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NON-MODEL-BASED POSITION CONTROL OF A PLANAR MULTI-LINK FLEXIBLE ROBOT

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This paper presents a class of non-model-based position controllers for a planar multi-link flexible robot. One can achieve not only the closed-loop stability of the original distributed parameter system, but also the asymptotic stability of the truncated system, which is obtained through representing the deflection of each link by an arbitrary finite number of flexible modes. The system dynamics (which are very complicated in the case of a multi-link flexible robot) are not explicitly involved in the controller design and stability proof. Instead, only a very basic system energy relationship and the dynamic equations of a single-link flexible robot are utilised. The controllers possess several remarkable advantages over traditional model-based ones. Numerical simulations are carried out on a two-link flexible robot and satisfactory results are obtained.

1. INTRODUCTION

Robots with flexible links possess many advantages over conventional rigid-link robots, such as lower arm weight, better energy efficiency, higher operating speed and improved portability [1]. These advantages greatly motivate the research in modeling and control of flexible link robots.

Modeling and control of flexible link manipulators are very difficult due to the infinite dimensionality of the system. A multi-rigid-link robot possesses only finite dimensions, while a robot with only one flexible link, described by partial differential equations (PDEs), is a distributed parameter system of infinite dimensions. Further, the inherent non-minimum phase characteristic from the base actuator to the end effector of flexible robots makes it very difficult to achieve a high level of performance whilst retaining robustness [2].

One way to deal with flexible robot system is to obtain an approximated finite dimensional model. The truncation in system dimensions is usually done based on model analysis, in which the infinite dimensions of the system are represented by the motion of joints (rigid modes) and an infinite number of assumed flexible modes. There are two methods to obtain the assumed flexible modes: the unconstrained and constrained modes methods [3]. Although the former is more accurate and has been adopted in [4] for a one-link unloaded flexible robot, it is difficult to extend it to a multi-link case. The latter, used by most of researchers, is obtained by assuming there is no acceleration of joints motion. In both methods, the finite dimensional model is derived by truncating the infinite number of modes, i.e. the modes with comparatively higher frequencies are neglected. It should be pointed out that the boundary value problem (and hence the corresponding mode shape functions) associated with the constrained modes method can be defined

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accurately only for a single-link flexible robot. In the multi-link case, even when the acceleration of joints motion is assumed to be zero, the boundary conditions of flexible links may not be obtainable. For this, further assumptions must be made to solve for the mode shape functions from the boundary value problem. For instance, in [5], the upper link of the the two-link flexible robot was assumed to have constant moment of inertia with respect to the middle joint and thus made the boundary value problem of the first link solvable. These assumptions will further reduce the accuracy of the model and may cause extra problems at the stage of controller design. When a model with finite dimensions is obtained, various kinds of controller design approaches for finite dimensional systems are then applicable [6–11]. Although the controllers in these papers were shown to be able to stabilise the truncated system by theoretical proof/simulations as well as laboratory experiments, the stability of the original infinite dimensional system is not obvious. The problems arise from the truncation procedure and other assumptions in the modeling should be carefully treated in practical applications. Some of these problems that have been highlighted in the literature are: (i) the order of controller needs to be increased with the number of the included flexible modes to achieve high accuracy of the performance; (ii) control and observation spillovers may occur due to the ignored high frequency dynamics [12]; and (iii) the controllers may be difficult to implement from the engineering point of view since full states measurements/observers are often required.

In an attempt to overcome the above shortcomings of the truncated model-based controller design approaches, an alternative method has been developed which has received increasing attention in recent years. In the method, system dynamics analysis and controller design are carried out directly based on the PDEs of the flexible robot system. Consequently, the stability analysis and resulting controllers are valid for the original distributed parameter system. Two examples can be found in [12] and [13]. In [12], direct strain feedback (DSFB) at the base of the single flexible link was considered to enhance the performance of the joint PID controller, and satisfactory experimental results were provided. However, the servo system of the motor drive, called a 'speed reference type motor drive', requires a very large internal gain. This extra requirement may make the system quite sensitive to noise and thus is undesirable when the robot is operated in a noisy environment. On the other hand, it is more convenient to use a conventional torque control-type motor drive in real-time implementation. The other paper [13] deals with the tracking of a Euler–Bernoulli beam. To make the boundary value problem solvable, an extra assumption is made in that there is no hub inertia, i.e. $I_h = 0$, which is not practically realisable. Also, both papers consider only a single-link flexible robot. In multi-link case, due to the highly complicated dynamics (PDEs), the system analysis, and subsequently the controller design, will be extremely difficult.

In this paper, a class of non-model-based controllers are presented for a planar multi-link flexible robot. Flexible robots with collocated actuators/sensors are energy dissipative, and hence are stable and possesses a certain degree of robustness [2]. The method is 'non-model-based' because the system dynamics is not involved explicitly in the controller design and stability proof. Actually, only a very basic system energy relationship and the dynamic equations of a one-link flexible beam are utilised. All feedback signals can be chosen to be measurable, because the control law possesses a form through which various kinds of feedback (including those from non-collocated sensors) can be introduced without destabilising the (distributed parameter) closed-loop system. This shows that the energy dissipative configuration is introduced in a certain way. Further, without the need to increase the order of the controller, the asymptotic stability of the truncated system with an arbitrary finite number of flexible modes can also be achieved. These properties imply that the method is free from problems (i)–(iii) mentioned above. The controller design based on the system energy relationship has been proposed in previous work [14]. However, that controller does not ensure asymptotic stability for the truncated system, and may suffer from the problem of local minimum, as shown in Section 3. The current work redesigns the controller and improves the results.

The rest part of this paper is organised as follows: in Section 2, the geometry of the planar multi-link flexible robot is briefly introduced; the non-model-based controller design approach is presented in Section 3; computer simulations are carried out on a two-link flexible robot to verify the effectiveness of the controllers in Section 3, followed by the conclusion in Section 5.

2. PLANAR MULTI-LINK FLEXIBLE ROBOT

Consider a planar N-link flexible robot which is operated in the horizontal plane with the effect of gravity being ignored. The geometry of the robot is shown in Fig. 1. The N links are connected by N motors. Motor 1 is located at the origin of the inertia frame $X_1O_1Y_1$. The remaining motors are movable. The free tip of the last link has a concentrated mass payload attached. There are $2N$ frames in total which describe the system i.e. $X_iO_iY_i$ and $x_iO_iy_i$, $i = 1, 2, \dots, N$. Apart from frame $X_1O_1Y_1$ (the inertia frame) all the other frames are local reference frames attached to the corresponding joints, specifically axis O_iX_i ($i = 2, 3, \dots, N$) is defined as the tangent to the end tip of link $i-1$, and axis O_ix_i ($i = 1, 2, \dots, N$) is tangent to link i at the base. The angular position of the ith link is denoted by θ_i measured in local frame $X_iO_iY_i$. θ_i is actually the angular difference between frames $x_iO_iy_i$ and $X_iO_iY_i$.

