[18] O. J. M. Smith, "Closer control of loops with dead time," *Chem. Eng. Progr.*, vol. 53, no. 5, pp. 217–219, 1957.
[19] M. Morari and E. Zafiriou, *Robust Process Control*. Upper Saddle River, NJ: Prentice-Hall, 1989.
[20] L. Mirkin, "Are distributed-delay control laws intrinsically unapproximable?," presented at the 4th IFAC Workshop Time Delay Systems, Rocquencourt, France, 2003.
[21] C. N. Nett, C. A. Jacobson, and M. J. Balas, "A connection between state-space and doubly coprime fractional representations," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 831–832, Sept. 1984.

# Adaptive Neural Network Control of Nonlinear Systems With Unknown Time Delays

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***Abstract*—In this note, adaptive neural control is presented for a class of strict-feedback nonlinear systems with unknown time delays. Using appropriate Lyapunov–Krasovskii functionals, the uncertainties of unknown time delays are compensated for such that iterative backstepping design can be carried out. In addition, controller singularity problems are solved by using the integral Lyapunov function and employing practical robust neural network control. The feasibility of neural network approximation of unknown system functions is guaranteed over practical compact sets. It is proved that the proposed systematic backstepping design method is able to guarantee semiglobally uniformly ultimate boundedness of all the signals in the closed-loop system and the tracking error is proven to converge to a small neighborhood of the origin.**

***Index Terms*—Adaptive control, neural networks, nonlinear time-delay systems.**

## I. Introduction

Recent years have witnessed great progress in adaptive control of nonlinear systems, which plays an important role due to its ability to compensate for parametric uncertainties. To obtain global stability, some restrictions had to be made to system nonlinearities such as matching conditions [1], extended matching conditions [2], or growth conditions [3]. To overcome these restrictions, recursive and systematic backstepping design was developed in [4]. The overparametrization problem was then removed in [5] by introducing the concept of tuning function. Several adaptive approaches for nonlinear systems with triangular structures have been proposed in [6], [7]. Robust adaptive backstepping control has been studied for uncertain nonlinear systems with unknown nonlinear functions in [8] and [9], and among others.

Recently, stabilization of nonlinear systems with time delays is receiving much attention [10], [11]. The existence of time delays may make the task more complicated and challenging, especially when the delays are not perfectly known. One way to ensure stability robustness with respect to this uncertainty is to seek for delay-independent solutions. Lyapunov–Krasovskii functionals have been used early as checking criteria for time-delay systems' stability in [12] and [13]. In [13], the system considered was a class of nonlinear time-delay systems with a so-called "triangular structure", which was later commented that it could not be "constructively obtained" in [14]. The uncertainties from unknown parameters or unknown nonlinear functions are yet to be discussed, especially when the virtual control coefficients are unknown nonlinear functions of states with known signs. With the aid of neural networks parameterization, stable controllers have been constructed in [15] and [16] (to name just a few) and a family of novel integral Lyapunov functions have been developed in [17] and [18] to avoid controller singularity problem commonly encountered in adaptive feedback linearization control.

In this note, we present an adaptive neural controller for a class of strict-feedback nonlinear systems with unknown time delays and unknown virtual control coefficients. Both integral functions and Lyaponov–Krasovskii functionals are used to construct the Lyapunov function candidates such that the controller singularity problem is avoided and the uncertainties from unknown time delays are removed. Semiglobally uniformly ultimate boundedness (SGUUB) of all the signals in the closed-loop system can be guaranteed and the control performance can be improved by appropriately choosing the design parameters. The proposed method expands the class of nonlinear systems that can be handled using adaptive control. The main contributions of the note lie in: 1) the use of integral Lyapunov functions to avoid controller singularity problem commonly encountered in feedback linearization control; 2) the introduction of appropriate Lyaponov–Krasovskii functionals to compensate for the unknown time delays for a time-delay independent controller design; 3) the introduction of practical robust control to avoid possible singularity problem due to the appearance of $1/z_i$ in the controller; and 4) the use of neural networks as function approximators with its feasibility being guaranteed over the practical compact sets for the first time in the literature.

## II. Problem Formulation and Preliminaries

Consider a class of single-input–single-output (SISO) nonlinear time-delay systems

$$\begin{aligned}\dot{x}_i(t) =& g_i(\bar{x}_i(t))x_{i+1}(t) + f_i(\bar{x}_i(t)) + +h_i(\bar{x}_i(t-\tau_i))\\ & 1 \le i \le n-1\\ \dot{x}_n(t) =& g_n(\bar{x}_n(t))u + f_n(\bar{x}_n(t)) + h_n(\bar{x}_n(t-\tau_n))\end{aligned} \tag{1}$$

where $\bar{x}_i = [x_1, x_2, \ldots, x_i]^T$, $x = [x_1, x_2, \ldots, x_n]^T \in R^n$ and $u \in R$ are the state variables and system input respectively, $g_i(\cdot)$, $f_i(\cdot)$, $h_i(\cdot)$ are unknown smooth functions, and $\tau_i$ are unknown time delays of the states, $i = 1, \ldots, n$. The control objective is to design an adaptive controller for (1) such that the state $x_1(t)$ follows a desired reference signal $y_d(t)$, while all signals in the closed-loop system are bounded. Define the desired trajectory $\bar{x}_{d(i+1)} = [y_d, \dot{y}_d, \ldots, y_d^{(i)}]^T$, $i = 1, \ldots, n-1$, which is a vector of $y_d$ up to its $i$th time derivative $y_d^{(i)}$. We have the following assumptions for the system's signals, unknown functions and reference signals.

