Adaptive Neural Network Control for a Class of MIMO Nonlinear Systems With Disturbances in Discrete-Time

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***Abstract*—In this paper, adaptive neural network (NN) control is investigated for a class of multiinput and multioutput (MIMO) nonlinear systems with unknown bounded disturbances in discrete-time domain. The MIMO system under study consists of several subsystems with each subsystem in strict feedback form. The inputs of the MIMO system are in triangular form. First, through a coordinate transformation, the MIMO system is transformed into a sequential decrease cascade form (SDCF). Then, by using high-order neural networks (HONN) as emulators of the desired controls, an effective neural network control scheme with adaptation laws is developed. Through embedded backstepping, stability of the closed-loop system is proved based on Lyapunov synthesis. The output tracking errors are guaranteed to converge to a residue whose size is adjustable. Simulation results show the effectiveness of the proposed control scheme.**

***Index Terms*—Discrete-time systems, high-order neural networks, MIMO systems, neural networks.**

I. Introduction

Neural networks (NNs) play an important role in control engineering, especially in nonlinear system control. Owing to their universal approximation abilities, learning, and adaptation abilities, they are used to approximate unknown nonlinear functions. This makes them one of the effective tools in nonlinear control system design. Active research has been carried out in neural network control by using the fact that neural networks can approximate a wide range of nonlinear functions to any desired degree of accuracy under certain conditions. Several stable NN control approaches have been proposed based on Lyapunov's stability theory [1]–[5]. One main advantage of these schemes is that the adaptive laws are derived based on Lyapunov synthesis and therefore, system stability is guaranteed without the requirement for offline training. In nonlinear control, radial basis function (RBF) neural networks [5], [6], high-order neural networks (HONNs) [7] and multilayer neural networks (MNNs) [5], [6] are three kinds of widely used NNs. For simplicity, HONNs are used to construct the stable adaptive control for a class of discrete-time nonlinear multiinput and multioutput (MIMO) systems.

In recent years, there have been many significant developments in nonlinear adaptive control for continuous-time systems. Many remarkable methods have been synthesized, including feedback linearization techniques [8], adaptive backstepping design [9], neural network control [4]–[6], and fuzzy logic control [10]. Owing to the complexity of nonlinear MIMO systems, most of the techniques developed for single input and single output (SISO) systems cannot be directly extended to MIMO systems. One of the main difficulties in MIMO nonlinear system control is input coupling. Based on feedback linearization, some results have been obtained for linearizable MIMO systems [11]–[13]. In order to decouple the inputs, usually an estimation of the "decoupling matrix" is needed and it is required to be invertible. However, it is difficult to guarantee the nonsingular property of the "decoupling matrix." In [14] and [15], decoupling control was investigated for MIMO systems. The accurate mathematical model of the controlled MIMO system and all the state variables are needed, which are too restrictive to be obtained in practical applications. By exploiting the triangular input structure, elegant adaptive control has been proposed for different classes of continuous-time MIMO systems with triangular form inputs without the requirement for a "decoupling matrix" [16], [17]. In [16], integral-type Lyapunov functions are used to solve for the possible control singularity problem in adaptive control for systems with triangular control inputs, while quadratic Lyapunov functions are investigated for MIMO nonlinear systems with complex interconnections and embedded inputs [17].

Most of those elegant methods mentioned above were developed for continuous-time systems. For discrete-time systems, especially nonlinear MIMO discrete-time systems, the control problem is more complex due to the couplings among subsystems, inputs, and outputs. Few results are available in the literature in comparison with those in continuous-time domain. Besides the difficulty of input coupling, the noncausal problem is another difficulty that is to be solved when constructing stable adaptive controllers for discrete-time systems in strict feedback form [18]. Due to these difficulties, the control of discrete-time nonlinear MIMO systems is not only challenging, but also of academic interest. In [19] and [20], two-layer neural networks and multilayer neural networks were used, respectively, to construct stable controls for a special class of discrete-time nonlinear MIMO systems. Improved weight tuning algorithms were derived, which removes the need for a persistent exciting (PE) condition for parameter convergence [21]. Though the methods proposed are effective, they are only applicable to a special class of discrete-time nonlinear MIMO systems, which can be represented in the form of $X(k+1) = F(X(k)) + \Lambda U(k)$, with Λ being a diagonal constant matrix. This is a very special class of discrete-time MIMO nonlinear systems without any

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input interconnections between subsystems. Another effective neural network control scheme was developed for a class of discrete-time nonlinear MIMO systems based on input–output model in [22]. The MIMO system studied is nonlinear auto regressive moving average with eXogenous inputs (NARMAX) model [23] and only past input and output data are used to construct stable NN control. For a class of MIMO sampled-data nonlinear systems under the assumption that all the states are available, an NN based adaptive control approach is studied in [24]. The discrete-time nonlinear model structure in [24] is derived from discretizing the original continuous-time nonlinear model by a second-order approximation. It still needs further investigation to show that this discrete-time model structure can indeed represent the original system. In this paper, we are considering a class of more challenging discrete-time MIMO nonlinear system in state-space description. Comparing with the systems studied in [19] and [20], the control inputs of the system studied in this paper are in triangular form that can only be represented as $X(k+1) = F(X(k), U(k))$ instead of $X(k+1) = F(X(k)) + G(X(k))U(k)$. Therefore, feedback linearization method is not applicable.

In [18], an effective HONN control scheme was proposed for a class of strict feedback discrete-time nonlinear SISO systems. Motivated by the design procedure in [18], and the classes of systems studied in continuous time in [5], [16], and [17], we investigate a class of MIMO nonlinear discrete-time systems with unknown bounded disturbances here, thereby extending the results obtained in [18]. There are n subsystems in the MIMO system under study, with each subsystem in strict feedback form. States interconnections between different subsystems only appear in the last equations of each subsystems, where the corresponding controls also appear. By transforming the MIMO system into a sequential decrease cascade form, the noncausal problem is avoided. The main contributions of this paper are that

- an effective neural network control scheme is proposed for a class of nonlinear MIMO systems with triangular form inputs, for which feedback linearization cannot be applied;
- by using neural networks as the emulators of the desired virtual controls and desired practical controls, the closed-loop system is proved to be SGUUB in the presence of unknown bounded disturbances.

The paper is organized as follows. System dynamics and some stability notions are proposed in Section II. Section III presents the structure and properties of HONN's used in controller design. The causality analysis and system transformation are proposed in Section IV. Adaptive NN control and stability analysis are studied in Section V via backstepping. Simulation results are given in Section VI to show the effectiveness of the proposed control scheme. Finally, conclusions are made in Section VII.

II. MIMO System Dynamics

Consider the following n inputs n outputs discrete-time MIMO nonlinear system with triangular form input, as shown in (1) at the bottom of the page, where $X(k) = \left[x_1^T(k), x_2^T(k)\ldots, x_n^T(k)\right]^T$ with $x_j(k) = [x_{j,1}(k), x_{j,2}(k), \ldots, x_{j,n_j}(k)]^T \in R^{n_j}$, $u(k) = [u_1(k), \ldots, u_n(k)]^T \in R^n$ and $y(k) = [y_1(k), \ldots, y_n(k)]^T \in R^n$ are the state variables, the system inputs and outputs, respectively; $\bar{u}_{j-1}(k) = [u_1(k), \ldots, u_{j-1}(k)] (j = 2, \ldots, n)$; $d(k) = [d_1(k), \ldots, d_n(k)]^T$ is the bounded disturbance vector; $\bar{x}_{j,i_j}(k) = [x_{j,1}(k), \ldots, x_{j,i_j}(k)]^T \in R^{i_j}$ denote the first i_j states of the jth subsystem; $f_{j,i_j}(\cdot)$ and $g_{j,i_j}(\cdot)$ are smooth nonlinear functions; and j, i_j, and n_j are positive integers. It can be seen that each subsystem of (1) is in strict feedback form, which makes the use of backstepping design technique possible. Furthermore, noting that the control inputs of the whole system are in triangular form, we may then use backstepping in a nested manner to design stable controls for this class of systems as that for continuous-time systems in [5].

Remark 1: It should be noted that, unlike the triangular form inputs discrete-time MIMO nonlinear system studied in [19] and [20], whose inputs can be written into feedback linearizable form

$$\begin{aligned} X(k+1) &= F(X(k)) + G(X(k))U(k) \\ U(k) &= [u_1(k), \ldots, u_n(k)]^T \end{aligned} \tag{2}$$

system (1), studied in this paper, cannot be written into the form of (2). Instead, it is in the following form:

$$\begin{aligned} X(k+1) &= F(X(k), \bar{U}_{n-1}(k)) + G(X(k))U(k) \qquad (3) \\ U(k) &= [u_1(k), \ldots, u_n(k)]^T \\ \bar{U}_{n-1}(k) &= [u_1(k), \ldots, u_{n-1}(k)]^T. \end{aligned}$$

It is obvious that feedback linearization method is not applicable for system (3). It is much more challenging to construct stable controls for this class of system which is not feedback linearizable.

$$\begin{aligned}
&\Sigma_1 \begin{cases} x_{1,i_1}(k+1) = f_{1,i_1}(\bar{x}_{1,i_1}(k)) + g_{1,i_1}(\bar{x}_{1,i_1}(k))x_{1,i_1+1}(k) & 1 \le i_1 \le n_1 - 1 \\ x_{1,n_1}(k+1) = f_{1,n_1}(X(k)) + g_{1,n_1}(X(k))u_1(k) + d_1(k) \end{cases} \\
&\vdots \\
&\Sigma_j \begin{cases} x_{j,i_j}(k+1) = f_{j,i_j}(\bar{x}_{j,i_j}(k)) + g_{j,i_j}(\bar{x}_{j,i_j}(k))x_{j,i_j+1}(k) & 1 \le i_j \le n_j - 1 \\ x_{j,n_j}(k+1) = f_{j,n_j}(X(k), \bar{u}_{j-1}(k)) + g_{j,n_j}(X(k))u_j(k) + d_j(k) \end{cases} \\
&\vdots \\
&\Sigma_n \begin{cases} x_{n,i_n}(k+1) = f_{n,i_n}(\bar{x}_{n,i_n}(k)) + g_{n,i_n}(\bar{x}_{n,i_n}(k))x_{n,i_n+1}(k) & 1 \le i_n \le n_n - 1 \\ x_{n,n_n}(k+1) = f_{n,n_n}(X(k), \bar{u}_{n-1}(k)) + g_{n,n_n}(X(k))u_n(k) + d_n(k) \end{cases} \\
&y_j(k) = x_{j,1}(k), \quad 1 \le j \le n
\end{aligned} \tag{1}$$

In order to use the backstepping design technique, it is required that the gains of virtual controls are not equal to zero. Therefore, the following assumption should be made:

Assumption 1: The sign of $g_{j,i_j}(\cdot)$ ($j = 1,\ldots,n$, $i_j = 1,\ldots,n_j$), are known and there exist two constants $\underline{g}_{j,i_j}, \bar{g}_{j,i_j} > 0$ such that $\underline{g}_{j,i_j} \le |g_{j,i_j}(\cdot)| \le \bar{g}_{j,i_j}, \forall\, X(k) \in \Omega \subset R^{\sum_{i=1}^n n_i}$.

Without losing generality, we shall assume that $g_{j,i_j}(\cdot)$ is positive in this paper. The control objective is to design control input $u(k) = [u_1(k),\ldots,u_n(k)]^T$ to make the system output $y(k) = [y_1(k),\ldots,y_n(k)]^T$ follow a known and bounded trajectory $y_d(k) = [y_{d_1}(k),\ldots,y_{d_n}(k)]^T$. Thus, the following assumption should be made.

Assumption 2: The desired trajectory $y_d(k) \in \Omega_y, \forall\, k > 0$ is smooth and known, where $\Omega_y \triangleq \{\chi | \chi = y(k)\}$.

In [25] and [26], the definition of uniform ultimate boundedness (UUB) for continuous-time system was given. A standard Lyapunov theorem extension proposed in [27] provides a method on how to judge the UUB stability. For completeness, it is cited here.

Theorem 1: Let $V(x)$ be a Lyapunov function of a continuous-time system that satisfies the following properties:

$$\gamma_1(\|x\|) \le V(x) \le \gamma_2(\|x\|)$$
$$\dot{V}(x) \le -\gamma_3(\|x\|) + \gamma_3(\eta)$$

where η is a positive constant, $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are continuous, strictly increasing functions, and $\gamma_3(\cdot)$ is a continuous, nondecreasing function. Thus, if

$$\dot{V}(x) < 0, \text{ for } \|x\| > \eta$$

then $x(t)$ is uniformly ultimately bounded. In addition, if $x(0) = 0$, $x(t)$ is uniformly bounded [27].

Similar to the definition of UUB for continuous-time system, its counterpart in discrete-time system is as follows:

Definition 1: The solution of (1) is semiglobally uniformly ultimately bounded (SGUUB), if for any Ω, a compact subset of $R^{\sum_{i=1}^n n_i}$ and all $X(k_0) \in \Omega$, there exists an $\epsilon > 0$, and a number $N(\epsilon, X(k_0))$ such that $\|X(k)\| < \epsilon$ for all $k \ge k_0 + N$. In other words, the solution of (1) is said to be SGUUB if, for any *a priori* given (arbitrarily large) bounded set Ω and any *a priori* given (arbitrarily small) set Ω_0, which contains (0,0) as an interior point, there exist a control u, such that every trajectory of the closed-loop system starting from Ω enters the set Ω_0 in a finite time and remains in it thereafter, [25].

