

# Persistent Dwell-Time Switched Nonlinear Systems: Variation Paradigm and Gauge Design

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**Abstract**—Asymptotic gain and adaptive control are studied for persistent dwell-time switched systems. Ultimate variations of auxiliary functions are considered for existence of asymptotic gain and a gauge design is introduced for switching-uniform adaptive control by partial state feedback and output feedback of switched systems subject to unmeasured dynamics and persistent dwell-time switching. The usage of the controlled dynamics as a gauge for the instability mode of the unmeasured dynamics makes it possible to design a control rendering the evolution of the overall system interchangeably driven by the stable modes of the controlled and unmeasured dynamics. Unmeasured-state dependent control gains are dealt with and unknown time-varying parameters are attenuated via asymptotic gain. Verification of asymptotic gain conditions is based on the relation between dissipation rates of unmeasured dynamics and timing characterizations  $\tau_p$  and  $T_p$  of switching sequences.

**Index Terms**—Adaptive control, gauge design, gauge Lyapunov function, output feedback control, switched systems, switching-uniform control, unmeasured dynamics.

## I. INTRODUCTION

**S**WITCHED systems are dynamical systems whose evolutions are described as successions of finite-time evolutions of elemental driving systems. Study of such systems is motivated by their theoretical interest, practical embrace, and technical advances in a variety of applications [1]–[7]. This paper addresses asymptotic behavior and control of switched systems via timing properties of the successions and structural properties of the driving dynamics.

The successions of evolutions can be described in terms of switching sequence, i.e., the sequence of pairs of system index and evolution time. As a driving variable, the switching sequence may act as either a control to be designed [3]–[5], [8], [9] or a destabilizing force to be suppressed. Of practical relevance, the latter role is played in applications, e.g., where unscheduled changes in system structure may occur due to failures or oper-

ation needs [1]. Thus, in switched systems, it is worth studying the problem of switching-uniform control, i.e., achieving the control objective uniformly with respect to a class of switching sequences.

Endeavor has been made for switching-uniform control of switched systems whose driving systems are described by differential equations in linear [10] or in Byrnes-Isidori canonical forms [11], and switched input-to-state stable systems subject to dwell-time switching [12]. In this paper, we are interested in the problems of switching-uniform control for the more general classes of persistent dwell-time switched systems subject to both dynamic and parametric uncertainties, which shall be formulated in Section III. The underlying difficulty lies in the unamenability of the existing stability theories, e.g., [13]–[18], to control design of switched systems.

A primary concern in stability of switched systems subject to persistent dwell-time switching and uncertainties lies in the possible diverging behavior of Lyapunov functions. As the evolution times can be arbitrarily short, a change of the driving system may occur at the instant either the respective Lyapunov function of the current driving system has not sufficiently decreased or the respective Lyapunov function of the driving system taking effect remains undesirably large. In addition, disturbances also raise possible increments in Lyapunov functions during inactive periods of their respective driving systems. These together make the usage of the classical paradigm of Lyapunov stability theory through *switching decreasing condition* [13], [16]–[18], i.e., Lyapunov functions are consistently decreasing on the whole evolution times of their respective driving systems, becomes expensive.

Motivated by the above considerations, we introduce in Section II a variation paradigm for verifying converging behavior of persistent dwell-time switched systems via auxiliary functions. We no longer rely on the switching decreasing condition. Instead, we consider ultimate variations of auxiliary functions for asymptotic gain. From the observation that small variations in auxiliary functions are achievable if the increments of these functions on diverging periods are satisfactorily compensated on dwell-time intervals, we study the property of *small-variation small-state* for permission of both converging and diverging behaviors of auxiliary functions. Based on the fact that the guaranteed minimum running times of dwell-time switching events shall produce large variations for large states, we show that the property is achievable with persistent dwell-time switching. The variation usage does not only enhance switched systems, but also present a paradigm for studying asymptotic behavior of dynamical systems.

Transforming the system model to a well-understood form is essential in nonlinear control design [19], [20]. The transformation may result in systems with zero-dynamics due to low rel-

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ative degrees [19], [20] or systems with unmeasured dynamics due to limited modeling capability [21]. Difficulties in control design of such systems are twofold. First, we have freedom to control a limited number of state variables and must leave the rest to evolve autonomously. This raises the issue on stability of the zero-dynamics in switched systems. Secondly, unmeasured dynamics pose the problem of feasibility of switching sequence generated via full-state feedback [22], [23] and computation of Lyapunov functions [24].

In addition, the existing paradigms for control design of systems whose unmeasured state enters the controlled dynamics are usually based on small-gain theorem [25], Lyapunov theory [26], and dissipativity theory [27]. While the paradigm based on small-gain theorem makes use of linear gains conditions, the latter two paradigms adopt the changing supply function technique [28] to suppress the unmeasured state-dependent quantities. In switched systems, due to possible non-negative cross-supply functions, any change of supply functions for large decreasing rates on evolution times also gives rise to large growth rates on inactive periods.

In Section IV, we shall introduce the gauge design overcoming the above difficulties. The underlying principle is to use unmeasured dynamics and controlled dynamics as gauges of each other. This usage is possible because whenever the state of the controlled dynamics is dominated by the unmeasured state, the desired behavior of the overall system is automatically guaranteed by the converging behavior of the unmeasured dynamics, and on the contrary, i.e., the unmeasured state is dominated by the controlled state, estimates of functions of the unmeasured state in terms of the controlled state are available so that a control making the controlled dynamics the driving dynamics of the overall system can be design to be independent of unmeasured states. The method allows the unknown time-varying parameters to be lumped as an input disturbance to be attenuated.

Switching sequences of arbitrarily short evolution times and uncontrolled switches also lay challenging obstacles and hence motivate development in output feedback control of switched systems. Under such switching sequences, it is not possible to switch among a set of observers designed *a priori* [24] as well as to design an observer providing state estimates for the whole time. In Section V, we propose a solution to these difficulties. Following the gauge design paradigm, we aim at making the dynamics of the whole system interchangeably driven by the stable modes of the dynamics of error variables and the unmeasured dynamics. It turns out that state estimates of the controlled dynamics are needed only in unstable modes of the unmeasured dynamics. Fortunately, in these modes, estimates of functions of unmeasured state in terms of errors variables and known variables are available for observation. We present a reduced-order observer independent of switching sequence and then design a control guaranteeing convergence in unstable modes of the unmeasured dynamics. It turns out that, due to discrepancy between control gains, the observer's parameters are no longer to be assigned freely as in switching-free systems. Instead, these parameters are designed taking account of the bounds of control gains. In this way, we are able to deal with not only unknown control gain variations but also full-state dependent control gains.

*Notations:*  $\mathbb{R}$ ,  $\mathbb{R}^+$ , and  $\mathbb{N}$  are the sets of real numbers, non-negative real numbers, and nonnegative integers, respectively.  $|\cdot|$  is the absolute value of scalars.  $\|\cdot\|$  is the Euclidean norm if  $\cdot$  is a vector and is the essential supremum norm if  $\cdot$  is a function. The restriction of a function  $u$  to a set  $I$  is denoted by  $u|_I$ . As usual,  $\mathcal{K}$ ,  $\mathcal{KK}$ ,  $\mathcal{KL}$  [29], and  $\mathcal{K}_\infty$  [30] are classes of comparison functions. If  $\lceil i \rceil$  is any symbol containing  $i$ , then  $\{\lceil i \rceil\}_i$  is the infinite sequence  $\{\lceil i \rceil\}_{i=0}^\infty$ .

## II. SYSTEMS MODEL AND ASYMPTOTIC GAIN

In this section, we study asymptotic gain of switched systems using their symbolic model. Using switching sequences instead of including discrete dynamics, we obtain a model fitting the context of hybrid systems [17], [18], [31] and providing accessibility of timing information of elementary evolutions. Such information is desired in our paradigm of analyzing variations auxiliary functions on different time stages of switching sequences.

### A. Switched Systems With Input

The basic elements of a switched system with input are systems with input which can be defined via transition mappings. Let  $X$  and  $U$  be topological spaces which shall be referred to as the spaces of the system state and input, respectively. Let  $x^\circ$  be a fictitious point and  $\mathcal{U}$  be the set of all functions from  $\mathbb{R}$  to  $U$ . A *system on  $X$  with input in  $U$*  is a triple  $(X, U, \mathcal{T})$ , where  $\mathcal{T} : \mathbb{R}^+ \times \mathbb{R} \times X \times \mathcal{U} \rightarrow X \cup \{x^\circ\}$  is the *transition mapping* satisfying i)  $\mathcal{T}(0, t, x, u) = x, \forall (t, x, u)$ , ii) if  $\mathcal{T}(s, t_0, x_0, u) = x^\circ$  for some  $(s, t_0, x_0, u)$ , then  $\mathcal{T}(s', t_0, x_0, u) = x^\circ, \forall s' \geq s$ , iii)  $\mathcal{T}(s_1 + s_2, t_0, x_0, u) = \mathcal{T}(s_2, t_0 + s_1, \mathcal{T}(s_1, t_0, x_0, u), u)$ ,  $\forall s_1, s_2 \in \mathbb{R}^+, (t_0, x_0, u) \in \mathbb{R} \times X \times \mathcal{U}$ , and iv) if  $u_1|_{[a,b]} = u_2|_{[a,b]}$  for some  $[a, b] \subset \mathbb{R}$ , then  $\mathcal{T}(s, a, x, u_1) = \mathcal{T}(s, a, x, u_2), \forall s \in [0, b - a]$ . By  $\mathcal{T}(s, t_0, x_0, u)$ , we obtain the system state reached for a duration of  $s$  of evolution under the input  $u$  since the initiation in the state  $x_0$  at the time  $t_0$ . The expression  $\mathcal{T}(s, t_0, x_0, u) = x^\circ$  indicates that the system is not evolvable for a time of  $s$  from  $(t_0, x_0)$  under the input  $u$ .

Let  $q^\circ \in \mathbb{N}$  be a non-zero number fixed *a priori* and  $\mathbb{Q}$  be the set  $\{1, \dots, q^\circ\}$ . In symbolics, a *switched system on  $X$  with input in  $U$*  is a quadruple

$$\Sigma = (X, U, \{\Sigma_q\}_{q \in \mathbb{Q}}, \mathcal{S}) \quad (1)$$

where  $\Sigma_q = (X, U, \mathcal{T}_q)$ ,  $q \in \mathbb{Q}$  are systems with input, and  $\mathcal{S}$  is a set of sequences in  $\mathbb{Q} \times \mathbb{R}^+$ .

Infinite sequences in  $\mathbb{Q} \times \mathbb{R}^+$  shall be called the switching sequences. Elements of a switching sequence  $\sigma$  are called switching events of  $\sigma$ . The  $i$ -th switching event of  $\sigma$ ,  $i \in \mathbb{N}$  is specified by the point in  $\mathbb{Q} \times \mathbb{R}^+$  denoted as  $(q_{\sigma,i}, \Delta\tau_{\sigma,i})$ . The system  $\Sigma_{q_{\sigma,i}}$  and the number  $\Delta\tau_{\sigma,i}$  are then respectively referred to as the driving system and the evolution time of the  $i$ -th switching event of  $\sigma$ . The number  $\tau_{\sigma,i}$ ,  $i \in \mathbb{N}$  determined recursively by  $\tau_{\sigma,i+1} = \tau_{\sigma,i} + \Delta\tau_{\sigma,i}$ ,  $\tau_{\sigma,0} = 0$  is referred to as the  $i$ -th switching time of  $\sigma$ . The time  $t_{\sigma,i} = t_0 + \tau_{\sigma,i}$  is referred to as either the starting time of the  $i$ -th switching event or the end time of the  $(i-1)$ -th switching event. For a  $s \in \mathbb{R}^+$ ,  $i_\sigma^-(s)$  is the largest integer satisfying  $\tau_{\sigma, i_\sigma^-(s)} \leq s$ , i.e.,  $i_\sigma^-(s) = \max\{i \in \mathbb{N} : \tau_{\sigma,i} \leq s\}$ . We shall call  $i_\sigma^-$  the *transition indicator* whose role is to determine the driving system  $\Sigma_{q_\sigma^-(s)}$

at a time  $t_0 + s$  and its respective time of evolution  $\Delta\tau_{\sigma,i\bar{\sigma}}(s)$  [32]. A switching sequence  $\sigma$  is non-Zeno if the sum of evolution times of any infinitely many consecutive switching events in  $\sigma$  is infinite.

Given  $(x_0, u, \sigma) \in \mathbb{R}^n \times \mathcal{U} \times \mathbb{S}$ , the function  $x : [t_0, \infty) \rightarrow X \cup \{x^\circ\}$  defined as  $x(t_0) = x_0$  and  $x(t) = \mathcal{T}_{q_{\sigma,i}}(t - t_{\sigma,i}, t_{\sigma,i}, x(t_{\sigma,i}), u), \forall t \in [t_{\sigma,i}, t_{\sigma,i+1}]$ ,  $i \in \mathbb{N}$  is said to be the *evolution* from  $(t_0, x_0)$  under the input  $u$  and switching sequence  $\sigma$  of  $\Sigma$ . We shall write  $x(t_0, x_0, u, \sigma)$  to indicate the dependence on  $(t_0, x_0, u, \sigma)$ . An evolution  $x$  of  $\Sigma$  is said to be nontrivial if there is  $t > t_0$  such that  $x(t) \neq x^\circ$  and forward complete if  $x(t) \neq x^\circ, \forall t \in [t_0, \infty)$ . The system (1) is said to be forward complete if so are all of its evolutions.

In the case  $X \subset \mathbb{R}^n$  and  $U \subset \mathbb{R}^m$ , let  $f : X \times U \rightarrow X$  be a continuous function and let  $u \in \mathcal{U}$  be a measurable locally essentially bounded function. Defining solutions in terms of Lebesgue integral, one knows that for each  $(t_0, x_0)$ , there is a locally absolutely continuous function  $x_f(t_0, x_0, u) : [t_0, t_0 + T) \rightarrow X$  which is the maximal solution to the initial value problem (IVP)  $\dot{x}(t) = f(x(t), u(t)), x(t_0) = x_0$  and is continuously dependent on parameters [30]. Due to the uniqueness and continuation of solutions of IVPs [30, Theorem 54], the mapping  $\mathcal{T}_f$  defined as  $\mathcal{T}_f(s, t_0, x_0, u) = x_f(t_0, x_0, u)(t_0 + s)$ ,  $s \in [0, T)$  and  $\mathcal{T}_f(s, t_0, x_0, u) = x^\circ, s \geq T$  satisfies properties of transition mapping. Thus, the triple  $(X, U, \mathcal{T}_f)$  well defines a system which we shall call the system described by the equation  $\dot{x} = f(x, u)$ .

Let  $(\tau_p, T_p)$  be a pair of positive real numbers fixed *a priori*. According to [15], a switching sequence  $\sigma$  is said to have the persistent dwell-time  $\tau_p$  with the period of persistence  $T_p$  if it has an infinite number of switching events of the evolution times no smaller than  $\tau_p$  and, for every two consecutive switching events of this property  $(q_{\sigma,i}, \Delta\tau_{\sigma,i})$  and  $(q_{\sigma,j}, \Delta\tau_{\sigma,j}), i < j$ , it holds that  $\tau_{\sigma,j} - \tau_{\sigma,i+1} \leq T_p$ . Switching events of  $\sigma$  of evolution times no smaller than  $\tau_p$  are then called dwell-time switching events of  $\sigma$ . The sequence of all dwell-time switching events of  $\sigma$  is denoted by  $\{(q_{\sigma,i_j^d}, \Delta\tau_{\sigma,i_j^d})\}_j$ . The time periods  $\mathcal{I}_j^d = [t_{\sigma,i_j^d}, t_{\sigma,i_{j+1}^d}]$  and  $\mathcal{I}_j^p = [t_{\sigma,i_{j+1}^d}, t_{\sigma,i_{j+1}^d}]$  are respectively called the dwell-time intervals and periods of persistence of  $\sigma$ .

*Assumption 2.1:* All switching sequences in  $\mathbb{S}$  are non-Zeno and have the same persistent dwell-time  $\tau_p$  with the same period of persistence  $T_p$ . The evolution times of switching events are all non-zero and bounded by a number  $\varpi > 0$ .

*Assumption 2.2:* The sets  $X \subset \mathbb{R}^n$  and  $U \subset \mathbb{R}^m$  are open and connected. For each  $(t_0, x_0, u) \in \mathbb{R} \times X \times \mathcal{U}$ , the mappings  $\mathcal{T}_q(\cdot, t_0, x_0, u) : [0, \varpi] \rightarrow X, q \in \mathbb{Q}$  are continuous.

In this paper, we are interested in switched systems satisfying Assumption 2.1 to which we refer as persistent dwell-time switched systems with persistence pair  $(\tau_p, T_p)$ .

### B. Asymptotic Gain

Asymptotic gain [33] is a mathematical characterization for the property of *small-input small-attractor* of systems with input. In this subsection, we shall study sufficient conditions for the existence of asymptotic gains in switched systems.

*Definition 2.1:* Let  $\gamma$  be a class- $\mathcal{K}_\infty$  function. The switched system (1) with  $X \subset \mathbb{R}^n$  and  $U \subset \mathbb{R}^m$  is said to have the asymptotic gain  $\gamma$  if for every essentially bounded input  $u \in \mathcal{U}$

and every  $(t_0, x_0, \sigma) \in \mathbb{R} \times X \times \mathbb{S}$ , the evolution  $x(t_0, x_0, u, \sigma)$  is forward complete and satisfies

$$\limsup_{t \rightarrow \infty} \|x(t_0, x_0, u, \sigma)(t)\| \leq \gamma(\|u\|). \quad (2)$$

The property of consistently decreasing along system trajectories of auxiliary (Lyapunov) functions is usually sought for stability-like behavior of dynamical systems. Though the switching decreasing condition lends itself to this property, it might not be satisfiable in systems exposing arbitrarily short evolution times. By this observation, we twist to study, for the first time, the use of the property of small-variation small-state of auxiliary functions for converging behavior of dynamical systems. Ultimate variations of auxiliary functions shall be considered so that diverging behavior of these functions is possible.

Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$  be a continuous function. The derivative of  $V$  along an evolution  $x = x(t_0, x_0, u, \sigma)$  of  $\Sigma$  is [29]

$$DV(x(t)) = \limsup_{h \rightarrow 0} \frac{1}{h} (V(x(t+h)) - V(x(t))) \quad (3)$$

and the variation of  $V$  between  $t_1$  and  $t_2$  along  $x$  is

$$\text{Var}_{t_1}^{t_2} V(x) = |V(x(t_1)) - V(x(t_2))|. \quad (4)$$

Variation  $\text{Var}_a^b V(x(t))$  indicates the deviation of  $V(x(t))$  achieved for a duration of  $b - a$  of evolution from the time  $a$ . This notion is different from the notion of total variation in real analysis [34]. Let  $V_1$  and  $V_2$  be functions from  $\mathbb{R}^n$  to  $\mathbb{R}^+$ . The relative variation between  $V_1$  and  $V_2$  at  $t_1$  and  $t_2$  along the evolution  $x$  is

$$\text{Var}_{t_1}^{t_2} [V_1, V_2](x) = |V_1(x(t_1)) - V_2(x(t_2))|. \quad (5)$$

For brevity, let  $\tilde{t} = t - t_0$  and  $q_{\sigma}^-(\tilde{t}) = q_{\sigma, i_{\sigma}^-}(\tilde{t})$ .

*Theorem 2.1:* Consider the switched system (1) satisfying Assumptions 2.1 and 2.2, the class- $\mathcal{K}_\infty$  functions  $\underline{\alpha}, \bar{\alpha}, \alpha, \gamma_1$ , and  $\gamma_2$ , and the continuous functions  $V_q : \mathbb{R}^n \rightarrow \mathbb{R}^+, q \in \mathbb{Q}$ . Suppose that

$$\underline{\alpha}(\|x\|) \leq V_q(x) \leq \bar{\alpha}(\|x\|), \forall x \in \mathbb{R}^n, q \in \mathbb{Q} \quad (6)$$

and for every (essentially) bounded input  $u \in \mathcal{U}$ , switching sequence  $\sigma \in \mathbb{S}$ , and starting point  $(t_0, x_0) \in \mathbb{R} \times X$ , the corresponding evolution  $x = x(t_0, x_0, u, \sigma)$  satisfies the following properties:

- i) for each  $t \in [t_{\sigma,i}, t_{\sigma,i+1}], i \in \mathbb{N}$ , if  $V_{q_{\sigma,i}}(x(t)) \geq \gamma_1(\|u\|)$ , then  $DV_{q_{\sigma,i}}(x(t)) \leq -\alpha(V_{q_{\sigma,i}}(x(t)))$ ;
- ii) the relative variations among  $V_q$ 's on periods of persistence  $\mathcal{I}_j^p = [t_{\sigma,i_j^d+1}, t_{\sigma,i_{j+1}^d}]$  satisfy

$$\limsup_{j \rightarrow \infty} \max_{t \in \mathcal{I}_j^p} \text{Var}_{t_{\sigma,i_j^d+1}}^t \left[ V_{q_{\sigma,i_j^d}}, V_{q_{\sigma,i_{j+1}^-}}(\tilde{t}) \right] (x) \leq \gamma_2(\|u\|). \quad (7)$$

Then, the switched system (1) has an asymptotic gain.

*Proof:* By Assumptions 2.1 and 2.2, the evolution  $x$  is forward complete. Let  $\gamma_d(s) = \max\{\tau_p \alpha(\gamma_1(s)), \gamma_2(s)\}$ . For an  $\epsilon \geq 0$ , we have the sets  $\mathcal{B}_q(\epsilon) = \{\zeta \in \mathbb{R}^n : \tau_p \alpha(V_q(\zeta)) \leq \gamma_d(\|u\|) + \epsilon\}, q \in \mathbb{Q}$ . We shall abbreviate  $\mathcal{B}_q(0)$  to  $\mathcal{B}_q$ .

For each  $\epsilon \geq 0$ , we have the following claim.

*Claim C*( $\epsilon$ ): For each  $j \in \mathbb{N}$ , if  $x(t_j^\circ) \in \mathcal{B}_{q_{\sigma, i_j^d}}(\epsilon)$  for some  $t_j^\circ \in [t_{\sigma, i_j^d}, t_{\sigma, i_{j+1}^d}]$ , then  $x(t) \in \mathcal{B}_{q_{\sigma, i_j^d}}(\epsilon), \forall t \in [t_j^\circ, t_{\sigma, i_{j+1}^d}]$ .

We shall prove this claim in the paradigm of [35, Lemma 2.14]. Suppose that the claim is not true. Then, there are  $t \in (t_j^\circ, t_{\sigma, i_{j+1}^d}]$  and  $\epsilon' > \epsilon/2$  such that  $\tau_p \alpha(V_{q_{\sigma, i_j^d}}(x(t))) > \gamma_d(\|u\|) + \epsilon'$ . Accordingly,  $t_j^* = \inf\{t \in (t_j^\circ, t_{\sigma, i_{j+1}^d}] : \tau_p \alpha(V_{q_{\sigma, i_j^d}}(x(t))) > \gamma_d(\|u\|) + \epsilon'\}$  exists and  $\tau_p \alpha(V_{q_{\sigma, i_j^d}}(x(t_j^*))) \geq \gamma_d(\|u\|) + \epsilon' > \tau_p \alpha(\gamma_1(\|u\|))$ . Thus, i) applies and  $DV_{q_{\sigma, i_j^d}}(x(t_j^*)) \leq -\alpha(V_{q_{\sigma, i_j^d}}(x(t_j^*))) < -\epsilon'/\tau_p$ . Hence,  $V_{q_{\sigma, i_j^d}}(x(t_j^*)) \leq V_{q_{\sigma, i_j^d}}(x(s))$  for some  $s < t_j^*$ . This contradicts to the minimality of  $t_j^*$ . Thus, the claim holds true.

Let us define  $t_j^\circ$  to be  $t_{\sigma, i_{j+1}^d}$  if there is no  $t \in [t_{\sigma, i_j^d}, t_{\sigma, i_{j+1}^d}]$  at which  $\tau_p \alpha(V_{q_{\sigma, i_j^d}}(x(t))) \leq \gamma_d(\|u\|)$  and to be  $\inf\{t \in [t_{\sigma, i_j^d}, t_{\sigma, i_{j+1}^d}] : \tau_p \alpha(V_{q_{\sigma, i_j^d}}(x(t))) \leq \gamma_d(\|u\|)\}$  if such  $t$  exists. According to the above claim applied for  $\epsilon = 0$ , we have

$$\tau_p \alpha \left( V_{q_{\sigma, i_j^d}} \left( x \left( t_j^\circ \right) \right) \right) \geq \gamma_d(\|u\|), \forall t \in [t_{\sigma, i_j^d}, t_j^\circ]. \quad (8)$$

We shall show that

$$\limsup_{j \rightarrow \infty} \text{Var}_{t_{\sigma, i_j^d}^{t_j^\circ}} V_{q_{\sigma, i_j^d}}(x) \leq \gamma_d(\|u\|). \quad (9)$$

Indeed, suppose that (9) does not hold. Then, there is a number  $\epsilon > 0$  and an infinite sequence  $\{t_{\sigma, i_{j_k}^d}\}_k$  such that

$$\left| V_{q_{\sigma, i_{j_k}^d}} \left( x \left( t_{\sigma, i_{j_k}^d} \right) \right) - V_{q_{\sigma, i_{j_k}^d}} \left( x \left( t_{j_k}^\circ \right) \right) \right| > \gamma_d(\|u\|) + \epsilon. \quad (10)$$

As  $\alpha \in \mathcal{K}_\infty$  which is unbounded and continuous, there is a number  $\delta = \delta(\epsilon) > 0$  such that

$$\tau_p \alpha(s) \leq \gamma_d(\|u\|) + \delta \Rightarrow s \leq \alpha^{-1}(\gamma_d(\|u\|)/\tau_p) + \epsilon/3. \quad (11)$$

As  $\gamma_d(s) \geq \gamma_2(s)$  and  $\text{Var}_{t_{\sigma, i_{j+1}^d}^{t_{\sigma, i_j^d}}} [V_{q_{\sigma, i_j^d}}, V_{q_{\sigma, i_{j+1}^d}}](x) \leq \max\{\text{Var}_{t_{\sigma, i_{j+1}^d}^t} [V_{q_{\sigma, i_j^d}}, V_{q_{\sigma, i_{j+1}^d}}](x) : t \in [t_{\sigma, i_j^d}, t_{\sigma, i_{j+1}^d}]\}$ , condition (7) implies that

$$\limsup_{j \rightarrow \infty} \left| V_{q_{\sigma, i_j^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) - V_{q_{\sigma, i_{j+1}^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) \right| \leq \gamma_d(\|u\|). \quad (12)$$

Hence, there is an  $N_\epsilon \in \mathbb{N}$  such that for all  $j > N_\epsilon$ , we have

$$V_{q_{\sigma, i_{j+1}^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) \leq V_{q_{\sigma, i_j^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) + \gamma_d(\|u\|) + \epsilon^* \quad (13)$$

where  $\epsilon^* = \min\{\delta(\epsilon), \epsilon/3\}$ .

Consider the case where there is a number  $p > N_\epsilon$  such that

$$\tau_p \alpha \left( V_{q_{\sigma, i_j^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) \right) \leq \gamma_d(\|u\|) + \epsilon^* \quad (14)$$

holds at  $j = p$ . We shall show that (14) also holds for all  $j \geq p$ . Indeed, from the above claim we have either  $x(t_1) \in \mathcal{B}_{q_{\sigma, i_{p+1}^d}}(\epsilon^*)$  for some  $t_1 \in \mathcal{I}_{p+1}^d = [t_{\sigma, i_{p+1}^d}, t_{\sigma, i_{p+1}^d}]$  and hence  $x(t) \in \mathcal{B}_{q_{\sigma, i_{p+1}^d}}(\epsilon^*), \forall t \in [t_1, t_{\sigma, i_{p+1}^d}]$  or

$\tau_p \alpha(V_{q_{\sigma, i_{p+1}^d}}(x(t))) > \gamma_d(\|u\|) + \epsilon^*, \forall t \in \mathcal{I}_{p+1}^d$ . Clearly,  $\tau_p \alpha(V_{q_{\sigma, i_{p+1}^d}}(x(t_{\sigma, i_{p+1}^d}))) \leq \gamma_d(\|u\|) + \epsilon^*$  in the former case. In the latter case, by definition of  $\gamma_d$ , we have  $\tau_p \alpha(V_{q_{\sigma, i_{p+1}^d}}(x(t))) > \gamma_d(\|u\|) + \epsilon^* \geq \tau_p \alpha(\gamma_1(\|u\|)), \forall t \in \mathcal{I}_{p+1}^d$ . Thus, from i) and  $t_{\sigma, i_{p+1}^d} - t_{\sigma, i_{p+1}^d} \geq \tau_p$ , we have

$$\begin{aligned} & V_{q_{\sigma, i_{p+1}^d}} \left( x \left( t_{\sigma, i_{p+1}^d} \right) \right) - V_{q_{\sigma, i_{p+1}^d}} \left( x \left( t_{\sigma, i_{p+1}^d} \right) \right) \\ & \geq \int_{t_{\sigma, i_{p+1}^d}^{t_{\sigma, i_{p+1}^d} + \tau_p}} \alpha \left( V_{q_{\sigma, i_{p+1}^d}} \left( x(t) \right) \right) dt \\ & \geq \tau_p \min_{t \in \mathcal{I}_{p+1}^d} \left\{ \alpha \left( V_{q_{\sigma, i_{p+1}^d}} \left( x(t) \right) \right) \right\} \geq \gamma_d(\|u\|) + \epsilon^*. \end{aligned} \quad (15)$$

Combining (15) and (13) applied at  $j = p$ , we obtain

$$V_{q_{\sigma, i_{p+1}^d}} \left( x \left( t_{\sigma, i_{p+1}^d} \right) \right) \leq V_{q_{\sigma, i_p^d}} \left( x \left( t_{\sigma, i_p^d} \right) \right). \quad (16)$$

Substituting (16) into (14) applied at  $j = p$ , it follows that (14) also holds true for  $j = p + 1$ .

In combination, (14) holds at  $j = p + 1$ . Continuation of this process shows that (14) holds for all  $j \geq p$ . As  $\epsilon^* \leq \delta(\epsilon)$ , it follows from (14) and (11) that

$$V_{q_{\sigma, i_j^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) \leq \alpha^{-1}(\gamma_d(\|u\|)/\tau_p) + \epsilon/3, \forall j > N_\epsilon. \quad (17)$$

Combining (13) and (17), we obtain

$$V_{q_{\sigma, i_{j+1}^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) \leq \alpha^{-1}(\gamma_d(\|u\|)/\tau_p) + \gamma_d(\|u\|) + 2\epsilon/3 \quad (18)$$

for all  $j > N_\epsilon$ . This coupled with (8) yields

$$\left| V_{q_{\sigma, i_{j+1}^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) - V_{q_{\sigma, i_{j+1}^d}} \left( x \left( t_{j+1}^\circ \right) \right) \right| \leq \gamma_d(\|u\|) + 2\epsilon/3 \quad (19)$$

for all  $j > N_\epsilon$ , which contradicts to (10).

In the case (14) does not hold for all  $j > N_\epsilon$ , we have

$$\tau_p \alpha \left( V_{q_{\sigma, i_{j+1}^d}} \left( x(t) \right) \right) > \gamma_d(\|u\|) + \epsilon^*, \forall t \in \mathcal{I}_{j+1}^d, j > N_\epsilon \quad (20)$$

which, through the argument leading to (15), leads to

$$V_{q_{\sigma, i_j^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) \leq V_{q_{\sigma, i_j^d}} \left( x \left( t_{\sigma, i_j^d} \right) \right) - \gamma_d(\|u\|) - \epsilon^*, \forall j > N_\epsilon. \quad (21)$$

Combining (21) and (13), we arrive at

$$V_{q_{\sigma, i_{j+1}^d}} \left( x \left( t_{\sigma, i_{j+1}^d} \right) \right) \leq V_{q_{\sigma, i_j^d}} \left( x \left( t_{\sigma, i_j^d} \right) \right), \forall j > N_\epsilon. \quad (22)$$

In addition, as  $V_{q_{\sigma, i_{j_k}^d}}(x(t_{\sigma, i_{j_k}^d} + 1)) \leq V_{q_{\sigma, i_{j_k}^d}}(x(t_{j_k}^\circ))$ , (13) and (10) imply that

$$\begin{aligned} & V_{q_{\sigma, i_{j_k}^d}} \left( x \left( t_{\sigma, i_{j_k}^d} \right) \right) \\ & \leq V_{q_{\sigma, i_{j_k}^d}} \left( x \left( t_{\sigma, i_{j_k}^d} + 1 \right) \right) + \gamma_d(\|u\|) + \epsilon^* \end{aligned}$$

$$\begin{aligned} &\leq V_{q_{\sigma, i_{j_k}^d}} \left( x \left( t_{i_{j_k}^d}^o \right) \right) + \gamma_d (\|u\|) + \epsilon^* \\ &\leq V_{q_{\sigma, i_{j_k}^d}} \left( x \left( t_{\sigma, i_{j_k}^d} \right) \right) - (\epsilon - \epsilon^*) \end{aligned} \quad (23)$$

for all  $k$  satisfying  $j_k > N_\epsilon$ . Using (23) and applying (22) successively from  $j_k$  to  $j_{k+1}$ , we obtain

$$V_{q_{\sigma, i_{j_{k+1}}^d}} \left( x \left( t_{\sigma, i_{j_{k+1}}^d} \right) \right) \leq V_{q_{\sigma, i_{j_k}^d}} \left( x \left( t_{\sigma, i_{j_k}^d} \right) \right) - 2\epsilon/3, \forall j_k > N_\epsilon. \quad (24)$$

Thus,  $V_{q_{\sigma, i_{j_k}^d}} (x(t_{\sigma, i_{j_k}^d})) < 0$  for sufficiently large  $k$ , which is a contradiction.

As a result, (9) holds true.

The proof of the convergence of  $x(t)$  is now in order. We first show the convergence of values of  $x$  at end times of dwell-time switching events, i.e.,

$$\limsup_{j \rightarrow \infty} \tau_P \alpha \left( V_{q_{\sigma, i_j^d}} \left( x \left( t_{\sigma, i_j^d} \right) \right) \right) \leq \gamma_d (\|u\|). \quad (25)$$

Suppose that (25) is not true. Then, there are  $\epsilon > 0$  and sequence  $\{j_k\}_k$  such that

$$\tau_P \alpha \left( V_{q_{\sigma, i_{j_k}^d}} \left( x \left( t_{\sigma, i_{j_k}^d} \right) \right) \right) > \gamma_d (\|u\|) + \epsilon, \forall k \in \mathbb{N}. \quad (26)$$

According to definition of  $t_j^o$  and claim  $\mathcal{C}(\epsilon)$ , we have  $t_{i_{j_k}^d}^o = t_{\sigma, i_{j_k}^d} + 1$  and hence (26) implies that

$$\tau_P \alpha \left( V_{q_{\sigma, i_{j_k}^d}} (x(t)) \right) > \gamma_d (\|u\|) + \epsilon, \forall t \in \mathcal{I}_{j_k}^d. \quad (27)$$

As  $\gamma_d(s) \geq \tau_P \alpha(\gamma_1(s))$ , (27) implies that  $V_{q_{\sigma, i_{j_k}^d}} (x(t)) > \gamma_1(\|u\|)$  and hence, by i), we have

$$DV_{q_{\sigma, i_{j_k}^d}} (x(t)) < -\alpha \left( V_{q_{\sigma, i_{j_k}^d}} (x(t)) \right), \forall t \in \mathcal{I}_{j_k}^d. \quad (28)$$

Taking integrals of both sides of (28) and using (27) yields

$$\begin{aligned} \text{Var}_{t_{\sigma, i_{j_k}^d}^o}^{t_{\sigma, i_{j_k}^d}^o + 1} V_{q_{\sigma, i_{j_k}^d}} (x(t)) &\geq \int_{t_{\sigma, i_{j_k}^d}^o}^{t_{\sigma, i_{j_k}^d}^o + 1} \alpha \left( V_{q_{\sigma, i_{j_k}^d}} (x(t)) \right) dt \\ &\geq \gamma_d (\|u\|) + \epsilon. \end{aligned} \quad (29)$$

Taking limits of both side of (29), we obtain

$$\limsup_{k \rightarrow \infty} \text{Var}_{t_{\sigma, i_{j_k}^d}^o}^{t_{\sigma, i_{j_k}^d}^o + 1} V_{q_{\sigma, i_{j_k}^d}} (x(t)) \geq \gamma_d (\|u\|) + \epsilon \quad (30)$$

which contradicts to (9). Thus, (25) holds true.

