

Variable Structure Control of a Distributed-Parameter Flexible Beam

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In this article, regulation of a distributed-parameter flexible beam is considered using variable structure control techniques. The proposed controller can stabilize the system exponentially and the converging speed can be set by the designer as desired. Different from existing variable structure controllers for flexible robots in the literature, the controller presented here is designed directly for the partial differential equations governing the motion of the distributed-parameter system. Thus, exponential stability holds for the original distributed-parameter system. Numerical simulations are also provided to verify the effectiveness of the approach presented. © 2001 John Wiley & Sons, Inc.

1. INTRODUCTION

Modelling and control of flexible link manipulators have been studied intensively in the literature. Since the vibration of a flexible link is governed by partial differential equations (PDEs), the system is a distributed-parameter system and possesses infinite number of dimensions, which makes it difficult to control. Further, the inherent nonminimum phase characteristic from the base actuator to the free tip of a flexible robot makes it very difficult to achieve

high level performance and robustness simultaneously.¹

Controller design for flexible link robots is in general based on an approximated finite-dimensional model. The model can be obtained by modal analysis, in which the vibration of a flexible link is represented by an infinite number of assumed flexible modes. The finite-dimensional model is derived by truncating the infinite number of modes to a finite number by neglecting the higher frequency modes. When a finite-dimensional model is obtained, some well-developed controller design approaches are then applicable.²⁻⁶ Although the con-

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trollers in these articles were shown to be able to stabilize the truncated system by theoretical proof—simulations as well as laboratory experiments, the stability of the original distributed-parameter system is not obvious. Some problems associated with the truncated finite-dimensional model that also are highlighted in the literature are: (i) the order of the controller increases as the number of flexible modes increases, (ii) control and observation spillovers may occur due to the ignored high frequency dynamics,⁷ and (iii) the controllers may be difficult to implement from the engineering point of view since full states measurements—observers are often required.

In an attempt to overcome these shortcomings, an alternative method has been developed for the control of flexible robots in recent years, where system analysis and controller design are carried out directly based on PDEs of flexible link robots. Consequently, stability analysis and the resulting controllers are valid for the original distributed-parameter system and the problems listed before are essentially avoided. Some design examples can be found in refs. 7–13. In this article, the same idea is used to design a variable structure controller for regulation of a flexible beam. Due to its simplicity and robustness to parametric uncertainties, variable structure control (VSC) is particularly attractive for nonlinear control problems, and is applied to a large range of practical systems.¹⁴ Some applications of VSC to flexible robots—structures can be found in refs. 15–17, which were all designed based on finite-dimensional truncated models. Accordingly, these controllers also suffer from the aforementioned problems due to model truncation. To avoid these pitfalls, controller design should be based on PDEs of the flexible robot system. It is pointed out in ref. 14 that one of the main difficulties in applying VSC to distributed-parameter systems is that there is a lack of general results for PDEs with discontinuous terms in the literature. However, by exploiting the properties of the physical system, we believe some special VSC controllers can be constructed. In this article, we first show that the finite time convergence of the system motion on the selected sliding mode provides an extra homogeneous boundary condition, and subsequently makes the boundary value problem of the system solvable using the traditional variable separation method. Then, exponential stability is achieved because the solution to the boundary value problem is trivial, which implies the final equilibrium position. The

exponential converging speed is determined by a design parameter of the controller. Note that the controller presented is applicable to this case only and is hard to generalize to multilink flexible robots. The implementation issues of the controller are also discussed. Finally, computer simulations are carried out to verify the presented approach.

The rest of this article is organized as follows: in Section 2, the dynamic equations (PDEs) of the system are obtained; the variable structure controller design is presented in Section 3; computer simulations are given in Section 4, followed by the conclusion in Section 5.

2. DYNAMICS OF THE SYSTEM

Let us consider a flexible beam which is clamped at its base on the rotor of a motor as shown in Figure 1. The flexible beam is rotating in the horizontal plane and the effect of gravity is neglected. Frame XOY is the fixed inertia frame and frame xOy is the local reference frame with axis Ox being tangent to the beam at the base. System parameters and variables are defined as follows:

- L : the length of the beam;
- EI the uniform flexural rigidity of the beam;
- ρ : the uniform mass per unit length of the beam;
- I_h : the hub inertia;
- τ : the control torque provided by the motor;
- θ : the joint angle;
- $y(x, t)$: the elastic deflection measured from the undeformed beam;
- $p(x, t)$: the approximated length-of-arc position of the beam.

Normally the deflection of the flexible beam is assumed to be small, then the position of a point on the beam can be expressed as $p(x, t) := x\theta + y(x, t)$.

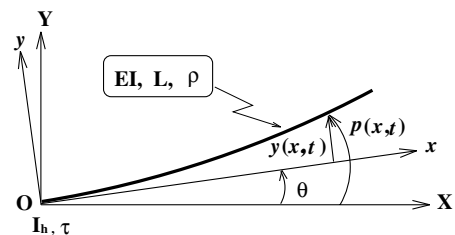


Figure 1. The flexible beam system.