*Assumption 2.1:* The system states $x(t)$ and their partial time derivatives, $\dot{\bar{x}}_{n-1}(t)$, are all available for feedback.

*Remark 2.1:* Note that the requirement for $\dot{\bar{x}}_{n-1}(t)$ is a constraint but realistic for many physical systems as we are not requiring $\dot{x}_n$ which is directly influenced by the control.

*Assumption 2.2:* The signs of $g_i(\bar{x}_i)$ are known, and there exist constants $g_{i0}$ and known smooth functions $\bar{g}_i(\bar{x}_i)$ such that $g_{i0} \le |g_i(\bar{x}_i)| \le \bar{g}_i(\bar{x}_i)\infty$, $\forall \bar{x}_i \in R^i$.

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*Assumption 2.3:* The desired trajectory vectors $\bar{x}_{di}$, $i = 2, \ldots, n$ are continuous and available, and $\bar{x}_{di} \in \Omega_{di} \subset R^i$ with $\Omega_{di}$ known compact sets.

*Assumption 2.4:* The unknown smooth functions $h_i(\bar{x}_i(t))$ satisfy the following inequality $|h_i(\bar{x}_i(t))| \le \sum_{j=1}^{i} |x_j(t)| \varrho_{ij}(\bar{x}_i(t))$ where $\varrho_{ij}(\cdot)$ are known smooth functions.

*Assumption 2.5:* The size of the unknown time delays is bounded by a known constant, i.e., $\tau_i \le \tau_{\max}$, $i = 1, \ldots, n$.

*Remark 2.2:* There are many physical processes which are governed by nonlinear differential equations of the form (1). Examples are recycled reactors, recycled storage tanks and cold rolling mills [19]. In general, most of the recycling processes inherit delays in their state equations.

A function approximator shall be used to approximate the unknown nonlinear functions. Among the two types of artificial neural networks, i.e, 1) linearly parametrized neural networks (LPNNs) [18] and 2) multilayer neural networks (MNN's), for simplicity of demonstrating the main idea, we shall use radial basis function (RBF) neural network (NN), a kind of LPNN, as an example in this note to approximate the continuous function $h(Z) : R^q \to R$ as

$$h_{nn}(Z) = W^T S(Z) \tag{2}$$

where the input vector $Z \in \Omega_Z \subset R^q$, weight vector $W \in R^l$, and basis function vector $S(Z) = [s_1(Z), s_2(Z), \ldots, s_l(Z)]^T \in R^l$, with $l$ being the NN node number and $s_i(Z)$ chosen as the commonly used Gaussian functions, which have the form $s_i(Z) = \exp[-(Z - \mu_i)^T (Z - \mu_i)/\eta_i^2]$, $i = 1, \ldots, l$ where $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^T$ is the center of the receptive field and $\eta_i$ is the width of the Gaussian function. Universal approximation results in [20] and [21] indicate that, if $l$ is chosen sufficiently large, $W^T S(Z)$ can approximate any continuous function to any desired accuracy over a compact set $\Omega_Z \subset R^q$ to arbitrary any accuracy as

$$h(Z) = W^{*T} S(Z) + \epsilon(Z), \forall Z \in \Omega_Z \subset R^q \tag{3}$$

where $W^*$ is the ideal constant weight vector, and $\epsilon(Z)$ is the approximation error which is bounded over the compact set, i.e., $|\epsilon(Z)| \le \epsilon^*$, $\forall Z \in \Omega_Z$ where $\epsilon^* > 0$ is an unknown constant. The ideal weight vector $W^*$ is an "artificial" quantity required for analytical purposes. $W^*$ is defined as the value of $W$ that minimizes $|\epsilon|$ for all $Z \in \Omega_Z \subset R^q$, i.e., $W^* := \arg\min_{W \in R^l} \left\{\sup_{Z \in \Omega_Z} |h(Z) - W^T S(Z)|\right\}$.

The following even function $p_i(\cdot) : R \to R$ is introduced for the purpose of the practical controller design in Section III:

$$p_i(x) = \begin{cases} 1, & |x| \ge c_{z_i} \\ 0, & |x| < c_{z_i} \end{cases} \quad \forall x \in R. \tag{4}$$

## III. Adaptive NN Controller Design

In this section, adaptive neural control is proposed for system (1) and the stability results of the closed-loop system are presented. The backstepping design procedure contains $n$ steps. The design of adaptive control laws is based on the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, \ldots, n$, where $\alpha_i(t)$ is an intermediate control which shall be developed for the corresponding $i$th subsystem based on an appropriate Lyapunov function $V_i(t)$. The control law $u(t)$ is designed in the last step to stabilize the whole closed-loop system based on the overall Lyapunov function $V_n$, which is the sum of the previous $V_i(t)$, $i = 1, \ldots, n-1$. For clarity, let $\tilde{W}_i = \hat{W}_i - W_i^*$, constant $c_i := 1/2\sigma_i \left\|W_i^* - W_i^0\right\|^2 + 1/2\epsilon_{z_i}^{*2}$, where $\hat{W}_i \in R^{l_i}$ is the estimate of ideal NN weight $W_i^* \in R^{l_i}$, $W_i^0 \in R^{l_i}$ is a design constant, $\epsilon_{z_i}^*$ is the upper bound of the NN approximation error, i.e., $\|\epsilon_i(Z_i)\| \le \epsilon_{z_i}^*$ with $Z_i$ being the corresponding inputs to be defined later, and small constant $\sigma_i > 0$ is to introduce $\sigma$-modification in adaptive control to be developed later. In this note, the following inequalities play an important role with $i = 1, \ldots, n$:

