to $t = T$ yields $(1/2)\int_0^{\rm T} \|e(t)\|_Q^2 dt \le W(e(0), \tilde{\Theta}_1(0), \tilde{\Theta}_2(0)) + (\rho^2/2)\int_0^{\rm T} \|d(t)\|^2 dt + \epsilon\Delta T$ where ΔT denotes the time interval with respect to the operation of the additional controller $u_r + u_h/\lambda_g^*$ which may be very short. The term $\epsilon\Delta T$ can be viewed as an extra disturbance and a kind of H^∞ tracking performance is also achieved. □

Remark 7: (i) In the previous literature [10], both adaptive state feedback and output feedback tracking controllers are proposed for a class of nonlinear SISO systems represented by an input–output model as in the form of (1). The result developed in this note can be viewed as an extension of the adaptive state feedback control design developed in [10] and possesses the following improvements: 1) we have extended to treat a larger class of nonlinear MIMO systems; 2) three types of uncertainty: parametric uncertainties, unmodeled perturbations and external disturbances, are simultaneously compensated; and 3) a nonlinear H^∞ tracking control is guaranteed. (ii) In the previous H^∞ tracking control schemes [1]–[3], [6], [14], the plant models must be assumed to be exactly known. In this study by incorporating both adaptive control and VSS control techniques the developed adaptive-robust H^∞ tracking control scheme can be extended to handle a broader class of nonlinear systems involving parametric uncertainties and unmodeled perturbations. On the other hand, compared with the existing tracking control schemes in [7]–[13], due to the incorporation of nonlinear H^∞ control, the developed adaptive-robust control scheme possesses a desired H^∞ performance and can be applied to uncertain nonlinear MIMO systems in the presence of finite-energy disturbances. (iii) The H^∞ tracking control developed here requires to solve an algebraic Riccati-like matrix (23) in Theorem 2 that includes the effect of the unmodeled perturbation $\Delta G(t,x)$. The smaller $\Delta G(t,x)$ implies the bounded value κ_e is smaller and so r^{-1} is also smaller. That is, there is a trade-off between the magnitude of $\Delta G(t,x)$ and the control gain in the robust H^∞ controller u_h. (iv) Result in Theorem 2 indicates that arbitrarily small attenuation level can be achieved via an adequate choice of control gain r^{-1} in u_h. The inequality $(1/\rho^2)I \le (1-\kappa_e/r)I$ provides us a sufficient condition to guarantee the solvability of $P = P^{\rm T} > 0$ of the Riccati-like (23) [1]. □

IV. Conclusion

This note has addressed the problem of designing robust tracking controls for a class of nonlinear MIMO systems involving parametric uncertainties, unmodeled dynamics and external disturbances. Hybrid adaptive-robust H^∞ tracking control schemes are developed to guarantee a transient and asymptotical output tracking performance in the sense that all the signals and states of the closed-loop system are bounded, the tracking error is UUB and an H^∞ tracking performance is achieved. The H^∞ tracking control relies only on the solution of a modified algebraic Riccati-like matrix equation and so the developed control law can easily be implemented. Consequently, compared with the existing H^∞ tracking control scheme and the robust/adaptive control scheme the developed adaptive-robust H^∞ tracking control design can be extended to handle a broader class of uncertain nonlinear MIMO systems.

Finally, how to extend to solve the adaptive H^∞ tracking control via output feedback for general uncertain nonlinear MIMO systems and to involve the effect of zero dynamics as in [10] will be the subject of future research.

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Reachability and Controllability of Switched Linear Discrete-Time Systems

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***Abstract*—This note investigates the reachability and controllability issues for switched linear discrete-time systems. Geometric characterization of controllability is presented. For reversible systems, the controllable sets and the reachable sets are identified in Wonham's geometric approach, and verifiable conditions for reachability and controllability are also presented.**

***Index Terms*—Controllability, observability, reachability, reconstructibility, switched discrete-time systems.**

I. Introduction

Switched systems are hybrid systems that consist of two or more subsystems and are controlled by switching laws. The switching law may be either supervised or unsupervised, time-driven or event-driven. In this note, we focus on the class of switched systems in which the

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switching laws are design parameters to be chosen online by a supervisor [6], [14], [17].