The total kinetic energy E_k and the potential energy E_p are given by

$$E_k = \frac{1}{2} I_h \dot{\theta}^2 + \frac{\rho}{2} \int_0^L \dot{p}^2(x, t) dx \quad (1)$$

$$E_p = \frac{EI}{2} \int_0^L [y''(x, t)]^2 dx \quad (2)$$

where the dots and the primes denote the derivatives with respect to time t and space variable x , respectively. Substituting (1) and (2) into the extended Hamilton's principle,

$$\int_{t_0}^{t_f} \delta(E_k - E_p + \tau\theta) dt = 0 \quad (3)$$

and noting that the bending moment of the beam at its base is

$$EIy''(0, t) = -\rho \int_0^L x\ddot{p}(x, t) dx$$

we obtain the following dynamic equations of the system,

$$I_h \ddot{\theta}(t) = \tau + EIy''(0, t) \quad (4)$$

$$\rho [x\ddot{\theta}(t) + \ddot{y}(x, t)] = -EIy''''(x, t) \quad (5)$$

Equation (4) is the moment balance equation at the base, and (5) describes the vibration of the beam. The corresponding boundary conditions are given by the following set of equations,

$$y(0, t) = 0 \quad (6)$$

$$y'(0, t) = 0 \quad (7)$$

$$y''(L, t) = 0 \quad (8)$$

$$y'''(L, t) = 0 \quad (9)$$

Equations (6) and (7) hold because frame xOy is selected such that axis Ox is tangent to the beam at the base. Boundary conditions (8) and (9) describe the fact that the bending moment and the shear force at the free tip are zero. Since the position of the flexible beam in the inertia frame XOY is represented by $p(x, t)$, we rewrite (5) with respect to $p(x, t)$. This leads to

$$\rho \ddot{p}(x, t) = -EI p''''(x, t) \quad (10)$$

which is in the same form as the Euler–Bernoulli's beam equation.¹⁸ The four boundary conditions become

$$p(0, t) = 0 \quad (11)$$

$$p'(0, t) = \theta \quad (12)$$

$$p''(L, t) = 0 \quad (13)$$

$$p'''(L, t) = 0 \quad (14)$$

Condition (12) is related to joint angle θ and thus is unhomogeneous. As shown in ref. 19, this leads to an unconstrained mode expansion, which approaches the constrained mode expansion when I_h is much larger than the inertia of the flexible beam, $\rho L^3/3$, with respect to the joint. In the next section, a variable structure controller will be constructed to provide a new boundary condition to replace condition (12) and to make it possible to solve the system dynamics using the traditional variable separation method.

3. VARIABLE STRUCTURE CONTROLLER

3.1. Controller Design

In this section, the variable structure controller is presented for the regulation of the flexible beam system described in Section 2. First, we need to choose a suitable sliding variable S . In ref. 15, the sliding variable was chosen based on the truncated model of the flexible robot. Consequently, the flexible modes and their first- and second-order time derivatives are needed for feedback. These signals are, in general, difficult to obtain. Bearing this in mind, we construct a sliding variable which leads to a controller that is relatively easy to implement. The implementation problems are discussed later in this section.

The sliding variable S is chosen as

$$S = \dot{W} + kW \quad (15)$$

where constant $k > 0$ is a design parameter, and

$$W = \theta + k_1 p'(L, t) \quad (16)$$

with scalar k_1 to be determined later. Note that when $k_1 = 0$, (15) reduces to the commonly used sliding variable for rigid body manipulators.²⁰ The term $k_1 p'(L, t)$ represents the flexibility of the beam.

The motivation for choosing such a sliding variable is to provide a suitable boundary condition for solving the beam equation (10) as shown later.

Theorem 3.1: For the system described by (4) and (10), if the variable structure controller is given by

$$\tau = -\text{sgn}(S) \left[b_1 |y''(0, t)| + b_2 k_1 |\ddot{p}'(L, t)| + kb_2 |\dot{\theta}| + kk_1 b_2 |\dot{p}'(L, t)| + \epsilon \right] \quad (17)$$

where $b_1 \geq EI$, $b_2 \geq I_h$, $\epsilon > 0$, $1 < k_1 < 3.83$ in (16) and

$$\text{sgn}(S) = \begin{cases} 1, & S > 0 \\ -1, & S < 0 \end{cases}$$

then the motion of the system will first reach the sliding mode $S = 0$ in a finite time, and consequently will converge to the equilibrium position $p(x, t) = 0$ exponentially with a time-constant $1/k$.