$$\sigma_i \left(\hat{W}_i - W_i^*\right)^T \left(\hat{W}_i - W_i^0\right) \le \frac{1}{2}\sigma_i \left\|\hat{W}_i - W_i^*\right\|^2 - \frac{1}{2}\sigma_i \left\|W_i^* - W_i^0\right\|^2 \tag{5}$$

$$\epsilon_i(Z_i) z_i(t) \le \frac{1}{2}\epsilon_{z_i}^{*2} + \frac{1}{2} z_i^2(t). \tag{6}$$

Define the integral Lyapunov function $V_{z_i}(t)$ [17], [18], the Lyapunov–Krasovskii functional $V_{U_i}(t)$ with the positive scalar function $U_i(\cdot)$, and the Lyapunov function candidates $V_i(t)$ as

$$V_{z_1}(t) = \int_0^{z_1} \sigma g_{1\gamma}^{-1}(\sigma + y_d) d\sigma \tag{7}$$

$$V_{z_i}(t) = \int_0^{z_i} \sigma g_{i\gamma}^{-1}(\bar{x}_{i-1}, \sigma + \alpha_{i-1}) d\sigma, \qquad i = 2, \ldots, n \tag{8}$$

$$V_{U_i}(t) = \int_{t-\tau_i}^{t} U_i(\bar{x}_i(\tau)) d\tau \tag{9}$$

$$V_i(t) = V_{i-1}(t) + V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}\tilde{W}_i^T(t)\Gamma_i^{-1}\tilde{W}_i(t) \tag{10}$$

where $g_{i\gamma}^{-1}(\bar{x}_i) = \gamma_i(\bar{x}_i)/g_i(\bar{x}_i)$ with $\gamma_i(\bar{x}_i) > 0$ being a smooth weighting function to be defined later and $(g_{i\gamma}^{-1})^2 = \gamma_i^2/g_i^2$, and $U_i(\bar{x}_i(t)) = 1/2\sum_{j=1}^{i} x_j^2(t)\varrho_{ij}^2(\bar{x}_i(t))$, $i = 1, \ldots, n$. The choice of weighting function $\gamma_i(\cdot)$ plays a key role in the controller design and results in different controllers with varying control performance [18]. The apparent and convenient choices for $\gamma_i(\cdot)$ are 1 and $\bar{g}_i(\bar{x}_i)$ for general nonlinear systems. Detailed explanations will be given based on $\gamma_i(\bar{x}_i) = \bar{g}_i(\bar{x}_i)$ in the following, while a remark will be given directly addressing the controller design and relevant stability and performance analysis for $\gamma_i(\bar{x}_i) = 1$ without derivation for conciseness.

By choosing $\gamma_i(\bar{x}_i) = \bar{g}_i(\bar{x}_i)$, we have $g_{i\gamma}^{-1}(\bar{x}_i) = \bar{g}_i(\bar{x}_i)/g_i(\bar{x}_i)$. From Assumption 2.2, we know that $g_{i\gamma}^{-1}(\bar{x}_i)$ are bounded by known functions as $1 < g_{i\gamma}^{-1}(\bar{x}_i) \le \bar{g}_i(\bar{x}_i)/g_{i0}$. Clearly, $V_{z_i}$ are positive–definite functions, $i = 1, \ldots, n$.

*Step 1)* Let us first consider the $z_1$-subsystem as

$$\dot{z}_1(t) = g_1(x_1(t))(z_2(t) + \alpha_1(t)) + f_1(x_1(t)) + h_1(x_1(t - \tau_1)) - \dot{y}_d(t). \tag{11}$$

By variable change $\sigma = \theta z_1$, we may rewrite $V_{z_1}$ in (7) as $V_{z_1} = z_1^2 \int_0^1 \theta g_{1\gamma}^{-1}(\theta z_1 + y_d) d\theta$. Noting that $1 \le g_{1\gamma}^{-1}(\theta z_1 + y_d) \le \bar{g}_1(\theta z_1 + y_d)/g_{10}$, we have

$$\frac{z_1^2}{2} \le V_{z_1} \le \frac{z_1^2}{g_{10}} \int_0^1 \theta \bar{g}_1(\theta z_1 + y_d) d\theta. \tag{12}$$

The time derivative of $V_{z_1}$ along (11) is

$$\dot{V}_{z_1}(t) = z_1(t)\left[\bar{g}_1(x_1(t))z_2(t) + \bar{g}_1(x_1(t))\alpha_1(t) + g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t)) + g_{1\gamma}^{-1}(x_1(t))h_1(x_1(t - \tau_1)) - \dot{y}_d(t)\int_0^1 g_{1\gamma}^{-1}(\theta z_1 + y_d) d\theta\right].$$