We now examine the converging behavior of the sequence  $\{x(t_j)\}_j$ , where  $\{t_j\}_j \subset [t_0, \infty)$  is an arbitrary divergent sequence. Let us divide  $\{t_j\}_j$  into two subsequences  $\{t_j^d\}_j$  and  $\{t_j^p\}_j$ , where the first subsequence consists of all elements of  $\{t_j\}_j$  that belong to dwell-time switching intervals and the second subsequence are the rest in  $\{t_j\}_j$ .

For a time  $t \in [t_0, \infty)$ , there is an interval  $[t_{\sigma, i_j^d}, t_{\sigma, i_{j+1}^d}]$  between starting times of two consecutive dwell-time switching

events that contains  $t$ . Then, we define  $i_{\mathcal{D}}(t) = i_j^d, \mathcal{I}^d(t) = [t_{\sigma, i_j^d}, t_{\sigma, i_{j+1}^d}]$ , and  $\mathcal{I}^p(t) = [t_{\sigma, i_{j+1}^d}, t_{\sigma, i_{j+1}^p}]$ . If  $t \in \mathcal{I}^d(t)$ , then we further define  $t_{\mathcal{D}}^o(t) = t_{\sigma, i_j^d}^o$ . Recall that  $\tilde{t} = t - t_0$ . For the sequence  $\{t_j^p\}_j$ , we have

$$\begin{aligned} &V_{q_{\sigma, i_{\sigma}^-}(i_j^p)} \left( x \left( t_j^p \right) \right) \\ &\leq V_{q_{\sigma, i_{\mathcal{D}}}(t_j^p)} \left( x \left( t_{\sigma, i_{\mathcal{D}}}(t_j^p) \right) \right) \\ &\quad + \left| V_{q_{\sigma, i_{\sigma}^-}(i_j^p)} \left( x \left( t_j^p \right) \right) - V_{q_{\sigma, i_{\mathcal{D}}}(t_j^p)} \left( x \left( t_{\sigma, i_{\mathcal{D}}}(t_j^p) \right) \right) \right| \\ &\leq V_{q_{\sigma, i_{\mathcal{D}}}(t_j^p)} \left( x \left( t_{\sigma, i_{\mathcal{D}}}(t_j^p) \right) \right) \\ &\quad + \max_{t \in \mathcal{I}^p(t_j^p)} \text{Var}_t^{t_{\sigma, i_{\mathcal{D}}}(t_j^p)} \left[ V_{q_{\sigma, i_{\mathcal{D}}}(t_j^p)}, V_{q_{\sigma, i_{\sigma}^-}(i_j^p)} \right] (x). \end{aligned} \quad (31)$$

Taking the limits of both sides of (31) and using (25) and (7), we obtain

$$\limsup_{j \rightarrow \infty} V_{q_{\sigma, i_{\sigma}^-}(i_j^p)} \left( x \left( t_j^p \right) \right) \leq \gamma_d (\|u\|) + \gamma_2 (\|u\|). \quad (32)$$

We now consider the sequence  $\{t_j^d\}_j$ . As we have shown,  $V_{q_{\sigma, i_{\mathcal{D}}}(t_j^d)} (x(t))$  is decreasing on  $[t_{\sigma, i_{\mathcal{D}}}(t_j^d), t_{\mathcal{D}}^o(t_j^d)]$  and is bounded by  $\alpha^{-1}(\gamma_d(\|u\|)/\tau_P)$  on  $[t_{\mathcal{D}}^o(t_j^d), t_{\sigma, i_{\mathcal{D}}}(t_{j+1}^d)]$ . Let  $\gamma_3(\|u\|) = \alpha^{-1}(\gamma_d(\|u\|)/\tau_P)$ . We have

$$\begin{aligned} &V_{q_{\sigma, i_{\mathcal{D}}}(t_j^d)} \left( x \left( t_j^d \right) \right) \\ &\leq \max \left\{ V_{q_{\sigma, i_{\mathcal{D}}}(t_j^d)} \left( x \left( t_{\sigma, i_{\mathcal{D}}}(t_j^d) \right) \right), \gamma_3 (\|u\|) \right\}. \end{aligned} \quad (33)$$

Since  $t_{\sigma, i_{\mathcal{D}}}(t_j^d)$  is the starting time of a dwell-time switching interval which is also the end time of a period of persistence, taking the limits of both sides of (33) and using (32), we obtain

$$\limsup_{j \rightarrow \infty} V_{q_{\sigma, i_{\mathcal{D}}}(t_j^d)} \left( x \left( t_j^d \right) \right) \leq \gamma^* (\|u\|) \quad (34)$$

where  $\gamma^*(s) = \max\{\gamma_d(s) + \gamma_2(s), \gamma_3(s)\}$ .

On the other hand, from (6), we have  $\underline{\alpha}(\|x(t)\|) \leq V_{\underline{q}}(x(t)), \forall t \in [t_0, \infty)$ . As  $\underline{\alpha}$  is continuous, combining (32) and (34), we arrive at

$$\begin{aligned} &\limsup_{j \rightarrow \infty} \|x(t_j)\| \\ &\leq \max \left\{ \underline{\alpha}^{-1} \left( \limsup_{j \rightarrow \infty} V_{q_{\sigma, i_{\sigma}^-}(i_j^p)} \left( x \left( t_j^p \right) \right) \right), \right. \\ &\quad \left. \underline{\alpha}^{-1} \left( \limsup_{j \rightarrow \infty} V_{q_{\sigma, i_{\mathcal{D}}}(t_j^d)} \left( x \left( t_j^d \right) \right) \right) \right\} \\ &\leq \underline{\alpha}^{-1} (\gamma^* (\|u\|)). \end{aligned} \quad (35)$$

Thus, defining the class- $\mathcal{K}_\infty$  function  $\gamma(s) = \underline{\alpha}^{-1}(\gamma^*(s))$ , we arrive at  $\limsup_{j \rightarrow \infty} \|x(t_j)\| \leq \gamma(\|u\|)$ . As  $\{t_j\}_j$  is arbitrary, this yields  $\limsup_{t \rightarrow \infty} \|x(t)\| \leq \gamma(\|u\|)$ . Finally, as  $\gamma$  is independent of  $x_0$  and  $\sigma$ , the conclusion of the theorem follows accordingly. ■

### III. CONTROL PROBLEMS

Models with parametric and dynamic uncertainties are essential in control of real systems [21]. While parametric uncertainty can be dealt with by adaptive control, certain stability properties

are desired to cope with dynamic uncertainty [20], [21], [36]. Here, we are interested in switched systems of the both uncertainties. A condition in terms of dissipation rates of unmeasured dynamics and parameters  $\tau_p$  and  $T_p$  of switching sequences is presented for dealing with dynamic uncertainty.

Consider the switched system (1) in which  $X = \mathbb{R}^{d+n}$ ,  $U = \mathbb{R} \times \Omega_\theta$  for some compact set  $\Omega_\theta \subset \mathbb{R}^{d_1}$ , and the driving dynamics  $\Sigma_q$ ,  $q \in \mathbb{Q}$  are described by equations of the form

$$\begin{aligned} \dot{z} &= Q_q(z, y, \theta) \\ \dot{x}_j &= g_{q,j}(z, \bar{x}_j, \theta)x_{j+1} + f_{q,j}(z, \bar{x}_j, \theta) \\ j &= 1, \dots, n \end{aligned} \quad (36)$$

where  $[z^T, x^T]^T \in \mathbb{R}^{d+n}$ ,  $z \in \mathbb{R}^d$  and  $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ , is the system state,  $\bar{x}_j = [x_1, \dots, x_j]^T$ ,  $x_{n+1} \stackrel{\text{def}}{=} u \in \mathbb{R}$  is the control input,  $\theta \in \mathbb{R}^{d_1}$  is the disturbance input,  $y = x_j$  is the controlled output, and  $Q_q$ ,  $g_{q,j}$ , and  $f_{q,j}$  are known continuous functions.

Let  $y_m = h_m(x)$  be the measured output of (1). Consider the following control structure

$$\dot{\zeta} = \Gamma(\zeta, y_m), \quad u = c(\zeta, y_m). \quad (37)$$

We have the following control problems:

**$\mathcal{P}_1$ ) Output regulation by partial state-feedback:** design a dynamic control (37) with  $y_m = x$  such that under any switching sequence  $\sigma \in \mathbb{S}$ , the output  $y(t)$  approaches to a small neighborhood of zero as  $t \rightarrow \infty$  while all signals in the closed-loop system remain bounded; and

**$\mathcal{P}_2$ ) Stabilization by output feedback:** design a dynamic control (37) with  $y_m = x_1$  such that under any switching signal  $\sigma \in \mathbb{S}$ , the trajectory  $x(t)$  approaches to a small neighborhood of the origin as  $t \rightarrow \infty$  while all signals in the closed-loop system remain bounded.

*Assumption 3.1:* There are positive definite and continuously differentiable functions  $U_q : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $q \in \mathbb{Q}$ , class- $\mathcal{K}_\infty$  functions  $\underline{\alpha}_U$ ,  $\bar{\alpha}_U$ ,  $\alpha_1$ ,  $\alpha_2$  and  $\rho$ , and a continuous function  $v : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $v(0) = 0$  such that  $U_{q_1}(z) \leq \rho(U_{q_2}(z))$ ,  $\underline{\alpha}_U(\|z\|) \leq U_q(z) \leq \bar{\alpha}_U(\|z\|)$ ,  $z \in \mathbb{R}^d$ , and

$$\frac{\partial U_{q_1}(z)}{\partial z} Q_{q_2}(z, y, \theta) \leq \alpha_{q_1 q_2}(U_{q_1}(z)) + v(y^2) \quad (38)$$

for all  $z \in \mathbb{R}^d$ ,  $q, q_1, q_2 \in \mathbb{Q}$  and  $y \in \mathbb{R}$ , where  $\alpha_{q_1 q_2}$  stands for  $-\alpha_1$  if  $q_1 = q_2$  and for  $\alpha_2$ , otherwise.

We shall call the dynamics of  $z$  either the unmeasured dynamics or the  $z$ -dynamics. The functions  $\alpha_1$  and  $\alpha_2$  are called the dissipation and cross-dissipation rates of the  $z$ -dynamics, respectively.

Let  $\mu$  be a  $C^1$  class- $\mathcal{K}_\infty$  function and define the functions  $\omega_k(a, b) = G_k^{-1}(G_k(a) + b)$ ,  $k = 1, 2$ , where

$$G_k(a) \stackrel{\text{def}}{=} \int_1^a \frac{ds}{\text{sgn}(k-1.5)\alpha_k(s) + \mu(s)}, \quad a \in \mathbb{R}^+. \quad (39)$$

*Assumption 3.2:* Both functions  $\alpha_1 - \mu$  and  $\alpha_2 + \mu$  are of class- $\mathcal{K}_\infty$  and  $(\partial\mu(s)/\partial s)\alpha_1(s) \rightarrow 0$  as  $s \rightarrow 0^+$ . There exist numbers  $R_p > 0$  and  $\tau_0 > 0$  and a class- $\mathcal{K}\mathcal{L}$  function  $\omega_0$  satisfying  $\omega_0(\omega_0(a, s), t) \leq \omega_0(a, s+t)$ ,  $\forall a, s, t \in \mathbb{R}^+$  such that  $\omega_2(s, t) < \infty$ ,  $\forall (s, t) \in [0, R_p] \times [0, T_p]$  and

$$\omega_1(\rho(\omega_2(s, T_p)), \tau_p) \leq \omega_0(s, \tau_0), \quad \forall s \in \mathbb{R}^+. \quad (40)$$

*Remark 3.1:* As  $\omega_1$  and  $\omega_2$  in Assumption 3.2 are, respectively, increasing and decreasing in their second argument [37], (40) holds if the dwell-time  $\tau_p$  is sufficiently large with respect to the period of persistence  $T_p$ . This agrees with the observation that the larger the dwell-time and the shorter the period of persistence are, the higher the attainability of a converging behavior will be.

*Remark 3.2:* In the linear case of  $\rho(s) = a_0 s$ ,  $\alpha_1(s) - \mu(s) = a_1 s$ , and  $\alpha_2(s) + \mu(s) = a_2 s$ , the satisfiability of Assumption 3.2 is obvious as (40) becomes  $\tau_0 \stackrel{\text{def}}{=} a_1 \tau_p - a_2 T_p > \ln a_0$  and a function  $\omega_0$  satisfying (40) is  $\omega_0(s, \tau_0) = s \exp(-\tau_0)$ . Also in this case,  $R_p = \infty$ .

*Assumption 3.3:* For each  $j \in \{1, \dots, n\}$ , there is a known positive function  $g_{\min, j}$  such that for all  $(z, \bar{x}_j, \theta) \in \mathbb{R}^d \times \mathbb{R}^j \times \Omega_\theta$  and  $q \in \mathbb{Q}$ , we have

$$|g_{q,j}(z, \bar{x}_j, \theta)| \geq g_{\min, j}(\bar{x}_j) > 0. \quad (41)$$

As  $g_{q,j}$ 's are continuous, from (41), we further assume that  $g_{q,j}$ 's are all positive without loss of generality.

*Rremark 3.3:* Assumption 3.3 is instrumental in dealing with the difficulty that the traditional cancelation design matching the control to nonlinearities  $f_{q,j}$ 's through the inverses  $1/g_{q,j}$  [20], [38] is no longer effective. The behind rationale is twofold: i) unmeasured state dependent control gains  $g_{q,j}$ 's are no longer computable, and ii) the discrepancies among  $g_{q,j}$  make matching control to all  $f_{q,j}$ 's impossible.

*Lemma 3.1 ([26]):* Let  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  be a continuous function. Then, there are smooth functions  $a : \mathbb{R}^n \rightarrow \mathbb{R}^+$ ,  $b : \mathbb{R}^m \rightarrow \mathbb{R}^+$ ,  $c : \mathbb{R}^n \rightarrow [1, \infty)$  and  $d : \mathbb{R}^m \rightarrow [1, \infty)$  such that

$$|f(x, y)| \leq a(x) + b(y) \text{ and } |f(x, y)| \leq c(x)d(y). \quad (42)$$

*Lemma 3.2 ([37]):* Let  $v(t)$  be a differentiable function in  $J = (a, b)$  such that

$$\dot{v}(t) \leq g(v(t)), \quad \forall t \in J \quad (43)$$

where  $g$  is a nonzero continuous function in  $I = (v_1, v_2)$ . Let  $t_0 \in J$  and  $v(t_0) = v_0 \in I$ , then

$$v(t) \leq G^{-1}(G(v_0) + t - t_0), \quad \forall t \in [t_0, b_1) = J_1 \quad (44)$$

where  $G(v) = \int_u^v (g(s))^{-1} ds$ ,  $u, v \in I$  and  $b_1 = \sup\{t \in [t_0, b) : G(v_0) + s - t_0 \in G(I), t_0 \leq s \leq t\}$ .

In addition, the function  $\omega(a, t) \stackrel{\text{def}}{=} G^{-1}(G(a) + t)$ ,  $a \in I$ ,  $t \in J$  satisfies

$$\omega(\beta(a, s), t) = \omega(a, s+t), \quad \forall a \in I, s, t \in J, s+t \in J. \quad (45)$$

#### IV. ADAPTIVE OUTPUT REGULATION

In this section, we present a *gauge design method* solving problem  $\mathcal{P}_1$  under Assumptions 3.1–3.3. The design makes use of the controlled (error) dynamics as a gauge for instability mode of the unmeasured dynamics to design a control preserving the converging behavior in this mode. In such way, the dynamics of the overall system is interchangeably driven by stable modes of these dynamics.

As  $v(0) = 0$ , there is a continuous function  $v_0$  such that  $v(s) = v_0(s)s$ ,  $s \geq 0$ . Let  $v_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nondecreasing and continuously differentiable function satisfying

$v_1(s) \geq \max\{v_0(s), 1\}$ ,  $s \in \mathbb{R}^+$ . For a  $q \in \mathbb{Q}$ , we have the following gauge along the evolution of  $z$ ,  $x$ :

$$\mu(U_q(z(t))) \leq 2V_g(\xi(t)) \stackrel{\text{def}}{=} v_1(\xi_1^2(t)) \xi_1^2(t) + \sum_{j=2}^n \xi_j^2(t) \quad (46)$$

where  $\xi_j = x_j - \alpha_{j-1}^o$ ,  $j = 1, \dots, n$  are error variables,  $\xi \stackrel{\text{def}}{=} [\xi_1, \dots, \xi_n]^T$ ,  $\alpha_0^o = 0, \alpha_1^o, \dots, \alpha_{n-1}^o$  are the so-called virtual controls to be designed. Let  $\bar{\xi}_j = [\xi_1, \dots, \xi_j]^T$  and  $\bar{\xi}_j^2 = [\xi_1^2, \dots, \xi_j^2]^T$ . The dynamics of  $\xi$  shall be referred to as the  $\xi$ -dynamics.

### A. Control Design

As long as the inverse of (46) holds for any  $q$ , the converging behavior of  $\xi$  is induced by the converging behavior of  $z$ . Accordingly, our design is to preserve the dissipation rates of the  $z$ -dynamics in the  $\xi$ -dynamics on periods (46) holds.

