solvable by means of a simple hybrid control law, i.e., it is possible to achieve global exponential stability of the zero equilibrium in the presence of (small) perturbations vanishing at the origin. The control law retains the basic properties of the discontinuous control laws proposed in [1], namely exponential convergence rate and lack of oscillatory behavior. The results presented in this note are based on the general theory developed in [15]. In this respect, the main contribution of this work is to show that, for a large class of nonholonomic systems, a robustly stabilizing control law can be explicitly designed, and it is possible to obtain explicit bounds on the admissible perturbations.

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Nonregular Feedback Linearization: A Nonsmooth Approach

Zhendong Sun and S. S. Ge

***Abstract*—In this note, we address the problem of exact linearization via nonsmooth nonregular feedback. A criterion of nonregular static state feedback linearizability is presented for a class of nonlinear affine systems with two control inputs, and its application to nonholonomic systems is briefly discussed.**

***Index Terms*—Nonlinear systems, nonregular feedback linearization, nonsmooth analysis.**

I. INTRODUCTION

Feedback linearization is a standard technique for control of many nonlinear systems. Since the pioneering work of Krener [20], which addressed linearization of nonlinear systems via state diffeomorphisms, the problem of linearization has been studied using increasingly more general transformations. The problem of regular static state feedback linearization was elegantly solved in [2] and [18]. The problem of regular dynamic state feedback linearization was first initiated in [7] and then extensively addressed in many references; see, for example, [6], [15], and the references therein. Dynamic feedback linearizability is closely related to the differential flatness of nonlinear systems [12], [13]. The problem of nonregular state feedback linearization was studied in [14] and [27].

Nonregular state feedback linearization is a rigorous design mechanism. In comparison with regular dynamic feedback linearization, this approach does not introduce any additional dynamics, while it is applicable to a broad class of practical engineering systems, such as robots with flexible joints [14]. By combining nonregular feedback linearization with backstepping design, the nonregular backstepping design approach provides a Lyapunov-function-based recursive design mechanism for a class of nonlinear systems [28]. This approach can avoid undesired cancellation of the beneficial nonlinearities and enhance robustness and softness through appropriate backstepping design of Lyapunov functions.

On the other hand, many practical systems do not admit any smooth static or dynamic state stabilizer due to the violation of the well-known necessary condition [3]. To cope with this difficulty, many innovative nonsmooth control approaches have been proposed in recent years. Among these, the problem of state equivalence for the singular case, i.e., the nested sequence of involutive distributions of the systems containing singular distributions was extensively investigated [4], [5]; a non-Lipschitz continuous feedback approach combining the theory of homogeneous systems and the idea of adding a power integrator was developed for global stabilization of several classes of nonlinear systems with uncontrollable unstable linearization [22], [25], [26]; and a generalized p-normal form was proposed which includes several

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known normal forms as special cases [8], [9]. The reader is referred to [10] for a recent development of nonsmooth analysis.

In this note, we propose a nonsmooth formulation for the problem of nonregular state feedback linearization. The main idea is to extend the nonregular feedback linearization scheme to include discontinuous state and input transformations, i.e., nonlinear systems are transformed into the Brunovsky form by transformations which may be singular or discontinuous on a lower dimensional submanifold of the state space. One advantage of this approach lies in the fact that it combines the idea of nonregular feedback linearization with nonsmooth analysis, thus provides additional flexibility in choosing linearizing transformations. Indeed, through the nonregular feedback control, we introduce the flexibility of reducing the number of external inputs; and by introducing nonsmooth transformations, it is possible to cope with nonlinear systems which do not admit any smooth stabilizer.

II. Problem Formulation

Let $\Re^n$ denote the nth-dimensional real field, and $\Omega \subset \Re^n$ denote a connected open set. For a map T defined on Ω, let $T(\Omega) \triangleq \{Tx : x \in \Omega\}$.