Applying Assumption 2.4, we have

$$\begin{aligned}\dot{V}_{z_1}(t) \le & z_1(t)\Bigg[\bar{g}_1(x_1(t))z_2(t)+\bar{g}_1(x_1(t))\alpha_1(t)\\ &+g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t))\\ &-\dot{y}_d\int_0^1 g_{1\gamma}^{-1}(\theta z_1+y_d)d\theta\Bigg]\\ &+|z_1(t)|g_{1\gamma}^{-1}(x_1(t))|x_1(t-\tau_1)|\\ &\times \varrho_1(x_1(t-\tau_1)).\end{aligned} \tag{13}$$

By using Young's Inequality, (13) becomes

$$\begin{aligned}\dot{V}_{z_1}(t) \le & z_1(t)\Bigg[\bar{g}_1(x_1(t))z_2(t)+\bar{g}_1(x_1(t))\alpha_1(t)\\ &+g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t))\\ &-\dot{y}_d\int_0^1 g_{1\gamma}^{-1}(\theta z_1+y_d)d\theta\Bigg]\\ &+\frac{1}{2}z_1^2(t)\left(g_{1\gamma}^{-1}\right)^2+\frac{1}{2}x_1^2(t-\tau_1)\\ &\times \varrho_1^2(x_1(t-\tau_1)).\end{aligned} \tag{14}$$

In standard iterative backstepping design, $\alpha_1(t)$ is usually designed to stabilize the $z_1$-subsystem except for the coupling term $\bar{g}_1 z_1 z_2$ to be dealt with in the next step. In doing so under the assumption of known functions, one more difficulty exists in the new problem setting. Although $\varrho_1(\cdot)$ is a known function, it cannot be utilized in designing $\alpha_1(t)$ as $x_1(t-\tau_1)$ is undetermined because of unknown time delay $\tau_1$. Intuitively, approximation methods such as neural networks can be used to approximate the unknown functions. The unknown functions $g_1(\cdot)$ and $f_1(\cdot)$ can be dealt with in this way without any problem. However, due to the existence of the unknown time delay $\tau_1$, functions of $x_1(t-\tau_1)$ are hard to be approximated using neural networks since the input $x_1(t-\tau_1)$ is undetermined because of the uncertain $\tau_1$. To overcome the design difficulties from the unknown time delay $\tau_1$, we consider the Lyapunov–Krasovskii functional $V_{U_1}(t)$ in (9) with its time derivative being

$$\dot{V}_{U_1}(t)=U_1(x_1(t))-U_1(x_1(t-\tau_1)) \tag{15}$$

which can be used to cancel the time-delay term on the right-hand side of (14) and thus eliminate the design difficulty from the unknown time delay $\tau_1$ without introducing any uncertainties to the system. Accordingly, we obtain

$$\begin{aligned}\dot{V}_{z_1}(t)+\dot{V}_{U_1}(t) \le & z_1(t)\Bigg[\bar{g}_1(x_1(t))z_2(t)+\bar{g}_1(x_1(t))\alpha_1(t)\\ &+g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t))\\ &-\dot{y}_d\int_0^1 g_{1\gamma}^{-1}(\theta z_1+y_d)d\theta\\ &+\frac{1}{2}z_1(t)\left(g_{1\gamma}^{-1}\right)^2\\ &+\frac{1}{2z_1(t)}x_1^2(t)\varrho_1^2(x_1(t))\Bigg].\end{aligned} \tag{16}$$

Comparing (16) with (14), it is found that the difficulty from the unknown time delay $\tau_1$ has been eliminated by introducing the Lyapunov–Krasovskii functional $V_{U_1}(t)$. By differentiating $V_{U_1}(t)$ with respect to time, the unknown time delay term $U_1(x_1(t-\tau_1)) = 1/2x_1^2(t-\tau_1)\varrho_1^2(x_1(t-\tau_1))$ appeared in (15) can be used for exact cancellation on the right-hand side of (14). The remaining term $U_1(x_1(t))$ from $\dot{V}_{U_1}(t)$ is a known function of known variables, which does not introduce any uncertainties to the system. Therefore, the design of intermediate control $\alpha_1(t)$ is free from unknown time delay $\tau_1$ at present stage.

For notation conciseness, we will omit the time variables $t$ and $t-\tau_1$ after time-delay terms have been eliminated. Under the assumption of exact knowledge, the certainty equivalent control is

$$\alpha_1^*=\frac{1}{\bar{g}_1(x_1)}\left[-k_1(t)z_1-Q_1(Z_1)-\frac{1}{2z_1}x_1^2\varrho_1^2(x_1)\right]$$

where

$$Q_1(Z_1)=g_{1\gamma}^{-1}(x_1)f_1(x_1)-\dot{y}_d\int_0^1 g_{1\gamma}^{-1}(\theta z_1+y_d)d\theta +\frac{1}{2}z_1\left(g_{1\gamma}^{-1}\right)^2$$

with $Z_1=[x_1,y_d,\dot{y}_d]^T\in\Omega_{Z_1}\subset R^3$ and $\Omega_{Z_1}:=\{z_1,\bar{x}_{d2}|z_1\in R,\bar{x}_{d2}\in\Omega_{d2}\}$. It is apparent that controller singularity may occur. In addition, it is certain that $\alpha_1^*$ is not an admissible control, since $\alpha_1^*$ is not well-defined when $z_1=0$ as $\lim_{z_1\to 0}2z_1=0$, $\lim_{z_1\to 0}x_1^2\varrho_1^2(x_1)\neq 0$ and L'Hopital's rule is not applicable to obtain the limit $\lim_{z_1\to 0}(x_1^2\varrho_1^2(x_1)/(2z_1))$. Therefore, care must be taken to guarantee the boundedness of the controller.