Switched systems deserve investigation for theoretical interest as well as for practical applications. Switching among different system structures is an essential feature of many engineering control applications such as power systems and power electronics [15], [10]. Control techniques based on switching between different controllers have been investigated in recent years, particularly in the context of adaptive control [8], [9]. The existence of systems that cannot be asymptotically stabilized by a single continuous feedback controller also motivates us to study switched systems [2]. Switched systems also arise naturally in the study of multirate sampled-data systems [11].

In the analysis and design of control systems, controllability and reachability are two fundamental concepts that need to be investigated. For switched continuous-time systems, the controllability and reachability issues have been addressed in several references. Studies for second-order switched linear systems can be found in [7], [17]. Geometric tests for reachability were presented for general switched linear control systems in [13]. For switched linear discrete-time systems, the set of points reachable from the origin were investigated in [12]. It was shown that this set (termed controllable set in [12]) is a subspace under certain hypothesis, but not always the case in general. Some further extension of this work can be found in [3], where the controllable set as the union of its maximal components was investigated.

In this note, the controllability and reachability issues are addressed for switched linear discrete-time systems. Section II formulates the problem and presents preliminary analysis. Geometric characterizations for the controllability and reachability sets are presented in Section III. In Section IV, two illustrating examples are analyzed in detail. Finally, some concluding remarks are made in Section V.

II. Definitions and Preliminaries

Consider a switched linear discrete-time control system given by

$$x_{k+1} = A_\sigma x_k + B_\sigma u_k \tag{1}$$

where

$x_k \in \Re^n$ and $u_k \in \Re^p$	states and inputs;
$\sigma : \{0, 1, \dots\} \to M = \{1, 2, \dots, m\}$	switching path to be designed;
matrix pair (A_k, B_k) for $k \in M$	subsystems of (1).

For clarity, for any positive integer k, set $\underline{k} = \{0, \dots, k-1\}$. It can be calculated that

$$\begin{aligned} x_k = \left(\Pi_{j=0}^{k-1} A_{i_j}\right) x_0 + \left(\Pi_{j=1}^{k-1} A_{i_j}\right) B_{i_0} u_0 + \cdots \\ + A_{i_{k-1}} B_{i_{k-2}} u_{k-2} + B_{i_{k-1}} u_{k-1} \end{aligned} \tag{2}$$

where $i_j = \sigma(j)$ for $j = 0, \dots, k-1$.

A state configuration x is said controllable, if it is transferable to the origin in finite time by appropriate choices of input u and switching path σ. The precise definitions of the relevant concepts are given as follows.

Definition 1: State $x \in \Re^n$ is controllable, if there exist a time instant $k > 0$, a switching path $\sigma : \underline{k} \to M$, and inputs $u : \underline{k} \to \Re^p$, such that $x_0 = x$ and $x_k = 0$.

Definition 2: The controllable set of system (1) is the set of states which are controllable.

Definition 3: System (1) is said to be (completely) controllable, if its controllable set is $\Re^n$.

For any matrices $A \in \Re^{n\times n}$ and $B \in \Re^{n\times p}$, set $\mathcal{B} = \operatorname{Im} B$ and $A^{-1}\mathcal{B} = \{x \in \Re^n : Ax \in \mathcal{B}\}$. Define

$$\begin{aligned} \mathcal{C}(i_0, \dots, i_k) = (A_{i_k} \dots A_{i_0})^{-1} \\ \times \left(A_{i_k} \dots A_{i_1}\mathcal{B}_{i_0} + \cdots + A_{i_k}\mathcal{B}_{i_{k-1}} + \mathcal{B}_{i_k}\right). \end{aligned} \tag{3}$$

Let $\mathcal{C}_k$ denote the set of points which can be transfered to the origin within k steps. It can be readily seen that

$$\mathcal{C}_k = \cup_{i_0,\dots,i_{k-1}\in M}\mathcal{C}(i_0, \dots, i_{k-1}) \tag{4}$$

and

$$\mathcal{C} = \cup_{k=1}^{\infty}\mathcal{C}_k \tag{5}$$

where $\mathcal{C}$ is the controllable set of system (1).