3. NON-MODEL-BASED POSITION CONTROLLER DESIGN

As stated in Section 1, obtaining the dynamic model of a multi-link flexible robot is very difficult due to its highly complicated dynamic behaviour. The traditional truncated model obtained by the constrained modes method suffers from many problems both in achieving stability for the original distributed parameter system, and in practical applications. Although the approaches developed in [12] and [13] are free from the undesirable

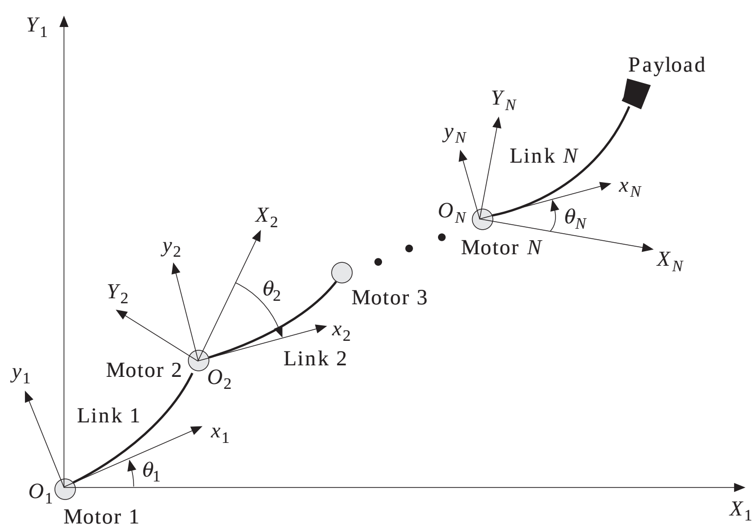

Figure 1. Geometry of the N-link flexible robot.

truncation procedure, only the single-link case was considered and some extra assumptions were made to the system.

In this section, a non-model-based controller design method for the multi-link flexible robot described in Section 2 is presented. The method utilises only a very basic system energy relationship and the dynamic equations of a one-link flexible robot which are much easier to obtain than that of the whole multi-link system (Appendix A). The resulting controllers possess many advantages over those model-based ones as will be shown.

First the collocated control of flexible robots is discussed. It is known that collocated actuators/sensors lead to an energy dissipative system, in which the stability and a certain degree of robustness are guaranteed [2]. The most simple collocated controller is the joint proportional-derivative (PD) controller, which uses only joint angle and joint velocity feedback and controls the flexible robot as a rigid one. The joint PD control leads to a stable closed-loop system. This is valid for not only the original system, but also the truncated model [15]. The problem of the joint PD control is that the motion of the end-effector of flexible robot exhibits very serious oscillation because there is no explicit control to suppress the flexible vibration. To introduce this kind of control, extra sensors must be used. However, as stated in [2], non-collocated sensors feedback may lead to instability. In [12], the extra feedback selected was the base strain of the flexible link. This feedback is also collocated since the base strain actually represents the bending moment of the beam at the joint. In this paper, it is shown that extra non-collocated feedback, if introduced in a certain way, can also be utilised without destroying the stability.

Referring to Fig. 1, the control objective is described as driving the N motors such that the tip payload is moved to a pre-defined position quickly, smoothly and accurately.

This paper assumes that any damping is neglected and the robot is operated without friction. Thus, the fact that the total change in system energy is equal to the work done by the N motor torques leads to the following basic energy relationship

$$E_k - E_{k0} + E_p - E_{p0} = \sum_{i=1}^{N} \int_{i=1}^{N} \tau_i \dot{\theta}_i(t)\, \mathrm{d}t, \tag{1}$$

where E_k and E_p are the total kinetic energy and total potential energy of the system, E_{k0} and E_{p0} are kinetic energy and potential energy at the initial moment $t = 0$, and τ_i is the torque provided by motor i. Differentiating with respect to time on both sides of equation (1) yields

$$\dot{E}_k + \dot{E}_p = \sum_{i=1}^{N} \tau_i \dot{\theta}_i(t). \tag{2}$$

Now the theorem on the closed-loop stability of the multi-link flexible robot system is discussed.

Theorem 1: the closed-loop multi-link flexible robot system is stable if the N control torques are given by

$$\tau_i = -k_{pi}[\theta_i(t) - \theta_{fi}] - k_{vi}\dot{\theta}_i - k_i \operatorname{sgn}(\dot{\theta}_i) f_i(t) \int_0^t |\dot{\theta}_i(s)| f_i(s)\, \mathrm{d}s, \qquad i = 1, 2, \ldots, N, \tag{3}$$

where k_{pi}, $k_{vi} > 0$, $k_i \geqslant 0$, θ_{fi} is the constant final position of the ith link, $\dot{\theta}_i$ is the time derivative of θ_i, $f_i(t)$ is the extra feedback introduced for link i and will be defined later, and the signum function is defined by

$$\operatorname{sgn}(\dot{\theta}_i) = \begin{cases} 1, & \dot{\theta}_i > 0 \\ 0, & \dot{\theta}_i = 0 \\ -1, & \dot{\theta}_i < 0 \end{cases}.$$

This can be proven by defining the following Lyapunov function candidate:

$$V = E_k + E_p + \frac{1}{2}\sum_{i=1}^{N} k_{pi}[\theta_i(t) - \theta_{fi}]^2 + \frac{1}{2}\sum_{i=1}^{N} k_i \left[\int^t |\dot{\theta}_i(s)| f_i(s)\, ds\right]^2. \quad (4)$$

Recalling equation (2), the time derivative of Lyapunov function V is given by

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{N} \tau_i \dot{\theta}_i(t) + \sum_{i=1}^{N} k_{pi}\dot{\theta}_i[\theta_i(t) - \theta_{fi}] \\ &\quad + \sum_{i=1}^{N} k_i |\dot{\theta}_i(t)| f_i(t) \int_0^t |\dot{\theta}_i(s)| f_i(s)\, ds \\ &= \sum_{i=1}^{N} \tau_i \dot{\theta}_i(t) + \sum_{i=1}^{N} k_{pi}\dot{\theta}_i[\theta_i(t) - \theta_{fi}] \\ &\quad + \sum_{i=1}^{N} k_i \dot{\theta}_i(t) \operatorname{sgn}(\dot{\theta}_i(t)) f_i(t) \int_0^t |\dot{\theta}_i(s)| f_i(s)\, ds. \end{aligned} \quad (5)$$

Substituting equation (3) into (5) yields

$$\dot{V} = -\sum_{i=1}^{N} k_{vi}\dot{\theta}_i^2 \leqslant 0. \quad (6)$$

Thus, the closed-loop system is energy dissipative and hence stable.

