[21] J. S. Shamma and D. Xiong, "Linear nonquadratic optimal control," *IEEE Trans. Automat. Contr.*, vol. 42, no. 6, pp. 875–879, 1997.

[22] M. Sznaier, "Norm based robust control of state-constrained discrete-time linear systems," *IEEE Trans. Automat. Contr.*, vol. 37, no. 7, pp. 1057–1062, July 1992.

[23] ——, "A set induced norm approach to the robust control of constrained systems," *Int. J. Contr. Optimiz.*, vol. 31, no. 3, pp. 733–746, May 1993.

[24] M. Sznaier and Z. Benzaid, "Robust control of systems under mixed time/frequency domain constraints via convex optimization," in *Proc. 31st Conf. Decision and Control*, Dec. 1992, pp. 2617–2622.

[25] M. Sznaier and F. Blanchini, "Robust control of constrained systems via convex optimization," *Int. J. Robust Nonlinear Contr.*, vol. 5, pp. 441–460, Aug. 1995.

[26] M. Sznaier and A. Sideris, "Norm based optimally robust control of constrained discrete time linear systems," in *Proc. 1991 Amer. Control Conf.*, June 1991, pp. 2710–2715.

[27] A. R. Teel, "Semi-global stabilizability of linear null controllable systems with input nonlinearities," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 96–100, 1995.

[28] A. Zheng and M. Morari, "Global stabilization of linear discrete-time systems with bounded controls—A model predictive control approach," in *Proc. 1994 Amer. Control Conf.*, vol. 3, Baltimore, MD, June 1994, pp. 2847–2851.

Adaptive Control of First-Order Systems with Nonlinear Parameterization

T. Zhang, S. S. Ge, C. C. Hang, and T. Y. Chai

***Abstract*—In this paper, an adaptive control method is presented for a class of first-order systems with nonlinear parameterization. The main features of the scheme are that a novel integral-type Lyapunov function is developed for constructing an asymptotically stable adaptive controller, and output tracking error bounds are provided to evaluate the control performance of the adaptive system. The design procedure and the effectiveness of the proposed controller are illustrated through an example study.**

***Index Terms*—Adaptive control, Lyapunov stability, nonlinear parameterization, transient performance.**

I. Introduction

In the past several years, adaptive control of nonlinear systems has received much attention, and many significant results ([1]–[5] and the references therein) have been obtained. One of the important assumptions in these adaptive schemes is that the system nonlinearities are linear in the unknown parameters. For many practical systems, nonlinear parameterization is common, e.g., fermentation processes [8], bioreactor processes [9], and friction dynamics [11]. Research on adaptive control for dynamic systems with nonlinear parameterization has been an important and challenging area.

In early work [6], adaptive control algorithms have been studied based on a speed-gradient approach. General stability conditions for these algorithms were obtained for a class of dynamic systems, which may be nonlinearly parametrized plants. Nevertheless, for a given system, no systematic scheme was provided to find an adaptive law for guaranteeing these stability conditions. Recently, several design methods were presented for different kinds of nonlinearly parametrized systems [7]–[14]. A globally stable adaptive output feedback controller was developed using high-gain adaptation for systems containing nonlinear parameterizations [7], which, however, is only applicable to set-point regulation control. For a class of nonlinearly parametrized plants similar to those arising in fermentation processes [8]–[10] an interesting design approach was provided by appropriately parameterizing the plant and choosing a suitable Lyapunov function with a cubic term for constructing a stable adaptive controller. The shortcomings of the approaches in [8] and [10] lie in several restrictive conditions' being needed for developing the adaptive controller. For example, the controller gain is required larger than a constant, which depends on system states, nonlinearities, and unknown parameters, and therefore usually has to be conservatively chosen in practical applications. Other requirements of implementing the scheme include a good knowledge of the upper and lower bounds of all unknown parameters and the reference signal to be $L1$.

Based on a min-max strategy, a new control scheme was investigated for nonlinear systems with convex/concave parameterization through the introduction of a tuning function [13], [14]. The proposed parameter tuning algorithm is distinct from the traditionally used gradient approach [6], [12] and guarantees the global stability of the closed-loop systems. To implement such an adaptive controller, one needs to solve a min-max optimization problem for obtaining the tuning function (which is in general difficult if the number of unknown parameters or the complexity of the nonlinear function increase), and the on-line implementation of the adaptive controller is challenging because of the requirement for the calculation of a function at which transitions of convexity and concavity occur. *A priori* knowledge about the unknown parameters and convex/concave sets is necessary to perform these calculations. In addition, the sliding-mode-type term is used in the controller and may result in a nonzero output tracking error under a continuous control action.