The reachability counterparts can be defined along the same line as follows.

Definition 4: State $x \in \Re^n$ is reachable, if there exist a time instant $k > 0$, a switching path $\sigma : \underline{k} \to M$, and inputs $u : \underline{k} \to \Re^p$, such that $x_0 = 0$ and $x_k = x$.

Definition 5: The reachable set of system (1) is the set of states which are reachable.

Definition 6: System (1) is said to be (completely) reachable, if its reachable set is $\Re^n$. Define

$$\mathcal{R}(i_0, \dots, i_k) = A_{i_k} \dots A_{i_1}\mathcal{B}_{i_0} + \cdots + A_{i_k}\mathcal{B}_{i_{k-1}} + \mathcal{B}_{i_k}$$

Let $\mathcal{R}_k$ denote the set of points which are reachable from the origin within k steps. It can be readily seen that

$$\mathcal{R}_k = \cup_{i_0,\dots,i_{k-1}\in M}\mathcal{R}(i_0, \dots, i_{k-1})$$

and

$$\mathcal{R} = \cup_{k=1}^{\infty}\mathcal{R}_k \tag{6}$$

where $\mathcal{R}$ is the reachable set of system (1).

Given a matrix A and a subspace $\mathcal{B} \in \Re^n$, let $\Gamma_A\mathcal{B}$ denote the minimal A-invariant subspace that contains $\mathcal{B}$, i.e.,

$$\Gamma_A\mathcal{B} = \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B}$$

This operation can be defined recursively as $\Gamma_{A_1}\Gamma_{A_2}\mathcal{B} = \Gamma_{A_1}(\Gamma_{A_2}\mathcal{B})$. For clarity, define the nested subspaces as

$$\begin{aligned} \mathcal{V}_1 &= \operatorname{Im} B_1 + \cdots + \operatorname{Im} B_m \\ \mathcal{V}_{i+1} &= \Gamma_{A_1}\mathcal{V}_i + \cdots + \Gamma_{A_m}\mathcal{V}_i \quad i = 1, 2, \dots \end{aligned} \tag{7}$$

and

$$\mathcal{V} = \sum_{k=1}^{\infty}\mathcal{V}_k.$$

Note that if $\dim\mathcal{V}_j = \dim\mathcal{V}_{j+1}$, then $\mathcal{V}_l = \mathcal{V}_j$ for $l > j$. This fact implies that $\mathcal{V}_n = \mathcal{V}$. It is readily seen that this subspace is the minimal subspace which is invariant under A_i, $i = 1, \dots, m$ and contains $\sum_{j=1}^{m}\mathcal{B}_j$. Subspace $\mathcal{V}$ plays an important role in the following derivations.

Because $A^i\operatorname{Im} B \subseteq \Gamma_A\operatorname{Im} B$ for all $A \in \Re^{n\times n}$, $B \in \Re^{n\times p}$ and $i \geq 0$, we know that the reachable set

$$\begin{aligned} \mathcal{R} \subseteq \cup_{k=0}^{\infty}\cup_{i_0,\dots,i_{k-1}\in M} \\ \left(\Gamma_{A_{i_{k-1}}}\cdots\Gamma_{A_{i_1}}\mathcal{B}_{i_0} + \cdots + \mathcal{B}_{i_{k-1}}\right) \subseteq \mathcal{V}. \end{aligned} \tag{8}$$

System (1) is said to be reversible, if all matrices A_i, $i = 1, \dots, m$ are nonsingular. For a reversible system, the controllable set

$$\begin{aligned} \mathcal{C} \subseteq \cup_{k=0}^{\infty}\cup_{i_0,\dots,i_{k-1}\in M} \\ \left(\Gamma_{A_{i_0}^{-1}}\mathcal{B}_{i_0} + \cdots + \Gamma_{A_{i_0}^{-1}}\dots\Gamma_{A_{i_{k-1}}^{-1}}\mathcal{B}_{i_{k-1}}\right) \subseteq \mathcal{V}. \end{aligned} \tag{9}$$

The above analysis is summarized in the following proposition.