1) *First Virtual Control Design:* Consider the first error variable  $\xi_1$ . As the evolution of  $x_1$  is driven by (36), the driving dynamics for  $\xi_1$  are described by the equations  $\dot{\xi}_1 = g_{q,1}(z, x_1, \theta)x_2 + f_{q,1}(z, x_1, \theta)$ ,  $q \in \mathbb{Q}$ . (47)

Consider the following function:

$$V_{g,1}(\xi_1) \stackrel{\text{def}}{=} \frac{1}{2} v_1(\xi_1^2) \xi_1^2 \quad (48)$$

which is said to be the first gauge Lyapunov function (GLF) candidate as it is a part of  $V_g$  (see (46)) which is used as a gauge for diverging behavior of the unmeasured dynamics. Let  $v_D(s) \stackrel{\text{def}}{=} (\partial v_1(s)/\partial s)s + v_1(s)$ . As  $V_{g,1}$  is continuously differentiable, the derivative of  $V_{g,1}$  along the evolution of  $\xi_1$  in terms of (3) satisfies

$$DV_{g,1}(\xi_1(t)) \leq \max_{p \in \mathbb{Q}} \{v_D(\xi_1^2(t)) (g_{p,1}(z(t), x_1(t), \theta(t)) \xi_1(t) \times (\xi_2(t) + \alpha_1^o(t)) + f_{p,1}(z(t), x_1(t), \theta(t))) \xi_1(t)\}. \quad (49)$$

Bearing in mind that computations are made along evolutions of state variables  $z$  and  $x$ , we shall often drop the time arguments of evolving variables for brevity. At this point, we aim at a virtual control  $\alpha_1^o$  making the RHS of (49) contain a negative functions of  $\xi_1^2$  whenever (46) holds true. To this end, let us estimate the  $z$ -dependent functions  $g_{p,1}$ ,  $f_{p,1}$ ,  $p \in \mathbb{Q}$  in terms of known variables  $\xi_j$ 's as follows.

Since  $g_{p,1}$ 's and  $f_{p,1}$ 's are continuous and  $U_p$ 's are radially unbounded by Assumption 3.1, for each  $p \in \mathbb{Q}$ , there is a continuous function  $\psi_{p,1}$  nondecreasing in the first argument such that  $g_{p,1}(\cdot)\xi_2 + f_{p,1}(\cdot) \leq \psi_{p,1}(U_q(z), x_1, \xi_2, \theta), \forall (z, x_1, \theta) \in \mathbb{R}^d \times \mathbb{R} \times \Omega_\theta, q \in \mathbb{Q}$ . Such a function  $\psi_{p,1}$  can be

$$\psi_{p,1}(s, x_1, \xi_2, \theta) = \sup \{g_{p,1}(\zeta, x_1, \theta)\xi_2 + f_{p,1}(\zeta, x_1, \theta) : \zeta \in \mathbb{R}^d, \|\zeta\| \leq \underline{\alpha}_U^{-1}(s)\}, s \in \mathbb{R}^+ \quad (50)$$

where  $\underline{\alpha}_U$  is given by Assumption 3.1. By Lemma 3.1, there are functions  $\psi_{p,1}^a$  and  $\psi_{p,1}^b$  such that

$$\begin{aligned} v_D(\xi_1^2) (g_{p,1}(\cdot)\xi_2 + f_{p,1}(\cdot)) \xi_1 &\leq |v_D(\xi_1^2)| \psi_{p,1}(U_q(z), \xi_1, \xi_2, \theta) |\xi_1| \\ &\leq |v_D(\xi_1^2)| \psi_{p,1}(\mu^{-1}(V_g(\xi)), \xi_1, \xi_2, \theta) |\xi_1| \\ &\leq \psi_{p,1}^a(\theta) \psi_{p,1}^b(\bar{\xi}_n) |\xi_1| \leq K_\Theta (\psi_{p,1}^b(\bar{\xi}_n))^2 \xi_1^2 \\ &\quad + \frac{(\psi_{p,1}^a(\theta))^2}{4K_\Theta} \end{aligned} \quad (51)$$

whenever (46) holds true, where  $K_\Theta > 0$  is a time-varying design parameter to be updated. As  $\mathbb{Q}$  is finite, there is a  $C^1$  positive function  $\psi_1$ , that is nondecreasing in each individual argument, such that  $\max\{\psi_{p,1}^b(\bar{\xi}_n)\} : p \in \mathbb{Q}\} \leq \psi_1(\bar{\xi}_n^2)$ . Such a function  $\psi_1$  can be selected as

$$\psi_1(s) \geq \sup \left\{ (\psi_{p,1}^b(\xi))^2 : \xi \in \mathbb{R}^n, \|\xi\|^2 \leq \sum_{j=1}^n s_j, p \in \mathbb{Q} \right\} \quad (52)$$

where  $s = [s_1, \dots, s_n]^T \in (\mathbb{R}^+)^n$ . Applying the identity [26]

$$f(\bar{x}_j) = \left( \int_0^1 \frac{\partial f(\bar{x}_{j-1}, s)}{\partial s} \Big|_{s=\beta x_i} d\beta \right) x_j + f(\bar{x}_{j-1}, 0) \stackrel{\text{def}}{=} A(\bar{x}_j)x_j + f(\bar{x}_{j-1}, 0) \quad (53)$$

to the function  $\psi_1$  recursively from  $j = n$  to  $j = 1$ , we obtain positive functions  $\varphi_{1,j}$ 's satisfying

$$\psi_1(\bar{\xi}_n^2) = \sum_{j=1}^n \varphi_{1,j}(\bar{\xi}_j^2) \xi_j^2 + \psi_1(0). \quad (54)$$

*Remark 4.1:* As  $\psi_1$  is nondecreasing in each single argument, its partial derivatives are nonnegative. Hence, the positiveness of  $\varphi_{1,j}(\cdot)$ 's in (54) is obvious.

Since  $\theta$  belongs to the compact set  $\Omega_\theta$ , there is an unknown constant  $\Theta_1$  such that  $(\psi_{q,1}^a(\theta))^2/4 \leq \Theta_1, \forall q \in \mathbb{Q}$ . This coupled with (51) and (54) yields

$$\begin{aligned} v_D(\xi_1^2) (g_{p,1}(z, x_1, \theta)\xi_2 + f_{p,1}(z, x_1, \theta)) \xi_1 &\leq K_\Theta \left( \sum_{j=1}^n \varphi_{1,j}(\bar{\xi}_j^2) \xi_j^2 + \psi_1(0) \right) \xi_1^2 + \frac{\Theta_1}{K_\Theta}, \quad \forall p \in \mathbb{Q} \end{aligned} \quad (55)$$

whenever (46) holds. Substituting (55) and  $x_2 = \xi_2 + \alpha_{u,1}$  into (49), we obtain upon the satisfaction of (46)

$$\begin{aligned} DV_{g,1}(\xi_1) &\leq \max \{v_D(\xi_1^2) g_{p,1}(z, x, \theta)\xi_1 \alpha_1^o : p \in \mathbb{Q}\} \\ &\quad + K_\Theta \left( \sum_{j=1}^n \varphi_{1,j}(\bar{\xi}_j^2) \xi_j^2 + \psi_1(0) \right) \xi_1^2 + \frac{\Theta_1}{K_\Theta}. \end{aligned} \quad (56)$$

*Remark 4.2:* The universal design of the functions  $\psi_1$  in (52) and subsequent  $\psi_j$ 's applies to the general functions  $f_{p,j}$ 's and  $g_{p,j}$ 's. In applications, the particular structures of these functions can be exploited for improving estimates in (55) and the subsequent inequality (70).

In view of (56), let us consider the following structure for the first virtual control  $\alpha_1^o$ :

$$\alpha_1^o = -\frac{1}{g_{\min,1}(x_1)} (\varrho_1(\xi_1) + K_\Theta \varphi_1(\xi_1)) \xi_1 \quad (57)$$

where  $\varrho_1$ , and  $\varphi_1$  are positive smooth functions to be specified.

By specification,  $v_1$  is a nondecreasing function which has nonnegative derivative and  $v_1(s) \geq 1, \forall s \in \mathbb{R}^+$ . Thus

$$v_D(s) = \frac{\partial v_1(s)}{\partial s} s + v_1(s) \geq v_1(s) \geq 1, \quad \forall s \in \mathbb{R}^+. \quad (58)$$

This coupled with the property  $g_{p,1}(\cdot)/g_{\min,1}(\cdot) \geq 1$  from Assumption 3.3 implies that

$$\begin{aligned} v_D(\xi_1^2) g_{p,1}(\cdot) \xi_1 \alpha_1^o &\leq -\frac{g_{p,1}(\cdot)}{g_{\min,1}(\cdot)} (\varrho_1(\xi_1) + K_\Theta \varphi_1(\xi_1)) \xi_1^2 \end{aligned}$$

$$\leq -\varrho_1(\xi_1)\xi_1^2 - K_\Theta\varphi_1(\xi_1)\xi_1^2, \quad \forall p \in \mathbb{Q}. \quad (59)$$

Let  $\varphi_1$  be a smooth positive function satisfying  $\varphi_1(\xi_1) \geq \varphi_{1,1}(\xi_1^2)\xi_1^2 + \psi_1(0)$ . Substituting (59) into (56) and using this property of  $\varphi_1$ , we arrive at

$$DV_{g,1}(\xi_1) \leq -\varrho_1(\xi_1)\xi_1^2 + K_\Theta \sum_{j=2}^n \varphi_{1,j}(\bar{\xi}_j^2) \xi_j^2 \xi_1^2 + \frac{\Theta_1}{K_\Theta} \quad (60)$$

whenever (46) holds true. This completes the first step.

*Remark 4.3:* Different from the usual backstepping design [20], the gauge inequality (46) leads to the domination function  $\psi_1(\cdot)$  that depend on all error variables which cannot be canceled all at once by  $\alpha_1^\circ$ . Through the novel decomposition of  $\psi_1$  into functions of square of error variables (54), it shall be sequentially canceled by the next virtual controls.

2) *Inductive Virtual Control Design:* The purpose of the inductive design is to augment  $V_{g,1}$  and design the virtual controls  $\alpha_j^\circ$ 's to propagate (60) in such a way that all the positive terms in the derivatives of the last augmented function are eliminated. We have the following inductive assumption.

*Inductive Assumption:* At a step  $s \geq 1$ , there are

i) gauge Lyapunov function candidates  $V_{g,j}$ 's given by

$$V_{g,j}(\bar{\xi}_j) = V_{g,j-1}(\bar{\xi}_{j-1}) + \frac{1}{2}\xi_j^2, \quad j = 2, \dots, s \quad (61)$$

with  $V_{g,1}$  given by (48);

ii) virtual controls

$$\alpha_j^\circ = -\frac{1}{g_{\min,j}(\bar{x}_j)} (\varrho_j(\xi_j) + K_\Theta\varphi_i(\bar{\xi}_j)) \xi_j, \quad j = 1, \dots, s \quad (62)$$

where  $g_{\min,j}$ 's are given by Assumption 3.3, and  $\varrho_j, \varphi_j, j = 1, \dots, s$  are design positive functions given by (74),  $i = j$  and (76) below; and

iii) an unknown constant  $\Theta_s > 0$  and nonnegative functions

$$\varphi_{l,j}, \quad l = 1, \dots, s, \quad j = s+1, \dots, n,$$

such that whenever (46) holds, the derivative of  $V_{g,s}$  along the evolution of  $\bar{\xi}_s$  satisfies

$$DV_{g,s}(\bar{\xi}_s(t)) \leq -\sum_{j=1}^s \varrho_j(\xi_j(t)) \xi_j^2(t) + K_\Theta(t) \sum_{j=s+1}^n \sum_{l=1}^s \varphi_{l,j}(\bar{\xi}_j^2(t)) \xi_j^2(t) \xi_l^2(t) + \frac{\Theta_s}{K_\Theta(t)}. \quad (63)$$

Suppose that the inductive assumption holds for  $s = k-1, k \geq 2$ . We will show that it also holds for  $s = k$ .

From (36), equations describing the driving dynamics of the evolution of the error variable  $\xi_k = x_k - \alpha_{k-1}^\circ$  are

$$\dot{\xi}_k = g_{q,k}(X_k, \theta)x_{k+1} + f_{q,k}(X_k, \theta) + R_{q,k}(X_k, K_\Theta, \theta) \quad (64)$$

where

$$R_{q,k}(\cdot) \stackrel{\text{def}}{=} -\sum_{j=1}^{k-1} \frac{\partial \alpha_{k-1}^\circ}{\partial x_j} (g_{q,j}(\cdot)x_{j+1} + f_{q,j}(\cdot)) - \frac{\partial \alpha_{k-1}^\circ}{\partial K_\Theta} \dot{K}_\Theta$$

$$X_k \stackrel{\text{def}}{=} [z^T, \bar{x}_k^T]^T, \quad q \in \mathbb{Q}. \quad (65)$$

*Remark 4.4:* As we shall design the update law for  $K_\Theta$  to be a continuous function of  $\xi_1$  (see (78) below), the functions  $R_{q,k}, q \in \mathbb{Q}$ 's in (64) are functions of  $z, \bar{x}_k, K_\Theta$ , and  $\theta$ .

Consider the  $k$ -th GLF candidate

$$V_{g,k}(\bar{\xi}_k) = V_{g,k-1}(\bar{\xi}_{k-1}) + \frac{1}{2}\xi_k^2. \quad (66)$$

The inductive assumption shows that (63) holds for  $s = k-1$ . This coupled with  $x_{k+1} = \xi_{k+1} + \alpha_k^\circ$  and (64) makes the derivative of  $V_{g,k}$  along the evolution of  $\xi_k$  satisfy

$$\begin{aligned} DV_{g,k}(\bar{\xi}_k) &\leq DV_{g,k-1}(\bar{\xi}_{k-1}) + D(\xi_k^2/2) \\ &\leq -\sum_{j=1}^{k-1} \varrho_j(\xi_j)\xi_j^2 + K_\Theta \sum_{j=k+1}^n \sum_{l=1}^{k-1} \varphi_{l,j}(\bar{\xi}_j^2) \xi_j^2 \xi_l^2 \\ &\quad + K_\Theta \left( \sum_{l=1}^{k-1} \varphi_{l,k}(\bar{\xi}_k^2) \xi_l^2 \right) \xi_k^2 + \frac{\Theta_{k-1}}{K_\Theta} \\ &\quad + \max_{q \in \mathbb{Q}} \{ (g_{q,k}(\cdot) (\xi_{k+1} + \alpha_k^\circ) \\ &\quad \quad + f_{q,k}(\cdot) + R_{q,k}(\cdot)) \xi_k \} \end{aligned} \quad (67)$$

whenever (46) holds true. We now seek for a virtual control  $\alpha_k^\circ$  that further cancels certain positive terms out and add desired negative terms to the RHS of (67). By the same type of reasoning leading to (51), we obtain smooth functions  $\psi_{p,k}^a$ 's and  $\psi_{p,k}^b$ 's such that whenever (46) holds, we have

$$(g_{p,k}(\cdot)\xi_{k+1} + f_{p,k}(\cdot) + R_{p,k}(\cdot)) \xi_k \leq K_\Theta (\psi_{p,k}^b(\bar{\xi}_n))^2 \xi_k^2 + (\psi_{p,k}^a(\theta))^2 / (4K_\Theta), \quad \forall p \in \mathbb{Q}. \quad (68)$$

Since  $\mathbb{Q}$  is finite, there is a  $C^1$  positive function  $\psi_k$  nondecreasing in each individual argument such that  $(\psi_{p,k}^b(\bar{\xi}_n))^2 \leq \psi_k(\bar{\xi}_n^2), \forall p \in \mathbb{Q}$ . Again, applying the identity (53) to the function  $\psi_k$  recursively from  $j = n$  to  $j = k$ , we obtain the functions  $\varphi_{k,j}, j = k, \dots, n$  such that

$$\psi_k(\bar{\xi}_n^2) = \sum_{j=k}^n \varphi_{k,j}(\bar{\xi}_j^2) \xi_j^2 + \psi_k(\bar{\xi}_{k-1}^2, 0, \dots, 0). \quad (69)$$

As  $\theta$  belongs to the compact set  $\Omega_\theta$  and the functions  $\psi_{p,k}^a$ 's are continuous, there is an unknown constant  $\Theta_k > 0$  such that  $\max\{\Theta_1, \dots, \Theta_{k-1}, \|(\psi_{p,k}^a(\theta(\cdot)))^2\|/4, p \in \mathbb{Q}\} \leq \Theta_k$ . Hence, from (68) and (69), we have

$$(g_{p,k}(\cdot)\xi_{k+1} + f_{p,k}(\cdot) + R_{p,k}(\cdot)) \xi_k \leq K_\Theta \left( \sum_{j=k}^n \varphi_{k,j}(\bar{\xi}_j^2) \xi_j^2 + \psi_k(\bar{\xi}_{k-1}^2, 0, \dots, 0) \right) \xi_k^2 + \frac{\Theta_k}{K_\Theta} \quad (70)$$

whenever (46) holds true. Substituting (70) into (67) yields

$$\begin{aligned} DV_{g,k}(\bar{\xi}_k) &\leq -\sum_{j=1}^{k-1} \varrho_j(\xi_j)\xi_j^2 \\ &\quad + K_\Theta \left( \sum_{l=1}^k \varphi_{l,k}(\bar{\xi}_k^2) \xi_l^2 + \psi_k(\bar{\xi}_{k-1}^2, 0, \dots, 0) \right) \xi_k^2 \\ &\quad + K_\Theta \sum_{j=k+1}^n \sum_{l=1}^k \varphi_{l,j}(\bar{\xi}_j^2) \xi_j^2 \xi_l^2 \\ &\quad + \max\{g_{p,k}(\cdot)\alpha_k^\circ \xi_k : p \in \mathbb{Q}\} + \Theta_k/K_\Theta \end{aligned} \quad (71)$$

whenever (46) holds true. Consider the virtual control

$$\alpha_k^\circ = -\frac{1}{g_{\min,k}(\bar{x}_k)} (\varrho_k(\xi_k) + K_\Theta\varphi_k(\bar{\xi}_k)) \xi_k \quad (72)$$



where  $\varrho_k$  and  $\varphi_k$  are  $C^1$  positive functions to be specified. As  $g_{p,k}(\cdot) \geq g_{\min,k}(\cdot), \forall p \in \mathbb{Q}$  by Assumption 3.3, we have

$$\begin{aligned} g_{p,k}(\cdot)\xi_k\alpha_k^\circ &= -\frac{g_{p,k}(\cdot)}{g_{\min,k}(\cdot)}(\varrho_k(\cdot) + K_\Theta\varphi_k(\cdot))\xi_k^2 \\ &\leq -\varrho_k(\cdot)\xi_k^2 - K_\Theta\varphi_k(\cdot)\xi_k^2, \quad \forall p \in \mathbb{Q}. \end{aligned} \quad (73)$$

Let  $\varphi_k$  be the  $C^1$  positive function satisfying

$$\varphi_k(\bar{\xi}_k) \geq \sum_{l=1}^k \varphi_{l,k}(\bar{\xi}_k^2)\xi_l^2 + \psi_k(\bar{\xi}_{k-1}^2, 0, \dots, 0). \quad (74)$$

Substituting (72) and (74) into (71), we arrive at

$$\begin{aligned} DV_{g,k}(\bar{\xi}_k) &\leq -\sum_{j=1}^k \varrho_j(\xi_j)\xi_j^2 \\ &\quad + K_\Theta \sum_{j=k+1}^n \sum_{l=1}^{k+1} \varphi_{l,j}(\bar{\xi}_j^2)\xi_j^2\xi_l^2 + \frac{\Theta_k}{K_\Theta} \end{aligned} \quad (75)$$

under (46), i.e., the inductive assumption holds for  $s = k$ .