Because of the complexities of the nonlinearly parameterized systems, investigation of adaptive control for general nonlinear plants is difficult, and usually results in a conservative design. In this paper, we shall restrict our attention to a class of first-order nonlinearly parametrized systems including the plants studied in [8] and [10]. By utilizing a good property of the studied systems, an integral-type Lyapunov function is introduced to construct a Lyapunov-based controller and parameter update laws. It is shown that globally asymptotic tracking can be achieved, and explicit transient bounds on the tracking error are also provided for different choices of Lyapunov functions.

This paper is organized as follows. Section II describes the studied systems and the control problem. In Section III, we introduce the novel Lyapunov function and derive a stable adaptive controller. The tracking performance of the systems and the method for choosing Lyapunov functions are discussed in Section IV. Finally, an example is given to illustrate the design procedure of the proposed controller in Section V.

II. System Description

Consider first-order nonlinear systems in the following form:

$$\dot{x} = \frac{f_n(x)}{f_d(x)} + \frac{g_n(x)}{g_d(x)}\, u \tag{1}$$

where $x \in R$ and $u \in R$ are state variable and system input, respectively, and $f_n(x)$, $g_n(x)$, $f_d(x) > 0$, and $g_d(x) > 0$ are continuous

Manuscript received January 22, 1998; revised June 30, 1999 and July 30, 1999. Recommended by Associate Editor, G. Bastin.

T. Zhang, S. S. Ge, and C. C. Hang are with the Department of Electrical Engineering, National University of Singapore, 119260 Singapore (e-mail: elegesz@nus.edu.sg).

T. Y. Chai is with the Department of Automatic Control, Northeastern University, Shenyang, Liaoning 110006, China.

Publisher Item Identifier S 0018-9286(00)06317-0.

functions containing unknown constant parameters. The control objective is to force x follow a desired trajectory x_d asymptotically.

Assumption 1: $g_n(x) \neq 0$ for all $x \in R$ and its sign is known.

Rewriting system (1), we have

$$\dot{x} = \frac{1}{f_d(x) g_d(x)} [f_n(x) g_d(x) + g_n(x) f_d(x) u]. \tag{2}$$

In this paper, we consider the systems satisfying the following parameterization conditions:

$$f_n(x) g_d(x) = \theta_a^T w_a(x), \qquad \theta_a \in R^{n_a}, \ w_a(x) \in R^{n_a} \tag{3}$$
$$g_n(x) f_d(x) = \theta_b^T w_b(x), \qquad \theta_b \in R^{n_b}, \ w_b(x) \in R^{n_b} \tag{4}$$
$$f_d(x) g_d(x) = \theta_c^T w_c(x), \qquad \theta_c \in R^{n_c}, \ w_c(x) \in R^{n_c} \tag{5}$$

where θ_j and $w_j(x)$ with $j = a, b, c$ are unknown constant parameter vectors and known regressor vectors, respectively, and n_j are positive integers.

Assumption 2: There exist a constant $b_0 > 0$ and a convex set Ω_θ such that $|\theta_b^T w_b(x)| \geq b_0, \forall \theta_b \in \Omega_\theta, x \in R$.

Remark 2.1: Since $f_d(x), g_d(x) > 0$, Assumption 1 implies that $g_n(x) f_d(x) \neq 0$, which is the controllability condition of the plant (2). Parameterization conditions (3)–(5) show that parameters θ_a and θ_b enter the parameterized plant (2) linearly, but θ_c appears nonlinearly. Some practical systems such as fermentation plants [8] and bioreactor processes [9] can be expressed in the parameterization form (2). As will be shown later, Assumption 2 is to be used for constructing a projection algorithm for avoiding controller singularity.

III. Lyapunov-Based Adaptive Controller Design

In Lyapunov-based control design, the most widely used Lyapunov function is in quadratic form, which plays an important role in adaptive controller design and stability analysis. In this section, a different kind of Lyapunov function shall be developed and utilized to construct a novel control structure and adaptive tuning law.

Define a scalar function

$$V_e(x, x_d) = \int_{x_d}^{x} (z - x_d) f_d(z) g_d(z) \alpha(z)\, dz \tag{6}$$

with $\alpha(x) > 0$ to be specified later. In order to analyze this function, we introduce the following lemma.

Lemma 3.1: If a function $F(x, y)$: $R \times R \to R$ is differentiable with respect to x, then for any two points $x_1, x_2 \in R$

$$F(x_1, y) = F(x_2, y) + \left[\int_0^1 \frac{dF(\phi_\lambda, y)}{d\phi_\lambda}\, d\lambda\right] (x_1 - x_2) \tag{7}$$

where $\phi_\lambda = \lambda x_1 + (1 - \lambda) x_2$.

Proof: From $\phi_\lambda = \lambda x_1 + (1 - \lambda) x_2$, we know that $(x_1 - x_2)\, d\lambda = d\phi_\lambda$. Therefore

$$\begin{aligned} \left[\int_0^1 \frac{dF(\phi_\lambda, y)}{d\phi_\lambda}\, d\lambda\right] (x_1 - x_2) &= \int_{x_2}^{x_1} dF(\phi_\lambda, y) \\ &= F(\phi_\lambda, y) \Big|_{x_2}^{x_1} \\ &= F(x_1, y) - F(x_2, y) \end{aligned}$$

which proves that (7) holds. Q.E.D.