Proof: Selecting Lyapunov function candidate $V = I_h S^2/2$, its time derivative is given by

$$\dot{V} = I_h S \dot{S} \quad (18)$$

From (15) and (16), we have

$$\begin{aligned} \dot{S} &= \ddot{W} + k\dot{W} \\ &= \ddot{\theta} + k_1 \ddot{p}'(L, t) + k\dot{\theta} + kk_1 \dot{p}'(L, t) \end{aligned} \quad (19)$$

Substituting (19) into (18) yields

$$\dot{V} = S \left[I_h \ddot{\theta} + I_h k_1 \ddot{p}'(L, t) + kI_h \dot{\theta} + kk_1 I_h \dot{p}'(L, t) \right] \quad (20)$$

Noting (4), we have

$$\begin{aligned} \dot{V} &= S \left[EIy''(0, t) + I_h k_1 \ddot{p}'(L, t) + kI_h \dot{\theta} \right. \\ &\quad \left. + kk_1 I_h \dot{p}'(L, t) + \tau \right] \end{aligned} \quad (21)$$

Substituting controller (17) into (21) yields

$$\begin{aligned} \dot{V} &= S \left[EIy''(0, t) + I_h k_1 \ddot{p}'(L, t) + kI_h \dot{\theta} \right. \\ &\quad \left. + kk_1 I_h \dot{p}'(L, t) \right] + S\tau \\ &= S \left[EIy''(0, t) + I_h k_1 \ddot{p}'(L, t) + kI_h \dot{\theta} \right. \\ &\quad \left. + kk_1 I_h \dot{p}'(L, t) \right] \\ &\quad - |S| \left[b_1 |y''(0, t)| + b_2 k_1 |\ddot{p}'(L, t)| + kb_2 |\dot{\theta}| \right. \\ &\quad \left. + kk_1 b_2 |\dot{p}'(L, t)| \right] - \epsilon |S| \\ &\leq -\epsilon |S| \end{aligned} \quad (22)$$

As shown in ref. 21, inequality (22) implies that the system will reach the sliding mode $S = 0$ in a finite time, which is smaller than $I_h |S(t=0)|/\epsilon$, and then will remain in the sliding mode. Therefore, from (15), the motion of the system, after reaching the sliding mode, will slide along $S = 0$ toward $W = 0$ exponentially with a time constant equal to $1/k$.

Since we have proven that the motion of the system converges to $W = 0$ exponentially, to prove the exponential stability of the closed-loop system, we only need to show that the flexible beam stops at the final equilibrium position $p(x, t) = 0$ provided that $W = 0$. Note that $W = 0$ implies

$$k_1 p'(L, t) = -\theta \quad (23)$$

Combining condition (12) and Eq. (23) yields the following homogeneous boundary condition,

$$p'(0, t) = -k_1 p'(L, t) \quad (24)$$

Thus, Eq. (10) can be solved under boundary conditions (11), (24), (13), and (14). Invoking the variable separation method,²² $p(x, t)$ can be represented by

$$p(x, t) = \Phi(x)Q(t)$$

Equation (10) can be written as

$$\frac{\Phi''''}{\Phi} \cdot \frac{EI}{\rho} = -\frac{\ddot{Q}}{Q} \quad (25)$$

Since the left-hand side of (25) is only space-dependent while the right-hand side is a purely time-varying function, it is obvious that both sides must equal a constant, say K . Accordingly, we have the following two ordinary differential equations,

$$\ddot{Q}(t) + KQ(t) = 0 \quad (26)$$

$$\Phi''''(x) = \frac{\rho}{EI} K\Phi(x) \quad (27)$$

Accordingly, the four boundary conditions (11), (24), (13), and (14) can then be written as

$$\Phi(0) = 0 \quad (28)$$

$$\Phi'(0) = -k_1 \Phi'(L) \quad (29)$$

$$\Phi''(L) = 0 \quad (30)$$

$$\Phi'''(L) = 0 \quad (31)$$

Equation (27) and conditions (28)–(31) describe the corresponding boundary value problem. To solve it, all possible K should be considered.

When $K = 0$, the solution to (27) possesses the following the general form,

$$\Phi(x) = C_1 x^3 + C_2 x^2 + C_3 x + C_4$$

Substituting it into (28)–(31) yields $C_1 = C_2 = C_3 = C_4 = 0$ provided that

$$k_1 \neq -1 \tag{32}$$

Therefore, we have $\Phi(x) = 0$ which leads to $p(x, t) = 0$.

When $K < 0$, letting $K = -\omega^2$ with ω being a nonzero number, Eq. (27) can be rewritten as

$$\Phi''''(x) = -\left(\frac{\beta}{L}\right)^4 \Phi(x) \tag{33}$$

where

$$\left(\frac{\beta}{L}\right)^4 = \frac{\rho}{EI} \omega^2 \tag{34}$$

The general solution to (33) is of the form,

$$\begin{aligned} \Phi(x) = & C_1 e^{ax} \sin(ax) + C_2 e^{ax} \cos(ax) \\ & + C_3 e^{-ax} \sin(ax) + C_4 e^{-ax} \cos(ax) \end{aligned} \tag{35}$$

where $a := \sqrt{2} \beta / 2L$. By substituting (35) into (28)–(31), and by letting $Z_1 = \sin(aL) + \cos(aL)$ and $Z_2 = \cos(aL) - \sin(aL)$, we obtain the following set of equations,