Consider an affine nonlinear system given by

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) = f(x) + g(x)u \tag{1}$$

where $x \in \Omega$ is the state, $u \in \Re^m$ is the input, the entries of $f(x)$ and $g(x)$ are analytic functions of x, and rank $g(x) = m, \forall x \in \Omega$. Without loss of generality, we assume that $0 \in \Omega$ and $f(0) = 0$.

Definition 1: Nonlinear control system (1) is said (nonsmooth) nonregular (static state) feedback linearizable, if there exist a state transformation

$$z = T(x) \tag{2}$$

and a nonregular state feedback

$$u(t) = \alpha(x) + \beta(x)v(t), \qquad v \in \Re^{m_0},\ m_0 \le m \tag{3}$$

where entries of $T(x)$, $\alpha(x)$ and $\beta(x)$ are analytic on an open and dense subset Ω_0 of Ω, and map $T : \Omega_0 \to T(\Omega_0)$ is a diffeomorphism, such that the transformed system with state z and input v is a controllable linear system.

Remark 1: Note that the state and input transformations are allowed to be nonsmooth in the sense that they are not necessarily well defined in the whole neighborhood of the origin. Instead, they may be singular or discontinuous on a "smaller" set which includes the origin. An important special case is the meromorphic transformations whose entries are meromorphic functions. In this case, the transformations are analytic everywhere except for on a lower dimensional submanifold of Ω.

Remark 2: In the nonregular static feedback (3), the gain matrix $\beta(x)$ is not necessarily square, and if it is square, it is not necessarily nonsingular at the origin. The use of nonregular state feedback is to extend the class of linearizable systems. However, the smooth scheme is only applicable to nonlinear systems which are stabilizable by smooth state feedback. To overcome the topological obstruction caused by smooth feedback, either continuous but non-Lipschitz transformations or discontinuous state feedback should be introduced. In the literature, continuous stabilizers were provided mainly based on the homogeneous system theory as in [25] and [26] for nonlinear systems with uncontrollable unstable linearization. On the other hand, numerous discontinuous stabilizers were proposed in the context of nonholonomic systems [1], [21]. Definition 1 permits discontinuous or singular state and input transformations and, hence, falls into the discontinuous feedback scheme.

III. Main Result

A. Feedback Linearizability

In this subsection, we present the main technical contribution which establishes the nonregular feedback linearizability for a class of nonlinear system with two inputs.

Theorem 1: For a two-input affine nonlinear system

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 \tag{4}$$

suppose there exist analytic vector fields $p(x)$ and $q(x)$ with $\mathrm{span}\{p(x), q(x)\} = \mathrm{span}\{g_1, g_2\}$, and a sequence of integers $0 \le \kappa_0 < \kappa_1 < \cdots < \kappa_l \le n-1$ with $l \ge 2$, such that the nested distributions defined by

$$\begin{aligned} G_0 &= \mathrm{span}\{q\} \\ G_i &= G_{i-1} + ad_f G_{i-1}, \qquad i \ge 1,\ i \ne \kappa_1, \ldots, \kappa_l \\ G_{\kappa_j} &= G_{\kappa_j - 1} + ad_{ad_f^{\kappa_0} p} G_{\kappa_j - 1}, \qquad j = 1, \ldots, l-1 \\ G_{\kappa_l} &= G_{\kappa_l - 1} + \mathrm{span}\{p\} \end{aligned} \tag{5}$$

satisfy the following conditions:

i) $\mathrm{rank} G_{n-1}(x) = n, \forall x \in \Omega$;
ii) $G_{\kappa_l - 2}$ and G_{n-2} are involutive;
iii) $ad_f G_{\kappa_j - 1} \subseteq G_{\kappa_j - 1}$ for $j = 1, \ldots, l$;
iv) $ad_{ad_j^f p} G_{i-1} \subseteq G_i$ for $j = 0, \ldots, \kappa_0 - 1$ and $i = 0, \ldots, n-2$;

then (4) is nonsmooth nonregular feedback linearizable.