3) *Actual Control Design*: From the initial design for  $s = 1$  in 1), applying the above inductive design successively until  $s = n$ , we obtain the  $n$ -th GLF candidate  $V_g \stackrel{\text{def}}{=} V_{g,n}$  given by (61),  $j = n$  and the virtual control  $\alpha_n^\circ$  given by (62),  $j = n$ . Let us select  $u = \alpha_n^\circ$ . The remaining designs are those of  $\varrho_j$ 's and the update law of  $K_\Theta$ .

As  $\mu'(s) = \partial\mu(s)/\partial s \geq 0, s > 0$  for  $\mu \in \mathcal{K}_\infty$  and  $\mu'(s)\alpha_1(s) \rightarrow 0$  as  $s \rightarrow 0^+$  by Assumption 3.2, there is a class- $\mathcal{K}$  function  $\alpha_g$  chosen to be  $C^1$  such that  $\alpha_g(\mu(s)) \geq \mu'(s)\alpha_1(s)$ . We choose  $\varrho_j$ 's to be  $C^1$  functions satisfying

$$\sum_{j=1}^n \varrho_j(\xi_j)\xi_j^2 \geq \alpha_g(2V_g(\bar{\xi}_n)). \quad (76)$$

Such functions  $\varrho_j$ 's exist as  $\alpha_g(2V_g(\bar{\xi}_n))$  can be expressed as

$$\begin{aligned} \alpha_g(2V_g(\bar{\xi}_n)) &= \alpha_g\left(\sum_{j=1}^n \tilde{v}_j(\xi_j^2)\xi_j^2\right) \leq \sum_{j=1}^n \alpha_g(n\tilde{v}_j(\xi_j^2)\xi_j^2) \\ &= \sum_{j=1}^n \left(\int_0^1 \frac{\partial\alpha_g(n\tilde{v}_j(s)s)}{\partial s} \Big|_{s=\beta\xi_j^2} d\beta\right) \xi_j^2 \end{aligned} \quad (77)$$

where  $\tilde{v}_j(\xi_j^2)$  is  $v_1(\xi_1^2)$  if  $j = 1$  and is 1, otherwise.

Let  $\epsilon_d > 0$  be a desired accuracy and  $k_K > 0$  be a tuning gain. We select the following update law for  $K_\Theta$  [26]:

$$\dot{K}_\Theta = \begin{cases} k_K(|\xi_1| - \epsilon_d) & \text{if } |\xi_1| \geq \epsilon_d \\ 0 & \text{if } |\xi_1| < \epsilon_d \end{cases}, \quad K_\Theta(0) = 1. \quad (78)$$

Finally, let  $w_\Theta(t) \stackrel{\text{def}}{=} \Theta_n/K_\Theta(t)$ . We shall estimate the derivative of  $U_g(\xi) \stackrel{\text{def}}{=} \mu^{-1}(2V_g(\xi))$  along the evolution of  $\xi$  as follows. As  $u = \alpha_n^\circ$  gives  $\xi_{n+1} = u - \alpha_n^\circ = 0$  and renders (63) hold for  $s = n$  under (46), using (76), we have

$$DV_g(\xi(t)) \leq -\alpha_g(2V_g(\xi(t))) + w_\Theta(t) \quad (79)$$

whenever (46) holds true. As  $\mu \in C^1$  and so is  $\mu^{-1}$ , we have

$$DU_g(\xi(t)) = 2(\mu'(\mu^{-1}(2V_g(\xi(t))))^{-1} DV_g(\xi(t)))$$

$$\begin{aligned} &\leq -2(\mu'(U_g(\xi(t))))^{-1} \\ &\quad \times (\alpha_g(\mu(U_g(\xi(t)))) - w_\Theta(t)) \end{aligned} \quad (80)$$

whenever (46) holds true. From (80) and the designated property  $\alpha_g(\mu(s)) \geq \mu'(s)\alpha_1(s)$ , if  $\alpha_g(\mu(U_g(\xi(t)))) \geq 2\|w_\Theta\| \geq 2w_\Theta(t)$ , then we further have

$$DU_g(\xi(t)) \leq -\alpha_1(U_g(\xi(t))) \quad (81)$$

whenever (46) holds true. This completes the design procedure.

*Remark 4.5*: As our design achieves (75) only when (46) holds true, stability of the resulting closed-loop system cannot be concluded from (75) with arbitrary  $\varrho_j$ 's as in the traditional Lyapunov-based control design. Here, condition (76) is presented to obtain (81) for stability analysis.

## B. Stability Analysis

In this section, we show that the control  $u = \alpha_n^\circ$  solves the problem  $\mathcal{P}_1$ . The main steps are as follows. We first show that the (switched) system of  $\tilde{x} = [z^T, \xi^T]^T$  has an asymptotic gain with respect to the input  $w_\Theta$ . Then, we show that the adaptation of the parameter  $K_\Theta$  will be stopped when the desired accuracy has been reached. Consider the functions

$$V_q(\tilde{x}) \stackrel{\text{def}}{=} \max\{U_q(z), \mu^{-1}(2V_g(\xi))\}, \quad q \in \mathbb{Q}. \quad (82)$$

As  $U_q, q \in \mathbb{Q}$ , and  $V_g$  are continuous functions and  $z(t)$  and  $\xi(t)$  are continuous in  $t$ , the functions  $V_q(\tilde{x}(t)), q \in \mathbb{Q}$  are continuous in  $t$  as well. Let  $\rho_\mu(s) = s + \mu(s)$  which is a class- $\mathcal{K}_\infty$  function. Let us verify that

$$V_q(\tilde{x}) \geq \rho_\mu^{-1}(U_q(z) + 2V_g(\xi)), \quad \forall q \in \mathbb{Q}. \quad (83)$$

Indeed, if  $\mu(U_q(z)) \geq 2V_g(\xi)$  then  $V_q(\tilde{x}) = U_q(z)$  and hence  $\rho_\mu(V_q(\tilde{x})) = U_q(z) + \mu U_q(z) \geq U_q(z) + 2V_g(\xi)$ . In the inverse case of  $\mu(U_q(z)) < 2V_g(\xi)$ , we have  $V_q(\tilde{x}) = \mu^{-1}(2V_g(\xi))$  and hence  $\rho_\mu(V_q(\tilde{x})) = \mu^{-1}(2V_g(\xi)) + 2V_g(\xi) \geq U_q(z) + 2V_g(\xi)$ . Combining both cases, we obtain (83).

Recall that  $\{(q_{\sigma,i_j^d}, \Delta\tau_{\sigma,i_j^d})\}_j$  is the sequence of dwell-time switching events of  $\sigma$  and, given the initial time  $t_0, t_{\sigma,i_j^d} = t_0 + \tau_{\sigma,i_j^d}$ . We have the following proposition.

*Proposition 4.1*: Under the input  $u = \alpha_n^\circ$ , the following properties holds along the resulting evolution  $\tilde{x}$ :

- i)  $V_{q_{\sigma,i_j^d}}(\tilde{x}(t)) \leq \max\{\omega_1(V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d}))), t - t_{\sigma,i_j^d}, \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|))\}, \forall t \in [t_{\sigma,i_j^d}, t_{\sigma,i_{j+1}^d}];$  and
- ii) if  $V_q(\tilde{x}(t_V)) < R_V$  for some  $t_V \in [t_{\sigma,i_{j-1}^d}, t_{\sigma,i_j^d}], q \in \mathbb{Q}$ , then  $V_q(\tilde{x}(t)) \leq \omega_2(V_q^w(t_V), T_p), \forall t \in [t_V, t_{\sigma,i_j^d}]$ ,

where  $\omega_1, \omega_2$ , and  $R_V$  are given in Assumption 3.2,  $V_q^w(s) = \max\{\mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|)), V_q(\tilde{x}(s))\}$ , and  $t_{\sigma,i_{-1}^d+1} \stackrel{\text{def}}{=} t_{\sigma,0}$ .

*Proof*: See Appendix A.  $\blacksquare$

Subject to the update law (78),  $K_\Theta(t)$  is nondecreasing and hence  $w_\Theta(t)$  is bounded. Define the constant  $V_{ini} = \max\{\chi(\|w_\Theta\|), V_{q_{\sigma,0}}(\tilde{x}(t_0))\}$  and the function  $\chi(s) = \mu^{-1}(\alpha_g^{-1}(2s)), s > 0$ . We have the following theorem.

*Theorem 4.1*: Consider the switched system (1) whose driving dynamics are described by (36). Suppose that Assumptions 2.1, 3.1–3.3 hold and  $V_{ini} < R_V$ . Then, under the control  $u = \alpha_n^\circ$  given by (62),  $j = n$  with  $\varrho_j$ 's satisfying (76), the resulting switched system of  $\tilde{x} = [z^T, \xi^T]^T$ , whose driving

dynamics are described by  $\dot{z} = Q_q(z, y, \theta)$  and (64) and whose input is  $w_\Theta$ , has an asymptotic gain.

*Proof:* We shall prove the theorem by showing that the (switched) system generating  $\tilde{x}$  satisfy conditions of Theorem 2.1 with the functions  $V_q, q \in \mathbb{Q}$  defined in (82).

As the driving dynamics of  $z$  and  $x$  are described by differential equations (36) and the resulting control  $u = \alpha_n^\circ$  is continuous, from the theory of differential equations [30], we know that the corresponding transition mappings  $\mathcal{T}_q$  are continuous in their domains of existence. We shall verify the forward completeness of  $\tilde{x}$  through its boundedness in the subsequent verification of condition ii) of Theorem 2.1.

From (83) and definition of  $V_q$ , we have

$$U_q(z) + \mu^{-1}(2V_g(\xi)) \geq V_q(\tilde{x}) \geq \rho_\mu^{-1}(U_q(z) + 2V_g(\xi)). \quad (84)$$

As  $\mu$  and  $\rho_\mu$  are class- $\mathcal{K}_\infty$  functions, this shows that (6) holds.

To verify condition i) of Theorem 2.1, we have the following cases at the time  $t \in [t_{\sigma,i}, t_{\sigma,i+1}]$ ,  $i \in \mathbb{N}$ .

*Case 1:* Inequality (46) does not hold for  $q = q_{\sigma,i}$ . In this case, we have  $V_{q_{\sigma,i}}(\tilde{x}(t)) = U_{q_{\sigma,i}}(z(t))$  and

$$v(y^2) \leq v_1(\xi_1^2(t)) \xi_1^2(t) \leq \mu(U_{q_{\sigma,i}}(z(t))). \quad (85)$$

From Assumption 3.1 and (85), we have

$$\begin{aligned} DV_{q_{\sigma,i}}(\tilde{x}(t)) &= \frac{\partial U_{q_{\sigma,i}}(z(t))}{\partial z} Q_{q_{\sigma,i}}(z(t), x_1(t), \theta(t)) \\ &\leq -\alpha_1(U_{q_{\sigma,i}}(z(t))) + \mu(U_{q_{\sigma,i}}(z(t))) \\ &\leq -\tilde{\alpha}_1(V_{q_{\sigma,i}}(\tilde{x}(t))) \end{aligned} \quad (86)$$

where we have defined  $\tilde{\alpha}_1(s) = \alpha_1(s) - \mu(s)$ .

*Case 2:* Inequality (46) holds for  $q = q_{\sigma,i}$ . In this case, we have  $V_{q_{\sigma,i}}(\tilde{x}(t)) = \mu^{-1}(2V_g(\xi(t))) = U_g(\xi(t))$ . By control design, (81) holds if  $V_{q_{\sigma,i}}(\tilde{x}(t)) = U_g(\xi(t)) \geq \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|))$ .

Both these cases show that  $DV_{q_{\sigma,i}}(\tilde{x}(t)) \leq -\tilde{\alpha}_1(V_{q_{\sigma,i}}(\tilde{x}(t)))$  if  $V_{q_{\sigma,i}}(\tilde{x}(t)) \geq \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|))$ . As  $\mu$ ,  $\alpha_g$ , and  $\tilde{\alpha}_1$  are class- $\mathcal{K}_\infty$  functions, this shows that the condition i) of Theorem 2.1 is satisfied.

We now verify condition ii) of Theorem 2.1. Suppose that  $V_q(\tilde{x}(t_j^*)) < R_V$  for some  $t_j^* \in [t_{\sigma,i_j^d-1+1}, t_{\sigma,i_j^d}]$ . According to ii) of Proposition 4.1,  $V_q(\tilde{x}(t))$  remains bounded on  $[t_j^*, t_{\sigma,i_j^d}]$  and hence, by (84), so is  $\tilde{x}(t)$ . In addition

$$\begin{aligned} V_q(\tilde{x}(t_{\sigma,i_j^d})) &\leq \max\{\omega_2(V_q(\tilde{x}(t_j^*)), T_p), \\ &\omega_2(\mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|)), T_p)\}. \end{aligned} \quad (87)$$

At the time  $t_{\sigma,i_j^d}$ , we have either a)  $V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d})) = U_{q_{\sigma,i_j^d}}(z(t_{\sigma,i_j^d}))$  which, by Assumption 3.1, is less than  $\rho(U_q(z(t_{\sigma,i_j^d})))$  which, in turn, is less than  $\rho(V_q(\tilde{x}(t_{\sigma,i_j^d})))$  or b)  $V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d})) = U_g(\xi(t_{\sigma,i_j^d})) \leq V_q(\tilde{x}(t_{\sigma,i_j^d}))$ . As  $\rho(s) \geq s$ , these together result in

$$V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d})) \leq \rho(V_q(\tilde{x}(t_{\sigma,i_j^d}))). \quad (88)$$

As  $\omega_1$  is nonincreasing in its second argument, the boundedness of  $V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d}))$  from (88) coupled with i) of

Proposition 4.1 implies that  $V_{q_{\sigma,i_j^d}}(\tilde{x}(t))$  remains bounded on  $[t_{\sigma,i_j^d}, t_{\sigma,i_j^d+1}]$  and hence so does  $\tilde{x}(t)$ . In conclusion,  $\tilde{x}(t)$  is bounded on  $[t_j^*, t_{\sigma,i_j^d+1}]$ .

Similarly, the boundedness of  $\tilde{x}(t)$  on the subsequent time period  $[t_{\sigma,i_j^d+1}, t_{\sigma,i_{j+1}^d+1}]$  is obtained if  $V_q(\tilde{x}(t_{\sigma,i_j^d+1})) < R_V$  for some  $q \in \mathbb{Q}$ . We shall show that this holds true with  $q = q_{\sigma,i_j^d}$ . Applying i) of Proposition 4.1, we have

$$V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d+1})) \leq \max\left\{\omega_1\left(V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d})), \tau_p\right), \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|))\right\}. \quad (89)$$

By stacking (87), (88), and (89) and using Assumption 3.2, we obtain

$$\begin{aligned} V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d+1})) &\leq \max\left\{\mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|)), \right. \\ &\omega_1(\rho(\omega_2(\mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|)), T_p)), \tau_p), \\ &\left.\omega_1(\rho(\omega_2(V_q(\tilde{x}(t_j^*)), T_p)), \tau_p)\right\} \\ &\leq \max\{\chi(\|w_\Theta\|), \omega_0(V_q(\tilde{x}(t_j^*)), \tau_0)\} \end{aligned} \quad (90)$$

where we have used the property (40) from Assumption 3.2. As  $\chi(\|w_\Theta\|) \leq V_{ini} \leq R_V$  and  $\omega_0(s, \tau_0) < s$ ,  $V_q(\tilde{x}(t_j^*)) \leq R_V$ , (90) shows that  $V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d+1})) < R_V$  and subsequently  $\tilde{x}(t)$  is bounded on  $[t_{\sigma,i_j^d+1}, t_{\sigma,i_{j+1}^d+1}]$ .

Now, let  $t_0^* = t_{\sigma,0} = t_0$  if  $t_{\sigma,0} < t_{\sigma,i_0^d}$  and  $t_0^* = t_{\sigma,1}$  if  $t_{\sigma,0} = t_{\sigma,i_0^d}$ . We claim that  $\tilde{x}(t)$  is bounded on  $[t_0, t_0^*]$  and  $V_q(\tilde{x}(t_0^*)) < R_V$  for some  $q \in \mathbb{Q}$ . This claim is obvious for  $t_0^* = t_{\sigma,0}$ . In the case  $t_0^* = t_{\sigma,1}$ , as the first switching event is of dwell-time and  $t_0^*$  is its end time, from i) of Proposition 4.1 and the property  $V_{q_{\sigma,i_0^d}}(\tilde{x}(t_{\sigma,i_0^d})) = V_{q_{\sigma,0}}(\tilde{x}(t_0)) < R_V$ , it follows that  $\tilde{x}(t)$  is bounded on  $[t_0, t_0^*]$ . In addition, at the time  $t_0^*$ , as  $i_0^d = 0$ , using i) of Proposition 4.1, we have either  $V_{q_{\sigma,0}}(\tilde{x}(t_0^*)) = V_{q_{\sigma,i_0^d}}(\tilde{x}(t_{\sigma,i_0^d+1})) \leq \chi(\|w_\Theta\|) < R_V$  or  $V_{q_{\sigma,i_0^d}}(\tilde{x}(t_{\sigma,i_0^d+1})) \leq \omega_1(V_{q_{\sigma,i_0^d}}(\tilde{x}(t_{\sigma,i_0^d})), \tau_p) \leq G_1^{-1}(G_1(V_{q_{\sigma,0}}(\tilde{x}(t_{\sigma,i_0^d}))) + 0) = V_{q_{\sigma,i_0^d}}(\tilde{x}(t_{\sigma,i_0^d})) = V_{q_{\sigma,0}}(\tilde{x}(t_{\sigma,0})) < R_V$  so that  $V_{q_{\sigma,0}}(\tilde{x}(t_0^*)) < R_V$ . In conclusion, our claim holds true.