$$\begin{aligned} C_2 + C_4 &= 0 \\ (Z_1 k_1 e^{aL} + 1)C_1 + (Z_2 k_1 e^{aL} + 1)C_2 \\ &+ (Z_2 k_1 e^{-aL} + 1)C_3 - (Z_1 k_1 e^{-aL} + 1)C_4 = 0 \\ e^{-aL} \cos(aL)C_1 - e^{aL} \sin(aL)C_2 - e^{-aL} \cos(aL)C_3 \\ &+ e^{aL} \sin(aL)C_4 = 0 \\ Z_2 e^{aL}C_1 - Z_1 e^{aL}C_2 + Z_1 e^{-aL}C_3 + Z_2 e^{-aL}C_4 &= 0 \end{aligned} \tag{36}$$

i.e.,

$$A_1 C = 0$$

where $C = [C_1 \ C_2 \ C_3 \ C_4]^T$ which is to be solved for

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ Z_1 k_1 e^{aL} + 1 & Z_2 k_1 e^{aL} + 1 & Z_2 k_1 e^{-aL} + 1 & -(Z_1 k_1 e^{-aL} + 1) \\ e^{aL} \cos(aL) & -e^{aL} \sin(aL) & -e^{-aL} \cos(aL) & e^{-aL} \sin(aL) \\ Z_2 e^{aL} & -Z_1 e^{aL} & Z_1 e^{-aL} & Z_2 e^{-aL} \end{bmatrix}$$

is the coefficient matrix of (36) whose determinant is a function of a and is given by

$$\begin{aligned} \Delta(A_1) = & 4[\cosh^2(aL) + 2k_1 \cosh(aL) \cos(aL) \\ & + \cos^2(aL)] \end{aligned} \tag{37}$$

To prove exponential stability, we need to show that Eq. (36) has only a trivial solution. This is true if $\Delta(A_1) \neq 0$. Now, the task is to decide the range of k_1 such that $\Delta(A_1) \neq 0$ holds for arbitrarily any a .

With the help of MATLAB (MathWorks, Inc.) and through some algebra derivations, it is found that (i) there does not exist any constant k_1 for which $\Delta(A_1) < 0$ always holds, (ii) if k_1 is chosen as

$$-1 < k_1 < 3.83 \tag{38}$$

then $\Delta(A_1) > 0$ always holds for arbitrarily any a .

Therefore, for the case $K < 0$, we have $\Phi(x) = 0$ and subsequently $p(x, t) = 0$ provided that k_1 is chosen according to (38).

The last choice for K is $K > 0$. Let $K = \omega^2$ with ω being a nonzero number. Recalling (34), Eq. (27) can be similarly rewritten as

$$\Phi''''(x) = \left(\frac{\beta}{L}\right)^4 \Phi(x) \quad (39)$$

The general solution to (39) is of the form,

$$\begin{aligned} \Phi(x) = & C_1 \cos \frac{\beta x}{L} + C_2 \cosh \frac{\beta x}{L} + C_3 \sin \frac{\beta x}{L} \\ & + C_4 \sinh \frac{\beta x}{L} \end{aligned} \quad (40)$$

From (28)–(31), we have

$$\begin{aligned} C_1 + C_2 &= 0 \\ k_1 \sin(\beta)C_1 - k_1 \sinh(\beta)C_2 - (1 + k_1 \cos(\beta))C_3 \\ &\quad - (1 + k_1 \cosh(\beta))C_4 = 0 \\ \cos(\beta)C_1 - \cosh(\beta)C_2 + \sin(\beta)C_3 - \sinh(\beta)C_4 &= 0 \\ \sin(\beta)C_1 - \sinh(\beta)C_2 + \cos(\beta)C_3 - \cosh(\beta)C_4 &= 0 \end{aligned} \quad (41)$$

i.e.,

$$A_2 C = 0$$

where

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ k_1 \sin(\beta) & -k_1 \sinh(\beta) & -(1 + k_1 \cos(\beta)) & -(1 + k_1 \cosh(\beta)) \\ \cos(\beta) & -\cosh(\beta) & \sin(\beta) & -\sinh(\beta) \\ \sin(\beta) & \sinh(\beta) & -\cos(\beta) & \cosh(\beta) \end{bmatrix}$$

is the coefficient matrix of (41). Similarly, Eq. (41) has only a trivial solution provided that the determinant of A_2 is not zero, i.e.,

$$\begin{aligned} \Delta(A_2) = & -2[\cos(\beta)\cosh(\beta) + k_1 \cos(\beta) \\ & + k_1 \cosh(\beta) + 1] \neq 0 \end{aligned}$$

Note that $\Delta(A_2)$ can be rewritten as

$$\begin{aligned} \Delta(A_2) = & -2[(1 + \cos(\beta))(1 + \cosh(\beta)) \\ & + (k_1 - 1)(\cos(\beta) + \cosh(\beta))] \end{aligned} \quad (42)$$

Because $\cosh(\beta) \geq 1$ and the equality sign holds only at $\beta = 0$, we can achieve $\Delta(A_2) < 0$ for all β if $k_1 > 1$. This leads to $\Phi(x) = 0$ and thus $p(x, t) = 0$ when $K > 0$. Recalling (32) and (38), one can see that k_1 should satisfy the following inequality,

$$1 < k_1 < 3.83 \quad (43)$$

which is the same as the condition given in Theorem 3.1. Now, we can conclude that $W = 0$ implies that the flexible beam stops at the final equilibrium position $p(x, t) = 0$. Consequently, because $W \rightarrow 0$

exponentially in the sliding motion, the system motion also converges to $p(x, t) = 0$ exponentially with time constant $1/k$. This completes the proof. ■

Remark 3.1: The range of k_1 given in (43) may not be the complete solution. Completely solving for k_1 from (37) and (42) is quite complicated. Consequently, Theorem 3.1 gives only a sufficient condition to stabilize the system.