Let $e = x - x_d$. Applying Lemma 3.1, (6) can be reexpressed as

$$V_e = e^2 \int_0^1 \lambda f_d(w_\lambda) g_d(w_\lambda) \alpha(w_\lambda)\, d\lambda \tag{8}$$

with $w_\lambda = \lambda x + (1 - \lambda) x_d$. Because $f_d(w_\lambda) g_d(w_\lambda) \alpha(w_\lambda) > 0$, $\forall\, w_\lambda \in R$, we know that V_e is a positive definite function with respect to e. In the following, we shall use V_e as a Lyapunov function candidate and derive a controller such that the time derivative $\dot{V}_e < 0$ for achieving asymptotic tracking control. For system (1) with nonlinearities satisfying (3)–(5), the error equation can be written as

$$\dot{e} = \frac{1}{\theta_c^T w_c(x)} \left[\theta_a^T w_a(x) + \theta_b^T w_b(x) u\right] - \dot{x}_d. \tag{9}$$

Letting $\sigma = z - x_d$ and noting (3)–(5), we may rewrite (6) as follows:

$$V_e = \int_0^e \sigma \theta_c^T w_c(\sigma + x_d) \alpha(\sigma + x_d)\, d\sigma. \tag{10}$$

Its time derivative is

$$\begin{aligned} \dot{V}_e = {} & e \left[\theta_c^T w_c(x)\right] \alpha(x) \dot{e} \\ & + \int_0^e \sigma \frac{d\left[\theta_c^T w_c(\sigma + x_d) \alpha(\sigma + x_d)\right]}{dx_d} \dot{x}_d\, d\sigma. \end{aligned} \tag{11}$$

Because $d[w_c(\sigma + x_d) \alpha(\sigma + x_d)]/dx_d = d[w_c(\sigma + x_d) \alpha(\sigma + x_d)]/d\sigma$, we have

$$\begin{aligned} \dot{V}_e &= e \left[\theta_c^T w_c(x)\right] \alpha(x) \dot{e} + \dot{x}_d \int_0^e \\ & \quad \cdot \sigma \frac{d\left[\theta_c^T w_c(\sigma + x_d) \alpha(\sigma + x_d)\right]}{d\sigma}\, d\sigma \\ &= e \left[\theta_c^T w_c(x)\right] \alpha(x) \dot{e} \\ & \quad + \dot{x}_d \left\{ \sigma \left[\theta_c^T w_c(\sigma + x_d) \alpha(\sigma + x_d)\right] \Big|_0^e \right. \\ & \qquad \left. - \int_0^e \left[\theta_c^T w_c(\sigma + x_d) \alpha(\sigma + x_d)\right] d\sigma \right\} \\ &= e \left[\theta_c^T w_c(x)\right] \alpha(x) (\dot{e} + \dot{x}_d) - \theta_c^T \left\{ \int_{x_d}^{x} w_c(z) \alpha(z)\, dz \right\} \dot{x}_d. \end{aligned}$$

Substituting (9) into the above equation, we obtain

$$\dot{V}_e = e \alpha(x) \left[\theta_a^T w_a(x) + \theta_b^T w_b(x) u - \theta_c^T \overline{w}_c(e, x_d) \frac{\dot{x}_d}{\alpha(x)}\right] \tag{12}$$

where

$$\overline{w}_c(e, x_d) = \frac{1}{e} \int_{x_d}^{x} w_c(z) \alpha(z)\, dz. \tag{13}$$

Utilizing Lemma 3.1, we can see that

$$\int_{x_d}^{x} w_c(z) \alpha(z)\, dz = e \int_0^1 w_c(\lambda e + x_d) \alpha(\lambda e + x_d)\, d\lambda.$$

Hence, $\overline{w}_c(e, x_d)$ is well defined for all $x, x_d \in R$. In view of (12), the following controller is chosen

$$u = \left[\hat{\theta}_b^T w_b(x)\right]^{-1} \left[-\frac{ke}{\alpha(x)} - \hat{\theta}_a^T w_a(x) + \hat{\theta}_c^T \overline{w}_c(e, x_d) \frac{\dot{x}_d}{\alpha(x)}\right] \tag{14}$$

where $k > 0$ is a design parameter, and $\hat{\theta}_a$, $\hat{\theta}_b$, and $\hat{\theta}_c$ are the estimates of θ_a, θ_b, and θ_c, respectively. Substituting (14) into (12), we have

$$\begin{aligned} \dot{V}_e = {} & -ke^2 + e\alpha(x) \\ & \cdot \left\{ \tilde{\theta}_a^T w_a(x) - \tilde{\theta}_b^T w_b(x) u + \tilde{\theta}_c^T \overline{w}_c(e, x_d) \frac{\dot{x}_d}{\alpha(x)} \right\} \end{aligned} \tag{15}$$