Lemma 1: Let $V(x(k))$ be a Lyapunov function of a discrete-time system that satisfies the following properties:

$$\begin{aligned}\gamma_1(\|x(k)\|) &\le V(x(k)) \le \gamma_2(\|x(k)\|)\\ V(x(k+1)) - V(x(k)) &= \Delta V(x(k))\\ &\le -\gamma_3(\|x(k)\|) + \gamma_3(\eta)\end{aligned} \tag{4}$$

where η is a positive constant, $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are strictly increasing functions, and $\gamma_3(\cdot)$ is a continuous, nondecreasing function. Thus, if

$$\Delta V(x(k)) < 0, \text{ for } \|x(k)\| > \eta$$

then $x(k)$ is uniformly ultimately bounded on a compact set, i.e., there exists a time instant k_T, such that $\|x(k)\| < \eta, \forall\, k > k_T$.

Remark 2: It should be noted that, the operator $\|\cdot\|$ in Lemma 1 can be any positively defined monotone increasing function or norm.

In the following, the definition of *persistent exciting (PE)* and *input to state stable* for discrete-time system are given, which will be used later.

Definition 2: The sequence $S(k)$ is said to be persistent exciting if there is $\bar{\lambda} > 0$ and integer $L > 0$ such that

$$\lambda_{\min}\left[\sum_{k=k_0}^{k_0+L-1} S(k)S^T(k)\right] \ge \bar{\lambda}, \quad \forall\, k_0 \ge 0 \tag{5}$$

where $\lambda_{\min}(M)$ denotes the smallest eigenvalue of M [21].

Lemma 2: Consider the linear time varying discrete-time system given by

$$x(k+1) = A(k)x(k) + Bu(k), \quad y(k) = Cx(k) \tag{6}$$

where $A(k)$, B, and C are appropriately dimensional matrices with B and C being constant matrices. Let $\Phi(k_1, k_0)$ be the state-transition matrix corresponding to $A(k)$ for system (6), i.e., $\Phi(k_1, k_0) = \prod_{k=k_0}^{k_1-1} A(k)$. If $\|\Phi(k_1, k_0)\| < 1, \forall k_1 > k_0 \ge 0$, then system (6) is 1) globally exponentially stable for the unforced system (i.e., $u(k) = 0$) and 2) bounded-input-bounded-output (BIBO) stable [28].

III. Function Approximation by HONN

In control engineering, NN is usually used as function approximator to emulate the unknown nonlinear ideal control u^*. For convenience, let us consider the high-order neural networks [5], [7]

$$\begin{aligned}\phi(W,z) &= W^T S(z), W \in R^{l\times p} \text{ and } S(z) \in R^l\\ S(z) &= [s_1(z), s_2(z), \ldots, s_l(z)]^T\\ s_i(z) &= \prod_{j\in I_i}[s(z_j)]^{d_j(i)}, \quad i = 1,2,\ldots,l\end{aligned}$$

where $z = [z_1, z_2, \ldots, z_q]^T \in \Omega_z \subset R^q$, positive integer l denotes the NN node number, p is the dimension of the function vector, $\{I_1, I_2, \ldots, I_l\}$ is a collection of l not-ordered subsets of $\{1,2,\ldots,q\}$ and $d_j(i)$ are nonnegative integers, W is an adjustable synaptic weight matrix, $s(z_j)$ is chosen as hyperbolic tangent function

$$s(z_j) = \frac{e^{z_j} - e^{-z_j}}{e^{z_j} + e^{-z_j}}.$$

For a desired function $u^*(z)$, there exists ideal weights W^* such that the smooth function u^* can be approximated by an ideal NN on a compact set $\Omega_z \subset R^q$

$$u^* = W^{*T} S(z) + \epsilon_z \tag{7}$$

where ϵ_z is the bounded NN approximation error satisfying $|\epsilon_z| \le \epsilon_0$ on the compact set, which can be reduced by increasing the number of the adjustable weights. The ideal weight matrix W^* is an "artificial" quantity required for analytical purpose, and is defined as that which minimizes $|\epsilon_z|$ for all $z \in \Omega_z \subset R^q$ in a compact region, i.e.,

$$W^* \triangleq \arg \min_{W \in R^{l \times m}} \left\{ \sup_{z \in \Omega_z} \left| u^* - W^T S(z) \right| \right\}, \quad \Omega_z \subset R^q. \tag{8}$$

In general, the ideal NN weight matrix, W^*, is unknown though constant, its estimate, $\hat{W}$, should be used for controller design which will be discussed later.

It should be noted that though HONNs are used for analysis in this paper, they may be replaced by any other linear approximators such as, radial basis function networks [29], spline functions [30], or fuzzy systems [31], which have the similar properties as the HONNs used, while the stability and performance properties of the adaptive system are still preserved.

IV. Causality Analysis and System Transformation

In this section, similarly as in [18], coordinate transformations are used to avoid the noncausal problem, which often appears in discrete-time nonlinear system control. We have assumed that each subsystem of system (1) is in strict feedback form. It seems that backstepping can be used to construct stable control. However, unlike in continuous-time systems, the causality contradiction [18] is one of the major problems that we will encounter when we construct controls for strict-feedback discrete-time nonlinear system through backstepping, as detailed in the following.

Consider the first subsystem Σ_1 in system (1), as shown in (9) at the bottom of the page. If we design the ideal fictitious control for the first equation in (9) as follows:

$$\alpha_{1,2}^*(k) = -\frac{1}{g_{1,1}(\bar{x}_{1,1}(k))} \left[f_{1,1}(\bar{x}_{1,1}(k)) - y_{d_1}(k+1) \right]$$

the first equation in (9) can be stabilized. Similarly, we can construct another ideal fictitious control

$$\alpha_{1,3}^*(k) = -\frac{1}{g_{1,2}(\bar{x}_{1,2}(k))} \left[f_{1,2}(\bar{x}_{1,2}(k)) - \alpha_{1,2}^*(k+1) \right] \tag{10}$$

to stabilize the second equation in (9). But unfortunately, $\alpha_{1,2}^*(k+1)$ in (10) is a fictitious control of the future. This means that the fictitious control $\alpha_{1,3}^*(k)$ is infeasible in practice. If we continue the process to construct the final desired control $u_1^*(k)$, we end up with a $u_1^*(k)$ that is infeasible due to unavailable future information. However, the above problem can be avoided if we transform the system equation into a special form which is suitable for backstepping design. The basic idea is as follows. If we consider the original system description as a one-step ahead predictor, then we can transform the one-step ahead predictor into an equivalent maximum n_1-step ahead predictor which can predict the future states, $x_{1,1}(k+n_1)$, $x_{1,2}(k+n_1-1), \ldots, x_{1,n_1}(k+1)$, then the causality contradiction is avoided when the controller is constructed based on the maximum n_1-step ahead predictor by backstepping. For the other $n-1$ subsystems, these transformations can similarly be constructed. The transformation procedure for the jth $(1 \le j \le n_j)$ subsystem is detailed as follows.

Consider the i_jth equation in jth subsystem of system (1)

$$x_{j,i_j}(k+1) = f_{j,i_j}(\bar{x}_{j,i_j}(k)) + g_{j,i_j}(\bar{x}_{j,i_j}(k)) x_{j,i_j+1}(k)$$
$$1 \le j \le n \text{ and } 1 \le i_j \le n_j - 1.$$

It can be easily obtained that $x_{j,i_j}(k+1)$ is a function of $\bar{x}_{j,i_j+1}(k)$. For convenience of analysis, we define

$$x_{j,i_j}(k+1) \triangleq f_{j,i_j}^{n_j}(\bar{x}_{j,i_j+1}(k)) \tag{11}$$

with

$$f_{j,i_j}^{n_j}(\bar{x}_{j,i_j+1}(k)) = f_{j,i_j}(\bar{x}_{j,i_j}(k)) + g_{j,i_j}(\bar{x}_{j,i_j}(k)) x_{j,i_j+1}(k).$$

Thus, we have

$$\bar{x}_{j,i_j}(k+1) = \begin{bmatrix} x_{j,1}(k+1) \\ \vdots \\ x_{j,i_j}(k+1) \end{bmatrix} = \begin{bmatrix} f_{j,1}^{n_j}(\bar{x}_{j,2}(k)) \\ \vdots \\ f_{j,i_j}^{n_j}(\bar{x}_{j,i_j+1}(k)) \end{bmatrix}$$
$$1 \le j \le n, 1 \le i_j \le n_j - 1.$$

It can be seen that $\bar{x}_{j,i_j}(k+1)$ is a function of $\bar{x}_{j,i_j+1}(k)$. Define function vector

$$\bar{x}_{j,i_j}(k+1) \triangleq F_{j,i_j}^{n_j}(\bar{x}_{j,i_j+1}(k)), \quad i_j = 1, \ldots, n_j - 1. \tag{12}$$

After one more step, the first $n_j - 1$ equations of each subsystem in (1) can be expressed as in (13) for $(1 \le j \le n)$, as shown at the bottom of the next page. Substituting (11) and (12) into (13), we can obtain (14) (see bottom of the next page) where

$$\begin{aligned} f_{j,i_j}^{n_j-1}(\bar{x}_{j,i_j+2}(k)) &= f_{j,i_j}(F_{j,i_j}^{n_j}(\bar{x}_{j,i_j+1}(k))) \\ &\quad + g_{j,i_j}(F_{j,i_j}^{n_j}(\bar{x}_{j,i_j+1}(k))) \\ &\quad \times f_{j,i_j+1}^{n_j}(\bar{x}_{j,i_j+2}(k)) \\ F_{j,n_j-1}(\bar{x}_{j,n_j}(k)) &= f_{j,n_j-1}(F_{j,n_j-1}^{n_j}(\bar{x}_{j,n_j}(k))) \\ G_{j,n_j-1}(\bar{x}_{j,n_j}(k)) &= g_{j,n_j-1}(F_{j,n_j-1}^{n_j}(\bar{x}_{j,n_j}(k))). \end{aligned}$$

$$\begin{cases} x_{1,1}(k+1) = f_{1,1}(\bar{x}_{1,1}(k)) + g_{1,1}(\bar{x}_{1,1}(k)) x_{1,2}(k) \\ x_{1,2}(k+1) = f_{1,2}(\bar{x}_{1,2}(k)) + g_{1,2}(\bar{x}_{1,2}(k)) x_{1,3}(k) \\ \vdots \\ x_{1,n_1-1}(k+1) = f_{1,n_1-1}(\bar{x}_{1,n_1-1}(k)) + g_{1,n_1-1}(\bar{x}_{1,n_1-1}(k)) x_{1,n_1}(k) \\ x_{1,n_1}(k+1) = f_{1,n_1}(X(k)) + g_{1,n_1}(X(k)) u_1(k) + d_1(k). \end{cases} \tag{9}$$

Following the same procedure, the first (n_j-2) equations in (14) of the jth subsystem of system (1) can be described by $(1\le i_j\le n_j-2)$

$$\bar{x}_{j,i_j}(k+2)=\begin{bmatrix}x_{j,1}(k+2)\\ \vdots\\ x_{j,i_j}(k+2)\end{bmatrix}=\begin{bmatrix}f_{j,1}^{n_j-1}(\bar{x}_{j,3}(k))\\ \vdots\\ f_{j,i_j}^{n_j-1}(\bar{x}_{j,i_j+2}(k))\end{bmatrix}$$

which is a function of $\bar{x}_{j,i_j+2}(k)$ and is denoted as

$$\bar{x}_{j,i_j}(k+2)=F_{j,i_j}^{n_j-1}(\bar{x}_{j,i_j+2}(k)),\quad i_j=1,\ldots,n_j-2.$$

Continuing the above procedure recursively, after (n_j-2) steps, the first two equations in the jth subsystem of (1) can be written as (15), shown at the bottom of the page, where

$$\begin{aligned}f_{j,1}^2(\bar{x}_{j,n_j}(k))&=f_{j,1}(F_{j,1}^3(\bar{x}_{j,n_j-1}(k)))\\&\quad+g_{j,1}(F_{j,1}^3(\bar{x}_{j,n_j-1}(k)))f_{j,2}^3(\bar{x}_{j,n_j}(k))\\F_{j,2}(\bar{x}_{j,n_j}(k))&=f_{j,2}(F_{j,2}^3(\bar{x}_{j,n_j}(k)))\\G_{j,2}(\bar{x}_{j,n_j}(k))&=g_{j,2}(F_{j,2}^3(\bar{x}_{j,n_j}(k))).\end{aligned}$$

After one more step, the first equations in the jth subsystem of (1) becomes

$$\begin{aligned}x_{j,1}(k+n_j)&=F_{j,1}(\bar{x}_{j,n_j}(k))\\&\quad+G_{j,1}(\bar{x}_{j,n_j}(k))x_{j,2}(k+n_j-1)\end{aligned}\tag{16}$$

where

$$\begin{aligned}F_{j,1}(\bar{x}_{j,n_j}(k))&=f_{j,1}(f_{j,1}^2(\bar{x}_{j,n_j}(k)))\\G_{j,1}(\bar{x}_{j,n_j}(k))&=g_{j,1}(f_{j,1}^2(\bar{x}_{j,n_j}(k))).\end{aligned}$$

Since (13) to (16) are all derived from the original system, the jth subsystem of original system (1) is equivalent to (17), shown at the bottom of the page.