Proposition 1: If the switched linear system (1) is reachable, or if system (1) is reversible and controllable, then

$$\mathcal{V} = \Re^n. \tag{10}$$

III. Main Results

A. Geometric Characterizations

In this subsection, we shall present criteria of controllability and reachability for switched linear systems.

Theorem 1: The switched linear system (1) is controllable if and only if there exist an integer $k<\infty$, and $i_0,\ldots,i_k$, such that

$$\mathrm{Im}(A_{i_k}\ldots A_{i_1}A_{i_0})\subseteq\mathcal{R}(i_0,\ldots,i_k). \tag{11}$$

Proof: From (3)–(5), the controllable set of system (1) is given by

$$\mathcal{C}=\cup_{k=1}^{\infty}\cup_{i_0,\ldots,i_{k-1}\in M}\Big((A_{i_{k-1}}\ldots A_{i_0})^{-1}(A_{i_{k-1}}\cdots A_{i_1}\mathcal{B}_{i_0}+\cdots+A_{i_{k-1}}\mathcal{B}_{i_{k-2}}+\mathcal{B}_{i_{k-1}})\Big).$$

That is, the controllable set can be expressed as a countable union of subspaces of $\Re^n$. Because $\Re^n$ cannot be expressed as a countable union of lower-dimensional subspaces, to ensure controllability of system (1), it must be

$$\begin{aligned}\mathcal{C}(i_0,\ldots,i_k)=&(A_{i_k}\ldots A_{i_0})^{-1}\\&(A_{i_k}\ldots A_{i_1}\mathcal{B}_{i_0}+\cdots+A_{i_k}\mathcal{B}_{i_{k-1}}+\mathcal{B}_{i_k})\\=&\Re^n\end{aligned} \tag{12}$$

for some $k<\infty$ and $i_0,\ldots,i_k\in M$. That is

$$(A_{i_k}\ldots A_{i_1}\mathcal{B}_{i_0}+\cdots+A_{i_k}\mathcal{B}_{i_{k-1}}+\mathcal{B}_{i_k})\supseteq\mathrm{Im}(A_{i_k}\ldots A_{i_0})\quad\diamondsuit.$$

For reachability of switched linear systems, similar criterion can be found in [12, Corollary of Th. 2]. It is summarized in the following theorem for completeness.

Theorem 2: The switched linear system (1) is reachable if and only if there exist an integer $k<\infty$, and $i_0,\ldots,i_k\in M$, such that

$$\mathcal{R}(i_0,\ldots,i_k)=\Re^n. \tag{13}$$

Remark 1: It is interesting to notice the resemblance between reachability of a switched linear system and weak controllability of a jump linear system [5]. For a jump linear system [5]

$$x_{k+1}=A_{r_k}x_k+B_{r_k}u_k \tag{14}$$

where $r_k\in M=\{1,\ldots,m\}$, $k=1,2,\ldots$ form a finite-state discrete-time ergodic Markov chain, it is weakly controllable if and only if for some $r_{i_0}\in M$, there exists a possible transition sequence $i_0,\ldots,i_{T-1}$ with $T\le\infty$, such that

$$\mathrm{rank}[B_{i_{T-1}},A_{i_{T-1}}B_{i_{T-2}},\ldots,A_{i_{T-1}}\ldots A_{i_1}B_{i_0}]=n$$

which is equivalent to condition (13).

As proved in [4], any causal discrete-time (input–output) system can be realized by means of a reversible state variable representation. Accordingly, reversible system representation is very general and applicable to a large class of systems. In the sequel, we present verifiable criteria for controllability and reachability of reversible switched linear systems. Moreover, we prove that the reachable and controllable sets are nothing but subspace $\mathcal{V}$ in this case.

