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Adaptive robust control of a class of nonlinear strict-feedback discrete-time systems with unknown control directions

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ABSTRACT

In this paper, adaptive control is studied for a class of nonlinear discrete-time systems in strict-feedback form with unknown control directions. The system is transformed to an n-step ahead predictor, based on which an adaptive control employing the predicted future states has been proposed. The discrete Nussbaum gain is exploited to overcome the difficulty caused by unknown control directions. The proposed control guarantees the boundedness of all the closed-loop signals and the output tracking error can be made to converge to zero if the system is free of external disturbance. The effectiveness of the proposed control is demonstrated in the simulation.

1. Introduction

In recent decades, adaptive control of discrete-time systems has been studied extensively and lately, nonlinear discrete-time systems in the strict-feedback form have attracted much research interest. In a seminal work [7], it is proved that a class of continuous nonlinear systems can be transformed to parameter-strict-feedback form via parameter-independent diffeomorphisms. A similar result is obtained in the discrete-time [15], in which the geometric conditions for the systems transformable to the form are given and then the discrete-time backstepping design is proposed. Later, by exploiting the parameter projection properties, the discrete-time backstepping has been extended for robust control in the presence of nonparametric uncertainties [18], time-varying parameters [19], and further in [17] the overparameterization in discrete-time backstepping is overcome. In [20], a novel parameter estimation is proposed for this class of discrete-time systems and it guarantees the convergence of estimates to the real values in finite steps if the system is free of any disturbance or nonparametric uncertainties. Using neural network approximation, controlling strict-feedback systems with unknown system functions has been studied in [3] in which a system transformation was carried out before applying backstepping design. The result has also been extended to multi-input and multi-output (MIMO) systems in [4].

As indicated in the above mentioned papers, the Lyapunov design for nonlinear discrete-time systems becomes much more intractable than in the continuous-time. The reason lies in that the linearity property of the derivative of a Lyapunov function in continuous-time is not present in the difference of Lyapunov function in the discrete-time [13]. Many controls designed for continuous-time systems may be not suitable for discrete-time systems due to some inherent difficulties in discrete-time models.

As an effort to extend the results of adaptive control of parameter-strict-feedback nonlinear systems, in this paper we will study the tracking problem of a more general class of strict-feedback nonlinear systems in which the control gains are unknown constants. Strict-feedback systems in this form is first studied in continuous-time [14], in which it is indicated that a class of nonlinear triangular systems T_{1S} proposed in [12] is transformable to this form.

One challenge of controlling the systems in this form lies in the unknown signs of the control gains, which are normally required to be known *a priori* in the adaptive control literature. These signs, called control directions in [6], represent motion directions of the system under any control, and the knowledge of these signs makes the adaptive control design much easier. When the control directions are unknown, some control methods have been developed in the literature. In [9], the correction vector approach was proposed, and it has been extended to design adaptive control of first-order nonlinear systems with unknown control directions [1]. Nonlinear robust control has been proposed in [6], which can identify online the unknown control directions and can guarantee global stability of the closed-loop system. The Nussbaum gain was first proposed by Nussbaum [10] in continuous-time for adaptive control of first order systems and later it was adopted in the adaptive control of linear systems

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with nonlinear uncertainties [11] to counteract the lack of *a prior* knowledge of control directions. In [14], the Nussbaum gain has been successfully combined with backstepping design for strict-feedback systems, and then it becomes a systematical design approach for continuous-time counterpart of the discrete-time system studied in this paper. The discrete Nussbaum gain proposed in [8] will be exploited for controlling strict-feedback discrete-time systems for the first time. The discrete-time Nussbaum gain is very different from its continuous-time counterpart, and hence, the control design in continuous-time is not applicable to discrete-time.

The other challenge is that the elegantly devised coordinate mapping in [15,17–19] for discrete-time backstepping is not applicable when the control gains are unknown. Thus, in this paper, the original n-th order strict-feedback system is first transformed into an n-step ahead predictor, based on which the adaptive control can be constructed by the certainty equivalence principle rather than the backsteping. The n-step ahead predictor involves the future states and consequently, a states prediction has been constructed. A difficulty arises here that the prediction errors may cause instability of the closed-loop system. Therefore, an augmented error combining tracking error and prediction error is introduced in the adaptive control law to compensate the effect of prediction error.

To illustrate the control design, a disturbance-free case is studied first and then robust control is presented in the presence of bounded disturbance by dead-zone technique. It should be pointed that, even though the upper bound of the disturbance is unknown, the adaptive algorithm with dead-zone can still be constructed in this paper. However, *a priori* knowledge of the upper and lower bounds, which may not be easy to be obtained in practice, is usually required to be known in building the adaptive controls with dead-zone. All the closed-loop signals are guaranteed to be globally bounded and the tracking error will converge to zero in the absence of disturbance.

The main contributions of the paper lie in:

(i) The n-th order strict-feedback system is transformed into an n-step ahead predictor, then a systematic adaptive control design has been synthesized.

(ii) Discrete Nussbaum gain is for the first time exploited for strict-feedback systems to cope with unknown control gains and the proposed control structure is free of controller singularity.

(iii) The predicted future states are employed in the control law and the effects of the prediction errors are compensated by introducing an augmented error.

Throughout this paper, the following notations are used in order

- $\|\cdot\|$ denotes the Euclidean norm of vectors and induced norm of matrices.
- $A := B$ means that B is defined as A.
- $[\,]^{\mathrm{T}}$ represents the transpose of vector.
- $\mathbf{0}_{[p]}$ stands for p-dimension zero vector.
- $(\hat{\cdot})$ and $(\tilde{\cdot})$ denote the estimate of parameters and estimation error, respectively.
- N^+ denotes the set of all nonnegative integers.

2. Problem formulation and preliminaries

2.1. System representation

Consider a class of strict-feedback nonlinear discrete-time systems in the following form:

$$\begin{cases}\xi_1(k+1) = \Theta_1^{\mathrm{T}}\Phi_1(\bar{\xi}_1(k)) + g_1\xi_2(k)\\ \xi_2(k+1) = \Theta_2^{\mathrm{T}}\Phi_2(\bar{\xi}_2(k)) + g_2\xi_3(k)\\ \vdots\\ \xi_n(k+1) = \Theta_n^{\mathrm{T}}\Phi_n(\bar{\xi}_n(k)) + g_n u(k) + d(k)\\ y(k) = \xi_1(k)\end{cases} \tag{1}$$

where $\bar{\xi}_i(k) = [\xi_1(k), \xi_2(k), \ldots, \xi_i(k)]^{\mathrm{T}}$ are system states, $\Theta_i \in R^{p_i}$, $g_i \in R$, $i = 1, 2, \ldots, n$, are unknown system parameters (p_i's are positive integers), $\Phi_i(\bar{\xi}_i(k)) : R^i \to R^{p_i}$ are known vector-valued functions, and $d(k)$ is the external disturbance which is bounded, i.e., $|d(k)| \le \bar{d}$. The value of $\bar{d}$, is not required to be known and it is only used in the analysis. The control objective is to make the output $y(k)$ track a bounded reference trajectory $y_d(k)$ and to guarantee the boundedness of all the closed-loop signals.