Definition 3: The form in (17) is said to be the sequential decrease cascade form (SDCF).

For convenience of analysis, define $(1\le j\le n$ and $1\le i_j\le n_j-1)$

$$\begin{aligned}&F_{j,i_j}(k)\triangleq F_{j,i_j}(\bar{x}_{j,n_j}(k)),\quad G_{j,i_j}(k)\triangleq, G_{j,i_j}(\bar{x}_{j,n_j}(k))\\&f_{j,n_j}(k)\triangleq f_{j,n_j}(X,\bar{u}_{j-1}(k)),\quad g_{j,n_j}(k)\triangleq g_{j,n_j}(X)\end{aligned}$$

then system (17) can be written as

$$\begin{cases}x_{j,1}(k+n_j)=F_{j,1}(k)+G_{j,1}(k)x_{j,2}(k+n_j-1)\\ \vdots\\ x_{j,n_j-1}(k+2)=F_{j,n_j-1}(k)+G_{j,n_j-1}(k)x_{j,n_j}(k+1)\\ x_{j,n_j}(k+1)=f_{j,n_j}(k)+g_{j,n_j}(k)u_j(k)+d_j(k)\\ y_j(k)=x_{j,1}(k).\end{cases}\tag{18}$$

Now, we can define the desired virtual controls and the ideal practical controls for each subsystem, as shown in (19) at the

$$\begin{cases}x_{j,i_j}(k+2)=f_{j,i_j}(\bar{x}_{j,i_j}(k+1))+g_{j,i_j}(\bar{x}_{j,i_j}(k+1))x_{j,i_j+1}(k+1) & i_j=1,2,\ldots,n_j-2\\ x_{j,n_j-1}(k+2)=f_{j,n_j-1}(\bar{x}_{j,n_j-1}(k+1))+g_{j,n_j-1}(\bar{x}_{j,n_j-1}(k+1))x_{j,n_j}(k+1)\end{cases}\tag{13}$$

$$\begin{cases}x_{j,i_j}(k+2)=f_{j,i_j}(F_{j,i_j}^{n_j}(\bar{x}_{j,i_j+1}(k)))+g_{j,i_j}(F_{j,i_j}^{n_j}(\bar{x}_{j,i_j+1}(k)))f_{j,i_j+1}^{n_j}(\bar{x}_{j,i_j+2}(k))\\ \qquad\triangleq f_{j,i_j}^{n_j-1}(\bar{x}_{j,i_j+2}(k)) & i_j=1,2,\ldots,n_j-2\\ x_{j,n_j-1}(k+2)=f_{j,n_j-1}(F_{j,n_j-1}^{n_j}(\bar{x}_{j,n_j}(k)))+g_{j,n_j-1}(F_{j,n_j-1}^{n_j}(\bar{x}_{j,n_j}(k)))x_{j,n_j}(k+1)\\ \qquad\triangleq F_{j,n_j-1}(\bar{x}_{j,n_j}(k))+G_{j,n_j-1}(\bar{x}_{j,n_j}(k))x_{j,n_j}(k+1) & 1\le j\le n\end{cases}\tag{14}$$

$$\begin{cases}x_{j,1}(k+n_j-1)=f_{j,1}^2(\bar{x}_{j,n_j}(k))\\ x_{j,2}(k+n_j-1)=F_{j,2}(\bar{x}_{j,n_j}(k))+G_{j,2}(\bar{x}_{j,n_j}(k))x_{j,3}(k+n_j-2)\end{cases}\tag{15}$$

$$\begin{cases}x_{j,1}(k+n_j)=F_{j,1}(\bar{x}_{j,n_j}(k))+G_{j,1}(\bar{x}_{j,n_j}(k))x_{j,2}(k+n_j-1)\\ \vdots\\ x_{j,n_j-1}(k+2)=F_{j,n_j-1}(\bar{x}_{j,n_j}(k))+G_{j,n_j-1}(\bar{x}_{j,n_j}(k))x_{j,n_j}(k+1)\\ x_{j,n_j}(k+1)=f_{j,n_j}(X,\bar{u}_{j-1}(k))+g_{j,n_j}(X)u_j(k)+d_j(k)\\ y_j(k)=x_{j,1}(k).\end{cases}\tag{17}$$

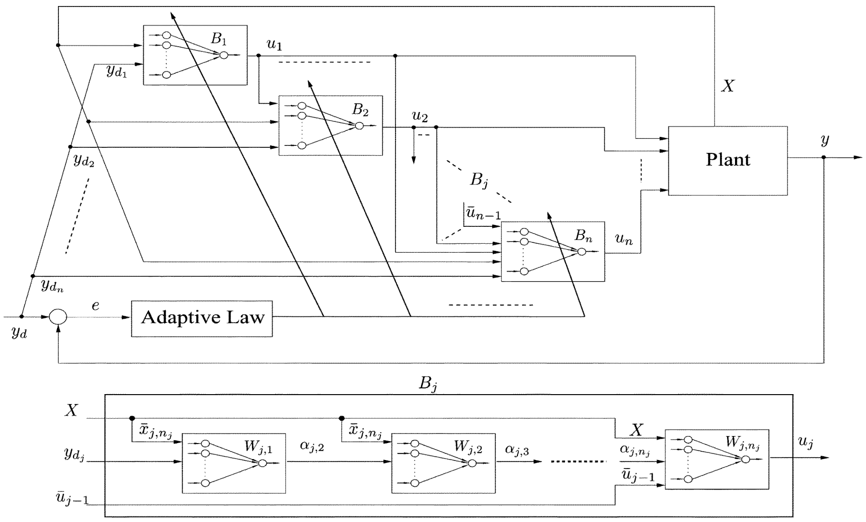

Fig. 1. Control system structure.

bottom of the page, which can stabilize the system in each step without the causality problem. Equation (19) can be further written as

$$\begin{cases} \alpha_{j,2}^*(k) \triangleq \varphi_{j,1}(\bar{x}_{j,n_j}(k), y_{d_j}(k+n_j)) \\ \alpha_{j,3}^*(k) \triangleq \varphi_{j,2}(\bar{x}_{j,n_j}(k), \alpha_{j,2}^*(k)) \\ \vdots \\ \alpha_{j,n_j}^*(k) \triangleq \varphi_{j,n_j-1}(\bar{x}_{j,n_j}(k), \alpha_{j,n_j-1}^*(k)) \\ u_j^*(k) \triangleq \varphi_{j,n_j}(X, \bar{u}_{j-1}(k), \alpha_{j,n_j}^*(k)) \\ y_j(k) = x_{j,1}(k) \end{cases} \tag{20}$$

where, $\varphi_{j,1}(\cdot), \ldots, \varphi_{j,n_j}(\cdot)$, $(1 \leq j \leq n)$ are nonlinear functions. It is obvious that the desired virtual controls $\alpha_{j,2}^*(k), \ldots, \alpha_{j,n_j}^*(k)$ and the ideal control $u_j^*(k)$ will drive the output of the jth subsystem to track $y_{d_j}(k+n_j)$ exactly provided that: 1) the exact system model is known, and 2) the disturbance $d_j(k) = 0$. However, in practical applications, usually these two conditions cannot be satisfied. In the following, neural networks will be used to emulate the desired virtual controls, as well as the desired practical controls when the exact system model is unknown. Using the Lyapunov synthesis, the closed-loop system is also shown to be SGUUB even in the presence of unknown bounded disturbances.

Detailed design procedure will be described in Section V. It should be noted that, unlike the procedure in [18], embedded backstepping is used to construct the neural network controllers due to the complexity structure of the MIMO system. The procedure can be divided into the two following steps [5]:

- for each subsystem, by using the backstepping design, the first $n_j - 1$ $(1 \leq j < n)$ equations can be stabilized if the corresponding virtual controls are properly chosen;
- by considering the last equations of each subsystem, we can see that the MIMO system is in strict feedback form relative to the control inputs $u_1(k), \ldots, u_n(k)$. Thus, by embedded backstepping design, the stability of the whole closed-loop system can be guaranteed.

V. Controller Design and Stability Analysis

The closed-loop system structure is shown in Fig. 1. For each subsystem of system (1), it can be transformed into the form of (18). Therefore, we can construct the controls via embedded backstepping without causality contradiction.

Choose the practical virtual controls and practical controls as follows:

$$\begin{aligned} \alpha_{j,i_j}(k) &= \hat{W}_{j,i_j-1}^T S_{j,i_j-1}(z_{j,i_j-1}(k)), \quad i_j = 2, \ldots, n_j \\ u_j(k) &= \hat{W}_{j,n_j}^T(k) S_{j,n_j}(z_{j,n_j}(k)) \end{aligned} \tag{21}$$

with

$$\begin{aligned} z_{j,1}(k) &= \left[\bar{x}_{j,n_j}^T(k), y_{d_j}(k+n_j)\right]^T \\ z_{j,i_j}(k) &= \left[\bar{x}_{j,n_j}^T(k), \alpha_{j,i_j}(k)\right]^T, \quad i_j = 2, \ldots, n_j - 1 \\ z_{j,n_j}(k) &= [X, \bar{u}_{j-1}(k), \alpha_{j,n_j}(k)]^T \end{aligned}$$

$$\begin{cases} \alpha_{j,2}^*(k) \triangleq x_{j,2}(k+n_j-1) = \frac{1}{G_{j,1}(k)}\left[y_{d_j}(k+n_j) - F_{j,1}(k)\right] \\ \alpha_{j,3}^*(k) \triangleq x_{j,3}(k+n_j-2) = \frac{1}{G_{j,2}(k)}\left[\alpha_{j,2}^*(k) - F_{j,2}(k)\right] \\ \vdots \\ \alpha_{j,n_j}^*(k) \triangleq x_{j,n_j}(k+1) = \frac{1}{G_{j,n_j-1}(k)}\left[\alpha_{j,n_j-1}^*(k) - F_{j,n_j-1}(k)\right] \\ u_j^*(k) \triangleq \frac{1}{g_{j,n_j}(k)}\left[\alpha_{j,n_j}^*(k) - f_{j,n_j}(k)\right] \\ y_j(k) = x_{j,1}(k) \end{cases} \tag{19}$$

where $\hat{W}_{j,i_j}$ denotes the estimation of ideal constant W^*_{j,i_j} $(1 \le j \le n, 1 \le i_j \le n_j)$, which will be specifically discussed in the proof of Theorem 2, and $S_{j,i_j}(\cdot)$ denotes the hyperbolic tangent function defined in Section III. Throughout this paper, we define

$$\tilde{W}_{j,i_j} = \hat{W}_{j,i_j} - W^*_{j,i_j}.$$

The corresponding weights updating laws are chosen as

$$\begin{aligned}\hat{W}_{j,i_j}(k+1) =& \hat{W}_{j,i_j}(k_{i_j}) \\ &- \Gamma_{j,i_j}\left[S(z_{j,i_j}(k_{i_j}))e_{j,i_j}(k+1) \right. \\ &\left. + \sigma_{j,i_j}\hat{W}_{j,i_j}(k_{i_j})\right] \\ k_{i_j} =& k - n_j + i_j, \quad i_j = 1, 2, \ldots, n_j \end{aligned} \tag{22}$$

where $\Gamma_{j,i_j} = \gamma_{j,i_j} I > 0$ is the adaptation gain, $\gamma_{j,i_j} > 0$, $\sigma_{j,i_j} > 0$ are positive constants and $0 < \gamma_{j,i_j}\sigma_{j,i_j} < 1$. The error vector is defined as $e_j(k) = [e_{j,1}(k), e_{j,2}(k), \ldots, e_{j,i_j}(k), \ldots, e_{j,n_j}(k)]^T$ with $e_{j,i_j}(k)$ denotes the error of each step defined as follows:

$$\begin{aligned} e_{j,1}(k) &= x_{j,1}(k) - y_{d_1}(k) \\ e_{j,2}(k) &= x_{j,2}(k) - \alpha_{j,2}(k - n_j + 1) \\ &\vdots \\ e_{j,n_j}(k) &= x_{j,n_j}(k) - \alpha_{j,n_j}(k-1). \end{aligned}$$

It should be noted that, in the neural network weights update, σ-modification is used to improve the robustness of the proposed control scheme [32].

The stability of the closed-loop system is summarized in Theorem 2.

Theorem 2: For the closed-loop nonlinear MIMO system (1) consisting of control (21) and adaptive law (22), there exists a semi-globally uniformly ultimately bounded equilibrium at $[e_{1,1}(k), \ldots, e_{n,1}(k)]^T = 0$, provided that the design parameters are properly chosen. This guarantees that all the signals, including the states $X(k)$, the control input $u(k)$ and NN weight estimates $\hat{W}_{j,i_j}(k)$ $(j = 1, \ldots, n,\ i_j = 1, \ldots, n_j)$, are all bounded, subsequently

$$\lim_{k\to\infty} \|y(k) - y_d(k)\| \le \varepsilon$$

where ε is a positive number.