Proof: It follows from the construction of the distributions that

$$\begin{aligned} G_k &= \mathrm{span}\left\{q, \ldots, ad_f^k q\right\}, \qquad k = 0, \ldots, \kappa_1 - 1 \\ G_{\kappa_j + k} &= G_{\kappa_j - 1} + \mathrm{span}\left\{\chi_j, \ldots, ad_f^k \chi_j\right\} \\ &\qquad j = 1, \ldots, l-1, \qquad k = 0, \ldots, \kappa_{j+1} - \kappa_j - 1 \\ G_{\kappa_l + k} &= G_{\kappa_l - 1} + \mathrm{span}\left\{p, \ldots, ad_f^k p\right\}, \qquad k = 0, 1, \ldots \end{aligned} \tag{6}$$

where χ_j, $j = 1, \ldots, l-1$, are given recursively by

$$\begin{aligned} \chi_1 &= ad_{ad_f^{\kappa_0} p}\left(ad_f^{\kappa_1 - 1} q\right) \\ \chi_j &= ad_{ad_f^{\kappa_0} p}\left(ad_f^{\kappa_j - \kappa_{j-1} - 1} \chi_{j-1}\right), \qquad j = 2, \ldots, l-1. \end{aligned}$$

Accordingly, $\dim G_i \le \dim G_{i-1} + 1$ for $i \ge 1$. It follows from condition i) and $\dim G_0 = 1$ that

$$\dim G_i(x) = i + 1 \qquad \forall x \in \Omega, \qquad i = 0, 1, \ldots, n-1. \tag{7}$$

Because $\mathrm{span}\{p(x), q(x)\} = \mathrm{span}\{g_1(x), g_2(x)\}$, one can express vector fields $p(x)$ and $q(x)$ in terms of $g_1(x)$ and $g_2(x)$ as follows:

$$\begin{aligned} p(x) &= \bar{\beta}_{1,1}(x) g_1(x) + \bar{\beta}_{1,2}(x) g_2(x) \\ q(x) &= \bar{\beta}_{2,1}(x) g_1(x) + \bar{\beta}_{2,2}(x) g_2(x) \end{aligned}$$

where $\bar{\beta}_{i,j}(x)$ are analytic real-valued functions and

$$\mathrm{rank} \begin{bmatrix} \bar{\beta}_{1,1}(x) & \bar{\beta}_{1,2}(x) \\ \bar{\beta}_{2,1}(x) & \bar{\beta}_{2,2}(x) \end{bmatrix} = 2.$$

Let $\bar{\beta}(x) = \begin{bmatrix} \bar{\beta}_{1,1}(x) & \bar{\beta}_{1,2}(x) \\ \bar{\beta}_{2,1}(x) & \bar{\beta}_{2,2}(x) \end{bmatrix}$. Applying the input transformation

$$u = \bar{\beta}(x) v \tag{8}$$

to (4) yields

$$\dot{x} = f(x) + p(x)v_0 + q(x)v_1 \tag{9}$$

where $v = [v_0, v_1]^T$ is the new input to be designed.

It is readily seen that, if the transformed system (9) is nonregular feedback linearizable, then the original system (4) is nonregular feedback linearizable, too.

From the Frobenius Theorem, there exists a real-valued function $\phi(x)$ such that

$$d\phi \perp G_{\kappa_l - 2} \quad d\phi \not\perp G_{\kappa_l - 1}. \tag{10}$$

As a matter of fact, $\phi(x)$ can be replaced by any of its nonzero constant multiplication, $c\phi(x)$, $c \neq 0$, without violating (10). This flexibility in choosing ϕ will be utilized in the following derivations.