Thus, by the preceding argument, we conclude that  $\tilde{x}(t)$  is bounded on  $t \in [t_0, \infty)$  and hence the forward completeness of  $\tilde{x}$  and satisfaction of Assumption 2.2 follow.

From the properties  $\omega_0(s, \tau_0) < s$  and  $\omega_0(\omega_0(a, s), t) \leq \omega_0(a, s + t)$  as given in Assumption 3.2, applying (90) successively from  $t_{\sigma,i_j^d+1}$  back to  $t_0^*$ , we have

$$\begin{aligned} V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d+1})) &\leq \max\left\{\chi(\|w_\Theta\|), \omega_0\left(V_{q_{\sigma,i_j^d-1}}(\tilde{x}(t_{\sigma,i_j^d-1+1})), \tau_0\right)\right\} \\ &\leq \dots \\ &\leq \max\{\chi(\|w_\Theta\|), \omega_0(V_{q_{\sigma,0}}(\tilde{x}(t_0^*)), (j-1)\tau_0)\}. \end{aligned} \quad (91)$$

As  $\omega_0 \in \mathcal{KL}$ , taking the limits of  $V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d+1}))$  as  $j \rightarrow \infty$  we obtain

$$\limsup_{j \rightarrow \infty} V_{q_{\sigma,i_j^d}}(\tilde{x}(t_{\sigma,i_j^d+1})) \leq \chi(\|w_\Theta\|). \quad (92)$$

In addition, since  $V_{q_{\sigma, i_j^d}}(\tilde{x}(t_{\sigma, i_j^d+1})) < R_V$ , from ii) of Proposition 4.1, we have  $V_{q_{\sigma, i_j^d}}(x(t)) \leq \rho(V_{q_{\sigma, i_j^d}}(\tilde{x}(t))) \leq \rho(\omega_2(V_{q_{\sigma, i_j^d}}^w(t_{\sigma, i_j^d+1}), T_p)), \forall t \in [t_{\sigma, i_j^d+1}, t_{\sigma, i_j^d+1}]$ . Since  $V_q$ 's are nonnegative, this implies that

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \max_{t \in \mathcal{I}_j^p} \text{Var}_{t_{\sigma, i_j^d+1}}^t \left[ V_{q_{\sigma, i_j^d}}, V_{q_{\sigma, i_j^d}} \right] (x) \\ & \leq \limsup_{j \rightarrow \infty} \max_{t \in \mathcal{I}_j^p} V_{q_{\sigma, i_j^d}}(x(t)) \\ & \leq \limsup_{j \rightarrow \infty} \rho \left( \omega_2 \left( V_{q_{\sigma, i_j^d}}^w(t_{\sigma, i_j^d+1}), T_p \right) \right) \\ & \leq \max \left\{ \rho(\omega_2(\chi(\|w_\Theta\|), T_p)), \right. \\ & \quad \left. \limsup_{j \rightarrow \infty} V_{q_{\sigma, i_j^d}}(\tilde{x}(t_{\sigma, i_j^d+1})) \right\} \\ & \leq \rho(\omega_2(\chi(\|w_\Theta\|), T_p)) \stackrel{\text{def}}{=} \gamma_2(\|w_\Theta\|). \end{aligned} \quad (93)$$

Thus, condition ii) of Theorem 2.1 is satisfied.

As the satisfaction of the conditions of Theorem 2.1 is independent of switching signal, applying Theorem 2.1, the conclusion of the theorem follows. ■

**Theorem 4.2:** Suppose that the hypotheses of Theorem 4.1 hold. Then, under the control  $u = \alpha_n$  given by (62),  $j = n$  with  $\varrho_i$ 's satisfying (76), the output  $x_1(t)$  converge to the set  $B_{\epsilon_d} = \{s \in \mathbb{R} : |s| \leq \epsilon_d\}$  and the trajectory  $\tilde{x}(t)$  remains bounded.

*Proof:* As  $w_\Theta(t)$  is bounded, by Theorem 4.1, there is a class- $\mathcal{K}_\infty$  function  $\gamma$  independent of switching sequence  $\sigma \in \mathcal{S}$  such that  $\tilde{x}(t) \rightarrow \{\zeta \in \mathbb{R}^{d \times n} : \|\zeta\| \leq \gamma(\|w_\Theta\|)\}$ . Our first purpose is to show that  $K_\Theta(t)$  is bounded. Indeed, suppose that the converse holds. Then there is a divergent sequence  $\{t_h\}_j \subset [t_0, \infty)$  such that  $\dot{K}_\Theta(t_j) \neq 0, \forall j \in \mathbb{N}$ . From the update law (78), it must hold that  $|\xi_1(t_j)| \geq \epsilon_d, \forall j \in \mathbb{N}$ . Let  $\epsilon > 0$  and  $\delta > 0$  be numbers satisfying  $\gamma(\epsilon) + \delta < \epsilon_d$ .

As  $\Theta_n$  is bounded and  $K_\Theta$  is unbounded and nondecreasing, there is a time  $t(\epsilon)$  such that  $w_\Theta(t) = \Theta_n/K_\Theta(t) \leq \epsilon, \forall t \geq t(\epsilon)$ . Let  $t_{\sigma, i(\epsilon)}$  be the switching time of  $\sigma$  that is greater than  $t(\epsilon)$ . By Theorem 4.1, the state  $\tilde{x}(t)$  of the switched error system remains bounded for  $t \leq t_{\sigma, i(\epsilon)}$ . As a result, the trajectory  $\tilde{x}(t)$ ,  $t \geq t_{\sigma, i(\epsilon)}$  of the switched error system is the trajectory  $\tilde{x}'(t)$  of the same switched error system with initial state  $\tilde{x}'(0) = \tilde{x}(t_{\sigma, i(\epsilon)})$ , initial value of the time-varying parameter  $\theta(0) = \theta(t_{\sigma, i(\epsilon)})$ , input  $\tilde{w}_\Theta$  defined by  $\tilde{w}_\Theta(t) = w_\Theta(t + t_{\sigma, i(\epsilon)}), t \geq 0$ , and switching sequence  $\sigma(\epsilon)$  defined by  $(q_{\sigma(\epsilon), i}, \Delta\tau_{\sigma(\epsilon), i}) = (q_{\sigma, i+i(\epsilon)}, \Delta\tau_{\sigma, i+i(\epsilon)}), i \in \mathbb{N}$ . Obviously  $\sigma(\epsilon)$  also has the persistent dwell-time  $\tau_p$  with the period of persistent  $T_p$ . By Theorem 4.1,  $\limsup_{t \rightarrow \infty} \|\tilde{x}'(t)\| \leq \gamma(\|\tilde{w}_\Theta\|) \leq \gamma(\epsilon)$ . As  $\tilde{x}'(t) = \tilde{x}(t + t(\epsilon)), t \geq 0$  and  $\|\tilde{w}_\Theta\| \leq \epsilon$ , it follows that there is  $T_\delta \in \mathbb{R}^+$  such that  $\|\tilde{x}(t + t(\epsilon))\| = \|\tilde{x}'(t)\| \leq \gamma(\epsilon) + \delta < \epsilon_d, \forall t \geq T_\delta$ . Thus  $|\xi_1(t_j)| \leq \|\tilde{x}(t)\| \leq \gamma(\epsilon) + \delta < \epsilon_d$ , for sufficiently large  $j$  which is a contradiction. Therefore,  $K_\Theta(t)$  is bounded.

Finally, as  $\tilde{x}(t)$  is continuous and bounded by Theorem 4.1,  $\xi_1(t)$  and hence  $\dot{K}_\Theta(t)$  are uniformly continuous. Thus, the monotonicity from the update law (78) and the boundedness of  $K_\Theta(t)$  show that  $\lim_{t \rightarrow \infty} \int_0^t \dot{K}_\Theta(s) ds$  exists and is finite. By Barbalat's lemma, we have  $\lim_{t \rightarrow \infty} \dot{K}_\Theta(t) = 0$ . Therefore,  $\xi_1(t) \rightarrow B_{\epsilon_d}, t \rightarrow \infty$  as, otherwise  $\lim_{t \rightarrow \infty} K_\Theta(t) \neq 0$ , a contradiction. ■

## V. ADAPTIVE OUTPUT FEEDBACK STABILIZATION

Instead of utilizing dynamical properties of the unmeasured dynamics, output feedback control deal with restricted measurement by exploiting structural properties to estimate unmeasured variables. In this direction, the main goal is to develop separation principles [39], [40]. When the running times of driving systems are decision variables which can be made as long as desired, a separation principle has been introduced for switched systems [24]. In persistent dwell-time switched systems exhibiting arbitrarily fast switching, it is practically impossible to synchronize switching in driving systems and switching in observers. As a result, a non-separation principle approach is more appealing. In this section, we introduce a gauge design approach for the problem  $\mathcal{P}_2$  of output feedback control of switched systems. The main novelty lies in the combination of the gauge design and the adaptive gain technique [41], [42] so that control gains dependent on unmeasured variables are allowed.

Consider the switched system (1) whose driving dynamics are described by (36). We have the following assumptions.

**Assumption 5.1:** There are known constants  $\Delta_G > 0, L_F > 0$ , and  $g_j^o, j = 1, \dots, n$ , such that for all  $z \in \mathbb{R}^d, \bar{x}_j \in \mathbb{R}^j, q \in \mathbb{Q}$ , and  $j = 1, \dots, n$ , we have

$$\begin{aligned} |g_{q,j}(z, \bar{x}_j) - g_j^o| & \leq \Delta_G \\ |f_{q,j}(z, \bar{x}_j)| & \leq L_F \left( \|z\|^{p/2} + |x_j| + \dots + |x_n| \right). \end{aligned} \quad (94)$$

**Assumption 5.2:** The system (1) satisfies Assumption 3.1 for  $\rho(s) = a_\rho s, \alpha_1(s) = a_{\alpha,1} s$ , and  $\alpha_2(s) = a_{\alpha,2} s$ , where  $a_\rho, a_{\alpha,1}$ , and  $a_{\alpha,2}$  are non-zero and positive constants. In addition,  $a_{\alpha,1} > a_{\alpha,2}, (a_{\alpha,1} - a_{\alpha,2})U_q(\zeta) \geq \|\zeta\|^p, \forall \zeta \in \mathbb{R}^d, q \in \mathbb{Q}$  and  $(a_{\alpha,1} - a_{\alpha,2})\tau_p - 2a_{\alpha,2}T_p > \ln a_\rho$ .

**Remark 5.1:** The Lipchitz-like condition in Assumption 5.1 on  $f_{q,j}$ 's is instrumental in output feedback control of nonlinear systems [39], [40], [43]. Our enhancement lies in the permission of control gains  $g_{q,j}$  depending on the unestimated state  $z$ .

Let  $P, Q$ , and  $\Lambda$  be positive symmetric matrices satisfying

$$A^T P + P A \leq -2Q, \quad DP + PD \geq 0, \quad (95)$$

$$\Gamma^T \Lambda + \Lambda \Gamma \leq -2I, \quad D\Lambda + \Lambda D \geq 0, \quad \text{and} \quad (96)$$

$$\Delta_G^* I \leq Q \quad (97)$$

where  $A = \{A_{j,l}\}, \Gamma = \{\Gamma_{j,l}\}$  are constant matrices whose elements are  $A_{j,1} = -a_j, \Gamma_{n,j} = -\gamma_j, j = 1, \dots, n, A_{j+1,j} = \Gamma_{j+1,j} = g_j^o, j = 1, \dots, n-1$ , and the rest are zero,  $D = \text{diag}\{1, \dots, n\}, I$  is the identity matrix, and

$$\Delta_G^* = \frac{\lambda_P^2}{2} \Delta_G^2 \left( 1 + \frac{\|\bar{\gamma}\|^2}{g_n^o} \right) \left( 1 + \frac{\lambda_\Lambda}{\lambda_P} \|\bar{a}\|^2 \right) + \lambda_P \Delta_G \quad (98)$$

where  $\lambda_P$  and  $\lambda_\Lambda$  are maximal eigenvalues of  $P$  and  $\Lambda$  respectively,  $\bar{a} = [a_1, \dots, a_n]^T$  and  $\bar{\gamma} = [\gamma_1, \dots, \gamma_n]^T$ .

**Remark 5.2:** In nonlinear systems, arbitrary  $A$  and  $Q$  in the solution set of (95) can be taken [39], [40]. By (97), we express that in switched systems, the poles of the observer might be sufficiently large in order to deal with the discrepancy between control gains. In general, this condition can be satisfied by adjusting  $A$  and  $Q$ . In the case  $\Delta_G = 0$ , i.e., the control gains of driving

systems are identical, the condition is obviously automatically satisfied.

For two vectors  $a, b \in \mathbb{R}^n$ , we have

$$\sum_{i=1}^n |a_i| \sum_{j=1}^i |b_j| = \sum_{i=1}^n \sum_{j=i}^n |a_j| |b_{j-i+1}| \leq n \|a\| \|b\|. \quad (99)$$

### A. Adaptive High-Gain Observer

Our goal is to estimate the controlled state  $x$ . Let  $\hat{x} = [\hat{x}_1, \dots, \hat{x}_n]^T$  denote the estimate of  $x$ , and let  $\lambda > 0$  be the time-varying observer's high-gain. We have the following reduced-order adaptive observer:

$$\dot{\hat{x}}_j = g_j \hat{x}_{j+1} + \lambda^j a_j (x_1 - \hat{x}_1), \quad j = 1, \dots, n \quad (100)$$

$$\lambda = k_\lambda e_1^2, \quad \lambda(0) = 1 \quad (101)$$

where  $\hat{x}_{n+1} \stackrel{\text{def}}{=} u$  and  $k_\lambda > 0$  is the tuning gain of  $\lambda$ .

Consider the following scaled variables

$$e_j = \frac{x_j - \hat{x}_j}{\lambda^j} \text{ and } \zeta_j = \frac{\hat{x}_j}{\lambda^j}, \quad j = 1, \dots, n. \quad (102)$$

According to (36) and (100), the equations describing the driving dynamics of  $e_j$ 's are

$$\begin{aligned} \dot{e}_j = & \frac{1}{\lambda^j} \left( (g_{q,j}(\cdot) - g_j^0) x_{j+1} + g_j^0 (x_{j+1} - \hat{x}_{j+1}) \right. \\ & \left. + f_{q,j}(z, \bar{x}_j, \theta) - \lambda^j a_j (x_1 - \hat{x}_1) \right) - j \frac{\dot{\lambda}}{\lambda} e_j. \end{aligned} \quad (103)$$

Let  $e = [e_1, \dots, e_n]^T$ ,  $\zeta = [\zeta_1, \dots, \zeta_n]^n$ . The driving dynamics of  $e$  are described by

$$\dot{e} = \lambda A e - \frac{\dot{\lambda}}{\lambda} D e + w_q^e(z, \bar{x}_{j+1}, \theta) \quad (104)$$

where  $w_q^e = [w_{q,1}^e, \dots, w_{q,n}^e]^T$ ,  $q \in \mathbb{Q}$  and

$$w_{q,j}^e = \frac{1}{\lambda^j} \left( (g_{q,j}(z, \bar{x}_j, \theta) - g_j^0) x_{j+1} + f_{q,j}(z, \bar{x}_j, \theta) \right), \quad j = 1, \dots, n, q \in \mathbb{Q}. \quad (105)$$

In view of (105), the error dynamics contain the functions  $w_q^e$ 's playing the role of destabilizing inputs. Accordingly, the stability of (104) is still in question. This introduces a distinction from output feedback control of nonlinear systems [41]–[43]. In the following, we present a gauge design adopting the adaptive high-gain technique in [42], [43].

### B. Control Design

Let  $r_0 = \lambda_\lambda^2 \|\bar{a}\|^2 / \lambda_P$  and define the functions

$$V_e(e) = (1 + r_0) e^T P e \quad \text{and} \quad V_\zeta(\zeta) = \zeta^T \Lambda \zeta. \quad (106)$$

Consider the gauge Lyapunov function  $V_g(e, \zeta) = V_e(e) + V_\zeta(\zeta)$  which shall be used as a gauge for the unestimated state  $z$ . Let  $a_\mu = a_{\alpha,2}$ . Along the evolution of  $\mathcal{E} = [e^T, \zeta^T]^T$ , we have the following gauges:

$$a_\mu U_q(z(t)) \leq V_g(\mathcal{E}(t)) = V_e(e(t)) + V_\zeta(\zeta(t)), \quad q \in \mathbb{Q}. \quad (107)$$

By Assumption 5.2, whenever (107) holds true, we have  $\|z\|^p \leq V_e + V_\zeta$  so that

$$\begin{aligned} \|z\|^{p/2} &= a_\mu^{-1/2} (a_\mu \|z\|^p)^{1/2} \leq a_\mu^{-1/2} (V_e + V_\zeta)^{1/2} \\ &\leq a_\mu^{-1/2} (V_e)^{1/2} + a_\mu^{-1/2} V_\zeta^{1/2}. \end{aligned} \quad (108)$$

The derivative  $DV_g(\mathcal{E}(t))$  can be computed through the derivatives of  $V_e$  and  $V_\zeta$  as follows.