Proof: The proof procedure is as follows.

1) For the jth $(1 \le j \le n)$ subsystem, use backstepping to prove its stability up to step $n_j - 1$, i.e., to guarantee the UUB stability for the first $n_j - 1$ equations.
2) For the last equations in each subsystem, noting that the practical control inputs are in strict feedback form, embedded backstepping is used to guarantee closed-loop system stability.

At time instant k, assume that $\bar{x}_{j,n_j}(k) \in \Omega$, then we will prove that $\bar{x}_{j,n_j}(k+1) \in \Omega$ and $u_j(k)$ are bounded by backstepping. Before proceeding, let $k_i = k - n_j + i, i = 1, 2, \ldots, n_j - 1$ for convenience of description.

Step 1: Considering the tracking error of the jth subsystem $(1 \le j \le n)$, $e_{j,1}(k) = x_{j,1}(k) - y_{d_j}(k)$, and noting the first equation in (18), we can obtain

$$\begin{aligned} e_{j,1}(k+n_j) =& x_{j,1}(k+n_j) - y_{d_j}(k+n_j) \\ =& F_{j,1}(k) + G_{j,1}(k)x_{j,2}(k+n_j-1) \\ &- y_{d_j}(k+n_j). \end{aligned} \tag{23}$$

Considering $x_{j,2}(k+n_j-1)$ as the fictitious control for (23), it is obvious that $e_{j,1}(k+n_j) = 0$ if we let

$$\begin{aligned} x_{j,2}(k+n_j-1) =& \alpha^*_{j,2}(k) \\ =& -\frac{1}{G_{j,1}(k)} \\ &\times \left[F_{j,1}(k) - y_{d_j}(k+n_j)\right]. \end{aligned} \tag{24}$$

Since $F_{j,1}(k)$ and $G_{j,1}(k)$ are unknown, they are not available for constructing the fictitious control $\alpha^*_{j,2}(k)$. However, $F_{j,1}(k)$ and $G_{j,1}(k)$ are functions of system state $\bar{x}_{j,n_j}(k)$, therefore, we can use HONNs to approximate $\alpha^*_{j,2}(k)$ as follows:

$$\begin{aligned} \alpha^*_{j,2}(k) &= W^{*T}_{j,1} S_{j,1}(z_{j,1}(k)) + \epsilon_{z_{j,1}}(z_{j,1}(k)) \\ z_{j,1}(k) &= \left[\bar{x}^T_{j,n_j}(k), y_{d_j}(k+n)\right]^T \in \Omega_{z_{j_1}} \subset R^{n_j+1}. \end{aligned} \tag{25}$$

Letting $\hat{W}_{j,1}$ be the estimate of $W^*_{j,1}$, the practical virtual control, $\alpha_{j,2}(k)$, is chosen as follows:

$$x_{j,2}(k+n_j-1) = \alpha_{j,2}(k) = \hat{W}^T_{j,1}(k) S_{j,1}(z_{j,1}(k)) \tag{26}$$

and the robust updating algorithm for NN weight is chosen as

$$\begin{aligned} \hat{W}_{j,1}(k+1) =& \hat{W}_{j,1}(k_1) - \Gamma_{j,1} \\ &\left[S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1) + \sigma_{j,1}\hat{W}_{j,1}(k_1)\right]. \end{aligned} \tag{27}$$

Substituting fictitious control (26) into (23), the error (23) is rewritten as

$$\begin{aligned} e_{j,1}(k+n_j) =& F_{j,1}(k) - y_{d_j}(k+n_j) \\ &+ G_{j,1}(k)\hat{W}^T_{j,1}(k)S_{j,1}(z_{j,1}(k)). \end{aligned} \tag{28}$$

Adding and subtracting $G_{j,1}(k)\alpha^*_{j,2}(k)$ to the right hand side of (28) and noting (25), we have $\forall\ z_{j,1}(k) \in \Omega_{z_{j_1}}$

$$\begin{aligned} e_{j,1}(k+n_j) =& F_{j,1}(k) - y_{d_1}(k+n) + G_{j,1}(k) \\ &\times \left[\hat{W}^T_{j,1}(k)S_{j,1}(z_{j,1}(k)) \right. \\ &- {W^*_{j,1}}^T S_{j,1}(z_{j,1}(k)) \\ &\left. - \epsilon_{z_{j,1}}(z_{j,1}(k))\right] \\ &+ G_{j,1}(k)\alpha^*_{j,2}(k). \end{aligned} \tag{29}$$

Substituting (24) into (29), we can obtain

$$e_{j,1}(k+n_j) = G_{j,1}(k)\left[\tilde{W}^T_{j,1}(k)S_{j,1}(z_{j,1}(k)) - \epsilon_{z_{j,1}}\right]. \tag{30}$$

Choose the Lyapunov function candidate

$$V_{j,1}(k) = \frac{1}{\bar{g}_{j,1}} e_{j,1}^2(k) + \sum_{p=0}^{n_j-1} \tilde{W}_{j,1}^T(k_1+p)\Gamma_1^{-1}\tilde{W}_{j,1}(k_1+p) \tag{31}$$

where $k_1 = k - n_j + 1$.

Noting the fact that $\tilde{W}_{j,1}^T(k_1)S_{j,1}(z_{j,1}(k_1)) = (e_{j,1}(k+1)/G_{j,1}(k_1)) + \epsilon_{z_{j,1}}$, the first difference of (31) along (27) and (30) is given by

$$\begin{aligned}
\Delta V_{j,1} =& V_{j,1}(k+1) - V_{j,1}(k)\\
=& \frac{1}{\bar{g}_{j,1}}\left[e_{j,1}^2(k+1) - e_{j,1}^2(k)\right]\\
&+ \tilde{W}_{j,1}^T(k+1)\Gamma_{j,1}^{-1}\tilde{W}_{j,1}(k+1)\\
&- \tilde{W}_{j,1}^T(k_1)\Gamma_{j,1}^{-1}\tilde{W}_{j,1}(k_1)\\
=& \frac{1}{\bar{g}_{j,1}}\left[e_{j,1}^2(k+1) - e_{j,1}^2(k)\right] - 2\tilde{W}_{j,1}^T(k_1)\\
&\times \left[S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1) + \sigma_{j,1}\hat{W}_{j,1}(k_1)\right]\\
&+ \left[S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1) + \sigma_{j,1}\hat{W}_{j,1}(k_1)\right]^T \Gamma_{j,1}\\
&\times \left[S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1) + \sigma_{j,1}\hat{W}_{j,1}(k_1)\right]\\
=& \frac{1}{\bar{g}_{j,1}}\left[e_{j,1}^2(k+1) - e_{j,1}^2(k)\right]\\
&- 2\tilde{W}_{j,1}^T(k_1)S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1)\\
&- 2\sigma_{j,1}\tilde{W}_{j,1}^T(k_1)\hat{W}_{j,1}(k_1)\\
&+ S_{j,1}^T(z_{j,1}(k_1))\Gamma_{j,1}S_{j,1}(z_{j,1}(k_1))e_{j,1}^2(k+1)\\
&+ 2\sigma_{j,1}\hat{W}_{j,1}^T(k_1)\Gamma_{j,1}S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1)\\
&+ \sigma_{j,1}^2\hat{W}_{j,1}^T(k_1)\Gamma_{j,1}\hat{W}_{j,1}(k_1)\\
\le& -\frac{1}{\bar{g}_{j,1}}e_{j,1}^2(k+1) - \frac{1}{\bar{g}_{j,1}}e_{j,1}^2(k)\\
&- 2\epsilon_{z_{j,1}}e_{j,1}(k+1)\\
&- 2\sigma_{j,1}\tilde{W}_{j,1}^T(k_1)\hat{W}_{j,1}(k_1)\\
&+ S_{j,1}^T(z_{j,1}(k_1))\Gamma_{j,1}S_{j,1}(z_{j,1}(k_1))e_{j,1}^2(k+1)\\
&+ 2\sigma_{j,1}\hat{W}_{j,1}^T(k_1)\Gamma_{j,1}S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1)\\
&+ \sigma_{j,1}^2\hat{W}_{j,1}^T(k_1)\Gamma_{j,1}\hat{W}_{j,1}(k_1).
\end{aligned}$$

Using the facts that

$$\begin{aligned}
&S_{j,1}^T(z_{j,1}(k_1))S_{j,1}(z_{j,1}(k_1)) < l_{j,1}\\
&S_{j,1}^T(z_{j,1}(k_1))\Gamma_{j,1}S_{j,1}(z_{j,1}(k_1)) \le \bar{\gamma}_{j,1}l_{j,1}\\
&2\epsilon_{z_{j,1}}e_{j,1}(k+1) \le \frac{\bar{\gamma}_{j,1}e_{j,1}^2(k+1)}{\bar{g}_{j,1}} + \frac{\bar{g}_{j,1}\epsilon_{z_{j,1}}^2}{\bar{\gamma}_{j,1}}\\
&2\sigma_{j,1}\hat{W}_{j,1}^T(k_1)\Gamma_{j,1}S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1)\\
&\quad \le \frac{\bar{\gamma}_{j,1}l_{j,1}e_{j,1}^2(k+1)}{\bar{g}_{j,1}} + \bar{g}_{j,1}\sigma_{j,1}^2\bar{\gamma}_{j,1}\left\|\hat{W}_{j,1}\right\|^2\\
&2\tilde{W}_{j,1}^T(k_1)\hat{W}_{j,1}(k_1)\\
&\quad = \left\|\tilde{W}_{j,1}(k_1)\right\|^2 + \left\|\hat{W}_{j,1}(k_1)\right\|^2 - \left\|W_{j,1}^*\right\|^2
\end{aligned}$$

we obtain

$$\begin{aligned}
\Delta V_{j,1} \le& -\frac{\rho_{j,1}}{\bar{g}_{j,1}}e_{j,1}^2(k+1) - \frac{1}{\bar{g}_{j,1}}e_{j,1}^2(k) + \beta_{j,1}\\
&-\sigma_{j,1}(1 - \sigma_{j,1}\bar{\gamma}_{j,1} - \bar{g}_{j,1}\sigma_{j,1}\bar{\gamma}_{j,1})\left\|\hat{W}_{j,1}(k_1)\right\|^2
\end{aligned}$$

where

$$\begin{aligned}
\rho_{j,1} =& 1 - \bar{\gamma}_{j,1} - \bar{\gamma}_{j,1}l_{j,1} - \bar{g}_{j,1}\bar{\gamma}_{j,1}l_{j,1}\\
\beta_{j,1} =& \frac{\bar{g}_{j,1}\epsilon_{z_{j,1}}^2}{\bar{\gamma}_{j,1}} + \sigma_{j,1}\left\|W_{j,1}^*\right\|^2.
\end{aligned}$$

If we choose the design parameters as follows:

$$\bar{\gamma}_{j,1} < \frac{1}{1 + l_{j,1} + \bar{g}_{j,1}l_{j,1}}, \quad \sigma_{j,1} < \frac{1}{(1+\bar{g}_{j,1})\bar{\gamma}_{j,1}} \tag{32}$$

then $\Delta V_{j,1} \le 0$, once the error $|e_{j,1}(k)|$ is larger than $\sqrt{\bar{g}_{j,1}\beta_{j,1}}$. This implies the boundedness of $V_{j,1}(k)$ for all $k \ge 0$, which leads to the boundedness of $e_{j,1}(k)$ because $V_{j,1}(k) = V_{j,1}(0) + \sum_{p=0}^{k}\Delta V_{j,1}(p) < \infty$. Furthermore, the tracking error $e_{j,1}(k)$ will asymptotically converge to the compact set denoted by $\Omega_{j,1} \subset R$, where $\Omega_{j,1} \triangleq \{\chi| \, |\chi| \le \sqrt{\bar{g}_{j,1}\beta_{j,1}}\}$.

The adaptation dynamics (27) can be written as

$$\begin{aligned}
\tilde{W}_{j,1}(k+1) =& (I - \Gamma_{j,1}\sigma_{j,1})\tilde{W}_{j,1}(k_1)\\
&- \Gamma_{j,1}\left[S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1) + \sigma_{j,1}W_{j,1}^*\right]\\
=& A_{j,1}(k)\tilde{W}_{j,1}(k_1)\\
&- \Gamma_{j,1}\left[S_{j,1}(z_{j,1}(k_1))e_{j,1}(k+1) + \sigma_{j,1}W_{j,1}^*\right].
\end{aligned}$$

Because $\gamma_{j,1} > 0, \sigma_{j,1} > 0$ and $0 < \sigma_{j,1}\,\gamma_{j,1} < 1$, we know that the transition matrix of $A_{j,1}(k)$ always satisfies $\|\Phi(k_1, k_0)\| < 1$. Furthermore, noting $S_{j,1}(z_{j,1}(k_1))$, $e_{j,1}(k+1)$ and $\sigma_{j,1}W_{j,1}^*$

are all bounded, by applying Lemma 2, $\tilde{W}_{j,1}(k)$ is bounded in a compact set denoted by $\Omega_{w_{j,1}}$, and hence, the boundedness of $\hat{W}_{j,1}(k)$ is assured.