Assumption 1. The system functions $\Phi_i(\bar{\xi}_i(k))$ are Lipschitz functions, i.e., $\|\Phi_i(\varepsilon_1)-\Phi_i(\varepsilon_2)\| \le L_i\|\varepsilon_1-\varepsilon_2\|, \forall\varepsilon_1, \varepsilon_2 \in R^i$, where L_i is the Lipschitz coefficient, and the control gains $g_i \ne 0, i = 1, 2, \ldots, n$.

Remark 1. When the control gains $g_i = 1$, system (1) becomes in parameter-strict-feedback form studied in [15,17–19]. But when g_i's are unknown, the control design for system (1) would be a challenge.

2.2. Useful Definitions and Lemmas

Definition 1 ([2]). Let $x_1(k)$ and $x_2(k)$ be two discrete-time scalar or vector signals, $\forall k \in N^+$.

- We denote $x_1(k) = O[x_2(k)]$, if there exist positive constants m_1, m_2 and k_0 such that $\|x_1(k)\| \le m_1 \max_{k'\le k}\|x_2(k')\| + m_2,\ \ \forall k > k_0$.
- We denote $x_1(k) = o[x_2(k)]$, if there exists a discrete-time function $\alpha(k)$ satisfying $\lim_{k\to\infty}\alpha(k) \to 0$ and a constant k_0 such that $\|x_1(k)\| \le \alpha(k)\max_{k'\le k}\|x_2(k')\|, \forall k > k_0$.
- We denote $x_1(k) \sim x_2(k)$ if they satisfy $x_1(k) = O[x_2(k)]$ and $x_2(k) = O[x_1(k)]$.

Lemma 1 ([5]). *For some given real scalar sequences $s(k)$, $b_1(k)$, $b_2(k)$ and vector sequence $\sigma(k)$, if the following conditions hold:*

(i) $\lim_{k\to\infty}\frac{s^2(k)}{b_1(k)+b_2(k)\sigma^{\mathrm{T}}(k)\sigma(k)} = 0$,

(ii) $0 < b_1(k) < K$ *and* $0 \le b_2(k) < K, \forall k \ge 1$, *with a finite* K,

(iii) $\sigma(k) = O[s(k)]$.

Then, we have

(a) $\lim_{k\to\infty} s(k) = 0$, *and* (b) $\sigma(k)$ *is bounded.*

Lemma 2. *Under Assumption 1, the states and input of system (1) satisfy*

$$\bar{\xi}_i(k) = O[y(k+i-1)],\quad i = 1, 2, \ldots, n$$
$$u(k) = O[y(k+n)]. \tag{2}$$

Proof. See Appendix A. ■

2.3. The discrete Nussbaum gain

The first discrete Nussbaum gain was proposed in [8], in which it is pointed that it is essential for the discrete sequence $x(k)$ to satisfy

$$x(0)=0, x(k)\geq 0, \quad |\Delta x(k)|\leq \delta_0, \quad \forall k \tag{3}$$

where $\Delta x(k) = x(k+1) - x(k)$, and δ_0 is a positive constant. Then, the discrete Nussbaum gain proposed in [8] is defined on the sequence $x(k)$ as

$$N(x(k)) = x_s(k)s_N(x(k)), \quad x_s(k)=\sup_{k'\leq k}\{x(k')\} \tag{4}$$

where $s_N(x(k))$ is the sign function of the discrete Nussbaum gain, i.e., $s_N(x(k)) = \pm 1$. The initial value is set as $s_N(x(0)) = +1$. Thereafter, the sign function $s_N(x(k))$ will be chosen by comparing the summation

$$S_N(x(k)) = \sum_{k'=0}^{k} N(x(k'))\Delta x(k') \tag{5}$$

with a pair of switching curves defined by $f(x_s(k)) = \pm x_s^{\frac{3}{2}}(k)$. The detail follows:

Step (a): At $k = k_1$, measure the output $y(k_1)$ and compute $\Delta x(k_1)$ and $x(k_1+1) = x(k_1) + \Delta x(k_1)$ and $S_N(x(k_1)) = S_N(x(k_1-1)) + N(x(k_1))\Delta x(k_1)$.

$$\text{Case } (s_N(x(k_1)) = +1): \begin{cases} \text{If } S_N(x(k_1)) \leq x_s^{\frac{3}{2}}(k_1), \text{ then go to Step (b)} \\ \text{If } S_N(x(k_1)) > x_s^{\frac{3}{2}}(k_1), \text{ then go to Step (c)} \end{cases}$$

$$\text{Case } (s_N(x(k_1)) = -1): \begin{cases} \text{If } S_N(x(k_1)) < -x_s^{\frac{3}{2}}(k_1), \\ \quad \text{then go to Step (b)} \\ \text{If } S_N(x(k_1)) \geq -x_s^{\frac{3}{2}}(k_1), \\ \quad \text{then go to Step (c)} \end{cases}$$

Step (b): Set $s_N(x(k_1+1)) = 1$, go to step (d).

Step (c): Set $s_N(x(k_1+1)) = -1$, go to step (d).

Step (d): Return to Step (a) and wait for the measurement of output.

Lemma 3 ([8]). *If $x_s(k)$ increases without bound, then*

$$\sup_{x_s(k)\geq\delta_0} \frac{1}{x_s(k)} S_N(x(k)) = +\infty, \qquad \inf_{x_s(k)\geq\delta_0} \frac{1}{x_s(k)} S_N(x(k)) = -\infty. \tag{6}$$

Lemma 4 ([8]). *If $x_s(k) \leq \delta_1$, then $|S_N(x(k))| \leq \delta_2$ where δ_1 and δ_2 are some positive constants.*

Lemma 5. *Let $V(k)$ be a positive definite function defined $\forall k$, $N(x(k))$ be the discrete Nussbaum gain proposed in [8], and θ be a nonzero constant. If the following inequality holds:*

$$V(k) \leq \sum_{k'=k_1}^{k} (c_1 + \theta N(x(k')))\Delta x(k') + c_2 x(k) + c_3, \quad \forall k \tag{7}$$

where c_1, c_2 and c_3 are some constants, k_1 is a positive integer, then $V(k)$, $x(k)$ and $\sum_{k'=k_1}^{k}(c_1 + \theta N(x(k')))\Delta x(k') + c_2 x_{(k)} + c_3$ must be bounded, $\forall k$.

Proof. Suppose that $x(k)$ is unbounded, then, because $x(k) \geq 0, \forall k$, $x_s(k)$ must increase without upper bound. Therefore, there must exist a k_0 such that $x_s(k) \geq \delta_0 \geq |\Delta x(k)|, \forall k \geq k_0$.