Theorem 3: Suppose the switched linear system (1) is reversible, then its reachable set is

$$\mathcal{R}=\mathcal{V}. \tag{15}$$

Proof: Let us prove it by contradiction. Suppose

$$\begin{aligned}&\dim\mathcal{R}(i_0,\ldots,i_k)\\&=\max\{\dim\mathcal{R}(l_0,\ldots,l_j){:}\,l_0,\ldots,l_j\in M,\ j=0,1,\ldots\}\\&<\dim\mathcal{V}.\end{aligned} \tag{16}$$

It follows from (16) that, for any arbitrary given integers $l_0,\ldots,l_j$, it has

$$\mathcal{R}(l_0,\ldots,l_j,i_0,\ldots,i_k)=\mathcal{R}(i_0,\ldots,i_k)$$

which implies that

$$(A_{i_k}\ldots A_{i_0})(A_{l_j}\ldots A_{l_1}\mathcal{B}_{l_0})\subseteq\mathcal{R}(i_0,\ldots,i_k). \tag{17}$$

On the other hand, note that

$$\mathcal{V}=\sum_{i_1,\ldots,i_n=1,\ldots,m}^{j_1,\ldots,j_n=0,1,\ldots,n-1}A_{i_n}^{j_n}\ldots A_{i_1}^{j_1}\mathcal{B}_{i_1}. \tag{18}$$

Since j and $l_0,\ldots,l_j$ in (17) can take arbitrarily any values, we have

$$(A_{i_k}\cdots A_{i_0})\mathcal{V}\subseteq\mathcal{R}(i_0,\ldots,i_k)$$

which is a contradiction because

$$\dim[(A_{i_k}\cdots A_{i_0})\mathcal{V}]=\dim\mathcal{V}>\dim\mathcal{R}(i_0,\ldots,i_k)$$

where the equality follows from the identity $\dim A\mathcal{V}=\dim\mathcal{V}$ for any nonsingular matrix $A\in\Re^{n\times n}$ and subspace $\mathcal{V}\subseteq\Re^n$. Accordingly, we have

$$\dim\mathcal{R}(i_0,\ldots,i_k)=\dim\mathcal{V}. \tag{19}$$

It follows from (8) that

$$\mathcal{R}=\mathcal{V}\quad\diamondsuit.$$

Theorem 4: Suppose the switched linear system (1) is reversible, then its controllable set is

$$\mathcal{C}=\mathcal{V}. \tag{20}$$

Proof: This theorem can be proven following the same argument of the proof of Theorem 3 and the details are omitted.

Corollary 1: For a reversible switched linear system, the following statements are equivalent:

i) the system is completely controllable;
ii) the system is completely reachable;
iii) $\mathcal{V}=\Re^n$.

Proof: Follows directly from Theorems 3 and 4.

Remark 2: For reversible switched linear systems, the reachable set, the controllable set and subspace $\mathcal{V}$ always coincide with each other. For nonreversible systems, however, these favorable properties do not hold any more as shown in Example 1 of Section IV.

Remark 3: The criterion for controllability and reachability in Corollary 1 is an extension of the well-known geometric criterion for reachability of linear system (A,B) [16]

$$\mathrm{Im}\,B+A\mathrm{Im}\,B+\cdots+A^{n-1}\mathrm{Im}\,B=\Re^n.$$

It is equivalent to the Kalman-type rank condition

$$\begin{aligned}\mathrm{rank}\Big[&B_1,\ldots,B_m,A_1B_1,\ldots,A_1B_m,\ldots,A_mB_1,\ldots\\&A_mB_m,\ldots,A_1^{n-1}B_1,\ldots\\&A_1^{n-1}B_m,A_1^{n-2}A_2B_1,\ldots,A_1^{n-2}A_2B_m,\ldots\\&A_m^{n-1}B_1,\ldots,A_m^{n-1}B_m\Big]=n\end{aligned} \tag{21}$$

which can be efficiently verified by polynomial-time algorithms [1].