The last term in equation (3) represents the extra feedback introduced into the basic joint PD control [the first two terms in equation (3)] in an attempt to enhance the vibration control. Note that there are no restrictions on choosing $f_i(t)$; $f_i(t)$ can be any variable related to the vibration of the ith link, or any combination of the bending variables. Some examples of bending variables of link i described in local frame $x_i O_i y_i$ are $y_i(x_i, t)$ (deflection at x_i), $y_i'(x_i, t)$ (rotation at x_i), $y_i''(x_i, t)$ (strain at x_i), $y_i'''(x_i, t)$ (shearing force at x_i). Obviously x_i should not exceed the length of the ith link. It is clear that choosing $f_i(t)$ as non-collocated sensor feedback is also permitted, i.e. non-collocated sensors feedback can be introduced in this way without destabilising the system.

Remark 1. Joint PD control is a special case of controller (3) when $k_i = 0$. Although joint PD control does not destabilise the system, the system performance is unsatisfactory because the vibration of the beam is not suppressed effectively. The introduction of the extra feedback allows the possibility to consider the vibration explicitly and provide the corresponding control.

Remark 2. The controller is independent of system parameters and thus possesses stability robustness to system parameter uncertainties. In fact, the closed-loop system is stable as long as k_{pi}, $k_{vi} > 0$ and $k_i \geqslant 0$.

Remark 3. The controller is easy to implement in practice. The joint angle θ_i and joint velocity $\dot{\theta}_i$ can be obtained by rotary encoder and tachometer attached to the rotor of motor i, and $f_i(t)$ can be decided from the available sensor facilities. Consequently, all feedback variables are practically measurable.

Remark 4. The stability proof is independent of the system dynamics, and thus the problem associated with model-based controllers mentioned in Section 1 are avoided.

Remark 5. If each $f_i(t)$ is chosen to be local feedback (i.e. described in local frame $x_iO_iy_i$, including collocated and non-collocated feedback), the controller is of the decentralised type, which has the advantages of requiring little computing power, and ease of implementation and tolerance to failure, since the N controllers in equation (3) can be implemented in parallel. When global feedback, for example acceleration feedback, is used, the controller is no longber decentralised, but the closed-loop stability of the system is not affected.

Although the closed-loop stability for the original distributed system has been given, it is difficult to achieve asymptotic stability. This is mainly due to the infinite dimensionality of the system. If the system is of finite dimensions, then LaSalle's theorem [16] can be used to prove the asymptotic stability, but the flexible robot system is infinite by dimensional. However, asymptotic stability can be achieved for the truncated system with arbitrary finite number of flexible modes.

In [14], the idea of controller design based on the system energy relationships (1) and (2) was proposed. In that paper, instead of the absolute value of $\dot{\theta}_i$, joint velocity itself was used in the last term in Lyapunov function candidate, i.e.

$$\bar{V} = E_k + E_p + \frac{1}{2}\sum_{i=1}^{N} k_{pi}[\theta_i(t) - \theta_{fi}]^2 + \frac{1}{2}\sum_{i=1}^{N} k_i\left[\int_0^t \dot{\theta}_i(s) f_i(s)\,\mathrm{d}s\right]^2.$$

The resulting controller is of the form

$$\bar{\tau}_i = -k_{pi}[\theta_i(t) - \theta_{fi}] - k_{vi}\dot{\theta}_i - k_i f_i(t)\int_0^t \dot{\theta}_i(s) f_i(s)\,\mathrm{d}s.$$

Although it possesses the same advantages as stated in the above remarks, this controller suffers from the problem of local minimum, i.e. the robot may stop before reaching the final position. This is possible if the ith link tends to stop at a position θ_{mi} (hence $\dot{\theta}_i \to 0$) which is not equal to θ_{fi}, then the first and the third terms in $\bar{\tau}_i$ approaches two constants, and the middle term approaches zero. If the two constants happen to cancel each other, $\bar{\tau}_i$ tends to zero and the ith link becomes stuck at position $\theta_i = \theta_{mi}$. This also implies that $\bar{\tau}_i$ cannot ensure asymptotic stability for the truncated system since the same local minimum problem also exists. The controller in equation (3) is free from this problem because τ_i will not be zero in the above situation. Suppose link i stops at θ_{mi}, control torque is $\tau_i = -k_{pi}[\theta_{mi} - \theta_{fi}]$, which is not zero unless $\theta_{mi} = \theta_{fi}$. Therefore, link i cannot stop before reaching the final position. In the following, it is shown that controller (3) stabilises the truncated system asymptotically by using only the dynamic equations of a one-link flexible robot given in Appendix A.

Theorem 2: the controller given by equation (3) can guarantee the asymptotic stability of the truncated system, which is obtained through representing the deflection of each link by an arbitrary finite number of flexible modes.

When the deflection of each link is represented by a finite number of flexible modes (higher frequency modes are neglected), the original system is reduced to a system of finite dimensions, to which LaSalle's theorem can be applied. Therefore, define the

$Z := \{(\theta, y) | \dot{V} = 0\}$ with $\theta = [\theta_1, \theta_2, \ldots, \theta_N]^T$ and $y = [y_1(x_1, t), y_2(x_2, t), \ldots, y_N(x_N, t)]^T$, and consider the motion of the system in the largest invariant set in set Z.

When $\dot{V} \equiv 0$, we have $\dot{\theta}_i \equiv 0$, and hence $\ddot{\theta}_i = 0$. The controller becomes

$$\tau_i = -k_{pi}[\theta_i - \theta_{fi}], \quad i = 1, 2, \ldots, N,$$

which is a constant.