Noting that $x(k+1) \leq x_s(k) + \delta_0$, we have the following inequality from (7), $\forall k \geq k_0$.

$$\begin{aligned} 0 &\leq \frac{V(k)}{x_s(k)} \leq \frac{\theta}{x_s(k)} \sum_{k'=0}^{k} N(x(k'))\Delta x(k') + c_1\frac{x(k+1)}{x_s(k)} + c_2\frac{x(k)}{x_s(k)} \\ &\quad + \frac{c_3}{x_s(k)} - \frac{1}{x_s(k)}\sum_{k'=0}^{k_1-1}(c_1 + \theta N(x(k')))\Delta x(k') \\ &\leq \frac{\theta}{x_s(k)} S_N(x(k)) + 2c_1 + c_2 + \frac{c_3}{\delta_0} + c_4 \end{aligned} \tag{8}$$

where $c_4 = \frac{1}{\delta_0}|\sum_{k'=0}^{k_1-1}(c_1 + \theta N(x(k')))\Delta x(k')|$ is some finite constant. According to Lemma 3, it yields a contradiction if $x(k)$ is unbounded, no matter $\theta > 0$ or $\theta < 0$. Therefore, $x(k)$ is bounded, as well as $x_s(k)$, $\forall k$. According to Lemma 4, $\sum_{k'=0}^{k}(c_1 + \theta N(x(k')))\Delta x(k') + c_2 x(k) + c_3$ and $V(k)$ are also bounded. ■

It should be mentioned that the counterpart of Lemma 5 in continuous-time has been obtained in [14].

3. Future states prediction and system transformation

As mentioned in Section 1, when the control gains are unknown, the coordinate transformation-based discrete-time backstepping in [15,17–19] is not applicable. In this paper, we will exploit an alternative adaptive control design for strict-feedback discrete-time systems in (1).

3.1. Future states prediction

It is noted in (1) that the future states $\bar{\xi}_i(k+n-i)$, $i = 1, 2, \ldots, n-1$, are deterministic at the k-th step because they are not dependent of control input. In this section, let us consider predicting these future states in the presence of the unknown parameters.

Let $\hat{\Theta}_i(k)$ and $\hat{g}_i(k)$ denote the estimates of Θ_i and g_i at the k-th step, respectively. For convenience, we denote $\hat{\bar{\Theta}}_i(k) = [\hat{\Theta}_i^{\mathrm{T}}(k), \hat{g}_i(k)]^{\mathrm{T}} \in R^{p_i+1}$. Denote parameter estimate errors as $\tilde{\Theta}_i(k) = \hat{\Theta}_i(k) - \Theta_i$, $\tilde{g}_i(k) = \hat{g}_i(k) - g_i$, and $\tilde{\bar{\Theta}}_i(k) = [\tilde{\Theta}_i^{\mathrm{T}}(k), \tilde{g}_i(k)]^{\mathrm{T}}$.

Define one-step prediction $\hat{\xi}_i(k+1|k)$, the estimate of $\xi_i(k+1)$ as follows:

$$\hat{\xi}_i(k+1|k) = \hat{\bar{\Theta}}_i^{\mathrm{T}}(k-n+2)\,\Psi_i(k), \quad i = 1, 2, \ldots, n-1 \tag{9}$$

where $\Psi_i(k) = [\Phi_i^{\mathrm{T}}(\bar{\xi}_i(k)), \xi_{i+1}(k)]^{\mathrm{T}} \in R^{p_i+1}$.

Define two-step prediction $\hat{\xi}_i(k+2|k)$, the estimate of $\xi_i(k+2)$ as follows:

$$\hat{\xi}_i(k+2|k) = \hat{\bar{\Theta}}_i^{\mathrm{T}}(k-n+3)\,\hat{\Psi}_i(k+1|k), \quad i = 1, 2, \ldots, n-2 \tag{10}$$

where

$$\begin{aligned} \hat{\Psi}_i(k+1|k) &= [\Phi_i^{\mathrm{T}}(\hat{\bar{\xi}}_i(k+1|k)), \hat{\xi}_{i+1}(k+1|k)]^{\mathrm{T}} \in R^{p_i+1} \\ \hat{\bar{\xi}}_i(k+1|k) &= [\hat{\xi}_1(k+1|k), \hat{\xi}_2(k+1|k), \ldots, \hat{\xi}_i(k+1|k)]^{\mathrm{T}}. \end{aligned} \tag{11}$$

Define j-step ($j = 3, 4, \ldots, n-1$) prediction $\hat{\xi}_i(k+j|k)$, the estimate of $\xi_i(k+j)$ as follows:

$$\hat{\xi}_i(k+j|k) = \hat{\bar{\Theta}}_i^{\mathrm{T}}(k-n+j+1)\,\hat{\Psi}_i(k+j-1|k), \quad i = 1, 2, \ldots, n-j \tag{12}$$

where

$$\begin{aligned} \hat{\Psi}(k+j-1|k) &= [\Phi_i^{\mathrm{T}}(\hat{\bar{\xi}}_i(k+j-1|k)), \hat{\xi}_{i+1}(k+j-1|k)]^{\mathrm{T}} \\ \hat{\bar{\xi}}_i(k+j-1|k) &= [\hat{\xi}_1(k+j-1|k), \hat{\xi}_2(k+j-1|k), \ldots, \\ &\quad \hat{\xi}_i(k+j-1|k)]^{\mathrm{T}}. \end{aligned} \tag{13}$$

The parameter estimates are calculated from the following update law:

$$\bar{\hat{\Theta}}_i(k+1) = \bar{\hat{\Theta}}_i(k-n+2) - \frac{\tilde{\hat{\xi}}_i(k+1|k)\,\Psi_i(k)}{1+\Psi_i^{\mathrm{T}}(k)\Psi_i(k)},\quad i=1,2,\ldots,n-1$$

$$\tilde{\hat{\xi}}_i(k+1|k) = \hat{\xi}_i(k+1|k) - \xi_i(k+1),\quad \bar{\hat{\Theta}}_i(k) = [\hat{\Theta}_i^T(k), \hat{g}_i(k)]^{\mathrm{T}}. \tag{14}$$

Lemma 6. *The parameter estimates, $\bar{\hat{\Theta}}_i(k)$, $i = 1,2,\ldots,n-1$ obtained from (14) are bounded and the estimate errors satisfy*

$$\tilde{\bar{\xi}}_i(k+n-i|k) = o[O[y(k+n-1)]]$$

where $\tilde{\bar{\xi}}_i(k+n-i|k) = \hat{\bar{\xi}}_i(k+n-i|k) - \bar{\xi}_i(k+n-i)$.

Proof. See Appendix B. ■

3.2. System transformation

Let us rewrite system (1) as

$$\begin{cases} y(k+n) = \Theta_1^{\mathrm{T}}\Phi_1(\bar{\xi}_1(k+n-1)) + g_1\xi_2(k+n-1)\\ \xi_2(k+n-1) = \Theta_2^{\mathrm{T}}\Phi_2(\bar{\xi}_2(k+n-2)) + g_2\xi_3(k+n-2)\\ \vdots\\ \xi_n(k+1) = \Theta_n^{\mathrm{T}}\Phi_n(\bar{\xi}_n(k)) + g_n u(k) + d(k)\end{cases} \tag{15}$$

and by iterative substitution, all the equations can be combined together as follows

$$y(k+n) = \sum_{i=1}^{n}\Theta_{f_i}^{\mathrm{T}}\Phi_i(\bar{\xi}_i(k+n-i)) + gu(k) + d_o(k) \tag{16}$$

where

$$\Theta_{f_1} = \Theta_1,\ \Theta_{f_i} = \left(\prod_{j=1}^{i-1} g_j\right)\Theta_i,\quad i=2,3,\ldots,n,\ g=\prod_{j=1}^{n} g_j,\ d_o(k) = \frac{g}{g_n}d(k) \tag{17}$$