Step 2: As defined before, $e_{j,2}(k) = x_{j,2}(k) - \alpha_{j,2}(k_1)$. Its $(n_j - 1)$ th difference is given by

$$\begin{aligned} e_{j,2}(k+n_j-1) &= x_{j,2}(k+n_j-1) - \alpha_{j,2}(k) \\ &= F_{j,2}(k) + G_{j,2}(k)x_{j,3}(k+n_j-2) \\ &\quad - \alpha_{j,2}(k). \end{aligned} \tag{33}$$

Similarly, consider $x_{j,3}(k+n_j-2)$ as a fictitious control for (33). It is obvious that $e_{j,2}(k+n_j-1) = 0$ if we choose

$$\begin{aligned} x_{j,3}(k+n_j-2) &= \alpha^*_{j,3}(k) \\ &= -\frac{1}{G_{j,2}(k)}\left[F_{j,2}(k) - \alpha_{j,2}(k)\right]. \end{aligned} \tag{34}$$

Accordingly, $\alpha^*_{j,3}(k)$ can be approximated by an ideal high-order neural network

$$\begin{aligned} \alpha^*_{j,3} &= W^{*T}_{j,2} S_{j,2}(z_{j,2}(k)) + \epsilon_{z_{j,2}}(z_{j,2}(k)) \\ z_{j,2}(k) &= [\bar{x}^T_{j,n_j}(k), \alpha_{j,2}(k)]^T \in \Omega_{z_{j,2}} \subset R^{n_j+1}. \end{aligned} \tag{35}$$

Consider the direct adaptive fictitious controller as

$$x_{j,3}(k+n_j-1) = \alpha_{j,3}(k) = \hat{W}^T_{j,2}(k) S_{j,2}(z_{j,2}(k)) \tag{36}$$

and the robust updating algorithm for NN weights as

$$\begin{aligned} \hat{W}_{j,2}(k+1) &= \hat{W}_{j,2}(k_2) \\ &\quad - \Gamma_{j,2}\left[S_{j,2}(z_{j,2}(k_2))e_{j,2}(k+1) \right. \\ &\quad \left. + \sigma_{j,2}\hat{W}_{j,2}(k_2)\right]. \end{aligned} \tag{37}$$

Following the same procedure in Step 1, we obtain the second step error equation

$$e_{j,2}(k+n_j-1) = G_{j,2}(k)\left[\tilde{W}^T_{j,2}(k)S_{j,2}(z_{j,2}(k)) - \epsilon_{z_{j,2}}\right]. \tag{38}$$

Choose the Lyapunov function candidate

$$\begin{aligned} V_{j,2}(k) &= V_{j,1}(k) + \frac{1}{\bar{g}_{j,2}} e^2_{j,2}(k) \\ &\quad + \sum_{p=0}^{n_j-2} \tilde{W}^T_{j,2}(k_2+p)\Gamma^{-1}_{j,2}\tilde{W}_{j,2}(k_2+p) \end{aligned} \tag{39}$$

where $k_2 = k - n_j + 2$. The first difference of (39) along (37) and (38) is given by

$$\begin{aligned} \Delta V_{j,2} \le &-\frac{\rho_{j,1}}{\bar{g}_{j,1}} e^2_{j,2}(k+1) - \frac{1}{\bar{g}_{j,1}} e^2_{j,1}(k) \\ &- \frac{\rho_{j,2}}{\bar{g}_{j,2}} e^2_{j,2}(k+1) - \frac{1}{\bar{g}_{j,2}} e^2_{j,2}(k) + \beta_{j,2} \\ &- \sigma_{j,2}(1 - \sigma_{j,2}\bar{\gamma}_{j,2} - \bar{g}_{j,2}\sigma_{j,2}\bar{\gamma}_{j,2}) \left\|\hat{W}_{j,2}(k_2)\right\|^2 \end{aligned}$$

where $\rho_{j,1}$ is defined as in Step 1, and $\rho_{j,2} = 1 - \bar{\gamma}_{j,2} - \bar{\gamma}_{j,2}l_{j,2} - \bar{g}_{j,2}\bar{\gamma}_{j,2}l_{j,2}$, $\beta_{j,2} = \beta_{j,1} + (\bar{g}_{j,2}\epsilon^2_{z_{j,2}}/\bar{\gamma}_{j,2}) + \sigma_{j,2}\left\|W^*_{j,2}\right\|^2$.

If we choose the design parameters as follows:

$$\bar{\gamma}_{j,2} < \frac{1}{1 + l_{j,2} + \bar{g}_{j,2}l_{j,2}}, \quad \sigma_{j,2} < \frac{1}{(1+\bar{g}_{j,2})\bar{\gamma}_{j,2}} \tag{40}$$

then $\Delta V_{j,2} \le 0$ once $|e_{j,1}(k)| > \sqrt{\bar{g}_{j,1}\beta_{j,2}}$ or $|e_{j,2}(k)| > \sqrt{\bar{g}_{j,2}\beta_{j,2}}$.

As explained in Step 1, $V_{j,2}(k)$ is bounded for all $k \ge 0$, and the tracking errors $e_{j,1}(k)$ and $e_{j,2}(k)$ are also bounded and will asymptotically converge to the compact set denoted by $\Omega_{j,2} \subset R^2$, where $\Omega_{j,2} \triangleq \{\chi | \chi = [\chi_1, \chi_2]^T, |\chi_1| \le \sqrt{\bar{g}_{j,1}\beta_{j,2}}, |\chi_2| \le \sqrt{\bar{g}_{j,2}\beta_{j,2}}\}$. The boundedness of $\hat{W}_{j,2}(k)$, or equivalently of $\tilde{W}_{j,2}(k)$ can be proved as in Step 1.

Step i $(2 < i < n_j)$: Following the same procedure as in Step 2, for $e_{j,i}(k) = x_{j,i}(k) - \alpha_{j,i}(k_{i-1})$, its $(n_j - i + 1)$th difference is

$$\begin{aligned} e_{j,i}(k+n_j-i+1) &= x_{j,i}(k+n_j-i+1) - \alpha_{j,i}(k) \\ &= F_{j,i}(k) + G_{j,i}(k)x_{j,i+1}(k+n_j-i) \\ &\quad - \alpha_{j,i}(k). \end{aligned}$$

Similarly, we have the direct adaptive fictitious controller and the robust updating algorithm for NN weights as follows:

$$\begin{aligned} x_{j,i+1}(k+n_j-i) &= \alpha_{j,i+1}(k) \\ &= \hat{W}^T_{j,i}(k) S_{j,i}(z_{j,i}(k)) \end{aligned} \tag{41}$$

$$\begin{aligned} \hat{W}_{j,i}(k+1) &= \hat{W}_{j,i}(k_i) \\ &\quad - \Gamma_{j,i}\left[S_{j,i}(z_{j,i}(k_i))e_{j,i}(k+1) \right. \\ &\quad \left. + \sigma_{j,i}\hat{W}_{j,i}(k_i)\right] \\ z_{j,i}(k) &= \left[\bar{x}^T_{j,n_j}(k), \alpha_{j,i}(k)\right]^T \\ &\in \Omega_{z_{j,i}} \subset R^{n_j+1}. \end{aligned} \tag{42}$$

Accordingly, we obtain the ith error equation

$$\begin{aligned} &e_{j,i}(k+n_j-i+1) \\ &\quad = G_{j,i}(k)\left[\tilde{W}^T_{j,i}(k)S_{j,i}(z_{j,i}(k)) - \epsilon_{z_{j,i}}\right]. \end{aligned} \tag{43}$$

Choose the Lyapunov function candidate

$$\begin{aligned} V_{j,i}(k) &= V_{j,i-1}(k) + \frac{1}{\bar{g}_{j,i}} e^2_{j,i}(k) \\ &\quad + \sum_{p=0}^{n_j-i} \tilde{W}^T_{j,i}(k_i+p)\Gamma^{-1}_{j,i}\tilde{W}_{j,i}(k_i+p) \end{aligned} \tag{44}$$

where $k_i = k - n_j + i$. The first difference of (44) along (42) and (43) is given

$$\begin{aligned} \Delta V_{j,i} \le &-\sum_{p=1}^{i} \frac{\rho_{j,p}}{\bar{g}_{j,p}} e^2_{j,p}(k+1) - \sum_{p=1}^{i} \frac{1}{\bar{g}_{j,p}} e^2_{j,p}(k) + \beta_{j,i} \\ &-\sigma_{j,i}(1 - \sigma_{j,i}\bar{\gamma}_{j,i} - \bar{g}_{j,i}\sigma_{j,i}\bar{\gamma}_{j,i}) \left\|\hat{W}_{j,i}(k_i)\right\|^2 \end{aligned}$$

where $\rho_{j,p}$, $p = 1, 2, \ldots, i-1$, are defined in previous $(i-1)$ steps, $\rho_{j,i} = 1 - \bar{\gamma}_{j,i} - \bar{\gamma}_{j,i} l_{j,i} - \bar{g}_{j,i}\bar{\gamma}_{j,i} l_{j,i}$ and $\beta_{j,i} = \beta_{j,i-1} + (\bar{g}_{j,i}\epsilon^2_{z_{j,i}}/\bar{\gamma}_{j,i}) + \sigma_{j,i}\left\|W^*_{j,i}\right\|^2$. If we choose the design parameters as follows:

$$\bar{\gamma}_{j,i} < \frac{1}{1 + l_{j,i} + \bar{g}_{j,i} l_{j,i}}, \quad \sigma_{j,i} < \frac{1}{(1+\bar{g}_{j,i})\bar{\gamma}_{j,i}} \tag{45}$$

then $\Delta V_{j,i} \le 0$ once any one of the i errors satisfies $|e_{j,p}(k)| > \sqrt{\bar{g}_{j,p}\beta_{j,i}}$, $p = 1, 2, \ldots, i$. This demonstrates that the tracking error $e_{j,1}(k), e_{j,2}(k), \ldots, e_{j,i}(k)$ are bounded for all $k \ge 0$, and will asymptotically converge to the compact set denoted by $\Omega_{j,i} \subset R^i$, where $\Omega_{j,i} \triangleq \{\chi | \chi = [\chi_1, \chi_2, \ldots, \chi_i]^T, \chi_p \le \sqrt{\bar{g}_{j,p}\beta_{j,i}}\ p = 1, 2, \ldots, i\}$. The boundedness of $\hat{W}_{j,i}(k)$, or equivalently of $\tilde{W}_{j,i}(k)$ can be proved as in Step 1.

Step n_j: By now, we have shown that for the first $n_j - 1$ equations of each subsystem, by suitably choosing the virtual controls' design parameters, the equations can be stabilized by the virtual controls. By carefully examining the last equations of all the subsystems, we can see that they are in strict feedback form relative to the practical control inputs, $u_1(k)$, $u_2(k), \ldots, u_n(k)$. This motivates us to use the backstepping design technique again to guarantee the stability of the whole closed-loop system.

Substep 1: Considering the first subsystem of system (1), according to (18), it can be written as

$$\begin{cases} x_{1,1}(k+n_1) = F_{1,1}(k) + G_{1,1}(k)x_{1,2}(k+n_1-1) \\ \vdots \\ x_{1,n_1-1}(k+2) = F_{1,n_1-1}(k) + G_{1,n_1-1}(k)x_{1,n_1}(k+1) \\ x_{1,n_1}(k+1) = f_{1,n_1}(k) + g_{1,n_1}(k)u_1(k) + d_1(k) \\ y_1(k) = x_{1,1}(k). \end{cases} \tag{46}$$

For the first $n_1 - 1$ equations of (46), we have shown that their stability can be guaranteed by suitabley chosen the virtual control design parameters. Let us consider the last equation. The error $e_{1,n_1}(k)$ can be written as $e_{1,n_1}(k) = x_{1,n_1}(k) - \alpha_{1,n_1}(k-1)$, its first difference is given by

$$\begin{aligned} e_{1,n_1}(k+1) &= x_{1,n_1}(k+1) - \alpha_{1,n_1}(k) \\ &= f_{1,n_1}(k) + g_{1,n_1}(k)u_1(k) + d_1(k) - \alpha_{1,n_1}(k). \end{aligned}$$

It is obvious that $e_{1,n_1}(k+1) = 0$ if we choose

$$u_1(k) = u_1^*(k) = -\frac{1}{g_{1,n_1}(k)}[f_{1,n_1}(k) - \alpha_{1,n_1}(k)]$$

and there are no disturbances, i.e., $d_1(k) = 0$. If $d_1(k) \ne 0$, we obtain $e_{1,n_1}(k+1) = d_1(k)$. Though exact tracking cannot be obtained, the error is bounded due to the boundedness of the disturbances. Similarly, $u_1^*(k)$ can be approximated by a high-order neural network

$$\begin{aligned} u_1^*(k) &= W^{*T}_{1,n_1} S_{1,n_1}(z_{1,n_1}(k)) + \epsilon_{z_{1,n_1}}(z_{1,n_1}(k)) \\ z_{1,n_1}(k) &= [X, \alpha_{1,n_1}(k)]^T \in \Omega_{z_{1,n_1}} \subset R^{1+\sum_{i=1}^n n_i}. \end{aligned}$$