where $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$, $i = a, b, c$. The next task is to find adaptive laws for achieving global tracking control. Choose an augmented Lyapunov function candidate

$$V = V_e + \tfrac{1}{2} \left(\tilde{\theta}_a^T \Gamma_a^{-1} \tilde{\theta}_a + \tilde{\theta}_b^T \Gamma_b^{-1} \tilde{\theta}_b + \tilde{\theta}_c^T \Gamma_c^{-1} \tilde{\theta}_c\right) \tag{16}$$

where $\Gamma_a = \Gamma_a^T > 0$, $\Gamma_b = \Gamma_b^T > 0$, and $\Gamma_c = \Gamma_c^T > 0$ are adaptation gain matrices. Our goal is to make $\dot{V}$ nonpositive. The time derivative of V is

$$\begin{aligned}\dot{V} &= \dot{V}_e + \tilde{\theta}_a^T \Gamma_a^{-1} \dot{\hat{\theta}}_a + \tilde{\theta}_b^T \Gamma_b^{-1} \dot{\hat{\theta}}_b + \tilde{\theta}_c^T \Gamma_c^{-1} \dot{\hat{\theta}}_c \\ &= -ke^2 + \tilde{\theta}_a^T \left[-w_a(x)\alpha(x)e + \Gamma_a^{-1}\dot{\hat{\theta}}_a\right] \\ &\quad + \tilde{\theta}_b^T \left[-w_b(x)\alpha(x)ue + \Gamma_b^{-1}\dot{\hat{\theta}}_b\right] \\ &\quad + \tilde{\theta}_c^T \left[\overline{w}_c(e, x_d)\dot{x}_d e + \Gamma_c^{-1}\dot{\hat{\theta}}_c\right]. \end{aligned} \tag{17}$$

To eliminate $\tilde{\theta}_a$, $\tilde{\theta}_b$, and $\tilde{\theta}_c$ from $\dot{V}$, we choose the adaptive laws

$$\dot{\hat{\theta}}_a = \Gamma_a w_a(x)\alpha(x)e \tag{18}$$

$$\dot{\hat{\theta}}_b = \text{Proj}\left\{\hat{\theta}_b,\ w_b(x)\alpha(x)ue\right\} \tag{19}$$

$$\dot{\hat{\theta}}_c = -\Gamma_c \overline{w}_c(e, x_d)\dot{x}_d e \tag{20}$$

where Proj$\{\cdot, \cdot\}$ denotes a suitable projection algorithm, which ensures that

$$\tilde{\theta}_b^T \left[-w_b(x)\alpha(x)ue + \Gamma_b^{-1}\dot{\hat{\theta}}_b\right] \le 0$$

and $\hat{\theta}_b \in \Omega_\theta$ with the convex Ω_θ defined in Assumption 2. Several standard techniques existing in the literature [15]–[19] can be used to construct this projection algorithm. Thus, during the adaptation, $|\hat{\theta}_b^T w_b(x)| \ge b_0$ and $\hat{\theta}_b \in \Omega_\theta$ are guaranteed and no singularity problem occurs to controller (14). As the main contribution of this paper is not on designing projection algorithm, we suppose that projection algorithm (19) has been carried out using the method given in [15]–[19]. Substituting (18)–(20) into (17), we obtain

$$\dot{V} \le -ke^2, \qquad t \ge 0. \tag{21}$$

Integrating (21) shows that $\int_0^t ke^2(\tau)\,d\tau \le V(0) - V(t) \le V(0) < \infty$, $\forall t \ge 0$. Therefore, $e \in L_2$ and $V(t) \le V(0)$. From (8) and (16), we know that $V_e(t), \hat{\theta}_j(t) \in L_\infty$, $j = a, b, c$. In view of $\int_0^1 \lambda f_d(w_\lambda) g_d(w_\lambda)\alpha(w_\lambda)\,d\lambda > 0$, we have $e \in L_\infty$. Since $\hat{\theta}_b^T w_b(x)$ is bounded away from zero ($\hat{\theta}_b \in \Omega_\theta$ guaranteed by the projection algorithm), the control input u is also bounded. It follows from (9) that $\dot{e}$ is bounded. Because $e \in L_2 \bigcap L_\infty$ and $\dot{e} \in L_\infty$, by Barbalat's lemma [20] we conclude that $\lim_{t\to\infty} e(t) = 0$ asymptotically. We summarize the above results into the following theorem.

Theorem 3.1: All the signals in the closed-loop adaptive system consisting of plant (1) satisfying Assumptions 1 and 2, controller (14), and adaptive laws (18)–(20) are globally uniformly bounded, and the asymptotic tracking is achieved, i.e., $\lim_{t\to\infty} x(t) = x_d(t)$.