Firstly, consider Link 1. The dynamic equations of a planar single-link flexible robot given by (A4–A5) (Appendix A) lead to the following equations for the first link

$$k_{p1}[\theta_1 - \theta_{f1}] = EI_1 y_1''(0, t) \tag{7}$$

$$\rho_1 \ddot{y}_1[(x_1, t)] = -EI_1 y_1''''(x_1, t), \quad 0 \leqslant x_1 \leqslant L_1, \tag{8}$$

where the subscript 1 is used to denote the corresponding quantities of link 1. To prove the asymptotic stability of the closed-loop system, equations (7) and (8) need to be solved. This can be done by invoking the variables separation $y_1(x_1, t) = \Phi(x_1)Q(t)$ and using four boundary conditions for equation (8). Two of the boundary conditions can be obtained directly from Fig. 1. By noting that axis Ox_1 is a tangent to link 1 at the base gives

$$y_1(0, t) = 0 \tag{9}$$

$$y_1'(0, \mathrm{t}) = 0. \tag{10}$$

The other two boundary conditions are given by

$$y_1''(0, t) = \frac{k_{p1}}{EI_1}[\theta_1 - \theta_{f1}] \tag{11}$$

$$y_1''(L_1, t) = \frac{k_{p2}}{EI_1}[\theta_2 - \theta_{f2}]. \tag{12}$$

Equation (11) is identical to equation (7), and equation (12) can be derived as follows. Because $\dot{\theta}_i \equiv 0$, the robot is operated as if all the motors are locked. Consequently, each motor can be taken simply as a concentrated mass. Hence, the bending moment at the tip of link $i-1$ should be equal to the base bending moment of link i, i.e.

$$EI_{i-1} y_{i-1}''(L_{i-1}, t) = EI_i y_i''(0, t), \quad i = 2, 3, \ldots, N. \tag{13}$$

Considering links 1 and 2, and noting that

$$\begin{aligned} EI_2 y_2''(0, t) &= -\tau_2 \\ &= k_{p2}[\theta_2 - \theta_{f2}], \end{aligned}$$

leads to equation (12). It is noted that the left sides of equations (11) and (12) are functions of time, while the right sides are constants. Here, it is assumed first that the two constants are zero. It is shown later that either constants being non-zero leads to invalid solutions. Now equations (11) and (12) can be rewritten as

$$y_1''(0, t) = 0 \tag{14}$$

$$y_1''(L_1, t) = 0. \tag{15}$$

Applying similar reasoning to the rest of the links leads to the following conclusions. All control torques are equal to zero and hence the joint angle of each link reaches the final value, i.e.

$$\tau_i = 0 \quad \text{and} \quad \theta_i = \theta_{fi}, \quad i = 1, 2, \ldots, N.$$

Also, the base bending moment and the bending moment at the tip of each link are zero (note the tip payload of link N is a concentrated mass), i.e.

$$y_i''(0, t) = 0 \quad \text{and} \quad y_i''(L_i, t) = 0, \quad i = 1, 2, \ldots, N.$$

Now equation (8) is solved under boundary conditions (9), (10), (14) and (15). Using $y_1(x_1, t) = \Phi(x_1)Q(t)$ leads to

$$\frac{\Phi''''}{\Phi}\frac{EI_1}{\rho_1} = -\frac{\ddot{Q}}{Q}. \tag{16}$$

Since the left side of equation (16) is only space dependent and the right side is a purely time-varying function, it is obvious that both sides must equal a constant. If the constant is denoted by K, two ordinary differential equations are obtained, namely,

$$\ddot{Q}(t) + KQ(t) = 0 \tag{17}$$

$$\Phi''''(x_1) = \frac{\rho_1}{EI_1} K\Phi(x_1). \tag{18}$$

The four boundary conditions (9) and (10) and (14) and (15) can then be written as

$$\Phi(0) = 0 \tag{19}$$

$$\Phi'(0) = 0 \tag{20}$$

$$\Phi''(0) = 0 \tag{21}$$

$$\Phi''(L_1) = 0. \tag{22}$$

Equation (18) and conditions (19–22) describe the corresponding boundary value problem; to sove it all possible K should be considered.

When $K = 0$, the solution to equation (18) has the general form of

$$\Phi(x_1) = C_1 x_1^3 + C_2 x_1^2 + C3x_1 + C_4.$$

Substituting this into (19–22) yields $C_1 = C_2 = C_3 = C_4 = 0$, i.e. $y_1(x_1, t) = 0$.

When $K < 0$, letting $K = -\omega^2$ with ω being a non-zero number, equation (18) can be rewritten as

$$\Phi''''(x_1) = -\left(\frac{\beta}{L_1}\right)^4 \Phi(x_1), \tag{23}$$

where

$$\left(\frac{\beta}{L_1}\right)^4 = \frac{\rho_1}{EI_1}\omega^2. \tag{24}$$

The general solution to (23) is of the form

$$\Phi(x_1) = C_1 \mathrm{e}^{ax_1} \sin(ax_1) + C_2 \mathrm{e}^{ax_1} \cos(ax_1) + C_3 \mathrm{e}^{-ax_1} \sin(ax_1) + C_4 \mathrm{e}^{-ax_1} \cos(ax_1), \tag{25}$$

where $a = \sqrt{2}\beta/2L_1$. Substituting equation (25) into (19–22) obtains the following set of equations

$$\begin{cases} C_2 + C_4 = 0 \\ C_1 + C_2 + C_3 - C_4 = 0 \\ C_1 - C_3 = 0 \\ (C_1 \mathrm{e}^{aL_1} - C_3 \mathrm{e}^{-aL_1}) \cos(aL_1) - (C_2 \mathrm{e}^{aL_1} - C_4 \mathrm{e}^{-aL_1}) \sin(aL_1) = 0, \end{cases} \tag{26}$$

which are solved for C_i ($i = 1, 2, 3, 4$). The determinant of the coefficient matrix of (26), which is a function of a, is given by

$$\text{Det} = 4\,(\sin(aL_1)\cosh(aL_1) + \cos(aL_1)\sinh(aL_1)).$$

To obtain the non-trivial solution of equation (26), we need Det $= 0$, which can be satisfied by an infinite number of a. Consider the non-negative solutions $a_i \geqslant 0$ with $i = 1, 2, \ldots$ (note that if a_i is a solution, then $-a_i$ is also a solution):

$$a_1 = 0;$$

$$a_i \to \infty \quad \text{as} \quad i \to \infty; \quad \text{and}$$

$$L_1(a_{a+1} - a_1) \to \pi \quad \text{as} \quad i \to \infty.$$

This leads to an infinite number of solutions to the boundary value problem

$$\phi_i(x_1) = C_1^i\,\mathrm{e}^{a_i x_1}\sin(a_i x_1) + C_2^i\,\mathrm{e}^{a_i x_1}\cos(a_i x_1) + C_3^i\,\mathrm{e}^{-a_i x_1}\sin(a_i x_1) + C_4^i\,\mathrm{e}^{-a_i x_1}\cos(a_i x_1),$$