Denote

$$v_0 = \phi(x). \tag{11}$$

System (9) can be rewritten as

$$\dot{x} = \bar{f}(x) + q(x)v_1 \tag{12}$$

where $\bar{f}(x) = f(x) + \phi(x)p(x)$.

In the sequel, we focus on (12) and prove its linearizability.

Because G_{n-2} is involutive and of dimension $n-1$, by the Frobenius Theorem, there exists a real-valued function $h(x)$ such that

$$\mathrm{span}\{dh\} = G_{n-2}^{\perp}. \tag{13}$$

Note that

$$q \in G_0 \tag{14}$$
$$ad_{\bar{f}}q = ad_f q + \phi ad_p q. \tag{15}$$

For convenience, for two sets S_1, S_2 and an element s, let $s \in S_1 - S_2$ if $s \in S_1$ and $s \notin S_2$. It follows from (15) and condition iv) that

$$ad_f q \in G_1 - G_0 \quad ad_p q \in G_1. \tag{16}$$

Suppose $ad_{\bar{f}}q \in G_0$, then it follows from (16) that $ad_p q \notin G_0$. In this case, $ad_f q + c\phi ad_p q \notin G_0$ for any constant $c \neq 1$. Accordingly, by appropriately choosing ϕ, it can be always made that

$$ad_{\bar{f}}q \in G_1 - G_0. \tag{17}$$

Suppose, for some $1 \leq i \leq n-2$, we have

$$ad_{\bar{f}}^j q \in G_j - G_{j-1}, \qquad j = 1, \ldots, i. \tag{18}$$

Then, it can be proven that

$$ad_{\bar{f}}^{i+1} q \in G_{i+1} - G_i. \tag{19}$$

To this end, compute

$$ad_{\bar{f}}^{i+1}q = ad_f\left(ad_{\bar{f}}^i q\right) + \phi ad_p\left(ad_{\bar{f}}^i q\right) + \left(L_{ad_{\bar{f}}^i q}\phi\right)p.$$

For $i < \kappa_l - 1$ and $i \neq \kappa_j - 1$, $j = 1, \ldots, l-2$, it follows from (18) that $L_{ad_{\bar{f}}^i q}\phi = 0$, and

$$ad_f\left(ad_{\bar{f}}^i q\right) \in G_{i+1} - G_i \quad ad_p\left(ad_{\bar{f}}^i q\right) \in G_{i+1}.$$

Therefore, up to a constant multiplication of ϕ, we have

$$ad_{\bar{f}}^{i+1}q = ad_f\left(ad_{\bar{f}}^i q\right) + \phi ad_p\left(ad_{\bar{f}}^i q\right) \in G_{i+1} - G_i.$$

For $i = \kappa_j - 1$ with $1 \leq j \leq l-1$, we have

$$ad_f\left(ad_{\bar{f}}^i q\right) \in G_{i+1}, ad_p\left(ad_{\bar{f}}^i q\right) \in G_{i+1}, L_{ad_{\bar{f}}^i q}\phi = 0.$$

Accordingly, we have

$$ad_{\bar{f}}^{i+1}q = ad_f\left(ad_{\bar{f}}^i q\right) + \phi ad_p\left(ad_{\bar{f}}^i q\right) \in G_{i+1}.$$

Suppose that

$$ad_f\left(ad_{\bar{f}}^i q\right) \in G_i \quad ad_p\left(ad_{\bar{f}}^i q\right) \in G_i \tag{20}$$

then by Jacobi identity [17, pp. 10], we have the following relationships:

$$ad_{ad_f p}\left(ad_{\bar{f}}^i q\right) = ad_f\left(ad_p\left(ad_{\bar{f}}^i q\right)\right) - ad_p\left(ad_f\left(ad_{\bar{f}}^i q\right)\right) \in G_i.$$