Remark 3.1: It has been shown that the choice of integral type Lyapunov function (8) plays an important role in the controller design. It is worth noting that, for a given system, a different function $\alpha(x)$ can be found to construct a different Lyapunov function V_e. Hence, the resulting controller is not unique, and the control performance is also affected by different choice. In Section IV, we will investigate the design of Lyapunov functions and their effects on the transient performance.

Remark 3.2: It is very interesting to note that when $\dot{x}_d = 0$, i.e., in adaptive regulation, controller (14) becomes

$$u = \left[\hat{\theta}_b^T w_b(x)\right]^{-1} \left[-\frac{ke}{\alpha(x)} - \hat{\theta}_a^T w_a(x)\right]$$

which is independent of the estimated parameter $\hat{\theta}_c$. This means that there is no need to estimate θ_c for adaptive regulation control.

IV. Control Performance

Theorem 3.1 only presents the boundedness of the signals in the closed-loop system and asymptotic tracking of the output; it does not show the transient response. In the following, we show that for a suitably chosen $\alpha(x)$, both L_2 and L_∞ performance for the tracking error can be obtained explicitly.

Theorem 4.1: For the closed-loop control system (2), (14), and (18)–(20):

i) the L_2 transient bound is given by

$$\begin{aligned}\int_0^\infty e^2(\tau)\,d\tau \le \frac{1}{k}\int_{x_d(0)}^{x(0)} [z - x_d(0)]\theta_c^T w_c(z)\alpha(z)\,dz \\ + \frac{1}{2k}\sum_{i=a}^{a,b,c}\left[\tilde{\theta}_i^T(0)\Gamma_i^{-1}\tilde{\theta}_i(0)\right]\end{aligned} \tag{22}$$

ii) if the $\alpha(x)$ is chosen such that $\theta_c^T w_c(x)\alpha(x) \ge c_0$ with constant $c_0 > 0$, then L_∞ tracking error bound

$$\begin{aligned}|e(t)| \le \left\{\frac{2}{c_0}\int_{x_d(0)}^{x(0)} [z - x_d(0)]\theta_c^T w_c(z)\alpha(z)\,dz\right\}^{(1/2)} \\ + \frac{1}{\sqrt{c_0}}\left\{\sum_{i=a}^{a,b,c} \tilde{\theta}_i^T(0)\Gamma_i^{-1}\tilde{\theta}_i(0)\right\}^{(1/2)}.\end{aligned} \tag{23}$$

Proof:

i) Integrating (21) over $[0, \infty)$, we have

$$\int_0^\infty ke^2(\tau)\,d\tau \le -\int_0^\infty \dot{V}\,d\tau = V(0) - V(\infty)$$

which leads to

$$\int_0^\infty e^2(\tau)\,d\tau \le \frac{1}{k}V(0) \le \frac{1}{k}V_e(0) + \frac{1}{2k}\sum_{i=a}^{a,b,c}\left[\tilde{\theta}_i^T(0)\Gamma_i^{-1}\tilde{\theta}_i(0)\right]. \tag{24}$$

It follows from (6) and (24) that (22) holds.

ii) For $\theta_c^T w_c(x)\alpha(x) \ge c_0$, it follows from (10) that

$$\begin{aligned}V_e &= \int_0^e \sigma\theta_c^T w_c(\sigma + x_d)\alpha(\sigma + x_d)\,d\sigma \\ &\ge c_0\int_0^e \sigma\,d\sigma = \frac{c_0}{2}e^2.\end{aligned} \tag{25}$$

Integrating (21) over $[0, t]$ leads to $\int_0^t \dot{V}(\tau)\,d\tau = V(t) - V(0) \le -\int_0^\infty ke^2(\tau)\,d\tau \le 0$. From (16) and (25), we obtain $c_0 e^2(t)/2 \le V_e(t) \le V(t) \le V(0)$, and therefore

$$\begin{aligned}e^2(t) \le \frac{2}{c_0}\int_{x_d(0)}^{x(0)} [z - x_d(0)]\theta_c^T w_c(z)\alpha(z)\,dz \\ + \frac{1}{c_0}\sum_{i=a}^{a,b,c}\left[\tilde{\theta}_i^T(0)\Gamma_i^{-1}\tilde{\theta}_i(0)\right].\end{aligned}$$

The above inequality implies that (23) holds. Q.E.D.

Remark 4.1: Two performance criteria (22) and (23) reveal that large initial errors $e(0)$ and $\tilde{\theta}_i(0)$, $i = a, b, c$ may lead to a large tracking error, and both L_2 and L_∞ performance can be improved by increasing the control parameter k and the adaptive gains Γ_i.