3.2. Implementation Consideration

To realize the variable structure controller (17), we need to feedback the following pieces of information: θ , $\dot{\theta}$, $y''(0, t)$, $p'(L, t)$, $\dot{p}'(L, t)$, and $\ddot{p}'(L, t)$. These measurements can all be obtained by using some currently available sensor facilities.

The joint angle θ and the joint velocity $\dot{\theta}$ can be obtained by the rotary encoder and tachometer, respectively. The base strain $y''(0, t)$ can be easily measured by attaching a strain gauge at the base of the flexible beam. Due to the existence of the first-order space derivatives, the signals $p'(L, t)$, $\dot{p}'(L, t)$,

and $\ddot{p}'(L, t)$ are not directly measurable. However, they can be estimated as

$$p'(L, t) \approx \frac{p(L, t) - p(L - H, t)}{H} \quad (44)$$

$$\dot{p}'(L, t) \approx \frac{d}{dt} \left[\frac{p(L, t) - p(L - H, t)}{H} \right] \quad (45)$$

$$\ddot{p}'(L, t) \approx \frac{\ddot{p}(L, t) - \ddot{p}(L - H, t)}{H} \quad (46)$$

where $H > 0$ is small. For $p'(L, t)$, as in refs. 2 and 23, a position detecting system can be used to obtain the position signals $p(L, t)$ and $p(L - H, t)$. The time derivative in (45) can be approximately calculated by a backward difference. For $\ddot{p}'(L, t)$, $\ddot{p}(L, t)$ and $\ddot{p}(L - H, t)$ in (46) are actually the acceleration signals which can be measured by attaching two accelerometers on the beam at $x = L$ (the tip) and $x = L - H$. All the required signals are therefore obtainable by currently available sensor facilities and the controller is thus realizable in practice. Construction of these signals in our simulation is discussed further in next section. Although the signals above may be quite inaccurate, it should be pointed out that the signals, either by measurement or by estimation, need not to be known very accurately, since robust sliding control can be achieved if ϵ is chosen large enough to cover the error existing in the measurement–signal estimation.¹⁵

Another problem that should be considered is the discontinuity of the controller, or the so-called chattering problem. To cope with it, several quas-sliding modes methods were discussed in ref. 14. In the next section, the controller is simulated with the signum function $\text{sgn}(\ast)$ as well as the saturation function $\text{sat}(\ast)$ shown in Figure 2. The latter leads

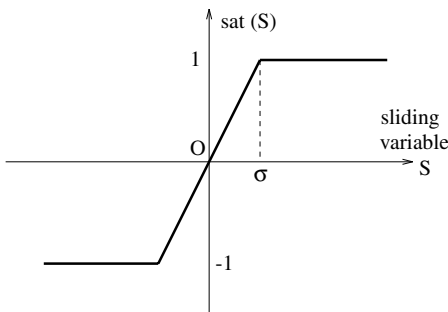


Figure 2. Saturation function.

to a continuous control but there is no sliding mode since the motion of the system will not stay on $S = 0$.¹⁴

4. NUMERICAL SIMULATIONS

This section presents some numerical simulations to demonstrate the effectiveness of the variable structure controller. In the simulations, the system parameters are chosen as: $L = 1.0$ m, $EI = 2.0$ N m², $\rho = 0.1$ Kg m⁻¹, and $I_h = 0.5$ Kg m². Controller parameters are selected to be $\epsilon = 0.5$, $k_1 = 1.01$, $k = 2.0$, $b_1 = EI$, and $b_2 = I_h$. The initial joint position is $\pi/9$ rad. A fourth-order Runge–Kutta–Merson program with adaptive steps is used to numerically solve the differential equations.

Remark 4.1: Although we are dealing with an infinite-dimensional system, it is impossible to use an infinite number of modes in the simulation. To make the simulation more meaningful, two measures are taken: (i) a relatively larger number of flexible modes is chosen in the simulation, and (ii) feedback signals associated with flexibility are assumed to be from the first mode only because the sensors available have low-pass bandwidth characteristics, and lower frequency component is dominant.

Simulation studies were carried out for different numbers of modes ranging from 4 to 10. It was found that there is no significant difference arising, which means four flexible modes is enough to approximate the plant and is representative.