B. Computational Issues

As stated in Theorems 3 and 4, the controllable (reachable) set for a reversible switched system is $\mathcal{V}$, which is defined recursively through (A_i,B_i), $i=1,\ldots,m$. It follows from (18) that $\mathcal{V}$ is summation of

$(mn)^n p$ items. It requires much computational effort to calculate this subspace if m and n are relatively large.

In this subsection, we provide a procedure to calculate $\mathcal{V}$ more efficiently.

Denote the nested subspaces as

$$\begin{aligned} \mathcal{W}_0 =& \operatorname{Im} B_1 + \cdots + \operatorname{Im} B_m \\ \mathcal{W}_i =& \mathcal{W}_{i-1} + A_1\mathcal{W}_{i-1} + \cdots + A_m\mathcal{W}_{i-1} \\ i =& 1, 2, \ldots \end{aligned} \tag{22}$$

and furthermore

$$\mathcal{W} = \sum_{i=0}^{\infty} \mathcal{W}_i.$$

It can be readily seen that $\mathcal{V}_n \subseteq \mathcal{W} \subseteq \mathcal{V}$. As a consequence, $\mathcal{W} = \mathcal{V}$. Let ρ denote the minimal integer j such that $\mathcal{W}_j = \mathcal{W}$. It can be readily seen that $\rho \le n-1$. Define natural numbers

$$n_0 = \dim \mathcal{W}_0 \quad n_i = \dim \mathcal{W}_i - \dim \mathcal{W}_{i-1} \quad i = 1, \ldots, \rho$$

and furthermore

$$\tau_i = \sum_{j=0}^{i} n_j, \quad i = 1, \ldots, \rho.$$

A basis of subspace $\mathcal{V}$ can be constructed according to the following steps.

1) Choose a group of base vectors $\gamma_1, \ldots, \gamma_{n_0}$ in $\mathcal{W}_0$.

2) Because

$$\begin{aligned} \mathcal{W}_1 =& \mathcal{W}_0 + \sum_{i=1}^{m} A_i\mathcal{W}_0 \\ =& \operatorname{span}\{\gamma_j, A_i\gamma_j,\ i = 1, \ldots, m,\ j = 1, \ldots, n_0\} \end{aligned}$$

we can find a basis of $\mathcal{W}_1$ by searching the set

$$\{\gamma_1, \ldots, \gamma_{n_0}, A_1\gamma_1, \ldots, A_1\gamma_{n_0}, \ldots, A_m\gamma_1, \ldots, A_m\gamma_{n_0}\}$$

from left to right for linearly independent column vectors. Denote this basis as

$$\gamma_1, \ldots, \gamma_{n_0}, \gamma_{n_0+1}, \ldots, \gamma_{\tau_1}.$$

3) Continuing this process, we can find a basis $\gamma_1, \ldots, \gamma_{n_0}, \ldots \gamma_{\tau_{l-1}+1}, \ldots, \gamma_{\tau_l}$ for $\mathcal{W}_l$. Because

$$\begin{aligned} \mathcal{W}_{l+1} =& \mathcal{W}_l + \sum_{i=1}^{m} A_i\mathcal{W}_l \\ =& \operatorname{span}\{\gamma_1, \ldots, \gamma_\tau, A_i\gamma_j,\ i = 1, \ldots, m \\ & j = \tau_{l-1}+1, \ldots, \tau_l\} \end{aligned}$$

by searching the set

$$\{\gamma_1, \ldots, \gamma_{\tau_l}, A_1\gamma_{\tau_{l-1}+1}, \ldots, A_1\gamma_{\tau_l}, \ldots, A_m\gamma_{\tau_{l-1}+1}, \ldots, A_m\gamma_{\tau_l}\}$$

from left to right for linearly independent column vectors, we can find a basis of $\mathcal{W}_{l+1}$

$$\gamma_1, \ldots, \gamma_{\tau_l}, \gamma_{\tau_l+1}, \ldots, \gamma_{\tau_{l+1}}.$$

4) Finally, we can find a basis $\gamma_1, \ldots, \gamma_{\tau_\rho}$ of subspace $\mathcal{W}$. That is

$$\mathcal{V} = \mathcal{W} = \operatorname{span}\{\gamma_1, \ldots, \gamma_{\tau_\rho}\}.$$

It involves not more than $mp + m\tau_{\rho-1}$ column vectors in the above procedure, which is only a small fraction of the original quantity, $(mn)^n p$.