Define

$$\Theta_f^{\mathrm{T}} = [\Theta_{f_1}^{\mathrm{T}},\ldots,\Theta_{f_n}^{\mathrm{T}}]^{\mathrm{T}} \in R^p$$

$$\Phi(k+n-1) = [\Phi_1^T(\bar{\xi}_1(k+n-1)), \Phi_2^T(\bar{\xi}_2(k+n-2)),\ldots, \Phi_n^T(\bar{\xi}_n(k))]^{\mathrm{T}} \in R^p \tag{18}$$

where $p=\sum_{i=1}^{n} p_i$. Then, Eq. (16) can be written in a compact form as

$$y(k+n) = \Theta_f^{\mathrm{T}}\Phi(k+n-1) + gu(k) + d_o(k). \tag{19}$$

4. Adaptive control without disturbance

In this section, we consider the adaptive control in the disturbance free case, i.e., $d_o(k) = 0$. From (19), to achieve the output tracking, one possible control structure is:

$$u(k) = \frac{1}{\hat{g}(k)}(-\hat{\Theta}_f^{\mathrm{T}}(k)\Phi(k+n-1) + y_d(k+n))$$

where $\hat{g}(k)$ and $\hat{\Theta}_f(k)$ are estimates of g and Θ_f, respectively. But this control structure is not well defined because it runs risk of singularity, i.e., $\hat{g}(k)$ may fall into a small neighborhood of zero. As indicated in [16], this problem is far more from trivial because in order to avoid singularity, the existing solutions to the control problem are usually given locally or assume *a priori* knowledge of the system, i.e., the sign and upper bound of the control gain g. In this paper, however, we estimate $\Theta_{fg} = g^{-1}\Theta_f$ and g^{-1} instead of Θ_f and g and thus, the resultant control is well defined. But in the parameter estimation update law, the sign of control gain g, the control direction, will be required to determine to which direction the estimation proceed. To overcome the difficulty caused by unknown control direction, the discrete Nussbaum gain is used in the update law.

It is also noted that the noncausal problem exists in the above control due to the future states depended function $\Phi(k+n-1)$ defined in (18). To solve the noncausal problem, let us consider predicting $\Phi(k+n-1)$ in the following manner:

$$\hat{\Phi}(k+n-1|k) = [\Phi_1^T(\hat{\bar{\xi}}_1(k+n-1|k)), \Phi_2^T(\hat{\bar{\xi}}_2(k+n-2|k)),\ldots,\Phi_n^T(\bar{\xi}_n(k))]^{\mathrm{T}} \tag{20}$$

where $\hat{\bar{\xi}}_i(k+n-i|k)$, $i=1,2,\ldots,n-1$, are defined in (11) and (13).

Lemma 7. *Denote $\tilde{\Phi}(k+n-1|k) = \hat{\Phi}(k+n-1|k) - \Phi(k+n-1)$, where $\hat{\Phi}(k+n-1|k)$ and $\Phi(k+n-1)$ are defined in (18) and (20). Then, we have $\tilde{\Phi}(k+n-1|k) = o[O[y(k+n-1)]]$.*

Proof. Noting the Lipschitz condition of $\Phi_i(\cdot)$, $i=1,2,\ldots,n$, one can easily derive it from the result that $\tilde{\bar{\xi}}_i(k+n-i|k) = o[O[y(k+n-1)]]$ in Lemma 6. ■

Using the predicted function $\hat{\Phi}(k+n-1|k)$, the following adaptive control is proposed

$$u(k) = -\hat{\Theta}_{fg}^{\mathrm{T}}(k)\hat{\Phi}(k+n-1|k) + \hat{g}_l(k)y_d(k+n) \tag{21}$$

where $\hat{\Theta}_{fg}^{\mathrm{T}}(k)$ and $\hat{g}_l(k)$ are the estimates of $\Theta_{fg} = g^{-1}\Theta_f$ and g^{-1}.

Substituting the adaptive control (21) into the n-step predictor (19) and subtracting $y_d(k+n)$ on both hand sides, we obtain the following error dynamics if $d_o(k)=0$

$$\begin{aligned} e(k+n) &= y(k+n) - y_d(k+n)\\ &= \Theta_f^{\mathrm{T}}\Phi(k+n-1) - g\hat{\Theta}_{fg}^{\mathrm{T}}(k)\hat{\Phi}(k+n-1|k)\\ &\quad + g\hat{g}_l(k)y_d(k+n) - y_d(k+n)\\ &= -g\tilde{\Theta}_{fg}^{\mathrm{T}}(k)\Phi(k+n-1)\\ &\quad + g\tilde{g}_l(k)y_d(k+n) - g\beta(k+n-1)\end{aligned} \tag{22}$$

where $\tilde{\Theta}_{fg}(k)$, $\tilde{g}_l(k)$ and $\beta(k+n-1)$ are defined as

$$\tilde{\Theta}_{fg}(k) = \hat{\Theta}_{fg}(k) - \Theta_{fg},\ \tilde{g}_l(k) = \hat{g}_l(k) - g^{-1},\ \beta(k+n-1) = \hat{\Theta}_{fg}^{\mathrm{T}}(k)\tilde{\Phi}(k+n-1|k).$$

The parameter estimates in the control law are updated by the following update law

$$\begin{aligned}
\epsilon(k) &= \frac{\gamma e(k) + N(x(k))\psi(k)\beta(k-1)}{G(k)}\\
\hat{\Theta}_{fg}(k) &= \hat{\Theta}_{fg}(k-n) + \gamma\frac{N(x(k))}{D(k)}\Phi(k-1)\epsilon(k),\quad \hat{\Theta}_{fg}(j) = \mathbf{0}_{[p_j]}\\
\hat{g}_l(k) &= \hat{g}_l(k-n) - \gamma\frac{N(x(k))}{D(k)}y_d(k)\epsilon(k),\\
&\quad \hat{g}_l(j) = 0,\quad j=0,-1,\ldots,-n+1\\
\Delta\psi(k) &= \psi(k+1) - \psi(k) = \frac{-N(x(k))\beta(k-1)\epsilon(k)}{D(k)}\\
\Delta z(k) &= z(k+1) - z(k) = \frac{G(k)\epsilon^2(k)}{D(k)},\quad z(0) = \psi(0) = 0\\
\beta(k-1) &= \hat{\Theta}_{fg}^{\mathrm{T}}(k-n)\tilde{\Phi}(k-1|k-n)\\
x(k) &= z(k) + \frac{\psi^2(k)}{2}\\
G(k) &= 1 + |N(x(k))|\\
D(k) &= (1+|\psi(k)|)(1+|N^3(x(k))|)(1+\|\Phi(k-1)\|^2 + y_d^2(k) + \beta^2(k-1) + \epsilon^2(k))
\end{aligned} \tag{23}$$

where $\epsilon(k)$ is introduced as an augmented error and the tuning parameter $\gamma > 0$ can be arbitrary constant specified by the designer. It should be mentioned that the requirement on sequence $x(k)$ in (3) is satisfied. It should be noted that $\beta(k-1)$ and $\Phi(k-1)$ used in the update law are available at the k-th step.