By recursion, we have

$$\begin{aligned} ad_{ad_f^j p}\left(ad_{\bar{f}}^i q\right) &= ad_f\left(ad_{ad_f^{j-1}p}\left(ad_{\bar{f}}^i q\right)\right), \\ &-ad_{ad_f^{j-1}p}\left(ad_f\left(ad_{\bar{f}}^i q\right)\right) \in G_i, \qquad j = 2, 3, \ldots \end{aligned}$$

which contradicts (7). Consequently, (20) must be invalid, and we have

$$ad_f\left(ad_{\bar{f}}^i q\right) \in G_{i+1} - G_i$$

or

$$ad_p\left(ad_{\bar{f}}^i q\right) \in G_{i+1} - G_i$$

which imply that, up to a constant multiplication of ϕ, we have

$$ad_{\bar{f}}^{i+1} q \notin G_i.$$

For $i = \kappa_l - 1$, we have

$$\begin{gathered} ad_f\left(ad_{\bar{f}}^i q\right) \in G_{i+1} \quad ad_p\left(ad_{\bar{f}}^i q\right) \in G_{i+1} \\ L_{ad_{\bar{f}}^i q}\phi \neq 0, \qquad p \in G_{i+1} - G_i. \end{gathered}$$

Therefore, up to a constant multiplication of ϕ, we have

$$ad_{\bar{f}}^{i+1}q = ad_f\left(ad_{\bar{f}}^i q\right) + \phi ad_p\left(ad_{\bar{f}}^i q\right) + \left(L_{ad_{\bar{f}}^i q}\phi\right)p \in G_{i+1} - G_i.$$

For $\kappa_l \leq i \leq n-2$, we have

$$ad_f\left(ad_{\bar{f}}^i q\right) \in G_{i+1} - G_i \quad ad_p\left(ad_{\bar{f}}^i q\right) \in G_{i+1}, \qquad p \in G_i.$$

Therefore, up to a constant multiplication of ϕ, we have

$$ad_{\bar{f}}^{i+1} q \in G_{i+1} - G_i.$$

These reasonings show that (18) implies (19) for $1 \leq i \leq n-2$. From the initial condition (17), it follows by induction that

$$\begin{aligned} &ad_{\bar{f}}^k q \in G_k, \qquad k = 0, 1, \ldots, n-2 \\ &ad_{\bar{f}}^{n-1} q \notin G_{n-2}. \end{aligned} \tag{21}$$

Accordingly, we have

$$\begin{aligned} &\left\langle dh, ad_{\bar{f}}^k q\right\rangle = 0, \qquad k = 0, 1, \ldots, n-2 \\ &\left\langle dh, ad_{\bar{f}}^{n-1} q\right\rangle \neq 0 \end{aligned}$$

which, by [17, Lemma 4.1.3], implies that

$$\begin{aligned} &L_q L_{\bar{f}}^k h = 0, \qquad k = 0, 1, \ldots, n-2 \\ &L_q L_{\bar{f}}^{n-1} h \neq 0. \end{aligned}$$

Define new coordinates z and new input w as

$$z = \left[h, L_{\bar{f}}h, \ldots, L_{\bar{f}}^{n-1}h\right]^T \tag{22}$$

$$w = L_{\bar{f}}^n h + \left(L_q L_{\bar{f}}^{n-1}h\right) v_1. \tag{23}$$

The state-space description of (12) in z-coordinate is then given by

$$\dot{z} = [z_2,\ z_3,\ \ldots\ ,\ z_n,\ w]^T \tag{24}$$

which is exactly the single-input Brunovsky canonical system.