C. Observability and Reconstructibility

In the above analysis, reference is made to reachability and controllability only. It should be noticed that the observability and reconstructibility counterparts can be addressed dualistically. In the sequel, we outline the relevant concepts and the corresponding criteria.

Consider a switched linear discrete-time control system with outputs given by

$$\begin{aligned} x_{k+1} =& A_\sigma x_k + B_\sigma u_k \\ y_k =& E_\sigma x_k \end{aligned} \tag{23}$$

where $x_k \in \Re^n$, $u_k \in \Re^p$ and $y_k \in \Re^r$ are the states, inputs and outputs, respectively, and $\sigma : \{0, 1, \ldots\} \to M = \{1, 2, \ldots, m\}$ is the switching path to be designed.

Definition 7: The switched linear system (23) is (completely) observable, if there exist an integer k and a switching path $\sigma : \underline{k} \to M$, such that knowledge of the output sequence $\{y_0, y_1, \ldots, y_k\}$ and the input sequence $\{u_0, u_1, \ldots, u_{k-1}\}$ is sufficient to determine x_0.

Definition 8: The switched linear system (23) is (completely) reconstructible, if there exist an integer k and a switching path $\sigma : \underline{k} \to M$, such that state x_k can be determined from knowledge of the output sequence $\{y_0, \ldots, y_k\}$ and the input sequence $\{u_0, \ldots, u_{k-1}\}$.

In view of Theorems 1–4 for reachability and controllability, the following criteria are readily obtained for observability and reconstructibility by using the principle of duality.

Theorem 5: The switched linear system (23) is observable if and only if there exist an integer $k < \infty$, and $i_0, \ldots, i_k$, such that

$$\mathcal{E}_{i_0} + A_{i_0}^{\mathrm{T}}\mathcal{E}_{i_1} + \cdots + A_{i_0}^{\mathrm{T}} \cdots A_{i_{k-1}}^{\mathrm{T}}\mathcal{E}_{i_k} = \Re^n \tag{24}$$

where $\mathcal{E}_i = \operatorname{Im} E_i^{\mathrm{T}}$ for $i = 1, \ldots, m$.

Moreover, if the system is reversible, then a necessary and sufficient condition for observability is

$$\sum_{i_1, \ldots, i_n = 1, \ldots, m}^{j_1, \ldots, j_n = 0, 1, \ldots, n-1} \left(A_{i_1}^{j_1}\right)^{\mathrm{T}} \cdots \left(A_{i_{n-1}}^{j_{n-1}}\right)^{\mathrm{T}} \mathcal{E}_{i_n} = \Re^n \tag{25}$$

Theorem 6: The switched linear system (23) is reconstructible if and only if there exist an integer $k < \infty$, and $i_0, \ldots, i_k$, such that

$$\mathcal{E}_{i_0} + A_{i_0}^{\mathrm{T}}\mathcal{E}_{i_1} + \cdots + A_{i_0}^{\mathrm{T}} \cdots A_{i_{k-1}}^{\mathrm{T}}\mathcal{E}_{i_k} \supseteq \operatorname{Im}\left(A_{i_0}^{\mathrm{T}} \cdots A_{i_k}^{\mathrm{T}}\right) \tag{26}$$

Moreover, if the system is reversible, then (25) is a necessary and sufficient condition for reconstructibility.