Remark 2. The adaptive control (21) employs predicted function, $\hat{\Phi}(k+n-1|k)$, which is based on the predicted future states $\bar{\xi}_i(k+n-i), i = 1, 2, \ldots, n-1$ that are defined in Section 3.1.

Theorem 1. *Consider the adaptive closed-loop system consisting of system (1) under Assumption 1, adaptive control (21) with parameters update law (23), predicted future states defined in from (9) to (12) with parameter estimation law (14). All the signals in the closed-loop system are guaranteed to be bounded and the tracking error $e(k)$ will converge to zero, if there is no external disturbance.*

Proof. Substituting the error dynamics (22) into the augmented error $\epsilon(k)$, one obtains

$$\gamma\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)\Phi(k-1) - \gamma\tilde{g}_l(k-n)y_d(k) = -\frac{1}{g}G(k)\epsilon(k) - \gamma\beta(k-1) + \frac{1}{g}N(x(k))\psi(k)\beta(k-1). \quad (24)$$

Consider a positive definite function $V(k)$ as

$$V(k) = \sum_{j=1}^{n}\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n+j)\tilde{\Theta}_{fg}(k-n+j) + \sum_{j=1}^{n}\tilde{g}_l^{2}(k-n+j). \quad (25)$$

The difference equation of $V(k)$ is given as

$$\begin{aligned}\Delta V(k) &= V(k) - V(k-1)\\ &= \tilde{\Theta}_{fg}^{\mathrm{T}}(k)\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)\tilde{\Theta}_{fg}(k-n) + \tilde{g}_l^{2}(k) - \tilde{g}_l^{2}(k-n)\\ &= (\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n))^{\mathrm{T}}(\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n))\\ &\quad + 2\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)(\tilde{\Theta}_{fg}(k) - \tilde{\Theta}_{fg}(k-n))\\ &\quad + (\tilde{g}_l(k) - \tilde{g}_l(k-n))^2 + 2\tilde{g}_l(k-n)(\tilde{g}_l(k) - \tilde{g}_l(k-n))\\ &= \gamma^2\frac{N^2(x(k))(\Phi^{\mathrm{T}}(k-1)\Phi(k-1) + y_d^2(k))}{D^2(k)}\epsilon^2(k)\\ &\quad + 2N(x(k))\frac{\gamma\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)\Phi(k-1)}{D(k)}\epsilon(k)\\ &\quad - 2N(x(k))\frac{\gamma\tilde{g}_l(k-n)y_d(k)}{D(k)}\epsilon(k)\end{aligned}$$

and note that

$$\begin{aligned}&\Delta x(k) = \Delta z(k) + \psi(k)\Delta\psi(k) + \frac{[\Delta\psi(k)]^2}{2}\\ &0 \le \Delta z(k) \le 1, \qquad 0 \le \Delta\psi(k) \le 1\\ &|N(x(k))|[\Delta\psi(k)]^2 \le \Delta z(k)\end{aligned} \quad (26)$$

we have

$$\begin{aligned}\Delta V(k) &\le \gamma^2\frac{G(k)\epsilon^2(k)}{D(k)} - 2\gamma\frac{N(x(k))\beta(k-1)\epsilon(k)}{D(k)}\\ &\quad - \frac{2}{g}N(x(k))\frac{G(k)\epsilon^2(k)}{D(k)}\\ &\quad + \frac{2}{g}N(x(k))\frac{N(x(k))\psi(k)\beta(k-1)\epsilon(k)}{D(k)}\\ &\le \gamma^2\Delta z(k) + 2\gamma\Delta\psi(k) - \frac{2}{g}N(x(k))\left(\Delta z(k) + \psi(k)\Delta\psi(k)\right.\\ &\quad \left. + \frac{[\Delta\psi(k)]^2}{2}\right) + \frac{1}{|g|}|N(x(k))|[\Delta\psi(k)]^2\\ &\le c_1\Delta z(k) + 2\gamma\Delta\psi(k) - \frac{2}{g}N(x(k))\Delta x(k)\end{aligned}$$

where $c_1 = \gamma^2 + \frac{1}{|g|}$. Taking summation of the above equation results

$$\begin{aligned}V(k) &\le -\frac{2}{g}\sum_{k'=0}^{k}N(x(k'))\Delta x(k')\\ &\quad + c_1 z(k) + c_1 + 2\gamma\psi(k) + 2\gamma + V(-1)\\ &\le -\frac{2}{g}\sum_{k'=0}^{k}N(x(k'))\Delta x(k') + c_1\left(z(k) + \frac{\psi^2(k)}{2}\right)\\ &\quad + c_1 + \frac{2\gamma^2}{c_1} + 2\gamma + V(-1)\\ &\le -\frac{2}{g}\sum_{k'=0}^{k}N(x(k'))\Delta x(k') + c_1 x(k) + c_2\end{aligned} \quad (27)$$

where $c_2 = c_1 + \frac{2\gamma^2}{c_1} + 2 + V(-1)$. Applying Lemma 5 to (27) results the boundedness of $V(k)$ and $x(k)$ and thus the boundedness of $z(k)$, which is a non-decreasing sequence. Further, this result implies the following conclusions:

(a) $\hat{\Theta}_{fg}(k), \hat{g}_l(k), G(k), N(x(k))$ and $\psi(k)$ are bounded, and
(b) $\sqrt{\Delta z(k)} \in L^2[0, \infty)$.

Notice that $y(k) = e(k) + y_d(k)$, where the reference signal $y_d(k)$ is bounded and thus we obtain $y(k) = O[e(k)]$. According to Lemma 2, we have

$$\begin{aligned}&\bar{\xi}_n(k) = O[y(k+n-1)] = O[e(k+n-1)]\\ &u(k) = O[y(k+n)] = O[e(k+n)]\end{aligned} \quad (28)$$

and according to Lemma 2, one can easily obtain $\Phi(k-1) = O[e(k-1)]$ from the Lipschitz condition of system functions $\Phi_i(\cdot)$, $i = 1, 2, \ldots, n$.

From the definition of $\beta(k+n-1)$ in (23), the boundedness of $\hat{\Theta}_{fg}(k)$, and according to Lemma 7, it is obvious that $\beta(k-1) = o[O[e(k-1)]]$. Then, from the boundedness of $N(x(k))$, $\psi(k)$, and $G(k)$, it is easy to deduce that $\epsilon(k) \sim e(k)$, and further, from the definition of $D(k)$ in (23), we have $D(k) = O[\epsilon^2(k)]$. The conclusion (b) implies that $\Delta z(k) = \frac{G(k)\epsilon^2(k)}{D(k)} \to 0$. Applying Lemma 1 and noting the boundedness of $G(k)$, we conclude that $\epsilon(k) \to 0$ and thus $e(k) \to 0$ and then the boundedness of states $\bar{\xi}_n(k)$ and control input is obvious according to (28). According to Lemma 6, we have the boundedness of the future states prediction and parameters estimates used in the prediction law. This completes the proof of the ultimately boundedness of all the closed-loop signals. ∎

5. Adaptive control with disturbance

In this section, we consider using dead zone method to deal with the external disturbance, which is bounded by an unknown constant.