Following the same procedure as in Step 1 or 2, we choose the direct adaptive controller and robust updating algorithm for NN weights as

$$u_1(k) = \hat{W}^T_{1,n_1}(k) S_{1,n_1}(z_{1,n_1}(k)) \tag{47}$$

$$\begin{aligned} \hat{W}_{1,n_1}(k+1) &= \hat{W}_{1,n_1}(k) \\ &\quad - \Gamma_{1,n_1}\Big[S_{1,n_1}(z_{1,n_1}(k))e_{1,n_1}(k+1) \\ &\quad + \sigma_{1,n_1}\hat{W}_{1,n_1}(k)\Big]. \end{aligned} \tag{48}$$

For the n_1th step error equation

$$\begin{aligned} e_{1,n_1}(k+1) &= g_{1,n_1}(k)\left[\tilde{W}^T_{1,n_1}(k)S(z_{1,n_1}(k)) - \epsilon_{z_{1,n_1}}\right] \\ &\quad + d_1(k) \\ &= g_{1,n_1}(k)\left[\tilde{W}^T_{1,n_1}(k)S(z_{1,n_1}(k)) - \epsilon_{z_{1,n_1}} + \frac{d_1(k)}{g_{1,n_1}(k)}\right] \\ &= g_{1,n_1}(k)\left[\tilde{W}^T_{1,n_1}(k)S(z_{1,n_1}(k)) - \epsilon'_{z_{1,n_1}}\right] \end{aligned} \tag{49}$$

with $\epsilon'_{z_{1,n_1}} = \epsilon_{z_{1,n_1}} - (d_1(k)/g_{1,n_1}(k))$. It is obvious that $\epsilon'_{z_{1,n_1}}$ is bounded because of the boundedness of $\epsilon_{z_{1,n_1}}$, $d_1(k)$ and $g_{1,n_1}(k)$. Choose the Lyapunov function candidate

$$\begin{aligned} V_{1,n_1}(k) = V_{1,n_1-1}(k) + \frac{1}{\bar{g}_{1,n_1}} e^2_{1,n_1}(k) \\ + \tilde{W}^T_{1,n_1}(k)\Gamma^{-1}_{1,n_1}\tilde{W}_{1,n_1}(k). \end{aligned} \tag{50}$$

The first difference of (50) along (48) and (49) is given

$$\begin{aligned} \Delta V_{1,n_1} \le &-\sum_{p=1}^{n_1}\frac{\rho_{1,p}}{\bar{g}_{1,p}}e^2_{1,p}(k+1) - \sum_{p=1}^{n_1}\frac{1}{\bar{g}_{1,p}}e^2_{1,p}(k) + \beta_{1,n_1} \\ &-\sigma_{1,n_1}(1 - \sigma_{1,n_1}\bar{\gamma}_{1,n_1} - \bar{g}_{1,n_1}\sigma_{1,n_1}\bar{\gamma}_{1,n_1})\left\|\hat{W}_{1,n_1}(k)\right\|^2 \end{aligned}$$

where $\rho_{1,p}, p = 1, 2, \ldots, n_1 - 1$, are defined as previous $(n_1 - 1)$ steps, and $\rho_{1,n_1} = 1 - \bar{\gamma}_{1,n_1} - \bar{\gamma}_{1,n_1} l_{1,n_1} - \bar{g}_{1,n_1}\bar{\gamma}_{1,n_1} l_{1,n_1}$, $\beta_{1,n_1} = \beta_{1,n_1-1} + (\bar{g}_{1,n_1}{\epsilon'}^2_{z_{1,n_1}}/\bar{\gamma}_{1,n_1})\sigma_{1,n_1}\left\|W^*_{1,n_1}\right\|^2$.

If we choose the design parameters as follows:

$$\bar{\gamma}_{1,n_1} < \frac{1}{1 + l_{1,n_1} + \bar{g}_{1,n_1} l_{1,n_1}}, \quad \sigma_{1,n_1} < \frac{1}{(1+\bar{g}_{1,n_1})\bar{\gamma}_{1,n_1}} \tag{51}$$

then $\Delta V_{1,n_1} \le 0$ once any one of the n_1 errors satisfies $|e_{1,p}(k)| > \sqrt{\bar{g}_{1,p}\beta_{1,n_1}}$, $p = 1, 2, \ldots, n_1$. This demonstrates that the tracking errors $e_{1,1}(k), e_{1,2}(k), \ldots, e_{1,n_1}(k)$ are bounded for all $k \ge 0$, and will asymptotically converge to the compact set denoted by Ω_{1,n_1}, where $\Omega_{1,n_1} \triangleq \{\chi | \chi = [\chi_1, \chi_2, \ldots, \chi_n]^T, |\chi_j| \le \sqrt{\bar{g}_{1,p}\beta_{1,n_1}}\ p = 1, 2, \ldots, n_1\}$. The boundedness of $\hat{W}_{1,n_1}(k)$, or equivalently of $\tilde{W}_{1,n_1}(k)$ can be proved as in Step 1.

Based on the procedure above, we can conclude that $\bar{x}_{1,n_1}(k+1) \in \Omega$ and $u_1(k)$ are bounded if $\bar{x}_{1,n_1}(k) \in \Omega$. Finally, if we initialize $\bar{x}_{1,n_1}(0) \in \Omega$, and choose the design parameters according to (32), (40), (45), and (51), we know there exists a $k^* > 0$, such that all errors $e_{1,1}(k)$, $e_{1,2}(k), \ldots, e_{1,n_1}(k)$ asymptotically converge to Ω_{1,n_1}. Furthermore, by applying Lemma 2 and following the same procedure in Step 1, the boundedness of the weights

$\hat{W}_{1,p}$ $(p = 1, 2, \ldots, n_1)$ can be proved. Thus, the closed-loop system is SGUUB and $\bar{x}_{1,n_1}(k) \in \Omega$ will hold for all $k > 0$.

Substep 2: For $e_{2,n_2}(k) = x_{2,n_2}(k) - \alpha_{2,n_2}(k-1)$, its first difference is given by

$$\begin{aligned} e_{2,n_2}(k+1) &= x_{2,n_2}(k+1) - \alpha_{2,n_2}(k) \\ &= f_{2,n_2}(k) + g_{2,n_2}(k)u_2(k) + d_2(k) - \alpha_{2,n_2}(k). \end{aligned}$$

It is obvious that $e_{2,n_2}(k+1) = 0$ if we choose

$$u_2(k) = u_2^*(k) = -\frac{1}{g_{2,n_2}(k)}[f_{2,n_2}(k) - \alpha_{2,n_2}(k)]$$

and there are no disturbances, i.e., $d_2(k) = 0$. If $d_2(k) \neq 0$, we obtain $e_{2,n_2}(k+1) = d_2(k)$. Though exact tracking cannot be obtained, the error is bounded due to the boundedness of the disturbances. Similarly, $u_2^*(k)$ can be approximated by an high-order neural network

$$\begin{aligned} u_2^*(k) &= W_{2,n_2}^{*T} S_{2,n_2}(z_{2,n_2}(k)) + \epsilon_{z_{2,n_2}}(z_{2,n_2}(k)) \\ z_{2,n_2}(k) &= [X, u_1(k), \alpha_{2,n_2}(k)]^T \in \Omega_{z_{2,n_2}} \subset R^{2+\sum_{i=1}^{n} n_i}. \end{aligned}$$

Following the same procedure in Substep 1, in this step, we will design control $u_2(k)$ to stabilize the first two subsystems of system (1). Choosing the following Lyapunov candidate

$$\begin{aligned} V_{2,n_2}(k) = V_{1,n_1}(k) + V_{2,n_2-1}(k) + \frac{1}{\bar{g}_{2,n_2}} e_{2,n_2}^2(k) \\ + \tilde{W}_{2,n_2}^T(k) \Gamma_{2,n_2}^{-1} \tilde{W}_{2,n_2}(k). \end{aligned}$$

By following the same procedure in Substep 1, we can obtain (for clarity of presentation, the details are omitted here) the first difference of $V_{2,n_2}(k)$ as follows:

$$\begin{aligned} \Delta V_{2,n_2}(k) \le & -\sum_{p=1}^{n_1} \frac{\rho_{1,p}}{\bar{g}_{1,p}} e_{1,p}^2(k+1) \\ & -\sum_{p=1}^{n_1} \frac{1}{\bar{g}_{1,p}} e_{1,p}^2(k) + \beta_{1,n_1} \\ & - \sigma_{1,n_1}(1 - \sigma_{1,n_1}\bar{\gamma}_{1,n_1} - \bar{g}_{1,n_1}\sigma_{1,n_1}\bar{\gamma}_{1,n_1}) \\ & \times \left\| \hat{W}_{1,n_1}(k) \right\|^2 \\ & - \sum_{p=1}^{n_2} \frac{\rho_{2,p}}{\bar{g}_{2,p}} e_{2,p}^2(k+1) - \sum_{p=1}^{n_2} \frac{1}{\bar{g}_{2,p}} e_{2,p}^2(k) \\ & + \beta_{2,n_2} - \sigma_{2,n_2} \\ & \times (1 - \sigma_{2,n_2}\bar{\gamma}_{2,n_2} - \bar{g}_{2,n_2}\sigma_{2,n_2}\bar{\gamma}_{2,n_2}) \\ & \times \left\| \hat{W}_{2,n_2}(k) \right\|^2 \end{aligned}$$

where $\rho_{2,n_2} = 1 - \bar{\gamma}_{2,n_2} - \bar{\gamma}_{2,n_2} l_{2,n_2} - \bar{g}_{2,n_2}\bar{\gamma}_{2,n_2} l_{2,n_2}$ and $\beta_{2,n_2} = \beta_{2,n_2-1} + (\bar{g}_{2,n_2}{\epsilon'}^2_{z_{2,n_2}}/\bar{\gamma}_{2,n_2}) + \sigma_{2,n_2}\|W_{2,n_2}^*\|^2$. By noting (51) and choosing $\bar{\gamma}_{2,n_2}$ and σ_{2,n_2} as follows:

$$\bar{\gamma}_{2,n_2} < \frac{1}{1 + l_{2,n_2} + \bar{g}_{2,n_2} l_{2,n_2}}, \quad \sigma_{2,n_2} < \frac{1}{(1 + \bar{g}_{2,n_2})\bar{\gamma}_{2,n_2}} \tag{52}$$

we obtain

$$\begin{aligned} \Delta V_{2,n_2}(k) \le -\sum_{p=1}^{n_1} \frac{1}{\bar{g}_{1,p}} e_{1,p}^2(k) - \sum_{p=1}^{n_2} \frac{1}{\bar{g}_{2,p}} e_{2,p}^2(k) \\ + \beta_{1,n_1} + \beta_{2,n_2}. \end{aligned} \tag{53}$$

It is obvious that for the first two subsystems of system (1), $\Delta V_{2,n_2}(k) \le 0$ once either

$$e_{1,p}^2 > \bar{g}_{1,p}(\beta_{1,n_1} + \beta_{2,n_2}), \quad p = 1, \ldots, n_1$$

or

$$e_{2,p}^2 > \bar{g}_{2,p}(\beta_{1,n_1} + \beta_{2,n_2}), \quad p = 1, \ldots, n_2.$$

It indicates that the errors $e_{1,p}^2$ $(p = 1, \ldots, n_1)$ and $e_{2,p}^2$ $(p = 1, \ldots, n_2)$ are all bounded in a compact set.