where $C_1^i \sim C_4^i$ denote the solution to equation (26) corresponding to a_i. Now the time-dependent ordinary differential equation (17) is considered. When $K = -\omega_i^2 = 4EI_1 a_i^4/\rho_1$ [using equation (24)], the solution to equation (17) is given by

$$q_i(t) = D_1^i\,\mathrm{e}^{\omega_i t} + D_2^i\,\mathrm{e}^{-\omega_i t}, \tag{27}$$

where D_1^i and D_2^i are related to the initial conditions of $y_1(x_1, t)$. Since the motion of the system when $\dot{V} \equiv 0$, the 'initial' above corresponds to the moment when the system enters the set of $\dot{V} \equiv 0$, instead of the initial operating moment. Then, from the superposition or linearity principle [18], a solution to equation (8) under boundary conditions (9), (10), (14) and (15) is given by

$$y_1(x_1, t) = \sum_{i=1}^{\infty} \phi_i(x_1) q_i(t). \tag{28}$$

Note that the ω_i in equation (27) can be either positive or negative, without loss of generality, if $\omega_i \geqslant 0$. This leads to $D_1^i = 0$ as follows. If $D_1^i \neq 0$, then $\lim_{t\to\infty} q_i(t) \to \infty$ and $\lim_{t\to\infty} \dot{q}_i(t) \to \infty$, and hence $\lim_{t\to\infty} \dot{y}_1(x_1, t) \to \infty$. This implies the kinetic energy E_k of the system approaches infinity, which contradicts the fact the V in equation (4) is actually bounded. Therefore, D_1^i must be zero. Consequently, when $K < 0$, the solution (28) approaches zero as time approaches infinity.

The last choice of K is $K > 0$. Let $K = \omega^2$ with ω being a non-zero number. Recalling equation (24), equation (18) can be rewritten similarly as

$$\Phi''''(x_1) = \left(\frac{\beta}{L_1}\right)^4 \Phi(x_1). \tag{29}$$

The general solution of (29) is of the form

$$\Phi(x_1) = C_1 \cos\frac{\beta x_1}{L_1} + C_2 \cosh\frac{\beta x_1}{L_1} + C_3 \sin\frac{\beta x_1}{L_1} + C_4 \sinh\frac{\beta x_1}{L_1}. \tag{30}$$

From conditions (19–22):

$$\begin{cases} C_1 + C_2 = 0 \\ C_3 + C_4 = 0 \\ -C_1 + C_2 = 0 \\ -C_1 \cos\beta + C_2 \cosh\beta - C_3 \sin\beta + C_4 \sinh\beta = 0 \end{cases} \tag{31}$$

which are solved for C_i $(i = 1,2,3,4)$. Similarly, a non-trivial solution to equation (31) exists provided that the determinant of the coefficient matrix is equal to zero, i.e.

$$-2\,(\sin(\beta) + \sinh(\beta)) = 0.$$

However, it is easy to check that this is possible only when $\beta = 0$, which in turn implies equation (29) is reduced to $\Phi''''(x_1) = 0$ which has been considered in the case of $K = 0$. This implies the solution $y_1(x_1, t)$ is zero when $K > 0$. Therefore, $\dot{V} \equiv 0$ implies either $y_1(x_1, t) = 0$ if $K \geqslant 0$, or $y_1(x_1, t) \to 0$ as $t \to \infty$ if $K < 0$.

In summary, $y_1(x_1, t) = 0$ provided that the system motion is in the largest invariant set in the set Z. Moreover, recalling that we already have $\theta_1 = \theta_{f1}$, we can further conclude that if the system motion is in the largest invariant set in Z, the first link must stop in the final position described by $\theta_1 = \theta_{f1}$ and $y_1(x_1, t) = 0$. Then, in this case, the local frame $X_2O_2Y_2$ is actually static with respect to the inertia frame $X_1O_1Y_1$. This allows us to take $X_2O_2Y_2$ as the inertia frame in which the second link is to be considered. From equations (41) and (42), the motion of link 2 described in this inertia frame is given by

$$k_{p2}[\theta_2 - \theta_{f2}] = EI_2 y_2''(0, t) \tag{32}$$

$$\rho_2 \ddot{y}_2[(x_2, t)] = -EI_2 y_2''''(x_2, t), \quad 0 \leqslant x_2 \leqslant L_2, \tag{33}$$

which is of exact the same form as equations (7) and (8). Moreover, from Fig. 1 and the above conclusions, a set of boundary conditions similar to (9), (10), (14) and (15) also exist for equation (33), i.e.

$$y_2(0, t) = 0$$

$$y_2'(0, t) = 0$$

$$y_2''(0, t) = 0$$

$$y_2''(L_2, t) = 0$$

Thus, carrying out similar analysis for link 2 leads to the conclusion that $y_2(x_2, t) = 0$ provided that the system motion is in the largest invariant set in Z. Also recalling $\theta_2 = \theta_{f2}$, link 2 must stop in its final position described by $\theta_2 = \theta_{f2}$ and $y_2(x_2, t) = 0$. This implies the local frame $X_3O_3Y_3$ can be taken as the inertia frame for link 3. Now, it is easy to see that repeating the same procedure until link N finally leads to the fact that if the system motion is in the largest invariant set in the set of $\dot{V} = 0$, then all links must stop in their final positions, i.e. $\theta_i = \theta_{fi}$ and $y_i(x_i, t) = 0$ $(i = 1, 2, \ldots, N)$.

Now, the case if the right side of either equation (11) or (12) is a non-zero constant is discussed. If this is true, then from $y_1(x_1, t) = \Phi(x_1)Q(t)$, $Q(t)$ and hence $y_1(x_1, t)$ must be constant. This means that the first link is static, which implies that the right sides of both equation (11) and (12) must equal the same non-zero constant (from the moment balance of a static bending beam). Subsequently frame $X_2O_2Y_2$ can be taken as the inertia frame for link 2. Because the base bending moment of link 2 is equal to the tip bending moment of link 1 (note the condition $\dot{V} \equiv 0$), the base bending moment of link 2 must also be the same constant. This, by assuming $y_2(x_2, t) = \Phi(x_2)Q(t)$, implies link 2 is also static. Repeating this for all links leads to a static bending N-link flexible robot. Then from the moment balance of the static robot, the base bending moment of link 1 should be equal to the tip bending moment of link N. Since the free tip of link N is loaded with a concentrated mass, the tip bending moment is zero. Therefore either the base bending moment of link 1 [equation (11)] or its tip bending moment [equation (12)] must be zero.