The control law still assume the form in (21) and the future states estimation law is still defined from (9) to (12). The deadzone method has been introduced into the parameter estimation laws as follows:

$$\begin{aligned}&\epsilon(k) = \frac{\gamma e(k) + N(x(k))\psi(k)\beta(k-1)}{G(k)}\\ &\hat{\Theta}_{fg}(k) = \hat{\Theta}_{fg}(k-n) + \gamma\frac{a(k)N(x(k))}{D(k)}\Phi(k-1)\epsilon(k), \quad \hat{\Theta}_{fg}(j) = \mathbf{0}_{[p_j]}\\ &\hat{g}_l(k) = \hat{g}_l(k-n) - \gamma\frac{a(k)N(x(k))}{D(k)}y_d(k)\epsilon(k),\\ &\qquad \hat{g}_l(j) = 0, j = 0, -1, \ldots, -n+1\\ &\Delta\psi(k) = \psi(k+1) - \psi(k) = \frac{-a(k)N(x(k))\beta(k-1)\epsilon(k)}{D(k)}\end{aligned}$$

$$\Delta z(k) = z(k+1) - z(k) = \frac{a(k)G(k)\epsilon^2(k)}{D(k)}, \quad z(0) = \psi(0) = 0$$
$$\beta(k-1) = \hat{\Theta}_{fg}^{\mathrm{T}}(k-n)\tilde{\Phi}(k-1|k-n)$$
$$x(k) = z(k) + \frac{\psi^2(k)}{2}$$
$$G(k) = 1 + |N(x(k))|$$
$$D(k) = (1+|\psi(k)|)(1+|N(x(k))|^3)(1+\|\Phi(k-1)\|^2 + y_d^2(k) + \beta^2(k-1) + \epsilon^2(k))$$
$$a(k) = \begin{cases} 1 & \text{if } |\epsilon(k)| > \lambda \\ 0 & \text{others} \end{cases} \tag{29}$$

where the tuning factor $\gamma > 0$ and threshold value $\lambda > 0$ can be arbitrary positive constants specified by the designer. In addition, it is obvious that requirement on sequence $x(k)$ in (3) is still satisfied. It should be mentioned that the proposed deadzone method does not require *a priori* knowledge of the upper bound of the disturbance, which is necessary in building the adaptive laws with dead-zones traditionally.

Theorem 2. *Consider the adaptive closed-loop system consisting of system (1), control (21) with parameter update law (29), predicted future state defined in from (9) to (12) with parameter estimation law (14). Under Assumption 1, all the signals in the closed-loop system are bounded and $G(k) = 1+|N(x(k))|$ will converge to a constant. Denote $C = \lim_{k\to\infty} G(k)$, then the tracking error satisfy $\lim_{k\to\infty}\sup|e(k)| < \frac{C\lambda}{\gamma}$, where γ and λ are the tuning factor and the threshold value specified by the designer.*

Proof. Substituting the error dynamics (22) into the augmented error $\epsilon(k)$ and considering $d_o(k) \neq 0$, one obtains

$$\begin{aligned}&\gamma\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)\Phi(k-1) - \gamma\tilde{g}_l(k-n)y_d(k)\\ &\quad = -\frac{1}{g}G(k)\epsilon(k) - \gamma\beta(k-1)\\ &\qquad + \frac{1}{g}\gamma d_o(k-n) + \frac{1}{g}N(x(k))\psi(k)\beta(k-1).\end{aligned} \tag{30}$$

Consider the positive definite function $V(k)$ same as in Section 4

$$V(k) = \sum_{j=1}^{n}\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n+j)\tilde{\Theta}_{fg}(k-n+j) + \sum_{j=1}^{n}\tilde{g}_l^{\,2}(k-n+j). \tag{31}$$

Note that

$$\frac{2}{g}a(k)N(x(k))d_o(k-n)\epsilon(k) \le a(k)\left|\frac{2\bar{d}}{g_n\lambda}\right||N(x(k))|\epsilon^2(k). \tag{32}$$

We have the difference equation of $V(k)$ by using the same technique in Section 4:

$$\begin{aligned}\Delta V(k) &= V(k) - V(k-1)\\ &= \frac{\gamma^2 a^2(k)N^2(x(k))(\Phi^{\mathrm{T}}(k-1)\Phi(k-1) + y_d^2(k))}{D^2(k)}G(k)\epsilon^2(k)\\ &\quad + 2N(x(k))\frac{a(k)\gamma\tilde{\Theta}_{fg}^{\mathrm{T}}(k-n)\Phi(k-1)}{D(k)}\epsilon(k)\\ &\quad - 2N(x(k))\frac{a(k)\gamma\tilde{g}_l(k-n)y_d(k)}{D(k)}\epsilon(k)\\ &\le \gamma^2\frac{a(k)G(k)\epsilon^2(k)}{D(k)} + \left|\frac{2\bar{d}}{g_n\lambda}\right|\frac{a(k)|N(x(k))|\epsilon^2(k)}{D(k)}\\ &\quad - 2\gamma\frac{a(k)N(x(k))\beta(k-1)\epsilon(k)}{D(k)}\\ &\quad - \frac{2}{g}N(x(k))\frac{a(k)G(k)\epsilon^2(k)}{D(k)}\\ &\quad + \frac{2}{g}N(x(k))\frac{a(k)N(x(k))\psi(k)\beta(k-1)\epsilon(k)}{D(k)}.\end{aligned} \tag{33}$$

Note that

$$\frac{a(k)|N(x(k))|\epsilon^2(k)}{D(k)} \le \Delta z(k)$$
$$|N(x(k))|[\Delta\psi(k)]^2 \le \Delta z(k).$$

Then, we have

$$\begin{aligned}\Delta V(k) \le{}& \left(\gamma^2 + \left|\frac{2\bar{d}}{g_n\lambda}\right|\right)\Delta z(k) + 2\gamma\Delta\psi(k) - \frac{2}{g}N(x(k))\\ &\times\left(\Delta z(k) + \psi(k)\Delta\psi(k) + \frac{[\Delta\psi(k)]^2}{2}\right)\\ &+ \frac{1}{|g|}|N(x(k))|[\Delta\psi(k)]^2\end{aligned}$$

which leads to

$$V(k) \le -\frac{2}{g}\sum_{k'=0}^{k}N(x(k'))\Delta x(k') + c_3 x(k) + c_4 \tag{34}$$

where c_3 and c_4 are some finite constants.