It is noted that point $z_1=0$ is not only an isolated point in $\Omega_{Z_1}$, but also the case that the system reaches the origin at this point. From a practical point of view, once the system reaches its origin, no control action should be taken for less power consumption. For ease of discussion, let us define sets $\Omega_{c_{z_1}}\subset\Omega_{Z_1}$ and $\Omega_{Z_1}^0$ as follows:

$$\Omega_{c_{z_1}}:=\{z_1\mid |z_1|<c_{z_1}\} \tag{17}$$

$$\Omega_{Z_1}^0:=\Omega_{Z_1}-\Omega_{c_{z_1}} \tag{18}$$

where $c_{z_1}$ is a constant that can be chosen arbitrarily small and "$-$" in (18) is used to denote the complement of set $B$ in set $A$ as $A-B:=\{x|x\in A \text{ and } x\notin B\}$.

Accordingly, the following practical control law is proposed:

$$\alpha_1^*=\frac{p_1(z_1)}{\bar{g}_1(x_1)}\left[-k_1(t)z_1-Q_1(Z_1)-\frac{1}{2z_1}x_1^2\varrho_1^2(x_1)\right] \tag{19}$$

where $p_i(\cdot)$ is defined in (4).

Since $f(\cdot)$ and $g(\cdot)$ are unknown smooth functions, the desired practical control $\alpha_1^*$ in (19) cannot be implemented in practice. Neural networks can be used to approximate the unknown function $Q_1(Z_1)$. Note that control action is only activated when $z_1\in\Omega_{Z_1}^0$, which means unknown function $Q_1(Z_1)$ is approximated by neural networks over the set $\Omega_{Z_1}^0$. According to the main result stated in [22], any real-valued continuous function can be arbitrarily closely approximated by a network of RBF type over a compact set. The compactness of set $\Omega_{Z_1}^0$ is a must to guarantee the feasibility of neural networks approximation, which is shown in the following lemma.

*Lemma 3.1:* Set $\Omega_{Z_1}^0$ is a compact set.

*Proof:* First, we show that $\Omega_{Z_1}^0$ is a *closed* set. From (18) and applying De Morgan's laws, we have

$$\Omega_{Z_1}^{0c}=\Omega_{Z_1}^c\cup\Omega_{c_{z_1}} \tag{20}$$

where $\Omega_{Z_1}^{0c}$ and $\Omega_{Z_1}^c$ denote the complements of $\Omega_{Z_1}^0$ and $\Omega_{Z_1}$ respectively. Since $\Omega_{Z_1}$ is a *compact* set, i.e., it is *closed* and *bounded*, $\Omega_{Z_1}^c$ is an *open* set. In addition, $\Omega_{c_{z_1}}$ is also an open set from its definition. From (20), we know that $\Omega_{Z_1}^{0c}$ is an open set, which means that its complement $\Omega_{Z_1}^0$ is a closed set. Second, from (18), we know that $\Omega_{Z_1}^0\subset\Omega_{Z_1}$. Since a closed subset of a compact set is compact, we can conclude that $\Omega_{Z_1}^0$ is a compact set. □

Based on Lemma 3.1, it is known that $Q_1(Z_1)$ is continuous and well-defined over compact set $\Omega^0_{Z_1}$ and can be approximated by neural networks to arbitrary any accuracy as follows $Q_1(Z_1) = W_1^{*T} S(Z_1) + \epsilon_1(Z_1)$ where $\epsilon_1(Z_1)$ is the approximation error. As the ideal weight $W_1^*$ is unknown, we shall use its estimate $\hat{W}_1$ instead, which forms the intermediate control $\alpha_1$ as

$$\alpha_1(t) = \frac{p_1(z_1)}{\bar{g}_1(x_1)} \left[ -k_1(t) z_1 - \hat{W}_1^T S(Z_1) - \frac{1}{2z_1} x_1^2 \varrho_1^2(x_1) \right]. \quad (21)$$

For uniformity of notation, we define sets $\Omega_{c_{z_i}} \subset \Omega_{Z_i}$ and $\Omega^0_{Z_i}$, $i = 2, \ldots, n$ as

$$\Omega_{c_{z_i}} := \{ z_i \mid |z_i| < c_{z_i} \} \quad (22)$$

$$\Omega^0_{Z_i} := \Omega_{Z_i} - \Omega_{c_{z_i}}. \quad (23)$$

Note that the control objective is to show that certain compact set $\Omega_S$ is domain of attraction in the sense that for all bounded initial conditions, there exists $\Omega_S$ such that all closed-loop signals will eventually converge to $\Omega_S$. i.e., all $Z_i(t)$ starting from within $\Omega^0_{Z_i}$ will enter into $\Omega_S$ and will stay within $\Omega_S$ thereafter.