For simplicity, the simulation results with four modes are presented. Simulation studies are carried out for both functions, $\text{sgn}(\ast)$ and $\text{sat}(\ast)$ with $\sigma = 0.01$. Figure 3 shows the tip deflections of the flexible beam. It is clear that the tip deflection of the controller with $\text{sgn}(\ast)$ converges much faster, and that with $\text{sat}(\ast)$ exhibits visible residue vibration. The residue vibration is expected since the system motion is not forced to stay on the $S = 0$ surface when $\text{sat}(\ast)$ is used. The convergences of the joint angle are shown in Figure 4. It is clear that the joint angle for the controller with $\text{sgn}(\ast)$ is much smoother and has less vibration than that with $\text{sat}(\ast)$. In the control of the flexible beam, what we care about most is the motion of the tip. Accordingly, the tip motion of the robot is given in Figure 5. It can be seen that the controller with $\text{sat}(\ast)$ exhibits visible

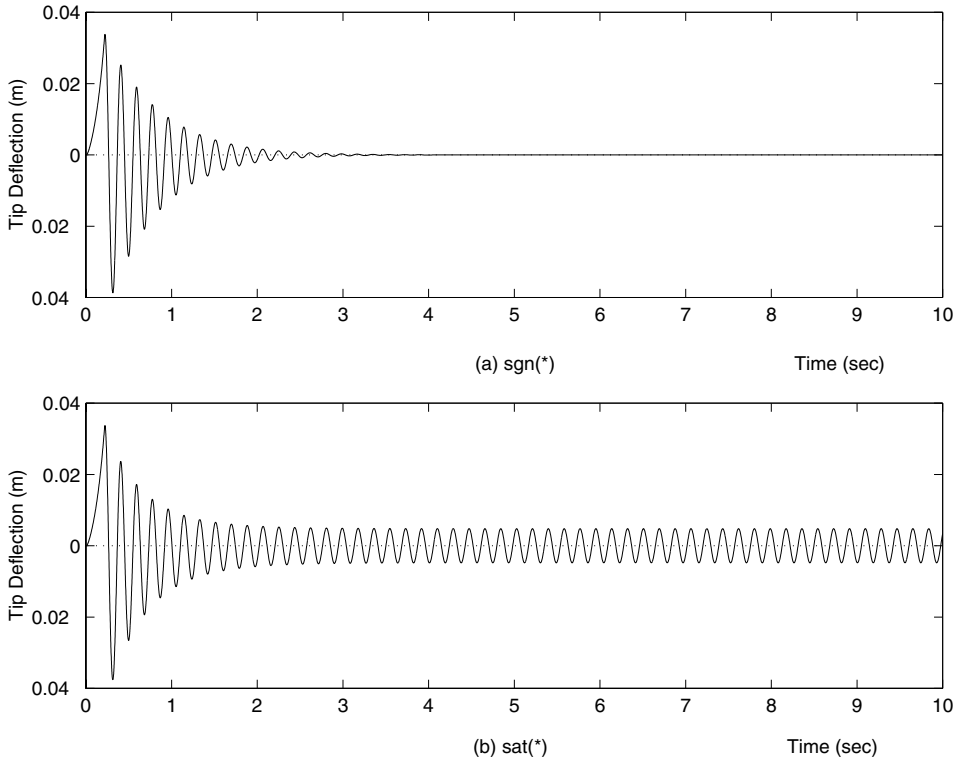


Figure 3. Tip deflection $y(L, t)$.

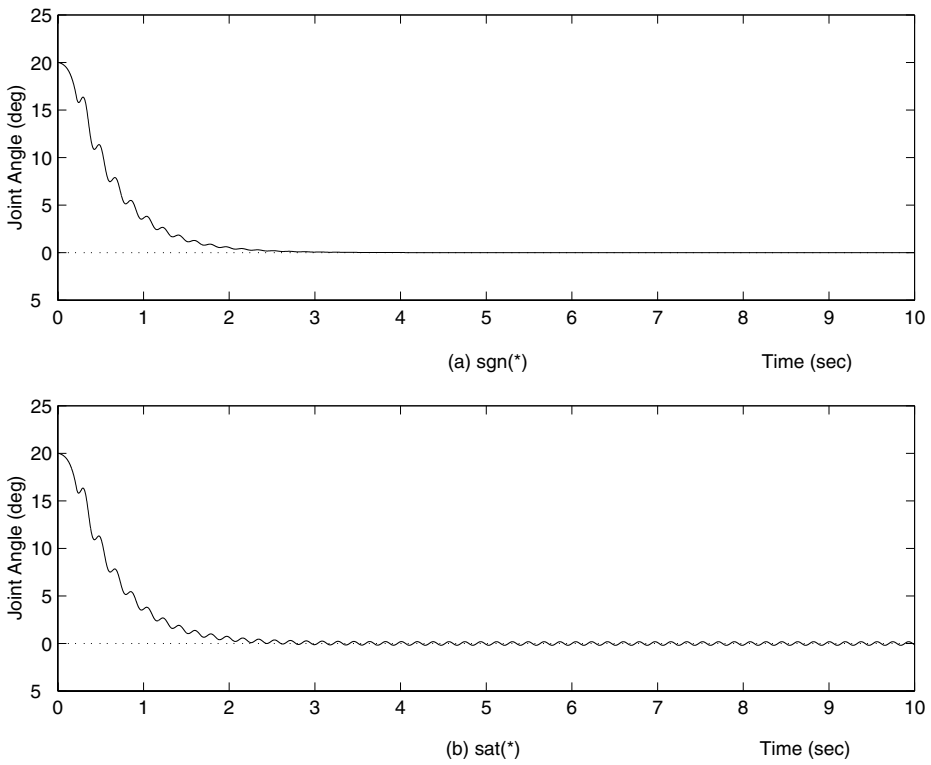


Figure 4. Joint motion angle θ .