Then, using the same analysis as in Section 4, we conclude the boundedness of $\hat{\Theta}_{fg}(k), \hat{g}_l(k), G(k), N(x(k))$ and $\psi(k)$. In addition, we have $\Delta z(k) \to 0$, which implies either $a(k) = 0$ or $\frac{G(k)\epsilon^2(k)}{D(k)} \to 0$ as $k \to \infty$. If the latter case is true, we have $\epsilon(k) \to 0$ by applying Lemma 1. If the former case is true, we have $\epsilon(k) \le \lambda$ as $k \to \infty$ from the definition of $a(k)$. In summary, we always have $\lim_{k\to\infty}\sup|\epsilon(k)| \le \lambda$ and $\lim_{k\to\infty}a(k) = 0$, such that $G(k) = 1+|N(x(k))|$ will converge to a constant, which is denoted as C. Noting that $\beta(k-1) = o[O[e(k-1)]] \to 0$, we derive from the definition of $\epsilon(k)$ in (29) that

$$\begin{aligned}\lim_{k\to\infty}\sup|\epsilon(k)| &= \lim_{k\to\infty}\sup\left\{\left|\frac{\gamma e(k) + N(x(k))\psi(k)\beta(k-1)}{G(k)}\right|\right\}\\ &= \lim_{k\to\infty}\sup\left\{\left|\frac{\gamma e(k)}{G(k)}\right|\right\} \le \lambda\end{aligned}$$

which implies

$$\lim_{k\to\infty}\sup|e(k)| \le \lim_{k\to\infty}\frac{G(k)\lambda}{\gamma} = \frac{C\lambda}{\gamma}. \tag{35}$$

Then, following the same procedure as in the previous section, the boundedness of other closed-loop signals can be concluded. This complete the proof of the boundedness of all the closed-loop signals. ■

6. Simulation results

The following second order nonlinear plant is used for simulation.

$$\begin{cases}\xi_1(k+1) = a_1\xi_1(k)\cos(\xi_1(k)) + a_2\xi_1(k)\sin(\xi_1(k)) + a_3\xi_2(k)\\ \xi_2(k+1) = b_1\xi_2(k)\dfrac{\xi_1(k)}{1+\xi_1^2(k)} + b_2\dfrac{\xi_2^3(k)}{2+\xi_2^2(k)} + b_3u(k) + d(k)\\ y(k) = \xi_1(k)\end{cases}$$

where $a_1 = 0.2$, $a_2 = 0.1$, $a_3 = 3$, $b_1 = 0.3$, $b_2 = -0.6$, $b_3 = -0.1$ and $d(k) = 0.2\cos(0.05k)\cos(\xi_1(k))$. The control objective is to make the output $y(k)$ track a desired reference trajectory $y_d(k) = 1.5\sin(\frac{\pi}{5}kT) + 1.5\cos(\frac{\pi}{10}kT)$, $T = 0.05$. The initial system states are $\bar{\xi}_2(j) = [1,\ 1]^{\mathrm{T}}, j = -1, 0$. The tuning factor and the threshold value are chosen as $\gamma = 6$ and $\lambda = 0.1$. The simulation results are presented in Figs. 1–4. Fig. 1 shows the output $y(k)$, the reference signal $y_d(k)$. Fig. 2 illustrates the boundedness of the control input $u(k)$, the estimated parameters $\hat{g}_l(k)$, and $\|\hat{\Theta}_{fg}(k)\|$. Fig. 3 shows the discrete sequence $x(k)$ and discrete Nussbaum gain $N(x(k))$. The discrete Nussbaum gain $N(x(k))$ adapts by searching alternately in the two directions such that it can been see that it turns from positive to negative in Fig. 3. Fig. 4 illustrates the terms $\beta(k)$ and $\psi(k)$ which are caused by prediction error.

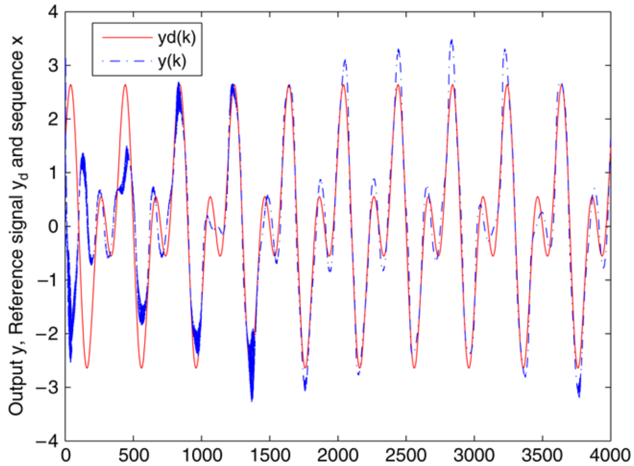

Fig. 1. Output and reference.

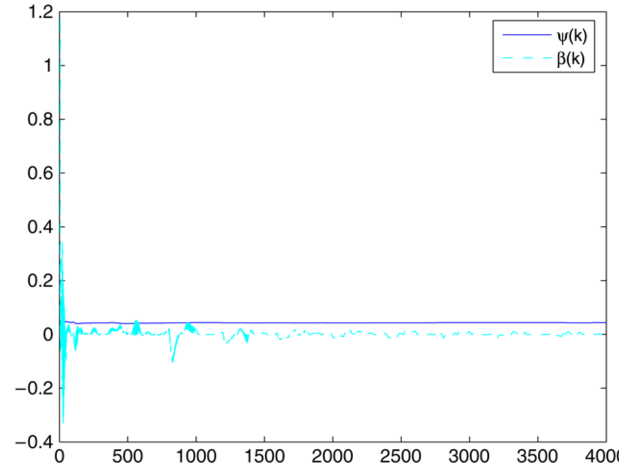

Fig. 4. $\beta(k)$ and $\psi(k)$

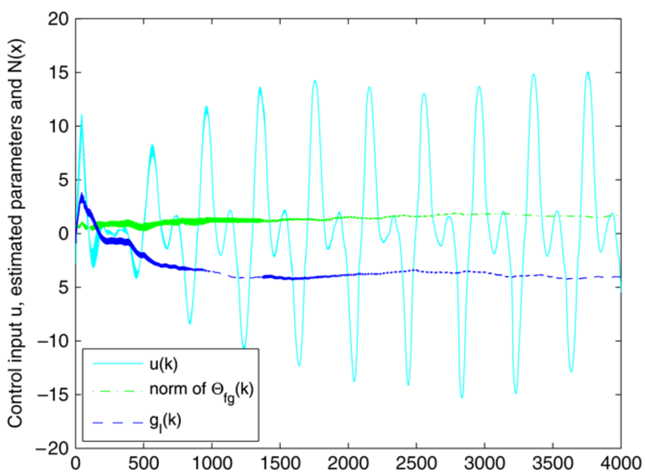

Fig. 2. Control and estimation

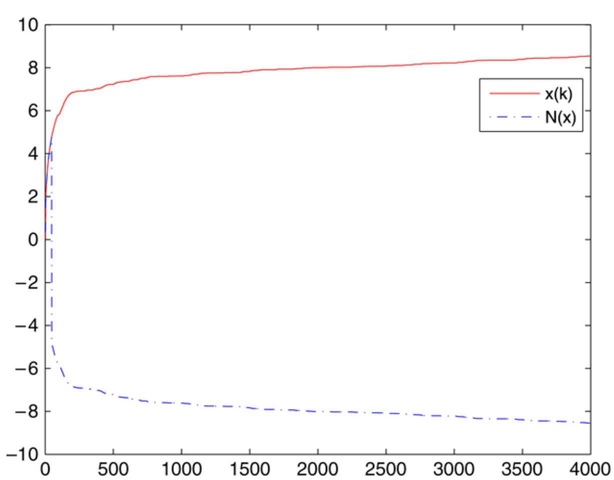

Fig. 3. Discrete Nussbaum gain.