In the following steps, the unknown functions $Q_i(Z_i)$, $i = 2, \ldots, n$ will be approximated by neural networks as

$$Q_i(Z_i) = W_i^{*T} S(Z_i) + \epsilon_i(Z_i) \qquad \forall Z_i \in \Omega^0_{Z_i}. \quad (24)$$

Consider the Lyapunov function candidate $V_1(t)$ in (10), whose time derivative along (16), (21) and (24) for $z_1 \in \Omega^0_{Z_1}$ is

$$\dot{V}_1 \leq -k_1(t) z_1^2 + \bar{g}_1(x_1) z_1 z_2 - \left( \hat{W}_1 - W_1^* \right)^T S(Z_1) z_1 + z_1 \epsilon_{z_1} + \left( \hat{W}_1 - W_1^* \right)^T \Gamma_1^{-1} \dot{\hat{W}}_1. \quad (25)$$

The following practical adaptive law is given for online tuning the NN weights

$$\dot{\hat{W}}_1 = p_1(z_1) \Gamma_1 \left[ S(Z_1) z_1 - \sigma_1 \left( \hat{W}_1 - W_1^0 \right) \right]. \quad (26)$$

Substituting (26) into (25), and using (5) and (6), we have

$$\dot{V}_1 \leq -k_1(t) z_1^2 - \frac{1}{2} \sigma_1 \left\| \hat{W}_1 - W_1^* \right\|^2 + \bar{g}_1(x_1) z_1 z_2 + c_1. \quad (27)$$

For $z_1 \in \Omega^0_{z_1}$, noting (12), and choosing

$$k_1(t) = \frac{1}{\varepsilon_{10}} \left[ 1 + \int_0^1 \theta \bar{g}_1(\theta z_1 + y_d) d\theta + \frac{1}{z_1^2} \int_{t-\tau_{\max}}^t \frac{1}{2} x_1^2(\tau) \varrho_1^2(x_1(\tau)) d\tau \right] \quad (28)$$

with $0 < \varepsilon_{10} \leq 2$, we have

$$\dot{V}_1 \leq -\frac{1}{\varepsilon_{10}} z_1^2 - \frac{g_{10}}{\varepsilon_{10}} V_{z_1} - \frac{1}{\varepsilon_{10}} \int_{t-\tau_1}^t \frac{1}{2} x_1^2(\tau) \varrho_1^2(x_1(\tau)) d\tau - \frac{1}{2} \sigma_1 \left\| \hat{W}_1 - W_1^* \right\|^2 + \bar{g}_1(x_1) z_1 z_2 + c_1. \quad (29)$$

Since $[t - \tau_1, t] \subset [t - \tau_{\max}, t]$, we have the inequality $\int_{t-\tau_1}^t 1/2 x_1^2(\tau) \varrho_1^2(x_1(\tau)) d\tau \leq \int_{t-\tau_{\max}}^t 1/2 x_1^2(\tau) \varrho_1^2(x_1(\tau)) d\tau$. Accordingly, (29) becomes

$$\dot{V}_1 \leq \bar{g}_1(x_1) z_1 z_2 - \frac{g_{10}}{\varepsilon_{10}} V_{z_1} - \frac{1}{\varepsilon_{10}} V_{U_1} - \frac{1}{2} \sigma_1 \left\| \hat{W}_1 - W_1^* \right\|^2 + c_1$$

where the coupling term $\bar{g}_1(x_1) z_1 z_2$ will be handled in the next step.

Remark 3.1: Applying Young's inequality, we know that $\overline{g}, z_1, z_2 \leq 1/2\, z_1^2 + 1/2\, \overline{g}_1^2 z_2^2$. The choice for $\varepsilon_{10}$ is to guarantee that $-(1/\varepsilon_{10} - 1/2) z_1^2 \leq 0$, so that the undesired destabilizing term $1/2\, z_1^2$ can be suppressed.

*Step* $i$ $(2 \leq i \leq n-1)$) Similar procedures are taken for $i = 2, \ldots, n-1$ as in Step 1).

The dynamics of $z_i$-subsystem is given by

$$\dot{z}_i(t) = g_i(\bar{x}_i(t))[z_{i+1}(t) + \alpha_i(t)] + f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t - \tau_i)) - \dot{\alpha}_{i-1}(t).$$

The time derivative of $V_{z_i}(t)$ in (8) is given by

$$\dot{V}_{z_i}(t) = z_i(t) \Bigg[ \bar{g}_i(\bar{x}_i(t))(z_{i+1}(t) + \alpha_i(t)) + g_{i\gamma}^{-1}(\bar{x}_i(t)) f_i(\bar{x}_i(t)) + g_{i\gamma}^{-1}(\bar{x}_i(t)) h_i(\bar{x}_i(t - \tau_i)) + \dot{\bar{x}}_{i-1} z_i(t) \int_0^1 \theta \frac{\partial g_{i\gamma}^{-1}(\bar{x}_{i-1}, \theta z_i + \alpha_{i-1})}{\partial \bar{x}_{i-1}} d\theta - \dot{\alpha}_{i-1} \int_0^1 g_{i\gamma}^{-1}(\bar{x}_{i-1}, \theta z_i + \alpha_{i-1}) d\theta \Bigg].$$