As  $g_{q,j}(\cdot)$ 's are positive, using Assumption 5.1, (108) and substituting  $x_j$  by  $\hat{x}_j + \lambda^j e_j$ , we obtain

$$\begin{aligned} |w_{q,j}^e| &\leq \frac{|g_j(\cdot) - g_j^0|}{\lambda^j} |\hat{x}_{j+1} + e_{j+1} \lambda^{j+1}| \\ &\quad + \frac{L_F}{\lambda^j} \left( \|z\|^{p/2} + \sum_{l=1}^j |\hat{x}_l + \lambda^l e_l| \right) \\ &\leq \frac{\Delta_G}{\lambda^j} |\hat{x}_{j+1}| + \Delta_G |e_{j+1}| \lambda \\ &\quad + \frac{L_F}{\lambda^j} \left( \sqrt{\frac{V_e}{a_\mu}} + \sqrt{\frac{V_\zeta}{a_\mu}} + \sum_{l=1}^j |\hat{x}_l + \lambda^l e_l| \right). \end{aligned} \quad (109)$$

Recall that  $\tilde{t} = t - t_0$  and the index of the driving dynamics at a time  $t$  is  $i_\sigma^-(\tilde{t})$ . From (96) and the designated positiveness of  $PD + DP$  in (95) and  $\dot{\lambda}/\lambda$  in (101), it follows that:

$$\begin{aligned} DV_e(e(t)) &= (1 + r_0) \\ &\quad \times \left( \lambda e^T (A^T P + P A) e - \frac{\dot{\lambda}}{\lambda} e^T (PD + DP) e \right) \\ &\quad + \sum_{j=1}^n e^T P_j w_{q, i_\sigma^-(\tilde{t}), j}^e \\ &\leq -2\lambda(1 + r_0) e^T Q e + (1 + r_0) \\ &\quad \times \left( \sum_{j=1}^n \frac{\Delta_G}{\lambda^j} |e^T P_j| |\hat{x}_{j+1}| \right. \\ &\quad \left. + \lambda \Delta_G \sum_{j=1}^n |e^T P_j| |e_{j+1}| \right. \\ &\quad \left. + \sum_{j=1}^n \frac{L_F |e^T P_j|}{\lambda^j} a_\mu^{-1/2} (\sqrt{V_e} + \sqrt{V_\zeta}) \right. \\ &\quad \left. + \sum_{j=1}^n |e^T P_j| \sum_{l=1}^j \frac{L_F |e_l|}{\lambda^{j-l}} \right. \\ &\quad \left. + \sum_{j=1}^n |e^T P_j| \sum_{l=1}^j \frac{L_F |\hat{x}_l|}{\lambda^j} \right) \end{aligned} \quad (110)$$

whenever (107) holds true, where  $P_j$  is the  $j$ -th column of  $P$ .

As  $\hat{x}_{n+1} = u$  shall be designed in the form (114), from (102), we have  $|\hat{x}_{j+1}|/\lambda^j = \lambda |\zeta_{j+1}|$ ,  $j = 1, \dots, n-1$  and  $|\hat{x}_{n+1}|/\lambda^n = \lambda |\gamma_1 \zeta_1 + \dots + \gamma_n \zeta_n|/g_n^0$  so that  $\|[\hat{x}_2/\lambda^2, \dots, \hat{x}_{n+1}/\lambda^{n+1}]^T\| \leq \lambda \|\zeta\| \sqrt{1 + \|\bar{\gamma}\|^2/g_n^0}$ . In addition, we have  $e_{n+1} = 0$  and  $\|e^T P\| \leq \lambda_P \|e\|$  due to the maximality of  $\lambda_P$ . These and (98) together lead to

$$\begin{aligned}
& \sum_{j=1}^n \Delta_G \lambda^{-j} |e^T P_j| |\hat{x}_{j+1}| + \lambda \Delta_G \sum_{j=1}^n |e^T P_j| |e_{j+1}| \\
& \leq \lambda \Delta_G \sqrt{1 + \|\bar{\gamma}\|^2 / g_n^2} |e^T P| \|\zeta\| + \lambda \Delta_G \|e^T P\| \|e\| \\
& \leq \lambda \lambda_P \Delta_G \left( \sqrt{1 + \|\bar{\gamma}\|^2 / g_n^2} \|e\| \|\zeta\| + \|e\|^2 \right) \\
& \leq \lambda \Delta_G^* \|e\|^2 + \lambda \|\zeta\|^2 / (2(1 + r_0)) \quad (111)
\end{aligned}$$

where we have used Young's inequality to decouple  $\lambda_P \Delta_G \sqrt{1 + \|\bar{\gamma}\|^2 / g_n^2} \|e\| \|\zeta\|$  into  $\lambda_P^2 \Delta_G^2 (1 + \|\bar{\gamma}\|^2 / g_n^2) (1 + r_0) \|e\|^2 / 2 + \lambda \|\zeta\|^2 / (2(1 + r_0))$ .

Let  $S_{(110)}$  be the sum of the last three terms in (110). Using inequalities  $V_e(e) \leq \lambda_P \|e\|^2$ ,  $V_\zeta(\zeta) \leq \lambda_\Lambda \|\zeta\|^2$ , and  $\|e^T P\| \leq \lambda_P \|e\|$ , (99) and Cauchy-Schwartz and Young's inequalities, it is straightforward that

$$\begin{aligned}
S_{(110)} & \leq \lambda^{-1} L_F \sqrt{n} a_\mu^{-1/2} (\sqrt{V_e} + \sqrt{V_\zeta}) \|e^T P\| \\
& \quad + n L_F \|e^T P\| \|e\| + n L_F \|e^T P\| \|\zeta\| \\
& \leq \frac{L_F \sqrt{n}}{\lambda a_\mu^{1/2}} \left( \lambda_P^2 \|e\|^2 + \frac{1}{2} (\lambda_P^2 \|e\|^2 + \lambda_\Lambda \|\zeta\|^2) \right) \\
& \quad + n L_F \lambda_P \|e\|^2 + n L_F (\lambda_P^2 \|e\|^2 + \|\zeta\|^2) / 2. \quad (112)
\end{aligned}$$

Substituting (111) and (112) into (110) and rearranging the obtained inequality, we have

$$DV_e(e) \leq -(1 + r_0)(\lambda - \ell) e^T Q e + (\lambda/2 + \ell(1 + r_0)) \|\zeta\|^2 \quad (113)$$

whenever (107) holds true, where  $\ell \geq \max\{\lambda^{-1} a_\mu^{-1/2} \lambda_P^2 \sqrt{n} + \lambda_P^2 (\sqrt{n} + n) / (2\lambda) + n \lambda_P \sqrt{n} \lambda_\Lambda / (2\lambda \sqrt{a_\mu}) + n/2\}$  is a constant. Such  $\ell$  exists since  $\lambda$  is non-decreasing by (101).

On the other hand, a direct computation from (100) and definition of  $\Gamma$  and  $D$  shows that under the control

$$u = -\lambda^{n+1} (\gamma_1 \zeta_1 + \dots + \gamma_n \zeta_n) / g_n^0 \quad (114)$$

the dynamics equation for  $\zeta$  is

$$\dot{\zeta} = \lambda \Gamma \zeta - \frac{\dot{\lambda}}{\lambda} D \zeta + \lambda \bar{a} e_1. \quad (115)$$

Thus, using (96) and noting that  $\dot{\lambda}/\lambda \geq 0$  by (101), we have

$$\begin{aligned}
DV_\zeta & \leq -2\lambda \|\zeta\|^2 - 2 \frac{\dot{\lambda}}{\lambda} \zeta^T \Lambda (D + bI) \zeta + 2\lambda \|\zeta^T \Lambda\| \|\bar{a}\| \|e_1\| \\
& \leq -2\lambda \|\zeta\|^2 + \lambda \|\zeta\|^2 + \lambda \lambda_{\max, \Lambda}^2 \|\bar{a}\|^2 \|e\|^2 \\
& \leq -\lambda \|\zeta\|^2 + \lambda r_0 e^T Q e. \quad (116)
\end{aligned}$$

From (113) and (116), we have the following inequality whenever (107) holds true:

$$DV_g(\mathcal{E}(t)) \leq -(\lambda - \ell(1 + r_0)) (e^T Q e + 0.5 \|\zeta\|^2). \quad (117)$$

**Theorem 5.1:** Under Assumptions 3.1, 5.1, and 5.2, the evolution  $\mathcal{E}$  of the switched system (1), whose driving dynamics are described by (36) and whose input is (114), satisfies  $\limsup_{t \rightarrow \infty} \|\mathcal{E}(t)\| = 0$ .

*Proof:* In view of (117) and Assumption 5.2, the dissipation rate of  $V_g$  can be matched to that of  $U_q$ 's when  $\lambda$  is sufficiently large. Therefore, the proof follows paradigms of Section IV-B. and [42]. Here, we shall provide a sketch of the proof whose details can be found in [32, Chapter 7].

We first verify Assumption 3.2. Let  $\tilde{a}_1 = (a_{\alpha,1} - a_{\alpha,2})$  and  $\tilde{a}_2 = 2a_{\alpha,2}$ . Define  $\mu(s) = \tilde{a}_1 s$ . By Assumption 5.2, the functions  $\omega_1$  and  $\omega_2$  are  $\omega_1(s, t) = s \exp(-\tilde{a}_1 t)$  and  $\omega_2(s, t) = s \exp(\tilde{a}_2 t)$ ,  $s, t \in \mathbb{R}^+$ . Hence,  $\omega_1(\rho(\omega_2(s, T_p)), \tau_p)$  becomes  $a_\rho \exp(\tilde{a}_2 T_p - \tilde{a}_1 \tau_p) s$  and a pair  $(\tau_0, \omega_0)$  satisfying Assumption 3.2 is  $\omega_0(s, t) = s \exp(-t)$ ,  $\tau_0 \leq \tilde{a}_1 \tau_p - \tilde{a}_2 T_p - \ln a_\rho$ . The satisfaction of the rest of Assumption 3.2 is obvious.

We now prove the boundedness of  $\lambda$ . Suppose that  $\lambda$  is unbounded. Then, there is a time  $t_\lambda > 0$  such that  $\lambda(t) - \ell(r_0 + 1) \geq a_1$ , the dissipation rate of  $z$ -dynamics. In this case, it follows from (117) that  $DV_g(\mathcal{E}(t)) \leq -a_1 V_g(\mathcal{E}(t))$  whenever (107) holds true. Thus, following the paradigm of the proof of Theorem 4.1, we conclude that the system  $(z, e, \zeta)$  satisfies conditions of Theorem 2.1 with auxiliary functions  $V_q(z, e, \zeta) = \max\{U_q(z), \tilde{a}_1^{-1} V_g(e, \zeta)\}$ ,  $q \in \mathbb{Q}$ . As a result,  $e_1(t) \rightarrow 0$ ,  $t \rightarrow \infty$ , which coupled with the boundedness and the continuity of  $e_1(t)$  further implies that  $\lambda(\infty) - \lambda(0) = \int_0^\infty e_1^2(t) dt = e_1^2(\infty) - e_1^2(0) < \infty$  which is a contradiction. Hence,  $\lambda$  is bounded.

Finally, in view of (100), the observer dynamics is independent of the switching sequence. Following the paradigm of [42], we have  $e(t), \zeta(t) \rightarrow 0, t \rightarrow \infty$ . ■

## VI. DESIGN EXAMPLES

### A. Adaptive Output Regulation

Consider the switched system with the driving dynamics

$$\Sigma_q : \dot{z} = Q_q(z, x_1, \theta), \quad \dot{x} = f_q(z, x, \theta) \quad (118)$$

where  $x = [x_1, x_2]^T$ ,  $z = [z_1, z_2]^T \in \mathbb{R}^2$ ,  $\theta = [\theta_1, \dots, \theta_5]^T \in \mathbb{R}^5$ ,  $q \in \{1, 2\}$ ,  $Q_1 = [-z_1(1 + z_1^4 + z_2^2) + (1/4)z_2(1 + \sin x_1), -z_2(1 + z_1^4 + z_2^2) - (1/4)z_1^3(1 + \sin x_1)]^T$ ,  $f_1 = [\theta_1 x_2 + e^{\theta_2 x_1} z_1^2, (2 + x_2^2 + \theta_1 z_2^2)u + \theta_2 x_1 z_2]^T$ ,  $Q_2 = [-8z_1 + 2z_2, -z_1 - 12z_2 + x_1]^T$ , and  $f_2 = [\theta_3 x_2 + 2\theta_4 \sqrt{z_1^2 + 3z_2^2}, (\theta_5 + x_2^2)u + \theta_4 x_2(z_1 + z_2)]$ . Let  $\theta_1(t) = 1 + \sin^2 t$ ,  $\theta_2(t) = \cos t$ ,  $\theta_3(t) = 2 + \cos t$ ,  $\theta_4(t) = \sin t$ , and  $\theta_5(t) = 2 - \sin t$ .

We have the following auxiliary functions:

$$U_1(z) = \frac{1}{4} z_1^4 + \frac{1}{2} z_2^2, \quad \text{and} \quad U_2(z) = z_1^2 + z_1 z_2 + 4z_2^2. \quad (119)$$

With the help of Young's inequality, we have

$$\begin{aligned}
\frac{\partial U_1}{\partial z} Q_1(\cdot) & \leq -4U_1^2(z) - 2U_1(z) \\
\frac{\partial U_1}{\partial z} Q_2(\cdot) & \leq \frac{U_1^2(z)}{150} + \frac{U_1(z)}{12} + \frac{x_1^2}{24} \\
\frac{\partial U_2}{\partial z} Q_1(\cdot) & \leq \frac{1}{2} U_2(z) \\
\frac{\partial U_2}{\partial z} Q_2(\cdot) & \leq -4U_2(z) + x_1^2. \quad (120)
\end{aligned}$$

From (119), a function  $\rho$  to satisfy Assumption 3.1 is  $\rho(s) = 3\sqrt{s} + 9s + s^2/4$ . Let the gauge inequality be  $U_q(z) \leq V_g(\xi)$ ,

i.e.,  $\mu(s) = s$ . Then, using (120), it can be computed that, in the ball  $U_1(z) \leq 150/4$ , the functions  $\omega_1$  and  $\omega_2$  in Assumption 3.1 are  $\omega_2(s, t) = se^{t/2}$  and  $\omega_1(s, t) = se^{-2t}$ . Hence, (40) becomes  $\rho(s)e^{-(2\tau_p - T_p/2)} \leq \omega_0(s, \tau_0)$ , which, given  $H > 0$ , can be satisfied for  $s \leq H$ ,  $2\tau_p - T_p/2 > \ln \rho(H)$ ,  $\omega_0(s, t) = se^t$ . In addition, the lower bounds of control gains are  $g_{\min,1} = 1$  and  $g_{\min,2} = 1 + x_2^2$ . Thus, conditions of Theorem 4.2 are satisfied. Following the design procedure in Section IV, we obtain the following control

$$\begin{aligned} u &= -\frac{1}{1+x_2^2} (k_3+k_4\xi_2^2+K_\Theta A^2 x_2^2 \\ &\quad + K_\Theta A^2 (e^{\xi_1^2}+1) (\xi_1^2+\xi_2^2) \\ &\quad + K_\Theta (\xi_1^2+\xi_2^2) x_2^2 + K_\Theta (e^{\xi_1^2}+4) \xi_1^2) \xi_2 \\ \alpha_1 &= -\left[ k_1+k_2\xi_1^2+K_\Theta (e^{\xi_1^2}+2) \xi_1^2 \right] \xi_1 \\ A &= k_1+3k_2\xi_1^2+3K_\Theta (e^{\xi_1^2}+2) \xi_1^2+2K_\Theta e^{\xi_1^2} \xi_1^4 \end{aligned} \quad (121)$$

where  $k_1, \dots, k_4$  are design parameters,  $\xi_1 = x_1$ ,  $\xi_2 = x_2 - \alpha_1$ , and  $K_\Theta$  is updated by (78). A value for the unknown constant  $\Theta_2$  is  $\Theta_2 = 7 > \sup\{\theta_1^2/4 + 2e^{\theta_2^2} + \theta_2^2/2, \theta_3^2/4 + 3\theta_4^2\}$ .

The simulation data are:  $k_1 = k_3 = 1$ ,  $k_2 = k_4 = 0.5$ ,  $(z(0), x(0)) = [1, -2, 0.4, -1]^T$ . The desired accuracy is  $\epsilon_d = 0.01$  and the tuning gain is  $k_K = 100$ . Timing parameters of the switching sequence are  $T_p = 0.2s$ ,  $\tau_p = 0.8s$ .

The simulation results are shown in Fig. 1. It can be seen from Fig. 1(a) that output regulation was well obtained. The peak points in control signal are due to the fast transient periods caused by changes of active subsystems. It is also observed from Fig. 1(b) that the adaptive gain  $K_\Theta$  converges to a fixed value and the remaining signals are bounded.

### B. Output Feedback

Consider the switched system whose driving dynamics are described by (118), where  $Q_1 = [-z_1 + z_2 z_1, -z_2(1 + z_2^2) - z_1^2 + z_2 x_1]^T$ ,  $f_1 = [\theta_1 x_2 + \ln(1 + \|z\|^2), (2 + \sin z_2^2)u + \theta_2 x_1]^T$ ,  $Q_2 = [-3z_1 + z_2, z_1 - 2z_2 + x_1]^T$ ,  $f_2 = [\theta_3 x_2 + 2\theta_4 x_1, \theta_5 u + \sqrt{z_1^2 + 3z_2^2}]^T$ . Let  $\theta_1(t) = 1 + \sin^2 t$ ,  $\theta_2(t) = \cos t$ ,  $\theta_3(t) = 1.5 + 0.6 \cos t$ ,  $\theta_4(t) = \sin t$ , and  $\theta_5(t) = 2.5 - 0.5 \sin t$ .