7. Conclusion

This paper has studied the adaptive control for a class of nonlinear discrete-time systems in strict-feedback form with employment of future states prediction. The discrete Nussbaum gain is exploited to counter the lack of knowledge of control directions. The effect of prediction error on the closed-loop stability is compensated by introducing an augmented error in the control parameters update law. All the signals in the closed-loop system are guaranteed bounded and the output tracking error is ultimately made to be zero in the absence of external disturbance. The robust control has also been studied for bounded disturbance with deadzone method. The boundedness of all the the closed-loop signals still hold and the output tracking error will be bounded in a neighborhood of zero.

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Appendix A

Proof. From system (1), we can see that

$$\xi_{i+1}(k) = \frac{1}{g_i}(\xi_i(k+1) - \Theta_i^{\mathrm{T}} \Phi_i(\bar{\xi}_i(k))).$$

Considering $i = 1$ and the Lipschitz condition of $\Phi_1(\cdot)$ in Assumption 1, we have $\xi_2(k) = O[\xi_1(k+1)] = O[y(k+1)]$ and further $\bar{\xi}_2(k) = O[\xi_1(k+1)] = O[y(k+1)]$.

Considering $i = 2$, we can deduce that $\xi_3(k) = O[\bar{\xi}_2(k+1)]$ and further $\bar{\xi}_3(k) = O[\bar{\xi}_2(k+1)] = O[\xi_1(k+2)] = O[y(k+2)]$.

Continuing the procedure, we have

$$\bar{\xi}_i(k) = O[\xi_1(k+i-1)], \quad i = 1, 2, \dots, n$$

and $\xi_n(k+1) = O[y(k+n)]$. For the control input, from (1) we have

$$u(k) = \frac{1}{g_n}(\xi_n(k+1) - \Theta_n^{\mathrm{T}} \Phi_n(\bar{\xi}_n(k))) = O[y(k+n)] \quad \blacksquare$$

Appendix B

Proof. Firstly, let us analyze the one-step prediction error, $\tilde{\xi}_i(k+1|k) = \hat{\xi}_i(k+1|k) - \xi_i(k+1)$, $i = 1, 2, \dots, n-1$. Noting that $\tilde{\xi}_i(k+1|k) = \tilde{\bar{\Theta}}_i^{\mathrm{T}}(k-n+2)\Psi_i(k)$ and considering a Lyapunov function $V_i(k) = \sum_{j=k-n+2}^{k} \|\tilde{\bar{\Theta}}_i(j)\|^2$, then, following the analysis of projection algorithm in [5], we can deduce from (14) that $\bar{\hat{\Theta}}_i(k)$ is bounded and

$$\frac{\tilde{\xi}_i(k+1|k)}{D_i(k)} := \alpha(k) \in L^2[0,\infty), \quad D_i(k) = [1 + \| \Psi_i(k)\|^2]^{1/2} = O[y(k+i)] \tag{36}$$

where the later equality is obtained according to Lemma 2 and the Lipschitz condition of $\Psi_i(\cdot)$. From (36), we can see

$$\tilde{\xi}_i(k+1|k) = \alpha(k)D_i(k) = o[O[y(k+i)]]$$
$$\bar{\tilde{\xi}}_i(k+1|k) = o[O[y(k+i)]] \quad i = 1, 2, \ldots, n-1. \tag{37}$$

Next, let us analyze the two-step prediction error, $\tilde{\xi}_i(k+2|k) = \hat{\xi}_i(k+2|k) - \xi_i(k+2)$, $i = 1, 2, \ldots, n-2$.

$$\tilde{\xi}_i(k+2|k) = \tilde{\xi}_i(k+2|k+1) + \check{\xi}_i(k+2|k)$$

where

$$\tilde{\xi}_i(k+2|k+1) = \hat{\xi}_i(k+2|k+1) - \xi_i(k+2) = o[O[y(k+i+1)]]$$
$$\begin{aligned}\check{\xi}_i(k+2|k) &= \hat{\xi}_i(k+2|k) - \hat{\xi}_i(k+2|k+1) \\ &= \bar{\hat{\Theta}}_i^{\mathrm{T}}(k-n+3)[\hat{\Psi}_i(k+1|k) - \Psi_i(k+1)].\end{aligned} \tag{38}$$

As a result of the Lipschitz condition of $\Psi_i(\cdot)$, we have $\| \hat{\Psi}_i(k+1|k) - \Psi_i(k+1)\| \le L_i \|\bar{\tilde{\xi}}_{i+1}(k+1|k)\| = o[O[y(k+i)]]$. Consider the boundedness of $\bar{\hat{\Theta}}_i^{\mathrm{T}}(k-n+3)$, we have $\check{\xi}_i(k+2|k) = o[O(y(k+i+1))]$.

Consequently, we have

$$\tilde{\xi}_i(k+2|k) = o[O[y(k+i+1)]]$$
$$\bar{\tilde{\xi}}_i(k+2|k) = o[O[y(k+i+1)]] \quad i = 1, 2, \ldots, n-2. \tag{39}$$

Similarly, for the j-step prediction error $\tilde{\xi}_i(k+j|k) = \hat{\xi}_i(k+j|k) - \xi_i(k+j)$, $i = 1, 2, \ldots, n-j$, $j = 3, 4, \ldots, n-1$, we have

$$\tilde{\xi}_i(k+j|k) = \tilde{\xi}_i(k+j|k+1) + \check{\xi}_i(k+j|k)$$

where

$$\begin{aligned}\tilde{\xi}_i(k+j|k+1) &= \hat{\xi}_i(k+j|k+1) - \xi_i(k+j) \\ &= o[O(y(k+i+j-1))]\end{aligned}$$

$$\begin{aligned}\check{\xi}_i(k+j|k) &= \hat{\xi}_i(k+j|k) - \hat{\xi}_i(k+j|k+1) \\ &= \bar{\hat{\Theta}}_i^{\mathrm{T}}(k-n+j+1)[\hat{\Psi}_i(k+j-1|k) \\ &\quad - \Psi_i(k+j-1|k+1)].\end{aligned} \tag{40}$$

Consider the Lipschitz condition of $\Psi_i(\cdot)$, we have $\| \hat{\Psi}_i(k+j-1|k) - \Psi_i(k+j-1|k+1)\| \le L_i \|\bar{\check{\xi}}_{i+1}(k+j-1|k)\| = o[O[y(k+i+j-1)]]$, where $\bar{\check{\xi}}_{i+1}(k+j|k) = [\check{\xi}_1(k+j|k), \check{\xi}_2(k+j|k), \ldots, \check{\xi}_{i+1}(k+j|k)]$. It together with the boundedness of $\bar{\hat{\Theta}}_i^{\mathrm{T}}(k-n+j-1)$ leads to

$$\tilde{\xi}_i(k+j|k) = o[O[y(k+i+j-1)]]. \tag{41}$$

Let $j = n-i$, we have the following result

$$\bar{\tilde{\xi}}_i(k+n-i|k) = o[O[y(k+n-1)]] \quad i = 1, 2, \ldots, n-j. \tag{42}$$

This completes the proof. ■

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