Noting Assumption 2.4, we have

$$\dot{V}_{z_i}(t) = z_i(t) \Bigg[ \bar{g}_i(\bar{x}_i(t))(z_{i+1}(t) + \alpha_i(t)) + g_{i\gamma}^{-1}(\bar{x}_i(t)) f_i(\bar{x}_i(t)) + \frac{1}{2} z_i(t) (g_{i\gamma}^{-1})^2 + \dot{\bar{x}}_{i-1} z_i(t) \int_0^1 \theta \frac{\partial g_{i\gamma}^{-1}(\bar{x}_{i-1}, \theta z_i + \alpha_{i-1})}{\partial \bar{x}_{i-1}} d\theta - \dot{\alpha}_{i-1} \int_0^1 g_{i\gamma}^{-1}(\bar{x}_{i-1}, \theta z_i + \alpha_{i-1}) d\theta \Bigg] + \frac{1}{2} \sum_{j=1}^{i} x_j^2(t - \tau_i) \varrho_{ij}^2(\bar{x}_i(t - \tau_i)). \quad (30)$$

In Step $i-1$), for $z_j \in \Omega^0_{Z_j}$, $j = 1, \ldots, i-1$, it has been obtained that

$$\dot{V}_{i-1} \leq \bar{g}_{i-1}(\bar{x}_{i-1}) z_{i-1} z_i + \sum_{j=1}^{i-1} \left( -\frac{g_{j0}}{\varepsilon_{j0}} V_{z_j} - \frac{1}{\varepsilon_{j0}} V_{U_j} - \frac{1}{2} \sigma_j \left\| \hat{W}_j - W_j^* \right\|^2 + c_j \right). \quad (31)$$

Consider $V_i(t)$ given in (10). For $z_j \in \Omega^0_{Z_j}$, $j = 1, \ldots, i$, its time derivative along (30) and (31) is

$$\dot{V}_i \leq \bar{g}_i(\bar{x}_i) z_i z_{i+1} + z_i \left[ \bar{g}_{i-1}(\bar{x}_{i-1}) z_{i-1} + \bar{g}_i(\bar{x}_i) \alpha_i + Q_i(Z_i) \right] + (\hat{W}_i - W_i^*)^T \Gamma_i^{-1} \dot{\hat{W}}_i + \sum_{j=1}^{i-1} \left( -\frac{g_{j0}}{\varepsilon_{j0}} V_{z_j} - \frac{1}{\varepsilon_{j0}} V_{U_j} - \frac{1}{2} \sigma_j \left\| \hat{W}_j - W_j^* \right\|^2 + c_j \right) \quad (32)$$

where

$$Q_i(Z_i) = g_{i\gamma}^{-1}(\bar{x}_i)f_i(\bar{x}_i) + \frac{1}{2}z_i(g_{i\gamma}^{-1})^2 + \dot{\bar{x}}_{i-1}z_i\int_0^1 \theta\frac{\partial g_{i\gamma}^{-1}(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial \bar{x}_{i-1}}d\theta - \dot{\alpha}_{i-1}\int_0^1 g_{i\gamma}^{-1}(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta$$

with $Z_i(t) = [\bar{x}_i, \dot{\bar{x}}_{i-1}, \alpha_{i-1}, \partial\alpha_{i-1}/\partial x_1, \partial\alpha_{i-1}/\partial x_2, \ldots, \partial\alpha_{i-1}/\partial x_{i-1}, \omega_{i-1}] \in \Omega_{Z_i}^0 \subset R^{3i}$, where

$$\dot{\alpha}_{i-1} = \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j + \omega_{i-1}, \omega_{i-1} = \frac{\partial\alpha_{i-1}}{\partial\bar{x}_{di}}\dot{\bar{x}}_{di} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j.$$

Similarly, we have the following intermediate control law:

$$\alpha_i = \frac{p_i(z_i)}{\bar{g}_i(\bar{x}_i)}\left[-\bar{g}_{i-1}(\bar{x}_{i-1})z_{i-1} - k_i(t)z_i - \hat{W}_i^T S(Z_i) - \frac{1}{2z_i}\sum_{j=1}^{i}x_j^2\varrho_{ij}^2(\bar{x}_i)\right] \quad (33)$$

$$\dot{\hat{W}}_i = p_i(z_i)\Gamma_i\left[S(Z_i)z_i - \sigma_i\left(\hat{W}_i - W_i^0\right)\right] \quad (34)$$

$$k_i(t) = \frac{1}{\varepsilon_{i0}}\left[1 + \int_0^1\theta\bar{g}_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta + \frac{1}{z_i^2}\int_{t-\tau_{\max}}^{t}\frac{1}{2}\sum_{j=1}^{i}x_j^2(\tau)\varrho_{ij}^2(\bar{x}_i(\tau))d\tau\right], \quad z_i \in \Omega_{Z_i}^0 \quad (35)$$

with $0 < \varepsilon_{i0} \le 2$. Substituting (33)–(35) into (32), and using (5), (6), and (12), we have

$$\dot{V}_i \le \bar{g}_i(\bar{x}_i)z_iz_{i+1} + \sum_{j=1}^{i}\left(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\left\|\hat{W}_j - W_j^*\right\|^2 + c_j\right)$$

where the coupling term $\bar{g}_i(\bar{x}_i)z_iz_{i+1}$ will be handled in the next step.

