} + \lambda_{n-1}s^{n-2} + \dots + \lambda_1$ is Hurwitz and, therefore, $\Upsilon(t) \rightarrow 0$ as $E(t) \rightarrow 0$. Then, the time derivative of the filtered tracking error can be written as

$$\dot{E} = F(X, U) - Y_d^{(n)}(t) + \Upsilon[0 \ \Lambda]^T. \tag{18}$$

Let $E = [e_1, \dots, e_m]^T$. Consider a continuous function

$$\text{sat}(e) = \begin{cases} 1 - \exp(-e/\gamma), & e > 0, \\ -1 + \exp(e/\gamma), & e \leq 0, \end{cases} \tag{19}$$

with γ being any small positive constant. As $\gamma \rightarrow 0$, $\text{sat}(e)$ approaches a step-transition from -1 at $e = 0^-$ to 1 at $e = 0^+$ continuously. We have the following result to establish the existence of an ideal control, U^* , that brings the output of the system to the desired trajectory.

Theorem 2. *Consider system (15) satisfying assumptions (A1)–(A3). There exists an ideal control input U^* such that*

$$\dot{E} = -k_v E - k_v \text{sat}(E), \tag{20}$$

where $\text{sat}(E) := [\text{sat}(e_1), \dots, \text{sat}(e_m)]^T$ and k_v is a positive constant. Subsequently, Eq. (20) leads to $\lim_{t \rightarrow \infty} |Y(t) - Y_d(t)| = 0$.

Proof. Plus and minus $k_v E + k_v \text{sat}(E)$ to the right-hand side of the error equation (18), which gives

$$\dot{E} = F(X, U) + v - k_v E - k_v \text{sat}(E), \tag{21}$$

where v is defined as

$$v = k_v E + k_v \text{sat}(E) - Y_d^{(n)}(t) + \Upsilon[0 \ \Lambda]^T. \tag{22}$$

From assumption (A2) and the fact that $\partial v / \partial U = 0$, we know

$$\left| \left[\frac{\partial(F(X, U) + v)}{\partial U} \right]_{ii} \right| - \sum_{j \neq i} \left| \left[\frac{\partial(F(X, U) + v)}{\partial U} \right]_{ij} \right| \geq d, \quad \forall (X, U) \in \mathbf{R}^{m(n+1)}.$$

By our global Implicit Function Theorem (Theorem 1), there exists a continuous ideal control input $U^*(Z)$ with $Z = [X, v]^T \in \mathbf{R}^{m(n+1)}$ such that

$$F(X, U^*(Z)) + v = 0. \tag{23}$$

Under the action of U^* , (21) and (23) imply that (20) holds.

Let $\rho = E^T E = \sum_{j=1}^m e_j^2$. If $E(t)$ is a solution of Eq. (20), then

$$\dot{\rho} = 2E^T \dot{E} = -2k_v \rho - 2k_v E^T \text{sat}(E) \leq -2k_v \rho,$$

because each $e_j \text{sat}(e_j) \geq 0$ and, therefore, $E^T \text{sat}(E) = \sum_{j=1}^m e_j \text{sat}(e_j) \geq 0$. By Comparison Theorem [12],

$$0 \leq \rho(t) \leq \rho(0) \exp(-2k_v t) \rightarrow 0$$

as $t \rightarrow +\infty$, implying that $|E(t)| \rightarrow 0$ as $t \rightarrow +\infty$. Hence $\lim_{t \rightarrow \infty} |Y(t) - Y_d(t)| = 0$ and the proof is completed. \square

5. Approximation of ideal control input

The contraction of map in the proof of Theorem 1 gives a method to approximate the ideal control input U^* . As defined in (22), v is a function of X and not dependent on F because both E and Υ depend on X . Once $X \in \mathbf{R}^{m \times n}$ is fixed, v is given. Moreover,

$$\begin{aligned} |\gamma| &\leq (k_v + 1)|\lambda||X - X_d| + k_v + M_Y \\ &\leq (k_v + 1)|\lambda||X| + (k_v + 1)|\lambda|nM_Y + k_v + M_Y, \end{aligned} \tag{24}$$

where $|\lambda| := \max\{|\lambda_1|, \dots, |\lambda_{n-1}|, 1\}$ and $M_Y > 0$ is the bound such that all $|Y_d|, |Y_d^{(1)}|, \dots, |Y_d^{(n-1)}|$ are less than or equal to M_Y .

Although we usually know less about F in control problems, sometimes it is still possible to assume that F is Lipschitzian, or strongly assume that

$$(A4) \quad \left| \frac{\partial}{\partial U} F(X, U) \right|_{ij} \leq M_1 \text{ for all } (X, U) \in \mathbf{R}^{m(n+1)}, \text{ where } M_1 > 0 \text{ is a constant.}$$

Construct a sequence $\{U_{(j)}(X)\}$ such that

$$\begin{cases} U_{(0)}(X) = 0, \\ U_{(j)}(X) = U_{(j-1)}(X) - \frac{1}{M_1+1}(F(X, U_{(j-1)}(X)) + v). \end{cases} \tag{25}$$

This sequence is convergent because the map \mathcal{T} in the proof of Theorem 1 is a contraction. As a consequence, we have

$$\lim_{j \rightarrow \infty} U_{(j)}(X) = U^*(X, v). \tag{26}$$

Furthermore, from (9), we have $|U_{(j+1)}(X) - U_{(j)}(X)| \leq (1 - \frac{d}{M_1+1})|U_{(j)}(X) - U_{(j-1)}(X)|$. Then by induction

$$|U_{(j+1)}(X) - U_{(j)}(X)| \leq \left(1 - \frac{d}{M_1+1}\right)^j |U_{(1)}(X) - U_{(0)}(X)|.$$

It implies inductively that

$$|U_{(j+p)}(X) - U_{(j)}(X)| \leq \frac{M_1+1}{d} \left(1 - \frac{d}{M_1+1}\right)^j \left\{1 - \left(1 - \frac{d}{M_1+1}\right)^p\right\} |U_{(1)}(X)|.$$

Then, letting $p \rightarrow \infty$, by (24), we obtain the estimate of error

$$\begin{aligned}
|U^*(X, v) - U_{(j)}(X)| &\leq \frac{M_1 + 1}{d} \left(1 - \frac{d}{M_1 + 1}\right)^j |U_{(1)}(X)| \\
&\leq \frac{M_0 + |\gamma|}{d} \left(1 - \frac{d}{M_1 + 1}\right)^j \\
&\leq \frac{(k_v + 1)|\lambda||X| + M_*}{d} \left(1 - \frac{d}{M_1 + 1}\right)^j, \tag{27}
\end{aligned}$$

where $M_* = M_0 + (k_v + 1)|\lambda|nM_Y + k_v + M_Y$, provided that we assume

(A5) F is bounded and $|F(X, U)| \leq M_0$ for all $(X, U) \in \mathbf{R}^{m(n+1)}$, where $M_0 > 0$ is a constant.

If we do not know so much information of F as in assumptions (A4) and (A5), the approximation of ideal control input U^* given by (26) and (27) may not work. In that case, [10] suggests that a neural network [11,21] can be taken as a function approximator which emulates a given nonlinear function up to a small error tolerance, where an analysis of Lyapunov functions for semi-globally uniformly ultimate boundedness will be involved.

Remark. The condition (A2) actually guarantees the existence of the ideal control input for a non-affine nonlinear MIMO system, no matter whether it is still considered to be restrictive in practices. We hope to make a progress with the new theorem of global implicit functions in the control problem of non-affine nonlinear MIMO system or start a new approach to the goal.

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