We have the auxiliary functions  $U_1(z) = U_2(z) = U(z) = z_1^2 + z_2^2$ . By a plain computation, it can be verified that

$$\frac{\partial U}{\partial z} Q_q(z, x_1) \leq -2U(z) + x_1^2, \quad \forall q \in \{1, 2\}. \quad (122)$$

Since  $U(z)$  is in quadratic form, Assumption 5.2 is satisfied for  $p = 2$ ,  $a_p = 1$ ,  $a_{\alpha,1} = 2$ ,  $a_{\alpha,2} = 1$  and the condition on  $\tau_p$  and  $T_p$  is  $\tau_p > 3T_p$ . In addition, from the given time-varying parameters, we have  $g_1^0 = 1.5$ ,  $g_2^0 = 2.5$  and  $\Delta_G = 0.6$ . Let us choose  $a_1 = 30$ ,  $a_2 = 25$ ,  $P = [2.65 \ -3; -3 \ 3.7]$ ,  $Q = [9 \ -1.475; -1.475 \ -9]$ ,  $\Gamma = [0 \ 1.5; -2 \ -5]$ ,  $\Lambda = [2.13 \ 0.5; 0.5 \ 0.35]$  which satisfy conditions (95), (96), and (97). In addition, the verification of the Lipchitz-like condition (94) is straightforward. Hence, conditions of Theorem 5.1 are satisfied so that the control  $u = -\lambda(2\lambda\hat{x}_1 + 5\lambda\hat{x}_2)$ , where  $\lambda$ ,  $\hat{x}_1$ , and  $\hat{x}_2$  are generated by (100) and (101), is valid.

The simulation data are:  $z(0) = [1, -2]^T$ ,  $x(0) = [-2, 5]^T$ ,  $\hat{x}(0) = 0$ ,  $\lambda(0) = 1$ , and  $k_\lambda = 10$ . The parameters of the switching sequence are:  $T_p = 0.2s$  and  $\tau_p = 0.7s$ .

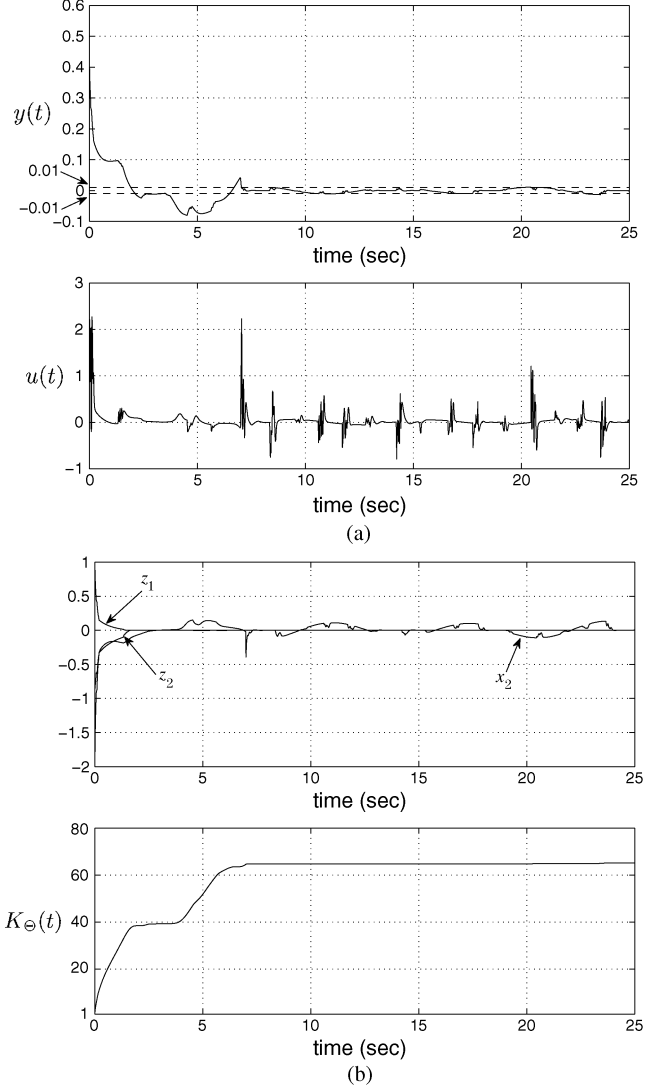


Fig. 1. Adaptive output regulation. (a) Output convergence and control input; (b)  $z_1(t)$ ,  $z_2(t)$ ,  $x_2(t)$ , and  $K_\Theta(t)$ .

The simulation results are shown in Fig. 2. It can be seen from Fig. 2(a) that the stabilization is well obtained. The peak points in control signal observed in Fig. 2(b) are due to the fast transient periods caused by changes of active subsystems. It is also observed from Fig. 2(b) that the adaptive observer's gain  $\lambda$  converges to a fixed value.

## VII. CONCLUSION

We introduced a variation paradigm and gauge design for asymptotic gain and adaptive control of persistent dwell-time switched systems. The generality of the introduced stability theory lies in the permission of growing for auxiliary functions. The property of small-variation small-state of auxiliary functions was utilized for deriving convergence. By gauge design, we aimed at making the overall system interchangeably driven by stable modes of component systems, which overcome the difficulties caused by positive cross-dissipation rates and unmeasured dynamics. Nonlinear relation in terms of timing parameters of switching sequence and dissipation rates of driving systems was introduced for switching-uniform

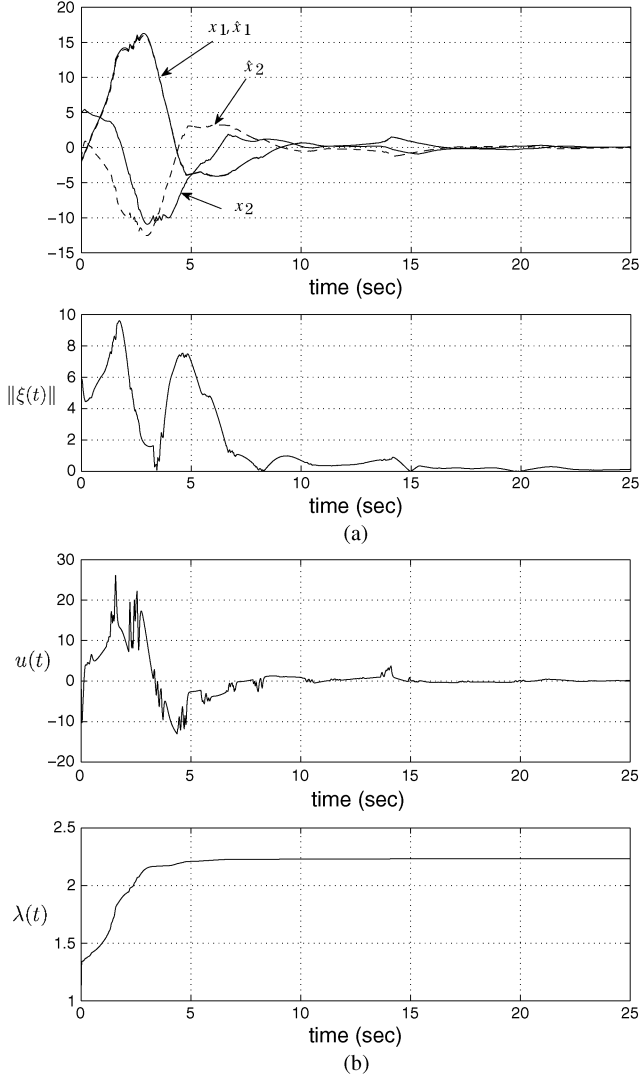


Fig. 2. Adaptive output feedback. (a) State convergence:  $\xi = [x_1 - \hat{x}_1, x_2 - \hat{x}_2]^T$ ; (b) control input  $u(t)$  and observer's gain  $\lambda(t)$ .

adaptive output regulation. By gauge design, adaptive output feedback stabilization was obtained for switched systems subject to unknown time-varying parameters, full-state dependent and non-identical control gains, and persistence dwell-time switching sequences.

#### APPENDIX A PROOF OF PROPOSITION 4.1

In this proof, a closed interval  $[t_1, t_2]$  (an open interval  $(t_1, t_2)$ , resp.) is said to be  $\xi - \text{domt}[q]$  ( $z - \text{domt}[q]$ , resp.) if (46) holds (does not hold, resp.) for all  $t$  in this interval. An interval of either these properties is said to be maximal in its corresponding property if it has no strict subinterval of the same property. We further denote  $q_{\sigma, i_j^d}$  by  $q_j^d$  for brevity.

Consider a dwell-time switching event  $(q_{\sigma, i_j^d}, \Delta\tau_{\sigma, i_j^d})$ ,  $j \in \mathbb{N}$ . We state that if the inequality

$$V_{q_j^d}(\tilde{x}(t)) \leq \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|)) \quad (123)$$

holds for some  $t = t_i \in [t_{\sigma, i_j^d}, t_{\sigma, i_j^d+1}]$ , then it also holds for all  $t \in [t_i, t_{\sigma, i_j^d+1}]$ .

Indeed, consider the case  $t_i$  belongs to a  $\xi - \text{domt}[q_j^d]$  interval  $[t_1, t_2] \subset [t_{\sigma, i_j^d}, t_{\sigma, i_j^d+1}]$ . As (46) holds on  $[t_1, t_2]$ , we have  $V_{q_j^d}(\tilde{x}(t)) = \mu^{-1}(2V_g(\xi(t)))$  and (79) holds for all  $t \in [t_1, t_2]$ . As  $\mu^{-1} \in \mathcal{K}_\infty$ , this coupled with the satisfaction of (123) at  $t = t_i$  implies that  $2V_g(\xi(t_i)) \leq \alpha_g^{-1}(2\|w_\Theta\|)$  and (79) holds on  $[t_1, t_2]$ . As such,  $2V_g(\xi(t)) \leq \alpha_g^{-1}(2\|w_\Theta\|), \forall t \in [t_1, t_2]$ . Since  $V_{q_j^d}(\tilde{x}(t)) = \mu^{-1}(2V_g(\xi(t)))$ ,  $t \in [t_1, t_2]$ , this shows that (123) holds on  $[t_1, t_2]$ .

Let  $(t_2, t_3)$  be the  $z - \text{domt}[q_j^d]$  next to  $[t_1, t_2]$ . On this interval, we have  $V_{q_j^d}(\tilde{x}(t)) = U_{q_j^d}(z(t))$  and the inverse of (46) holds for  $q = q_j^d$ . Thus, from Assumption 3.1 and the fact that the driving dynamics for  $z$  on  $[t_{\sigma, i_j^d}, t_{\sigma, i_j^d+1}]$  is that of the index  $q_{\sigma, i_j^d}$ , for  $t \in (t_2, t_3)$ , we have

$$\begin{aligned} DU_{q_j^d}(z(t)) &\leq -\alpha_1 \left( U_{q_j^d}(z(t)) \right) + v(y^2) \\ &\leq -\alpha_1 \left( U_{q_j^d}(z(t)) \right) + \mu \left( U_{q_j^d}(z(t)) \right) \\ &< 0 \end{aligned} \quad (124)$$

where we have used the property  $v(\xi_1^2) \leq 2V_g(\xi(t)) \leq \mu(U_{q_j^d}(z(t)))$  held on  $z - \text{domt}[q_j^d]$  intervals. Therefore,  $U_{q_j^d}(z(t))$  is decreasing on  $(t_2, t_3)$ . As we have  $U_{q_j^d}(z(t)) = \mu^{-1}(2V_g(\xi(t)))$  at the transition time  $t_2$  and (123) holds at  $t = t_2$ , this coupled with the continuity of  $U_{q_j^d}(z(t))$  further implies that  $V_{q_j^d}(\tilde{x}(t)) = U_{q_j^d}(z(t))$  is bounded by  $\mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|))$  on  $(t_2, t_3)$  as well. Continuing this process until  $t_{\sigma, i_j^d+1}$  is reached, we conclude that the statement is true.

In the case  $t_i$  belongs to a  $z - \text{domt}[q_j^d]$  interval, it is obvious from the above argument that the statement is true.

We now consider  $t_i$  to be minimal in the sense that there is no  $t_0 \in [t_{\sigma, i_j^d}, t_{\sigma, i_j^d+1}]$ ,  $t_0 < t_i$  such that (123) holds for  $t = t_0$ . For such  $t_i$ , we have  $V_{q_j^d}(\tilde{x}(t)) > \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|)), \forall t \in [t_{\sigma, i_j^d}, t_i)$  and hence (81) holds on  $\xi - \text{domt}[q_j^d]$  intervals contained in  $[t_{\sigma, i_j^d}, t_i)$ . Thus, on  $\xi - \text{domt}[q_j^d]$  intervals, as  $V_{q_j^d}(\tilde{x}(t)) = \mu^{-1}(2V_g(\xi(t))) = U_g(\xi(t))$ , we have  $DV_{q_j^d}(\tilde{x}(t)) \leq -\alpha_1(V_{q_j^d}(\tilde{x}(t)))$ . This coupled with (124) and the fact that  $V_{q_j^d}(\tilde{x}(t)) = U_{q_j^d}(z(t))$  on  $z - \text{domt}[q_j^d]$  intervals show that

$$DV_{q_j^d}(\tilde{x}(t)) \leq -\alpha_1 \left( V_{q_j^d}(\tilde{x}(t)) \right) + \mu \left( V_{q_j^d}(\tilde{x}(t)) \right), \quad \forall t \in [t_{\sigma, i_j^d}, t_i). \quad (125)$$

Since  $D^+V_{q_j^d}(\tilde{x}(t)) \leq DV_{q_j^d}(\tilde{x}(t))$ , using comparison principle [44] in combination with Lemma 3.2 for (125), we have  $V_{q_j^d}(\tilde{x}(t)) \leq \omega_1(V_{q_j^d}(\tilde{x}(t_{\sigma, i_j^d})), t - t_{\sigma, i_j^d}), \forall t \in [t_{\sigma, i_j^d}, t_i)$ . Combining this estimate with the above estimate (123) of  $V_{q_j^d}(\tilde{x}(t))$  on  $[t_i, t_{\sigma, i_j^d+1}]$ , we obtain

$$V_{q_j^d}(\tilde{x}(t)) \leq \max \left\{ \omega_1 \left( V_{q_j^d}(\tilde{x}(t_{\sigma, i_j^d})), t - t_{\sigma, i_j^d} \right), \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|)) \right\}, \quad \forall t \in [t_{\sigma, i_j^d}, t_{\sigma, i_j^d+1}] \quad (126)$$

i.e., the statement i) of the Proposition is true.

We prove the statement ii) of the Proposition by examining increments of  $V_q(\tilde{x}(t))$  on the interval  $[t_V, t_{\sigma, i_j^d}]$ . Consider the first maximal  $z - \text{domt}[q]$  subinterval  $(t_1, t_2)$  of  $[t_V, t_{\sigma, i_j^d}]$ . As  $v(y_2) \leq \mu(U_q(z(t)))$  and  $V_q(\tilde{x}(t)) = U_q(z(t))$  on  $z - \text{domt}[q]$  intervals, from Assumption 3.1, we have

$$DV_q(\tilde{x}(t)) \leq \alpha_2 (V_q(\tilde{x}(t))) + \mu(V_q(\tilde{x}(t))), \quad \forall t \in (t_1, t_2). \quad (127)$$

Again, applying comparison principle [44] in combination with Lemma 3.2 for (127), we obtain

$$V_q(\tilde{x}(t)) \leq \omega_2 (V_q(\tilde{x}(t_1)), t - t_1), \quad t \in (t_1, t_2). \quad (128)$$

In addition, as  $[t_V, t_1]$  (if not empty) is  $\xi - \text{domt}[q]$ , from the above proof of i), we have  $V_q(\tilde{x}(t_1)) \leq \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|))$  if  $V_q(\tilde{x}(t_i)) \leq \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|))$  for some  $t_i \in [t_V, t_1]$  and  $DV_q(\tilde{x}(t)) \leq -\alpha_1(V_q(\tilde{x}(t))), \forall t \in [t_V, t_1]$  implying that  $V_q(\tilde{x}(t)) \leq V_q(\tilde{x}(t_V)), \forall t \in [t_V, t_1]$  if there is no such  $t_i$ . Therefore, for any  $t \in [t_V, t_1]$ , we have

$$V_q(\tilde{x}(t)) \leq \max\{\mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|)), V_q(\tilde{x}(t_V))\} = V_q^w(t_V). \quad (129)$$

Since  $\alpha_2(s) + \mu(s) > 0$ ,  $\omega_2$  is nondecreasing in both arguments. Thus, combining (128) and (129), we obtain

$$V_q(\tilde{x}(t)) \leq \omega_2 (V_q^w(t_V), t - t_V), \quad \forall t \in [t_V, t_2]. \quad (130)$$

We now consider the next pair of  $\xi - \text{domt}[q]$  and  $z - \text{domt}[q]$  subintervals of  $[t_V, t_{\sigma, i_j^d}]$ , namely  $[t_2, t_3]$  and  $(t_3, t_4)$ , respectively. Since  $V_q^w(t_V) \geq \mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|))$ ,  $\omega_2$  is nondecreasing, and on  $\xi - \text{domt}[q]$  intervals,  $V_q(\tilde{x}(t))$  is decreasing as long as it is not smaller than  $\mu^{-1}(\alpha_g^{-1}(2\|w_\Theta\|))$ , the inequality in (130) holds for  $t \in [t_2, t_3]$  as well. Furthermore, as  $(t_3, t_4)$  is  $z - \text{domt}[q]$ , we also have (128) with  $(t_1, t_2)$  replaced by  $(t_3, t_4)$ . Thus, by the additive and nondecreasing properties of  $\omega_2$  (see Lemma 3.2), we have

$$\begin{aligned} V_q(\tilde{x}(t)) &\leq \omega_2 (V_q(\tilde{x}(t_3)), t - t_3) \\ &\leq \omega_2 (\omega_2 (V_q^w(t_V), t_3 - t_V), t - t_3) \\ &\leq \omega_2 (V_q^w(t_V), t - t_V), \quad \forall t \in (t_3, t_4). \end{aligned} \quad (131)$$

Continuing this process until  $t_{\sigma, i_j^d}$  is reached, we arrive at

$$V_q(\tilde{x}(t)) \leq \omega_2 (V_q^w(t_V), t - t_V), \quad \forall t \in [t_V, t_{\sigma, i_j^d}]. \quad (132)$$

As  $\omega_2$  is nondecreasing and  $t_{\sigma, i_j^d} - t_V \leq T_p$ , ii) follows (132) directly. ■

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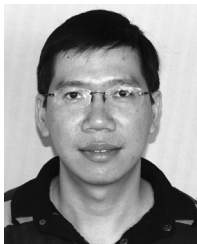
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