Substep j $(2 < j < n)$: For $e_{j,n_j}(k) = x_{j,n_j}(k) - \alpha_{j,n_j}(k-1)$, its first difference is given by

$$\begin{aligned} e_{j,n_j}(k+1) &= x_{j,n_j}(k+1) - \alpha_{j,n_j}(k) \\ &= f_{j,n_j}(k) + g_{j,n_j}(k)u_j(k) + d_j(k) \\ &\quad - \alpha_{j,n_j}(k). \end{aligned} \tag{54}$$

It is obvious that $e_{j,n_j}(k+1) = 0$ if we choose

$$u_j(k) = u_j^*(k) = -\frac{1}{g_{j,n_j}(k)}[f_{j,n_j}(k) - \alpha_{j,n_j}(k)] \tag{55}$$

and there are no disturbances, i.e., $d_j(k) = 0$. If $d_j(k) \neq 0$, we obtain $e_{j,n_j}(k+1) = d_j(k)$. Though exact tracking cannot be obtained, the error is bounded due to the boundedness of the disturbances. Similarly, $u_j^*(k)$ can be approximated by an high-order neural network

$$\begin{aligned} u_j^*(k) &= W_{j,n_j}^{*T} S_{j,n_j}(z_{j,n_j}(k)) + \epsilon_{z_{j,n_j}}(z_{j,n_j}(k)) \\ z_{j,n_j}(k) &= \left[X, \bar{u}_{j-1}(k), \alpha_{j,n_j}(k)\right]^T \\ &\in \Omega_{z_{j,n_j}} \subset R^{j+\sum_{i=1}^{n} n_i}. \end{aligned} \tag{56}$$

Following the same procedure as in Substep 1 or 2, we choose the direct adaptive controller and robust updating algorithm for NN weights as

$$u_j(k) = \hat{W}_{j,n_j}^T(k) S_{j,n_j}(z_{j,n_j}(k)) \tag{57}$$

$$\begin{aligned} \hat{W}_{j,n_j}(k+1) = \hat{W}_{j,n_j}(k) \\ - \Gamma_{j,n_j}\Big[S_{j,n_j}(z_{j,n_j}(k)) e_{j,n_j}(k+1) \\ + \sigma_{j,n_j}\hat{W}_{j,n_j}(k) \Big]. \end{aligned} \tag{58}$$

For the n_jth step error equation

$$\begin{aligned} e_{j,n_j}(k+1) &= g_{j,n_j}(k)\left[\tilde{W}^T_{j,n_j}(k)S(z_{j,n_j}(k)) - \epsilon_{z_{j,n_j}}\right] + d_j(k) \\ &= g_{j,n_j}(k)\left[\tilde{W}^T_{j,n_j}(k)S(z_{j,n_j}(k)) - \epsilon_{z_{j,n_j}} + \frac{d_j(k)}{g_{j,n_j}(k)}\right] \\ &= g_{j,n_j}(k)\left[\tilde{W}^T_{j,n_j}(k)S(z_{j,n_j}(k)) - \epsilon'_{z_{j,n_j}}\right] \end{aligned} \tag{59}$$

with $\epsilon'_{z_{j,n_j}} = \epsilon_{z_{j,n_j}} - (d_j(k)/g_{j,n_j}(k))$. It is obvious that $\epsilon'_{z_{j,n_j}}$ is bounded because of the boundedness of $\epsilon_{z_{j,n_j}}$, $d_j(k)$ and $g_{j,n_j}(k)$. Choose the Lyapunov function candidate

$$\begin{aligned} V_{j,n_j}(k) = V_{j-1}, n_j(k) + V_{j,n_j-1}(k) + \frac{1}{\bar{g}_{j,n_j}}e^2_{j,n_j}(k) \\ +\tilde{W}^T_{j,n_j}(k)\Gamma^{-1}_{j,n_j}\tilde{W}_{j,n_j}(k). \end{aligned} \tag{60}$$

It is obvious that $V_{j,n_j}(k)$ includes three parts. The first part, $V_{j-1}, n_j(k)$ corresponds to the summation of the first $j-1$ subsystems' Lyapunov functions, the second part $V_{j,n_j-1}(k)$ corresponds to the first n_j-1 equations of the jth subsystems, and $(1/\bar{g}_{j,n_j})e^2_{j,n_j}(k) + \tilde{W}^T_{j,n_j}(k)\Gamma^{-1}_{j,n_j}\tilde{W}_{j,n_j}(k)$ corresponds to the last equation of the jth subsystem.

The first difference of (60) along (58) and (59) is given as

$$\begin{aligned} \Delta V_{j,n_j} \le\, & \Delta V_{j-1}, n_j(k) - \sum_{p=1}^{n_j}\frac{\rho_{j,p}}{\bar{g}_{j,p}}e^2_{j,p}(k+1) \\ & - \sum_{p=1}^{n_j}\frac{1}{\bar{g}_{j,p}}e^2_{j,p}(k) + \beta_{j,n_j} \\ & - \sigma_{j,n_j}(1-\sigma_{j,n_j}\bar{\gamma}_{j,n_j} - \bar{g}_{j,n_j}\sigma_{j,n_j}\bar{\gamma}_{j,n_j}) \\ & \times \|\hat{W}_{j,n_j}(k)\|^2 \end{aligned} \tag{61}$$

where $\rho_{j,p}, p = 1,2,\ldots,n_j-1$, are defined as previous $(n_j - 1)$ steps, and $\rho_{j,n_j} = 1 - \bar{\gamma}_{j,n_j} - \bar{\gamma}_{j,n_j}l_{j,n_j} - \bar{g}_{j,n_j}\bar{\gamma}_{j,n_j}l_{j,n_j}$, $\beta_{j,n_j} = \beta_{j,n_j-1} + (\bar{g}_{j,n_j}\epsilon'^2_{z_{j,n_j}}/\bar{\gamma}_{j,n_j}) + \sigma_{j,n_j}\left\|W^*_{j,n_j}\right\|^2$.

Similar to the procedure in derivation of inequality (53), if we choose the design parameters as follows:

$$\bar{\gamma}_{j,n_j} < \frac{1}{1+l_{j,n_j}+\bar{g}_{j,n_j}l_{j,n_j}}, \quad \sigma_{j,n_j} < \frac{1}{(1+\bar{g}_{j,n_j})\bar{\gamma}_{j,n_j}} \tag{62}$$

then inequality (61) can be further written as

$$\begin{aligned} \Delta V_{j,n_j}(k) \le -\sum_{p=1}^{n_1}\frac{1}{\bar{g}_{1,p}}e^2_{1,p}(k) - \cdots - \sum_{p=1}^{n_j}\frac{1}{\bar{g}_{j,p}}e^2_{j,p}(k) \\ +\beta_{1,n_1} + \cdots + \beta_{j,n_j} \end{aligned}$$

then $\Delta V_{j,n_j} \le 0$ once any one of the errors

$$\begin{aligned} & e^2_{q,p}(k) > \bar{g}_{q,p}\left(\beta_{1,n_1} + \cdots + \beta_{j,n_j}\right) \\ & q = 1,\ldots,j \text{ and } p = 1,\ldots,n_q. \end{aligned}$$

This demonstrates that the errors $e_{q,p}$ $(q = 1,\ldots,j$, $p = 1,\ldots,n_q)$ are bounded for all $k \ge 0$, and will asymptotically converge to the compact set denoted by Ω_{j,n_j}. The boundedness of $\hat{W}_{j,n_j}(k)$, or equivalently of $\tilde{W}_{j,n_j}(k)$ can be proved as in Step 1.

Based on the procedure above, we can conclude that $\bar{x}_{j,n_j}(k+1) \in \Omega$ and $u_j(k)$ are bounded if $\bar{x}_{j,n_j}(k) \in \Omega$. Finally, if we initialize $\bar{x}_{j,n_j}(0) \in \Omega$, and choose the design parameters according to (32), (40), (45), (52), and (62), there exists a k^*, such that all errors asymptotically converge to Ω_{j,n_j}, and NN weight errors are all bounded. This implies that the closed-loop system is SGUUB. Then $\bar{x}_{j,n_j}(k) \in \Omega$, $\hat{W}_{j,p}$, $p = 1,2,\ldots,n_j$ will hold for all $k > 0$.

Substep n: Finally, in this step, by combining the Lyapunov functions of each subsystem to give the whole system's Lyapunov function candidate, we can claim that the closed-loop system is SGUUB.

For $e_{n,n_n}(k) = x_{n,n_n}(k) - \alpha_{n,n_n}(k-1)$, its first difference is given by

$$\begin{aligned} e_{n,n_n}(k+1) &= x_{n,n_n}(k+1) - \alpha_{n,n_n}(k) \\ &= f_{n,n_n}(k) + g_{n,n_n}(k)u_n(k) \\ &\quad + d_n(k) - \alpha_{n,n_n}(k). \end{aligned}$$

It is obvious that $e_{n,n_n}(k+1) = 0$ if we choose

$$u_n(k) = u^*_n(k) = -\frac{1}{g_{n,n_n}(k)}[f_{n,n_n}(k) - \alpha_{n,n_n}(k)] \tag{63}$$

and there are no disturbances, i.e., $d_n(k) = 0$. If $d_n(k) \ne 0$, we obtain $e_{n,n_n}(k+1) = d_n(k)$. Though exact tracking cannot be obtained, the error is bounded due to the boundedness of the disturbances. Similarly, $u^*_n(k)$ can be approximated by an high-order neural network

$$\begin{aligned} u^*_n(k) &= W^{*T}_{n,n_n}S_{n,n_n}(z_{n,n_n}(k)) + \epsilon_{z_{n,n_n}}(z_{n,n_n}(k)) \\ z_{n,n_n}(k) &= [X, \bar{u}_{n-1}(k), \alpha_{n,n_n}(k)]^T \\ &\in \Omega_{z_{n,n_n}} \subset R^{n+\sum_{i=1}^n n_i}. \end{aligned}$$

Choose the direct adaptive controller and robust updating algorithm for NN weights as

$$\begin{aligned} u_n(k) &= \hat{W}^T_{n,n_n}(k)S_{n,n_n}(z_{n,n_n}(k)) \\ \hat{W}_{n,n_n}(k+1) &= \hat{W}_{n,n_n}(k) \\ &\quad - \Gamma_{n,n_n}\left[S_{n,n_n}(z_{n,n_n}(k))e_{n,n_n}(k+1) + \sigma_{n,n_n}\hat{W}_{n,n_n}(k)\right]. \end{aligned}$$

Consider the following Lyapunov candidate

$$V_{n,n_n}(k) = \sum_{p=1}^{n-1} V_{p,n_p}(k) + V_{n,n_n-1}(k) + \frac{1}{\bar{g}_{n,n_n}} e_{n,n_n}^2(k) + \tilde{W}_{n,n_n}^T(k)\Gamma_{n,n_n}^{-1}\tilde{W}_{n,n_n}(k). \quad (64)$$

By following the same procedure in Substep $j(2 < j < n)$, if the design parameters are suitable chosen as

$$\begin{cases} \bar{\gamma}_{n,n_n} < \frac{1}{1+l_{n,n_n}+\bar{g}_{n,n_n}l_{n,n_n}} \\ \sigma_{n,n_n} < \frac{1}{(1+\bar{g}_{n,n_n})\bar{\gamma}_{n,n_n}} \end{cases} \quad (65)$$

we have

$$\Delta V_{n,n_n}(k) \le -\sum_{p=1}^{n_1} \frac{1}{\bar{g}_{1,p}} e_{1,p}^2(k) - \cdots - \sum_{p=1}^{n_n} \frac{1}{\bar{g}_{n,p}} e_{n,p}^2(k) + \beta_{1,n_1} + \cdots + \beta_{n,n_n}.$$

Define $\beta = \sum_{j=1}^{n} \beta_{j,n_j} = \beta_{1,n_1} + \cdots + \beta_{n,n_n}$, we obtain

$$\Delta V_{n,n_n}(k) \le \sum_{j=1}^{n} \left\{ -\sum_{i=1}^{n_j} \frac{1}{\bar{g}_{j,i}} e_{j,i}^2(k) \right\} + \beta \quad (66)$$

then $\Delta V(k)_{n,n_n} \le 0$ once any one of the $\sum_{j=1}^{n} n_j$ errors satisfies $|e_{j,i}(k)| > \sqrt{\bar{g}_{j,i}\beta}$, $j = 1,\ldots,n$ and $i = 1,\ldots,n_j$. This demonstrates that the tracking errors $e_{q,p}$ ($q = 1,\ldots,n$, $p = 1,\ldots,n_q$) are all bounded for all $k \ge 0$, and will asymptotically converge to the compact set denoted by Ω_{n,n_n}, where $\Omega_{n,n_n} \triangleq \{\chi | \chi = [\chi_{j,i}], j = 1,\ldots,n, i = 1,\ldots,n_j, |\chi_{j,i}| \le \sqrt{\bar{g}_{j,i}\beta}\}$. Now, we can conclude that all the errors are bounded.

Having proven that all the errors $e_{q,p}$ ($q = 1,\ldots,n$, $p = 1,\ldots,n_q$) are bounded in a compact set, we now further show that the neural network weights are also bounded. Considering the weights update law in (22), it can be rewritten as

$$\begin{aligned} \hat{W}_{j,i_j}(k+1) &= (I - \Gamma_{j,i_j}\sigma_{j,i_j})\hat{W}_{j,i_j}(k_{i_j}) - \Gamma_{j,i_j} S(z_{j,i_j}(k_{i_j}))e_{j,i_j}(k+1) \\ &\triangleq A_{j,i_j}\hat{W}_{j,i_j}(k_{i_j}) - \Gamma_{j,i_j} S(z_{j,i_j}(k_{i_j}))e_{j,i_j}(k+1) \\ k_{i_j} &= k - n_j + i_j, \quad i_j = 1,2,\ldots,n_j \end{aligned} \quad (67)$$

where $A_{j,i_j} = I - \Gamma_{j,i_j}\sigma_{j,i_j}$. Because the eigenvalues of matrix A_{j,i_j} are all in the unit circle, it is easy to obtain the eigenvalues of the transition matrix of system (67) to be within the unit circle too. By using Lemma 2, we conclude that the neural network weights are bounded.

In summary, the closed-loop nonlinear MIMO system (1), consisting of controller (21) and adaptive law (22), is semi-globally uniformly ultimately bounded, and has an equilibrium at $[e_{1,1}(k),\ldots,e_{n,1}(k)]^T = 0$, provided that the design parameters are properly chosen. All the signals, including the states $X(k)$, the control inputs $u_j(k)$ ($j = 1,\ldots,n$), the tracking errors $e_{j,1}(k)$ ($j = 1,\ldots,n$) and NN weight estimates $\hat{W}_{j,i_j}(k)$ ($j = 1,\ldots,n, i_j = 1,\ldots,n_j$)), are all bounded. ■

Remark 3: Considering the parameter conditions in (62), it can be seen that faster learning rate (increasing $\bar{\gamma}_{j,n_j}$) requires the neurons number l_{j,n_j} to decrease. Thus, the approximation accuracy will be affected. In practical applications, how to choose the adaptation gain $\bar{\gamma}_{j,n_j}$ and the neurons number l_{j,n_j} is a problem that needs to be dealt with carefully.