Remark 4.2: It is shown from (22) and (23) that the smaller the magnitude of $\alpha(x)$ is, the better the control performance that can be achieved. In particular, if an $\alpha(x)$ can be taken such that

$c_1 \geq \theta_c^T w_c(x)\alpha(x) \geq c_0$ with constant c_1, then L_2 performance (22) can be further written as

$$\int_0^\infty e^2(\tau)\, d\tau \leq \frac{c_1}{2k} e^2(0) + \frac{1}{2k} \sum_{i=a}^{a,\, b,\, c} \left[\tilde{\theta}_i^T(0)\Gamma_i^{-1}\tilde{\theta}_i(0)\right]$$

and L_∞ tracking error bound can be expressed as

$$|e(t)| \leq \sqrt{\frac{c_1}{c_0}}\, |e(0)| + \frac{1}{\sqrt{c_0}} \left\{ \sum_{i=a}^{a,\, b,\, c} \tilde{\theta}_i^T(0)\Gamma_i^{-1}\tilde{\theta}_i(0) \right\}^{1/2}. \quad (26)$$

Remark 4.3: As we have seen from the above discussions, the choice of $\alpha(x)$ affects the transient response of the tracking error significantly. Since $\theta_c^T w_c(x) > 0$ with θ_c the constant vector and $w_c(x)$ the known function vector, designing an $\alpha(x)$ such that $\theta_c^T w_c(x)\alpha(x) \leq c_1$ is not difficult. For instance, one such choice is

$$\alpha(x) = \frac{1}{\sqrt{n_c \sum_{i=1}^{n_c} w_{ci}^2(x)}} \quad (27)$$

where $w_{ci}(x)$ are the elements of vector $w_c(x)$, which results in $\theta_c^T w_c(x)\alpha(x) \leq \max[|\theta_{c1}|, |\theta_{c2}|, \cdots, |\theta_{cn_c}|]$. To find an $\alpha(x)$ to guarantee $\theta_c^T w_c(x)\alpha(x) \geq c_0$ is not obvious because the detailed structure of $w_c(x)$ is not given explicitly. However, if exact knowledge of $w_c(x)$ is available for a given system, choosing such an $\alpha(x)$ is in general not a difficult task. For example, if $\theta_{ci} w_{ci}(x) \geq 0$, $i = 1, 2, \cdots, n_c$, the $\alpha(x)$ given in (27) guarantees $\theta_c^T w_c(x)\alpha(x) \geq \min[|\theta_{c1}|, |\theta_{c2}|, \cdots, |\theta_{cn_c}|]/\sqrt{n_c}$.

Remark 4.4: The transient performance analysis provided in Theorem 4.1 makes it possible to relax the assumptions of the proposed approach. For instance, if Assumptions 1 and 2 hold only for $x \in \Omega$ with Ω a compact set, the developed scheme is still applicable if the controller parameters are chosen appropriately. The reason is that the upper bounds of the state x [derived from (23) or (26)] can be adjusted through tuning the design parameters k and Γ_i such that $x \in \Omega$ is guaranteed for all time if $x(0)$ is inside a subset $\Omega_0 \subset \Omega$ (not including the boundary of Ω).

Remark 4.5: Although the proposed scheme is developed for first-order systems, it can be extended to a class of nth-order systems with states accessible, e.g., nonlinear systems in a Brunovsky control form [16], [17] or the plants in [14] with nonlinearities taken in the form of this paper. It can be shown that by suitably choosing the augmented tracking error, the dynamic behavior of the nth-order system will be determined by a first-order differential equation, which is similar to the error equation (9), and controller design, stability, and performance analysis can be performed using the similar approach presented in this work.

V. Example Study

In this section, we shall apply the proposed scheme to the adaptive control problem for a class of fermentation processes [8] described by

$$\dot{x} = \frac{\rho k_1 x^2}{k_2 + k_3 x} + g(x)u \quad (28)$$

where $x \in [0, +\infty)$ represents a physical quantity, ρ can be either $+1$ or -1, constant parameters $k_1, k_2, k_3 > 0$, and $|g(x)| > 0, \forall x \in [0, +\infty)$. The control objective is to make the state x track a desired signal $x_d \in [0, +\infty)$. The error equation of the plant can be written in the form of (9) as

$$\dot{e} = \frac{1}{1 + \theta_1 x}\left[\theta_2 x^2 + (1 + \theta_1 x)g(x)u\right] - \dot{x}_d \quad (29)$$

with $\theta_1 = k_3/k_2$ and $\theta_2 = \rho k_1/k_2$. Now consider $\alpha(x) = 1$ and

$$V_e = \int_0^e \sigma[1 + \theta_1(\sigma + x_d)]\, d\sigma.$$

The time derivative of V_e along (29) is

$$\begin{aligned}\dot{V}_e &= e(1 + \theta_1 x)\dot{e} + \theta_1 \int_0^e \sigma \dot{x}_d\, d\sigma \\ &= e\left\{\theta_2 x^2 + (1 + \theta_1 x)[g(x)u - \dot{x}_d]\right\} + \theta_1 \dot{x}_d \frac{e^2}{2} \\ &= e\left\{\theta_2 x^2 + \theta_1 \dot{x}_d \frac{e}{2} + (1 + \theta_1 x)[g(x)u - \dot{x}_d]\right\}. \end{aligned} \quad (30)$$