residue vibration, while that with $\text{sgn}(\cdot)$ is quite good since the trajectory is smooth and there is very little vibration. In actual implementation, a trade-off has to be made between control performance and continuity of control signals. For completeness, the bounded control signals are shown in Figure 6 for both cases. It can be seen that the use of $\text{sat}(\cdot)$ eliminates the chattering and generates a continuous control, but the control performance degrades. Finally, the variations of S are shown in Figure 7. Since the S in the two cases are very close to each other as shown in Figure 7(a), they are replotted in Figure 7(b) in the time interval $1.0 < t < 10.0$ s. One can see that the S for $\text{sat}(\cdot)$ is actually oscillating around the $S = 0$ surface, rather than staying on it, while the S for $\text{sgn}(\cdot)$ almost (due to the numerical error) remains in the sliding mode.

Remark 4.2: Note that in the stability analysis of the proposed controller, the dynamics of the system have never been used, and no truncation procedure has been invoked which makes the controller (i) model-free, and (ii) different from the traditional approaches based on a truncated model. To test the

robustness of the controller, all the feedback signals for the controller are from the first mode only in the simulation.

Remark 4.3: In comparison with other methods in the literature, though the torque required is large, we found that the proposed controller can stabilize the system much faster (less than 3 s). In actual implementation, a trade-off has to be made between performance and torque magnitude. Reducing k decreases the control torque magnitude, but leads to a longer settling time. Note that we have chosen $b_1 = EI$ and $b_2 = I_h$, the lower bounds, in the simulation for a reduced torque magnitude. Increasing b_1 and b_2 will lead to shorter settling time, larger control torque magnitude, and bigger overshoot.

5. CONCLUSION

Regulation of a distributed-parameter flexible beam was investigated using the VSC technique in this article. A novel controller was developed based on the PDEs of the system, and thus the problems

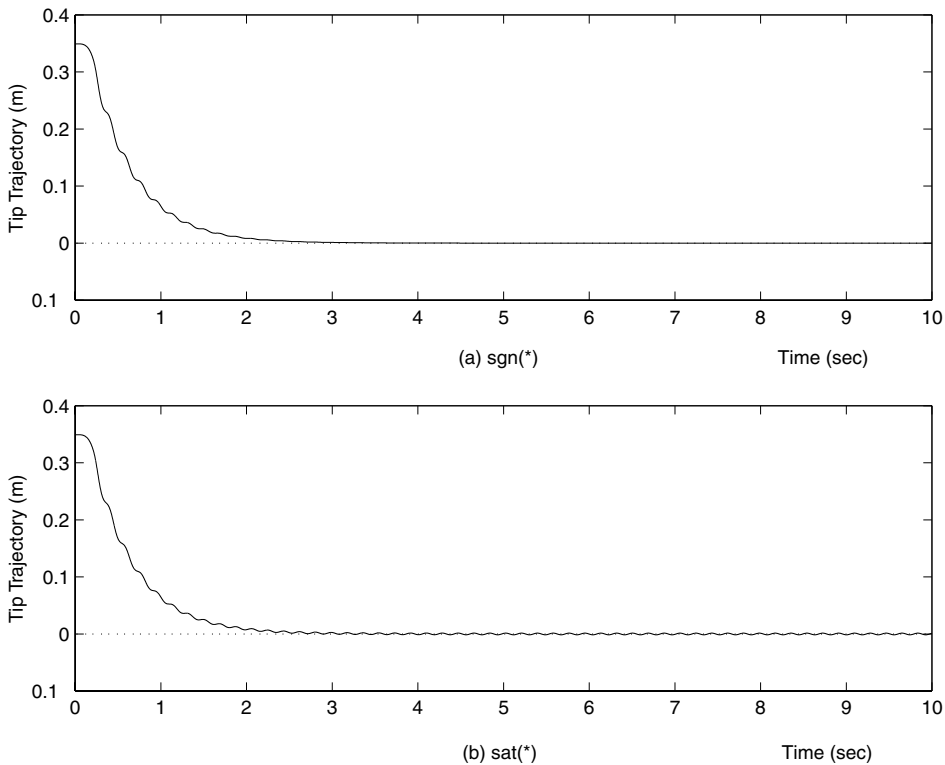


Figure 5. Tip trajectory $p(L, t)$.