Remark 4: In adaptive nonlinear system control, PE condition is important for parameter convergence and system robustness. However, it is usually very difficult to verify its existence in practical applications [32]. Noticing Definition 2, the definition of PE condition in discrete-time system, we can see that to check its existence is not an easy task. In this paper, by adding a standard σ-modification term [32] in the weight update laws (22), the need of PE condition for weights update is removed.

Remark 5: In Theorem 2, by using the neural network emulator (21) and the weight update laws (22), through Lyapunov analysis, we can only obtain the boundedness of the closed-loop signals, including the states, the outputs and the neural network weights.

VI. Simulation

To illustrate the effectiveness of the proposed schemes, simulation studies are carried out for the following MIMO discrete-time system with triangular form inputs, as shown at the bottom of the page, where

$$\begin{cases} f_{1,1}(\bar{x}_{1,1}(k)) = \frac{x_{1,1}^2(k)}{1+x_{1,1}^2(k)} \\ g_{1,1}(\bar{x}_{1,1}(k)) = 0.3 \\ f_{1,2}(\bar{x}_{1,2}(k)) = \frac{x_{1,1}^2(k)}{1+x_{1,2}^2(k)+x_{2,1}^2(k)+x_{2,2}^2(k)} \\ g_{1,2}(\bar{x}_{1,2}(k)) = 1 \\ d_1(k) = 0.1\cos(0.05k)\cos(x_{1,1}(k)) \end{cases}$$

$$\begin{cases} f_{2,1}(\bar{x}_{2,1}(k)) = \frac{x_{2,1}^2(k)}{1+x_{2,1}^2(k)} \\ g_{2,1}(\bar{x}_{2,1}(k)) = 0.2 \\ f_{2,2}(\bar{x}_{2,2}(k), u_1(k)) = \frac{x_{2,1}^2(k)}{1+x_{1,1}^2+x_{1,2}^2(k)+x_{2,2}^2(k)} u_1^2(k) \\ g_{2,2}(\bar{x}_{2,2}(k)) = 1 \\ d_2(k) = 0.1\cos(0.05k)\cos(x_{2,1}(k)). \end{cases}$$

$$\begin{cases} x_{1,1}(k+1) = f_{1,1}(\bar{x}_{1,1}(k)) + g_{1,1}(\bar{x}_{1,1}(k))x_{1,2}(k) \\ x_{1,2}(k+1) = f_{1,2}(\bar{x}_{1,2}(k)) + g_{1,2}(\bar{x}_{1,2}(k))u_1(k) + d_1(k) \\ x_{2,1}(k+1) = f_{2,1}(\bar{x}_{2,1}(k)) + g_{2,1}(\bar{x}_{2,1}(k))x_{2,2}(k) \\ x_{2,2}(k+1) = f_{2,2}(\bar{x}_{2,2}(k), u_1(k)) + g_{2,2}(\bar{x}_{2,2}(k))u_2(k) + d_2(k) \\ y_1(k) = x_{1,1}(k) \\ y_2(k) = x_{2,1}(k) \end{cases}$$

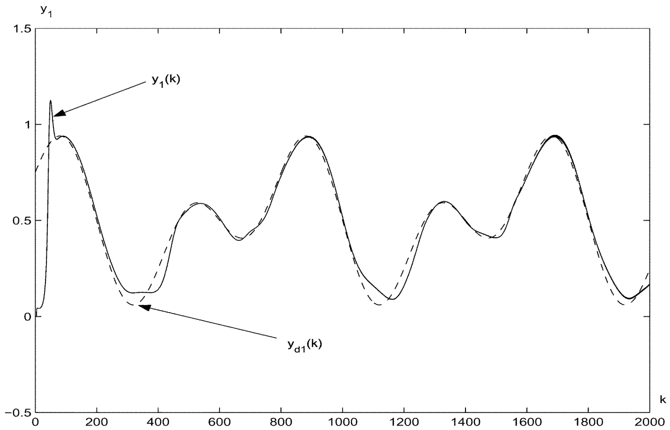

Fig. 2. Tracking performance $y_1(k)$ and $y_{d_1}(k)$.

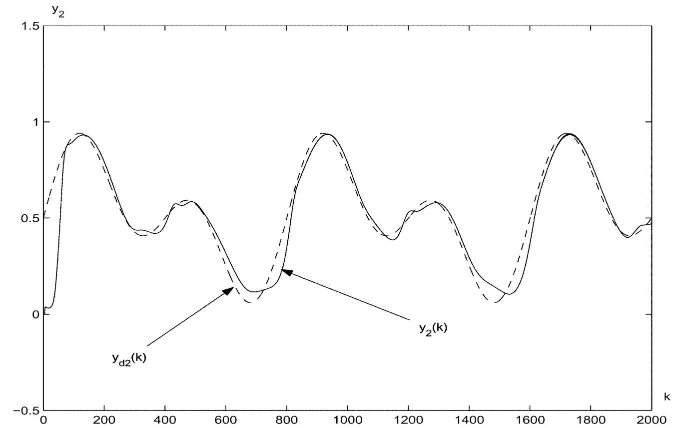

Fig. 3. Tracking performance with respect to $y_2(k)$ and $y_{d_2}(k)$.

The control objective is to drive the output $y(k) = [y_1(k), y_2(k)]^T$ of the system to follow desired reference signals

$$y_{d_1}(k) = 0.5 + \frac{1}{4}\cos\left(\frac{\pi T k}{4}\right) + \frac{1}{4}\sin\left(\frac{\pi T k}{2}\right)$$
$$y_{d_2}(k) = 0.5 + \frac{1}{4}\sin\left(\frac{\pi T k}{4}\right) + \frac{1}{4}\sin(\frac{\pi T k}{2})$$

with $T = 0.01$.

The initial condition for system states is $x_{1,1}(0) = 0$, $x_{1,2}(0) = 0$, $x_{2,1}(0) = 0$ and $x_{2,2}(0) = 0$. The neurons used are $l_{1,1} = 12$, $l_{1,2} = 20$, $l_{2,1} = 12$ and $l_{2,2} = 30$. All the elements of the neural network weights $\hat{W}_{1,1}(0)$, $\hat{W}_{1,2}(0)$, $\hat{W}_{2,1}(0)$, and $\hat{W}_{2,2}(0)$ are initialized to be random numbers between 0.00 and 0.01, and the active functions $S_{1,1}(0)$, $S_{1,2}(0)$, $S_{2,1}(0)$, and $S_{2,2}(0)$ are initialized to be random numbers between 0.00 and 0.02. The initial values of the virtual controls are $\alpha_{1,2}(0) = 0$ and $\alpha_{2,2}(0) = 0$. σ modification gains are $\sigma_{1,1} = \sigma_{1,2} = \sigma_{2,1} = \sigma_{2,2} = 0.01$, and adaptive gain matrices are $\Gamma_{1,1} = \Gamma_{1,2} = \Gamma_{2,1} = 0.025I$, and $\Gamma_{2,2} = 0.010I$.

For clarity, the formulas used in the simulation are listed here. The virtual controls and the practical controls are as follows $(i = 1, 2)$:

$$\Sigma_i : \begin{cases} \alpha_{i,2}(k) = \hat{W}_{i,1}(k) S_{i,1}(z_{i,1}(k)) \\ z_{i,1}(k) = [x_{i,1}(k), x_{i,2}(k), y_{d_i}(k+2)]^T \\ u_i(k) = \hat{W}_{i,2}(k) S_{i,2}(z_{i,2}(k)) \\ z_{i,2}(k) = [x_{1,1}(k), x_{1,2}(k), x_{2,1}(k), x_{2,2}(k), \alpha_{i,2}(k)]^T. \end{cases}$$

The error definitions are $(i = 1, 2)$

$$\Sigma_i : e_{i,1}(k) = y_i(k) - y_{d_i}(k)$$
$$e_{i,2}(k) = x_{i,2}(k) - \alpha_{i,2}(k-1).$$

The weights update law are shown at the bottom of the page $(i = 1, 2)$. Simulation results are shown in Figs. 2–6. Figs. 2 and 3 show the tracking performances of the first subsystem and the second subsystem, respectively. It can be seen that, in the initial period of simulation, the tracking errors are large. Then, as the time increases, the practical outputs converge to the neighborhoods of the desired signals. The control input trajectories $u_1(k) = \hat{W}_{1,2}(k) S_{1,2}(z_{1,2}(k))$ and $u_2(k) = \hat{W}_{2,2}(k) S_{2,2}(z_{2,2}(k))$ are shown in Fig. 4. Their corresponding neural network weights norms $\|\hat{W}_{1,2}(k)\|$ and $\|\hat{W}_{2,2}(k)\|$ are shown in Fig. 5. From Figs. 4 and 5, we can see

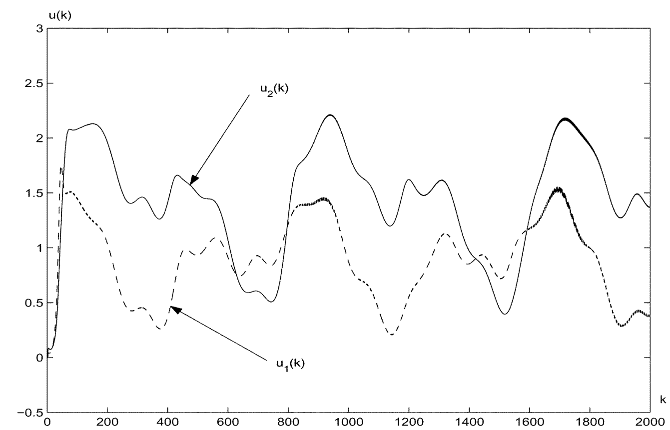

Fig. 4. Control inputs $u_1(k)$ and $u_2(k)$.

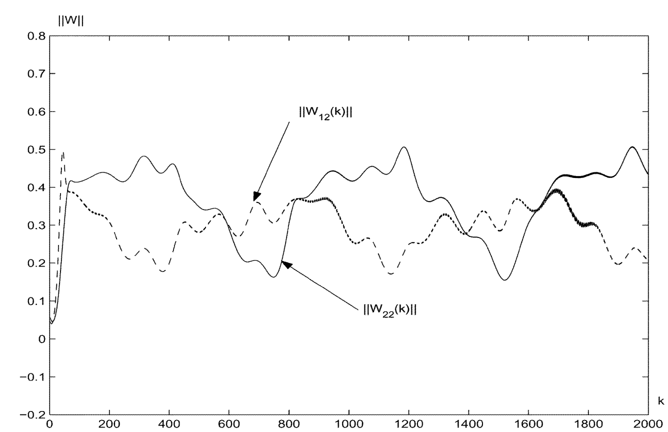

Fig. 5. Control input weights norms $\|\hat{W}_{12}(k)\|$ and $\|\hat{W}_{22}(k)\|$.

$$\Sigma_i : \begin{cases} \hat{W}_{i,1}(k) = \hat{W}_{i,1}(k-2) - \Gamma_{i,1}[S_{i,1}(z_{i,1}(k-2)) e_{i,1}(k) + \sigma_{i,1} W_{i,1}(k-2)] \\ \hat{W}_{i,2}(k) = \hat{W}_{i,2}(k-1) - \Gamma_{i,2}[S_{i,2}(z_{i,2}(k-1)) e_{i,2}(k) + \sigma_{i,2} W_{i,2}(k-1)]. \end{cases}$$

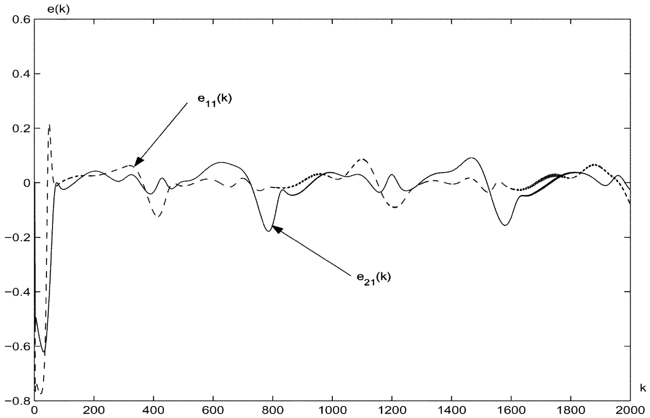

Fig. 6. Error dynamics.

that both the control inputs and their corresponding weights norms are all bounded. The dynamics of the tracking errors are shown in Fig. 6. It can be seen that the tracking errors are also bounded.

VII. Conclusion

In this paper, neural network control has been investigated for a class of discrete-time nonlinear MIMO system. In order to avoid the noncausal problem in backstepping design, the MIMO system under study was first transformed into sequential decrease cascade form, which completely solved the noncausal problem. Then, HONNs were used to approximate the desired controls. By using backstepping design in a nested manner, the closed-loop system was proved to be SGUUB based on Lyapunov analysis.

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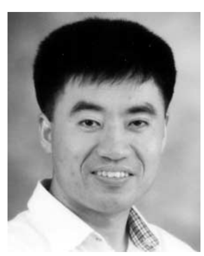

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