Therefore, if the controller is designed as

$$u = \frac{1}{g(x)}\left[\dot{x}_d - \frac{2\hat{\theta}_2 x^2 + \left(\hat{\theta}_1 \dot{x}_d + k\right)e}{2\left(1 + \hat{\theta}_1 x\right)}\right] \quad (31)$$

(30) becomes

$$\dot{V}_e = e\left\{-\frac{ke}{2} - \tilde{\theta}_2 x^2 - \tilde{\theta}_1\left[\dot{x}_d \frac{e}{2} + xg(x)u - x\dot{x}_d\right]\right\} \quad (32)$$

with $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$, $i = 1, 2$. Considering a positive definite function $V = V_e + (1/2)(\tilde{\theta}_1^2 + \tilde{\theta}_2^2)$, its time derivative is

$$\begin{aligned}\dot{V} = &-\frac{ke^2}{2} + \tilde{\theta}_2\left(-x^2 e + \dot{\hat{\theta}}_2\right) \\ &+ \tilde{\theta}_1\left\{-\left[xg(x)u - \frac{\dot{x}_d}{2}(x + x_d)\right]e + \dot{\hat{\theta}}_1\right\}. \end{aligned} \quad (33)$$

Accordingly, the adaptive laws are chosen as

$$\dot{\hat{\theta}}_1 = \begin{cases} \left[xg(x)u - \dfrac{\dot{x}_d}{2}(x + x_d)\right]e, & \\ \qquad \text{if } \hat{\theta}_1 > 0 \text{ or if } \hat{\theta}_1 = 0 \text{ and} & \\ \qquad \left[xg(x)u - \dfrac{\dot{x}_d}{2}(x + x_d)\right]e \geq 0 & \\ 0, \qquad \text{otherwise} & \end{cases} \quad (34)$$

$$\dot{\hat{\theta}}_2 = x^2 e. \quad (35)$$

From the projection algorithm (34) and the fact that $\theta_1 > 0$, we have $\tilde{\theta}_1\{-[xg(x)u - (\dot{x}_d/2)(x + x_d)]e + \dot{\hat{\theta}}_1\} \leq 0$ and $\hat{\theta}_1 \geq 0$. Substituting (34) and (35) into (33), we obtain $\dot{V} \leq -ke^2/2 \leq 0$. According to Theorem 3.1, the asymptotic tracking control of plant (28) can be achieved and all the signals in the system are bounded. Since $x \in [0, +\infty)$ and $\hat{\theta}_1 \geq 0$, we have $1 + \hat{\theta}_1 x \geq 1$, which ensures a well-defined controller (31), and thus all the signals in the system are bounded. Compared with the method in [8], an important advantage of adaptive controller (31), (34), and (35) is that *a priori* knowledge for the bounds of k_1, k_2, and k_3 is not required.

VI. Conclusion

A direct adaptive controller has been presented for a class of first-order systems with nonlinear parameterization. Both control structure and parameter tuning algorithms are developed through the newly chosen Lyapunov function. Global stability and asymptotic convergence are obtained, and transient performance is explicitly provided for evaluating the proposed adaptive control system.