It should be pointed out the above discussion is valid no matter how many (finite or infinite) flexible modes are considered, since the truncation assumption is not used. Because of the infinite dimensionality of the original distributed parameter system, it cannot be guaranteed that the system solution approaches the largest invariant set in Z as time approaches infinity (i.e. LaSalle's theorem is not applicable). Now invoking the truncation assumption, the elastic deflection of each link is assumed to be described by a finite number of flexible modes, and subsequently the system is of only finite dimensions. For this truncated system, because it has been proven already that the largest invariant set in Z is the final equilibrium position [i.e., $\theta_i = \theta_{fi}$ and $y_i(x_i, t) = 0, \quad i = 1, 2, \ldots, N$], the asymptotic stability directly follows the LaSalle's Theorem. This completes the proof of theorem 2.

The proof above does not need to be solved explicitly for the mode shape functions. Solving for mode shape functions is very difficult in the multi-link case, and some extra assumptions, such as the assumption of constant moment of inertia of the upper link must be made [5]. These extra assumptions will reduce further the accuracy of the resulting model, and should be taken into consideration in the controller design.

Remark 6. Asymptotic stability can be guaranteed for the truncated system of an arbitrary finite number of flexible modes without need to increase the order of the controller. This implies that the computational burden can be always kept light.

4. NUMERICAL SIMULATIONS ON A 2-LINK FLEXIBLE ROBOT

In this section, some numerical simulations are carried out on a two-link flexible robot. The plant is simulated by a finite element model, in which each link is divided into four elements of the same length. A fourth-order Runge–Kutta program with adaptive step, size is used to numerically solve the differential equations. The sampling interval is 0.01 s.

The system parameters are given in Table 1, in which M_{t1} denotes the mass of the second motor, and M_{t2} is the payload attached to the free tip of link 2.

The initial and final positions of the robot are described in Fig. 2. The initial joint positions are zero, and the set point values are $\theta_{f1} = 45^\circ$ and $\theta_{f2} = 30^\circ$.

TABLE 1

System parameters

	Link 1	Link 2
Length	$L_1 = 1.0$ m	$L_2 = 0.8$ m
Flexural rigidity	$EI_1 = 5.0$ Nm^2	$EI_2 = 3.0$ Nm^2
Linear density	$\rho_1 = 0.1$ kg/m	$\rho_2 = 0.1$ kg/m
Hub initial	$I_{h1} = 3.0$ kgm^2	$I_{h2} = 1.5$ kgm^2
Payload	$M_{t1} = 0.1$ kg	$M_{t2} = 0.05$ kg

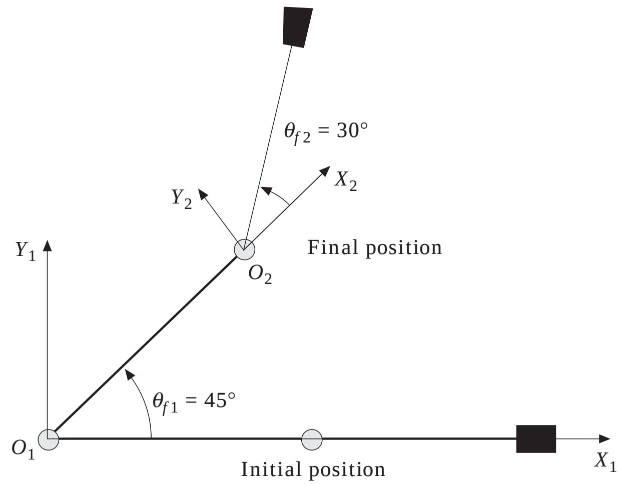

Figure 2. Two-link flexible robot.

4.1. Joint PD control

Firstly, consider the simulation results of the following joint PD controller

$$\tau_{PDi} = -k_{pi}(\theta_i - \theta_{fi}) - k_{vi}\dot{\theta}_i, \quad i = 1, 2.$$

From theorem 1, any $k_{pi}, k_{vi} > 0$ does not destabilise the closed-loop system. However, a different selection of k_{pi} and k_{vi} will lead to a very different performance. Here, they are determined as follows to make the equivalent rigid motion critically damped. If the flexible links are assumed to be rigid, using joint PD control leads to the following rigid motion error equations

$$I_{ei}\ddot{e}_i(t) + k_{vi}\dot{e}_i(t) + k_{pi}e_i = 0, \quad i = 1, 2,$$

where I_{ei} represents the equivalent inertia with respect to the ith joint, $e_i = \theta_i - \theta_{fi}$, and $\ddot{e}_i = \ddot{\theta}_i$ and $\dot{e}_i = \dot{\theta}_i$ since θ_{fi} is constant. It should be noted that because of the rotational movement of link 2 in its own local reference frame $X_iO_iY_i$, I_{e1} is not constnat even when the two links are all assumed to be rigid. For simplicity, in our simulations, I_{e1} is determined by further assuming that motor 2 is locked at $\theta_2 = 0$, i.e. the two links aligned. Subsequently, $I_{e1} = 3.46\,\mathrm{kgm}^2$ and $I_{e2} = 1.55\,\mathrm{kgm}^2$. For the second-order systems above, if critical damping is assumed ($\zeta = 1$), the PD feedback gains can be determined by

$$k_{pi} = I_{ei}\omega_{ni}^2$$

$$k_{vi} = 2I_{ei}\omega_{ni},$$

where ω_{ni} ($i = 1, 2$) are the corresponding natural frequencies. If $\omega_{n1} = 2.5$ and $\omega_{n2} = 2.3$, the two joint PD controllers are given by

$$\tau_{PD1} = -21.6(\theta_1 - \theta_{f1}) - 17.3\dot{\theta}_1 \quad (34)$$

$$\tau_{PD2} = -8.2(\theta_2 - \theta_{f2}) - 7.1\dot{\theta}_2. \quad (35)$$

The performance of the joint PD control are plotted by dashed lines in figures for comparison.

4.2. NON-MODEL-BASED CONTROLLER WITH BASE STRAIN FEEDBACK

From equation (3), one can see that in addition to joint PD control effort, the $f_i(t)$ term is introduced to enhance vibration suppression of flexible link i. As stated in Section 3, the introduction of the $f_i(t)$ term into the controller allows great freedom in feedback design according to actual instrumentation. Considering that strain gauges have been very widely used in control of flexible robots, it is assumed they are available and base strain of each link is used for feedback in the following simulations. Thus, the following controllers are used.

$$\tau_1 = \tau_{PD1} - 1400 y_1''(0, t)\,\mathrm{sgn}\,(\dot{\theta}_1) \int_0^t |\dot{\theta}_1(s)| y_1''(0, s)\,\mathrm{d}s \tag{36}$$

$$\tau_2 = \tau_{PD2} - 1400 y_2''(0, t)\,\mathrm{sgn}\,(\dot{\theta}_2) \int_0^t |\dot{\theta}_2(s)| y_2''(0, s)\,\mathrm{d}s, \tag{37}$$

where τ_{PD1} and τ_{PD2} are given in equations (34) and (35). From equation (3), $k_1 = k_2 = 1400$, and $f_1(t) = y_1''(0, t)$ and $f_2(t) = y_2''(0, t)$ are the base strains of links 1 and 2, respectively. It should be pointed out that the values of k_1 and k_2 may not be optimal. They are only used to show the effectiveness of the $f_i(t)$ feedback in supprressing elastic vibration.