IV. Illustrating Examples

Example 1: Consider system (1) with $n = 4, m = 2$ and

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{27}$$

Simple calculation gives

$$\begin{aligned} \mathcal{V} =& \operatorname{span}\{e_1, e_2\} \\ \mathcal{R} =& \operatorname{span}\{e_1\} \cup \operatorname{span}\{e_2\} \\ \mathcal{C} =& \operatorname{span}\{e_1, e_2, e_3\} \cup \operatorname{span}\{e_1, e_2, e_4\}. \end{aligned}$$

Note that neither the controllable set nor the reachable set is a subspace of the total space. Furthermore, $\mathcal{R} \subset \mathcal{V} \subset \mathcal{C}$, where the subset relationships are strict proper.

Example 2: Controllability of a multirate sampled-data system. Consider the linear continuous time-invariant system given by

$$\begin{aligned}\dot{x} &= Ax + Bu(t)\\ &= \begin{bmatrix} 0 & -100\pi & 0 & 0\\ 100\pi & 0 & 0 & 0\\ 0 & 0 & \frac{3}{2}\pi & 0\\ 0 & 0 & 0 & \frac{3}{2}\pi \end{bmatrix} \\ &\quad + \begin{bmatrix} 10 & 0\\ 10 & 0\\ 0 & 10\\ 0 & 10 \end{bmatrix} u(t) \end{aligned} \tag{28}$$

which can be verified to be controllable.

The corresponding sampled-data system is given by

$$x_{k+1} = A_T x_k + B_T u_k \tag{29}$$

where T is the sampling interval

$$\begin{aligned} x_k &= x(kT) \quad u_k = u(kT),\\ A_T &= e^{AT} \quad B_T = \int_0^{\mathrm{T}} e^{\tau A} d\tau B. \end{aligned}$$

If the sampling intervals are chosen as $T_1 = 0.01$ and $T_2 = 0.015$, we have the corresponding matrix pairs (A_{T_1}, B_{T_1}) and (A_{T_2}, B_{T_2}) given as

$$A_{T_1} = \begin{bmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -0.5 & -\sin\frac{3}{2}\pi\\ 0 & 0 & \sin\frac{3}{2}\pi & -0.5 \end{bmatrix}$$

$$B_{T_1} = \begin{bmatrix} 0.1 & 0\\ 0.1 & 0\\ 0 & -0.05 - 0.1\sin\frac{3}{2}\pi\\ 0 & -0.05 - 0.1\sin\frac{3}{2}\pi \end{bmatrix}$$

and

$$A_{T_2} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{bmatrix} \quad B_{T_2} = \begin{bmatrix} 0.15 & 0\\ -0.15 & 0\\ 0 & -0.15\\ 0 & -0.15 \end{bmatrix}.$$

It can be verified that

$$\operatorname{rank}\left[B_{T_i}, A_{T_i}B_{T_i}, \ldots, A_{T_i}^3 B_{T_i}\right] = 3, \quad i = 1, 2$$

which show that the corresponding sampled-data systems are not controllable.

Now we consider the multirate sampling of system (28) with sampling rate of either T_1 or T_2. A question naturally arises: does there exist a sampling strategy such that the resulted switched system is controllable? That is, is the switched system (1) with $A_i = A_{T_i}$, $B_i = B_{T_i}$, $i = 1, 2$ controllable or not?

Simple computation gives

$$\mathcal{V} \supseteq \operatorname{span}\{B_1, B_2, A_1B_1, A_2B_2\} = \Re^4.$$

From Corollary 1, the controllability follows. Moreover, it can be verified that

$$\mathcal{C}(2,1) = \mathcal{R}(2,1) = \Re^4.$$

Accordingly, the switched system is controllable from (and reachable to) any point within two steps by choosing subsystem (A_2, B_2) at the first step and then switching to subsystem (A_1, B_1) at the second step.

This example illustrates that switching among different sampling rates may avoid singularities caused by inappropriate choice of sampling rates.

V. Conclusion

Controllability and reachability issues have been addressed for switched linear discrete-time systems. Geometric characterizations for controllability and reachability were presented. For reversible systems, the controllable and reachable sets have been proven to be subspaces of the total space, and verifiable criteria for controllability and reachability have also been presented. Criteria for observability and reconstructibility have been obtained by duality.

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