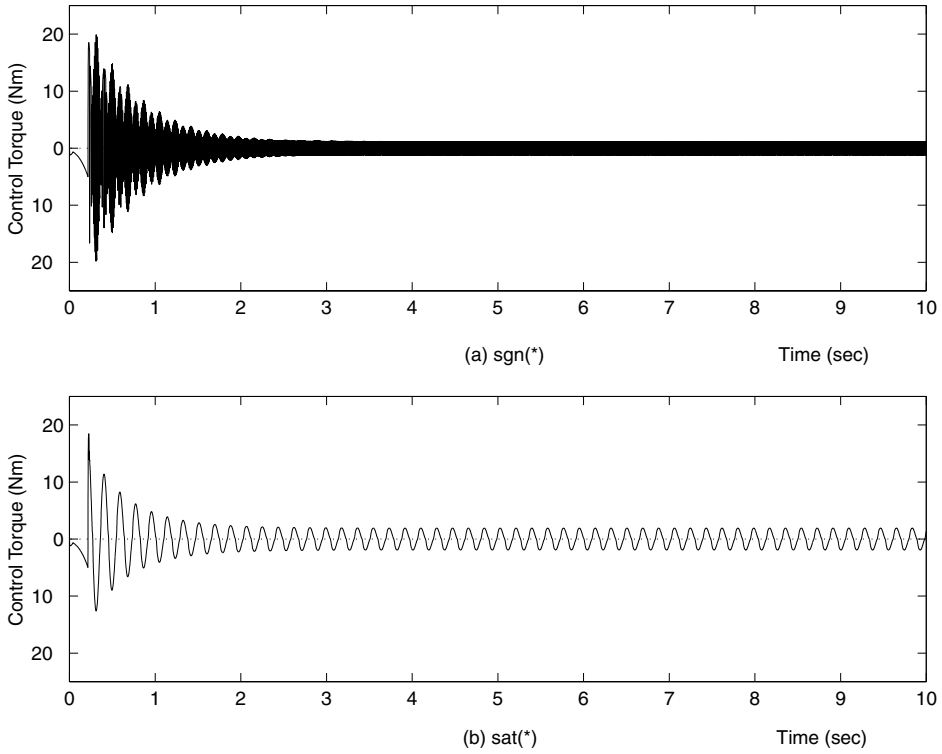


Figure 6. Control torque τ .

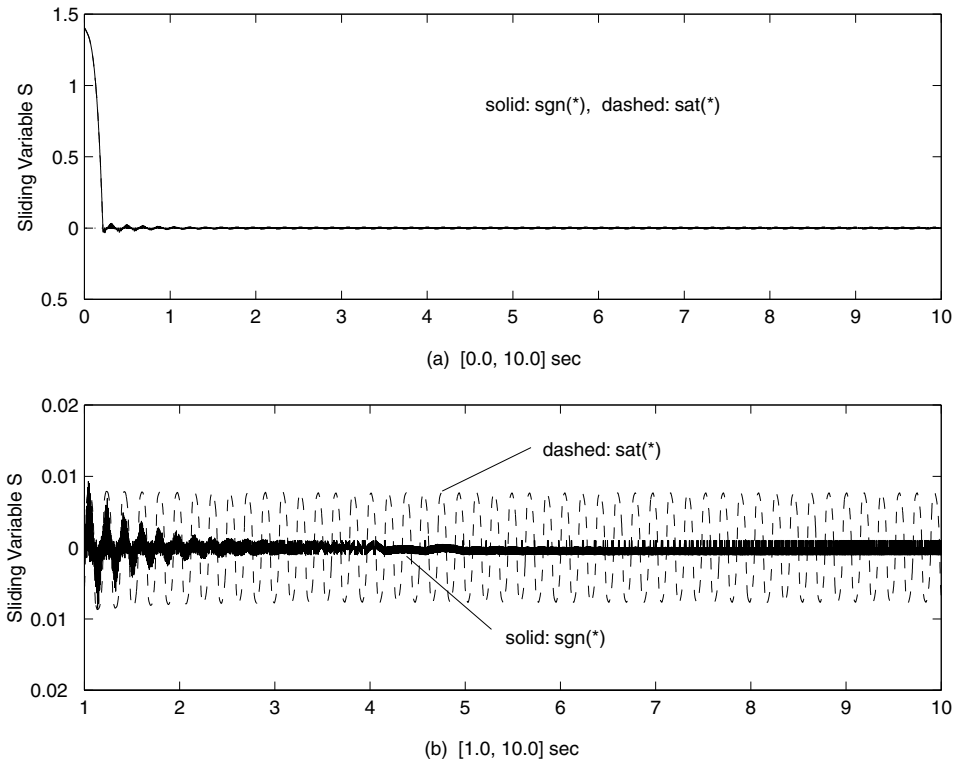


Figure 7. Sliding variable S.

associated with the truncated-model-based controllers are overcome. The controller can be easily implemented since all the required signals can be measured or estimated by some currently available sensor facilities. Numerical simulations are provided to verify the effectiveness of the presented controller.

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