REFERENCES

[1] S. S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 1123–1131, Nov. 1989.
[2] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Systematic design of adaptive controller for feedback linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 1241–1253, Nov. 1991.
[3] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.
[4] R. Marino and P. Tomei, *Nonlinear Adaptive Design: Geometric, Adaptive, and Robust*. London, U.K.: Prentice-Hall, 1995.
[5] T. A. Johansen and P. A. Ioannou, "Robust adaptive control of minimum phase non-linear systems," *Int. J. Adapt. Contr. Signal Process*, vol. 10, pp. 61–78, 1996.
[6] A. L. Fradkov, "Speed-gradient scheme and its application in adaptive control," *Automat. Remote Contr.*, vol. 40, no. 9, pp. 1333–1342, 1979.
[7] R. Marino and P. Tomei, "Global adaptive output-feedback control of nonlinear systems—Part II: Nonlinear parameterization," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 17–48, Jan. 1993.
[8] J. D. Boskovic, "Stable adaptive control of a class of first-order nonlinearly parametrized plants," *IEEE Trans. Automat. Contr.*, vol. 40, no. 2, pp. 347–350, 1995.
[9] ——, "Stable adaptive control of a class of Nonlinearly-parametrized bioreactor processes," in *Proc. Amer. Contr. Conf.*, Washington, DC, June 1995, pp. 1795–1799.
[10] ——, "Adaptive control of a class of nonlinearly parametrized plants," *IEEE Trans. Automat. Contr.*, vol. 44, no. 7, pp. 930–934, 1998.
[11] B. Armstrong-Helouvry, P. Dupont, and C. Canudas de Wit, "A survey of models, analysis tools and compensation methods for control of machines with friction," *Automatica*, vol. 30, no. 7, pp. 1083–1138, 1994.
[12] R. Ortega, "Some remarks on adaptive neuro-fuzzy systems," *Int. J. Adapt. Contr. Signal Process.*, vol. 10, pp. 79–83, 1996.
[13] A. P. Loh, A. M. Annaswamy, and F. P. Skanze, "Adaptive control of Dynamic systems with nonlinear parametrization," in *Proc. 4th Eur. Contr. Conf.*, Brussels, Belgium, 1997.
[14] A. M. Annaswamy, F. P. Skantze, and A. P. Loh, "Adaptive control of continuous time systems with convex/concave parametrization," *Automatica*, vol. 34, no. 1, pp. 33–49, 1998.
[15] G. C. Goodwin and D. Q. Mayne, "A parameter estimation perspective of continuous time model reference adaptive control," *Automatica*, vol. 23, no. 1, pp. 57–70, 1987.
[16] A. Yesidirek and F. L. Lewis, "Feedback linearization using neural networks," *Automatica*, vol. 31, no. 11, pp. 1659–1664, 1995.
[17] T. Zhang, S. S. Ge, and C. C. Hang, "Stable adaptive control for a class of nonlinear systems using a modified Lyapunov function," *IEEE Trans. Automat. Contr.*, vol. 45, no. 1, pp. 129–132, 2000.
[18] H. K. Khalil, "Adaptive output feedback control of nonlinear system represented by input–output models," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 177–188, 1996.
[19] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Englewood Cliffs, NJ: Prentice-Hall, 1996.
[20] M. V. Popov, *Hyperstability of Control Systems*. New York: Springer-Verlag, 1973.

A Parameter-Dependent Lyapunov Function for a Polytope of Matrices

T. Mori and H. Kokame

***Abstract*—A new sufficient condition for a polytope of matrices to be Hurwitz-stable is presented. The stability is a consequence of the existence of a parameter-dependent quadratic Lyapunov function, which is assured by a certain linear constraint for generating extreme matrices of the polytope. The condition can be regarded as a duality of the known extreme point result on quadratic stability of matrix polytopes, where a fixed quadratic Lyapunov function plays the role. The obtained results are applied to a polytope of second-degree polynomials for illustration.**

***Index Terms*—Hurwitz-stability, parameter-dependent Lyapunov function, polytope of matrices, quadratic stability.**

I. INTRODUCTION

Polytopes of matrices are now established as one of standard representations of uncertainties involved in state-space models of control systems [1], [2]. When the system matrices of uncertain systems are formulated by a polytope of matrices, a stability problem of the polytope naturally arises. It is known that one generally cannot expect the extreme point result on stability, Hurwitz or Schur alike, of polytopes of matrices. That is, stability of the generating extreme matrices does not necessarily imply that of every matrix in the polytope. This means that in order to assure stability of a polytope we have to impose additional constraints to the stability condition or stricter conditions than that for each extreme matrix. Considerable numbers of such conditions are currently in hand (see, e.g., the references in [3]), but each of them has its own demerit. For example, diagonal dominance-type conditions for Hurwitz stability require negativity of diagonal entries of the extreme matrices, an apparent restriction to their applicability.

This paper presents a new sufficient condition for Hurwitz-stability of a polytope of matrices, thus providing an alternative to the existing tools. The stability comes from a parameter-dependent quadratic Lyapunov function, the existence of which is ensured by a linear constraint for the extreme matrices. The obtained condition can be considered as a kind of dual of the quadratic stability result on a polytope of matrices, where, by contrast, a fixed quadratic Lyapunov function plays the role. As an illustrative example, we look into the Lyapunov function problems of a polytope of polynomials, which are connected to the matrix counterpart with a companion form. The contents of the paper are laid out as follows. In the next section, the main result is stated along with the quadratic stability result, which is known, but can also be proved in the context of the present approach. Section III includes an application of the results to a Lyapunov function problem for a polytope of second-degree polynomials. Discussions are given on which type of Lyapunov functions can cover the polytopes of the polynomials in the coefficient plane. Some comments are also made on the differences between the present results and existing analysis methods that utilize parameter-dependent Lyapunov functions. Finally, inSection IV, several remarks are given to conclude the paper.

Manuscript received July 20, 1998; revised January 28, 1999 and September 8, 1999. Recommended by Associate Editor, R. Tempo.

T. Mori is with the Department of Electronics & Information Science, Kyoto Institute of Technology, Matsugasaki, Sakyo, Kyoto 606-8585 Japan (e-mail: mori@dj.kit.ac.jp).

H. Kokame is with the Department of Electrical & Electronic Systems, Osaka Prefecture University, Gakuencho, Sakai, 599-8531 Japan (e-mail: kokame@ecs.ees.osakafu-u.ac.jp).

Publisher Item Identifier S 0018-9286(00)06074-8.