By Definition 1, (12) is nonregular feedback linearizable, which implies that the original system (4) is also nonregular feedback linearizable. In addition, the corresponding linearizing state and input transformations for (4) are (22) and

$$u = \bar{\beta}(x)\begin{bmatrix}\phi(x)\\ \frac{w-L_{\bar{f}}^n h}{L_q L_{\bar{f}}^{n-1}h}\end{bmatrix} \tag{25}$$

respectively. ♢

Remark 3: In the theorem, condition i) ensures the controllability of the nonlinear system, and condition ii) is also a standard condition for feedback linearizability. Conditions iii) and iv), however, impose additional requirements at the "jump" indexes. The indexes $0 \leq \kappa_0 < \kappa_1 < \cdots < \kappa_l \leq n-1$, although not given explicitly, are verifiable by exhausted searching of the finite possible candidates. We also need to find suitable vector fields $p(x)$ and $q(x)$. However, in many cases, we have either $p(x) = g_1(x)$, $q(x) = g_2(x)$ or $p(x) = g_2(x)$, $q(x) = g_1(x)$. Therefore, the verification of the theorem, though seemingly involved, may quite straightforward for many practical systems, especially those with specific physical structures.

Remark 4: The key idea of the proof is the introduction of the nonregular state feedback (11). Through the introduction of this nonregular state feedback, the two-input system (9) is transformed into the single-input system (12) which is feedback linearizable. By allowing the flexibility of reducing the number of external inputs, the system is linearizable although it may not be linearizable via any regular state feedback. Technically, the linearizability is achieved by avoiding singularities of the distributions. That is, when facing possible singular distributions, we seek for other vector fields to produce the required sequence of regular distributions. Similar ideas have been utilized to deal with the singular case for state equivalence of single-input nonlinear systems in [4].

Remark 5: Note that the analysis in the proof is essentially nonsmooth in that the transformations involved are not necessarily smooth. The state transformation (22) may be singular (i.e., not full rank) on a lower dimensional submanifold, and the input transformation (25) is meromorphic and hence not necessarily well defined in the whole neighborhood of the origin.

Remark 6: Once the linearizing output $h(x)$ and the function $\phi(x)$ (called singular input function) are determined, the linearizing state and input transformations (22) and (25) can be calculated easily. The determination of $h(x)$ and $\phi(x)$ involve the integration of a set of completely integrable systems, or equivalent, the solution of some solvable partial differential equations, which may not be obtainable as a routine. However, for many nonlinear systems with particular structures, the integration of the integrable systems is available. Thus, $h(x)$ and $\phi(x)$ can be explicitly obtained.

B. Control System Design

For a nonlinear system satisfying Theorem 1, control design for stabilization can be developed easily by following the readily available results. Owing to space limitation, the main ideas are outlined without any detailed derivation. As the transformed system is the single-input Brunovsky canonical system, either standard linear feedback design theory or the backstepping design technique provide stabilizing controllers for the transformed system. When the controllers are transformed into the original nonlinear system, we have stabilizing controllers which are well-defined if the system is initiated from an allowed initial set. The allowed initial set can be determined through standard linear analysis (cf. [14]). To make the original nonlinear system globally attractive, we only need to drive any initial state into the allowed initial set by an appropriate control input as has been discussed in many references [1], [14], [16].

IV. Linearizable Nonholonomic Systems

The criterion provided in the last section is quite general, and several forms of nonholonomic systems are nonregular feedback stabilizable as illustrated in the follows.

First, consider nonholonomic systems of the form

$$\begin{aligned} {y_1}^{(r_1)} &= u_1 \\ {y_i}^{(r_i)} &= \xi_i(\bar{y}^1, \ldots, \bar{y}^i, y_{i+1})u_1 \\ & \quad i = 2, \ldots, m-1 \\ {y_m}^{(r_m)} &= u_2 \end{aligned} \tag{26}$$

where $m \geq 3$, $r_i \geq 1$, $\bar{y}^i = [y_i, \ldots, y_i^{(r_i-1)}]^T$, $i = 1, \ldots, m$, and ξ_i, $i = 2, \ldots, m-1$ are analytic functions vanishing at the origin with

$$\frac{\partial \xi_i}{\partial y_{i+1}} \neq 0, \qquad i = 2, \ldots, m-1.$$

Note that model (26) includes the chained form [24] and the second-order chained form [11] as special cases.