Firstly, the joint motion is shown in Fig. 3. The joint motion of joint PD control (dashed lines), though quite fast, exhibits large vibration and overshoot. To speed up the joint motion further, one can use a larger natural frequency ω_{ni}; however, this leads to even larger vibration and overshoot. It can be seen that the new controller eliminates the vibration and overshoot in joint motion without slowing down the convergence.

The priority in tip positioning control is the tip trajectory, which is required to converge quickly with the smallest possible vibration/overshoot to improve positioning accuracy.

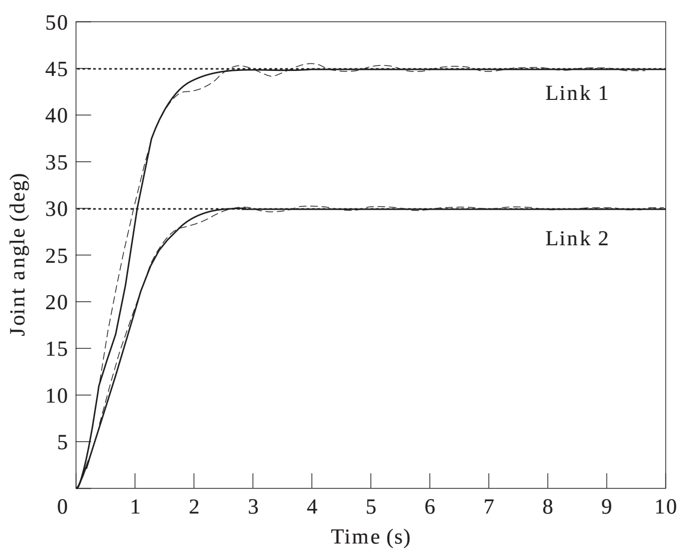

Figure 3. Joint motion. ----, τ_{PDi}; ——, τ_i.

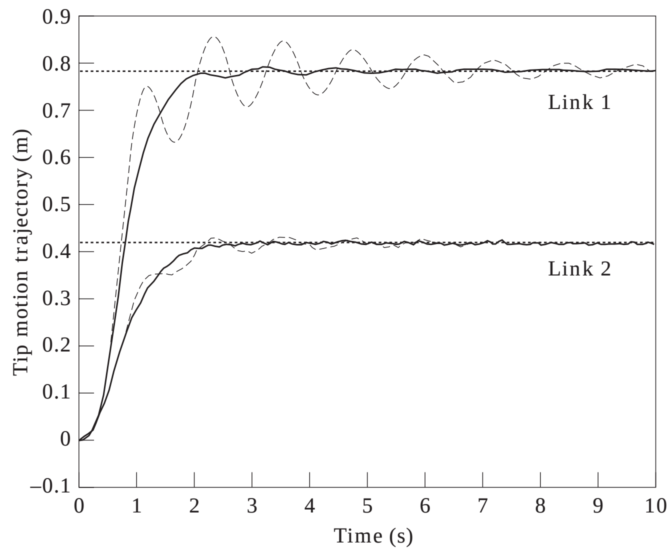

Figure 4. Tip motion $p_i = L_i\theta_i + y_i(L_i, t)$. ----, τ_{PDi}; ——, τ_i.

Under the assumption of small deflection, the tip position of the two links can be approximated by

$$p_1 = L_1\theta_1 + y_1(L_1, t)$$

$$p_2 = L_2\theta_2 + y_2(L_2, t),$$

in which the angular displacements θ_1 and θ_2 should be represented in radians rather than degrees. The tip positions p_1 and p_2 are plotted in Fig. 4. It can be seen that the joint PD control (dashed lines) exhibit serious vibration and large overshoot, which are very undesirable for accurate tip position control of flexible robots. Such results are not

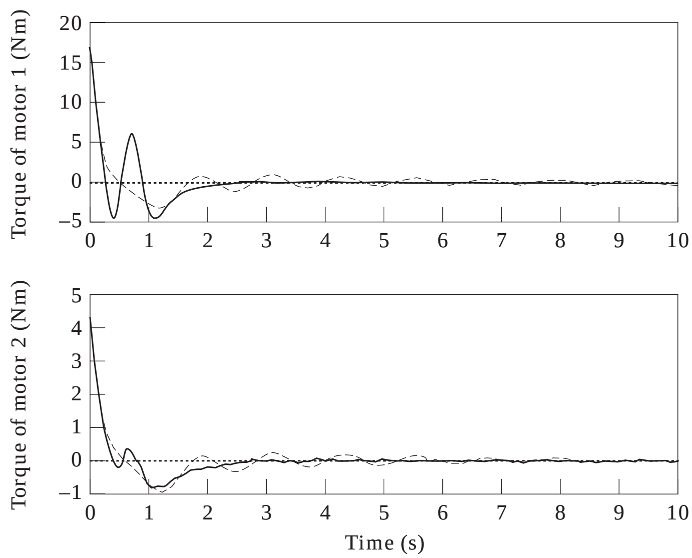

Figure 5. Control effort. ----, τ_{PDi}; ——, τ_i.

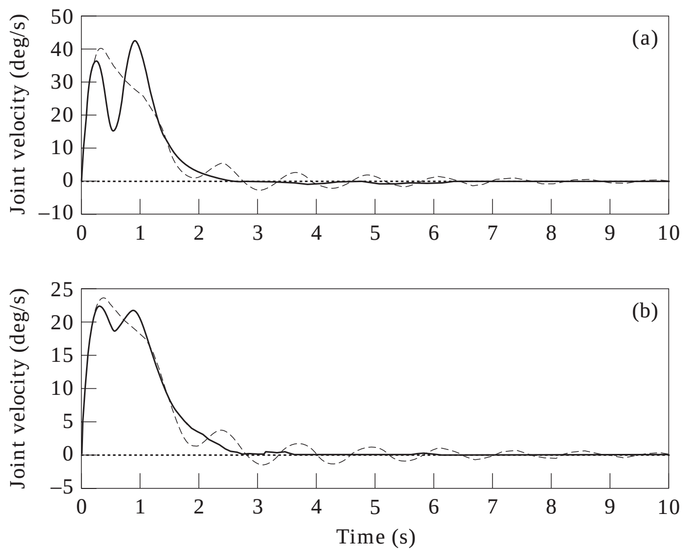

Figure 6. Joint velocity $\dot{\theta}_i$. (a) Link 1; (b) link 2. ----, τ_{PDi}; ——, τ_i.

surprising since there is no explicit control effort on suppressing flexible vibration. While the tip trajectories of the new controller are quite good since they are smooth, converging fast and there are only very little vibration and overshoot. From this figure, one can clearly see the effect of the $f_i(t)$ (which is the base strain here) feedback.

For completeness, Fig. 5 shows the control efforts. The joint velocities $\dot{\theta}_1$ and $\dot{\theta}_2$, which are used in both PD and the new controller, are plotted in Fig. 6. Finally, the feedback signals $f_1(t) = y_1''(0, t)$ and $f_2(t) = y_2''(0, t)$, together with the base strain in joint PD control (dashed lines) are given in Fig. 7. From Fig. 7, if base strain is used to represent the bending of the flexible links, one can also conclude that the introduction of $f_i(t)$ feedback is effective in controlling the link flexibility.

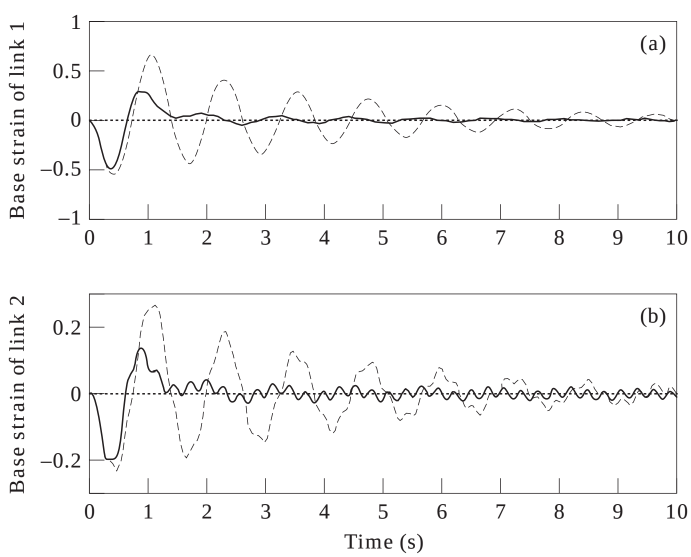

Figure 7. Base strain signal. (a) Link 1 [$y_1''(0, t)$]; (b) link 2 [$y_2''(0, t)$]. ----, τ_{PDi}; ——, τ_i.

In the above, through simulations, we have shown the effectiveness of the controller (3). It should be pointed out that other kinds of feedback $f_i(t)$ can also be considered depending on the available sensor facilities. The strain, which is very easy to measure in practice is used here only as an example.

5. CONCLUSION

This paper presents a non-model-based controller design approach for a multi-link flexible robot. In this method, the dynamics of the system is not explicitly needed. Instead, it makes use of only the very basic system energy relationship, and the dynamic equations of a single-link flexible robot.

Using this method, a general controller is constructed. The controller is independent of system parameters and subsequently possesses stability robustness to parameters variations. Furthermore, the controller is very simple, very flexible in its form and can be implemented easily according to actual instrumentation. Because the controller design is independent of system dynamics, the drawbacks/problems associated with the model-based controller design methods are essentially avoided.

Numerical simulations show that a very simple special case of the controller, in which only base strain feedback is used to represent the bending of each flexible link, can give quite satisfactory performance for a two-link flexible robot.

Currently, there are no guidelines on choosing $f_i(t)$ and the value of k_i to optimise the system performance. The effect of $f_i(t)$ feedback on the system dynamics shall be investigated in future research.

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APPENDIX A: DYNAMICS OF A PLANAR ONE-LINK FLEXIBLE ROBOT

A flexible link with length L, uniform flexural rigidity EI and mass per unit length ρ is clamped at its base to the rotor of a motor, and loaded by a concentrated mass M_t at the free tip, as shown in Fig. A1. The robot is operated in the horizontal plane without gravitational influence. Frame XOY is the inertia frame; frame xOy, with axis Ox being tangent to the beam at the base, is the local reference frame rotating with the robot.

By introducing $p(x, t) := x\theta(t) + y(x, t)$ to represent the position of a point on the beam, the kinetic energy of the system can be calculated by

$$E_k = \frac{1}{2} I_h \dot{\theta}^2 + \frac{\rho}{2} \int_0^L \dot{p}^2(x, t)\,\mathrm{d}x + \frac{1}{2} M_t p^2(L, t). \tag{A1}$$

The potential energy (strain energy of the bending beam) is given by

$$E_p = \frac{EI}{2} \int_0^L [y''(x, t)]^2\,\mathrm{d}x. \tag{A2}$$

By substituting equations (38) and (39) into the extended Hamilton's Principle:

$$\int_{t_0}^{t_f} \delta(E_k - E_p + \tau\theta)\,\mathrm{d}t = 0, \tag{A3}$$

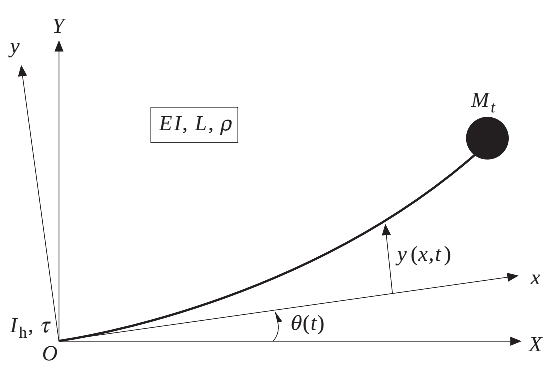

Figure A1. One-link flexible robot.

and noting that the bending moment of the flexible beam at the base can be calculated by

$$EIy''(0, t) = -\rho \int_0^L x\ddot{p}(x, t)\,\mathrm{d}x - M_t L[L\ddot{\theta}(t) + \ddot{y}(L, t)],$$

leads to the following dynamic equations

$$I_h\ddot{\theta}(t) = \tau + EIy''(0, t) \tag{A4}$$

$$\rho[x\ddot{\theta}(t) + \ddot{y}(x, t)] = -EIy''''(x, t), \quad 0 \leqslant x \leqslant L. \tag{A5}$$