By letting $p(x) = g_1(x)$, and $q(x) = g_2(x)$, it can be verified that Theorem 1 holds and the linearizing output and the singular input function could be explicitly constructed, say as

$$\begin{aligned} h &= y_1 \\ \phi &= \phi_1\left(y_1, \ldots, y_1^{(r_1)}, y_2\right) \quad \frac{\partial \phi_1}{\partial y_2} \neq 0. \end{aligned} \tag{27}$$

Second, consider nonholonomic systems of the form

$$\dot{x} = \begin{bmatrix}1\\0\\0\\ \vdots\\0\end{bmatrix} u_1 + \begin{bmatrix}0\\1\\ \xi_3(x_1)\\ \vdots\\ \xi_n(x_1)\end{bmatrix} u_2 \tag{28}$$

where $\xi_i(x)$, $i = 3, \ldots, n$ are analytic functions vanishing at the origin with

$$\frac{\partial^{i-2}\xi_i}{\partial x_1^{i-2}} \neq 0, \qquad i = 3, \ldots, n.$$

It can be verified that Theorem 1 holds with

$$\begin{aligned} h(x) &= x_1 \\ \phi(x) &= x_n - \sum_{i=2}^{n-1} \varphi_i(x_1)x_i \end{aligned} \tag{29}$$

where $\varphi_i(x_1), i = 2, \ldots, n-1$ satisfy

$$\begin{bmatrix} \varphi_3 \\ \vdots \\ \varphi_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi_3}{\partial x_1} & \cdots & \frac{\partial \xi_{n-1}}{\partial x_1} \\ & \ddots & \\ \frac{\partial^{n-3} \xi_3}{\partial x_1^{n-3}} & \cdots & \frac{\partial^{n-3} \xi_{n-1}}{\partial x_1^{n-3}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \xi_3}{\partial x_1} \\ \vdots \\ \frac{\partial^n \xi_n}{\partial x_1^{n-3}} \end{bmatrix}$$

$$\varphi_2 = \xi_n - \sum_{j=3}^{n-1} \varphi_j(x)\xi_j(x).$$

This model includes the power form [23] as a special case. In addition, as the extended nonholonomic integrator [16] is equivalent to (26) via a state diffeomorphism, it is nonregular state feedback linearizable.

Finally, consider dynamic nonholonomic systems of the form

$$\begin{aligned} \dot{y} &= g_1(y)v_1 + g_2(y)v_2 \\ v_1^{(r_1)} &= u_1 \\ v_2^{(r_2)} &= u_2 \end{aligned} \tag{30}$$

where $r_i \geq 1, i = 1, 2$, and $\dot{y} = g_1(y)v_1 + g_2(y)v_2$ is either a (high-order) chained system (26) or a generalized power system (28) when view y as the state and $v = [v_1, v_2]^T$ as the input.

It can be verified that (30) is nonregular feedback linearizable. Moreover, any linearizing output and singular input function for system

$$\dot{y} = g_1(y)v_1 + g_2(y)v_2$$

are also linearizing output and singular input function for (30).

This model includes the extended power form [19] as a special case.

Remark 7: As nonregular feedback linearizability is invariant under regular state feedback transformation, any nonholonomic system which is static feedback equivalent to one of the previous forms is also nonregular feedback linearizable. Note that the dynamic feedback equivalence among the above forms is well known (see, e.g., [13]). It seems that regular dynamic state feedback linearizability and nonregular static state feedback linearizability are the same for many systems. The relationship between them is very interesting and deserving further investigation.

V. Conclusion

In this note, the problem of linearization via nonsmooth nonregular static state feedback has been formulated and addressed. A new criterion has been presented for linearizability of a class of affine nonlinear systems with two inputs. We showed that several well-known nonholonomic forms are nonregular feedback linearizable as possible applications.

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