*Step n*) This is the final step, since the actual control $u$ appears in the dynamics of $z_n$-subsystem as given by

$$\dot{z}_n = g_n(\bar{x}_n(t))u + f_n(\bar{x}_n(t)) + h_n(\bar{x}_n(t-\tau_n)) - \dot{\alpha}_{n-1}(t).$$

Consider $V_{z_n}(t)$ given in (8). Noting Assumption 2.4, its time derivative is

$$\dot{V}_{z_n}(t) = z_n(t)\left[\bar{g}_n(\bar{x}_n(t))u(t) + g_{n\gamma}^{-1}(\bar{x}_n(t))f_n(\bar{x}_n(t)) + \frac{1}{2}z_n(t)(g_{n\gamma}^{-1})^2 + \dot{\bar{x}}_{n-1}z_n(t) \times \int_0^1\theta\frac{\partial g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial\bar{x}_{n-1}}d\theta - \dot{\alpha}_{n-1}\int_0^1 g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta\right] + \frac{1}{2}\sum_{j=1}^{n}x_j^2(t-\tau_n)\varrho_{nj}^2(\bar{x}_n(t-\tau_n)). \quad (36)$$

In Step $n-1$), for $z_i \in \Omega_{Z_i}^0$, $i = 1,\ldots,n-1$, it has been obtained that

$$\dot{V}_{n-1} \le \bar{g}_{n-1}(\bar{x}_{n-1})z_{n-1}z_n + \sum_{j=1}^{n-1}\left(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\|\hat{W}_j - W_j^*\|^2 + c_j\right). \quad (37)$$

Consider $V_n(t)$ given in (10). For $|z_i| \in \Omega_{Z_i}^0$, $i = 1,\ldots,n$, its time derivative along (36) and (37) is

$$\dot{V}_i \le \bar{g}_i(\bar{x}_i)z_iz_{i+1} + z_i\left[\bar{g}_{i-1}(\bar{x}_{i-1})z_{i-1} + \bar{g}_i(\bar{x}_i)\alpha_i + Q_i(Z_i)\right] + (\hat{W}_i - W_i^*)^T\Gamma_i^{-1}\dot{\hat{W}}_i + \sum_{j=1}^{i-1}\left(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\left\|\hat{W}_j - W_j^*\right\|^2 + c_j\right) \quad (38)$$

where

$$Q_n(Z_n) = g_{n\gamma}^{-1}(\bar{x}_n)f_n(\bar{x}_n) + \frac{1}{2}z_n\left(g_{n\gamma}^{-1}\right)^2 + \dot{\bar{x}}_{n-1}z_n\int_0^1\theta\frac{\partial g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial\bar{x}_{n-1}}d\theta - \dot{\alpha}_{n-1}\int_0^1 g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta$$

with $Z_n(t) = [\bar{x}_n, \dot{\bar{x}}_{n-1}, \alpha_{n-1}, \partial\alpha_{n-1}/\partial x_1, \partial\alpha_{n-1}/\partial x_2, \ldots, \partial\alpha_{n-1}/\partial x_{n-1}, \omega_{n-1}] \in \Omega_{Z_n}^0 \subset R^{3n}$, where

$$\dot{\alpha}_{n-1} = \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\dot{x}_j + \omega_{n-1}$$

$$\omega_{n-1} = \frac{\partial\alpha_{n-1}}{\partial\bar{x}_{dn}}\dot{\bar{x}}_{dn} + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j.$$

We construct the following adaptive neural control law:

$$u(t) = \frac{p_n(z_n)}{\bar{g}_n(\bar{x}_n)}\left[-\bar{g}_{n-1}(\bar{x}_{n-1})z_{n-1} - k_n(t)z_n - \hat{W}_n^T S(Z_n) - \frac{1}{2z_n}\sum_{j=1}^{n}x_j^2\varrho_{nj}^2(\bar{x}_n)\right] \quad (39)$$

$$\dot{\hat{W}}_n = p_n(z_n)\Gamma_i\left[S(Z_n)z_n - \sigma_i\left(\hat{W}_n - W_n^0\right)\right] \quad (40)$$

$$k_n(t) = \frac{1}{\varepsilon_{n0}}\left[1 + \int_0^1\theta\bar{g}_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta + \frac{1}{z_n^2}\int_{t-\tau_{\max}}^{t}\frac{1}{2}\sum_{j=1}^{n}x_j^2(\tau)\varrho_{nj}^2(\bar{x}_n(\tau))d\tau\right], \quad z_n \in \Omega_{Z_n}^0 \quad (41)$$

with $0 < \varepsilon_{n0} \le 2$. Substituting (39)–(41) into (38), and using (5), (6), and (12), we have

$$\dot{V}_n(t) \le \sum_{j=1}^{n}\left(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\left\|\hat{W}_j - W_j^*\right\|^2 + c_j\right). \quad (42)$$

*Theorem 3.1:* Consider the closed-loop system consisting of the plant (1) under Assumptions 2.1–2.5, the controller (39) and the NN weight updating law (40). For bounded initial conditions, the following properties hold.

i) All signals in the closed-loop system remain SGUUB and the vector $Z = [Z_1^T, \ldots, Z_n^T]^T$ remains in a compact set $\Omega_Z^0 := \Omega_{Z_1}^0 \cup \cdots \cup \Omega_{Z_n}^0$ specified as

$$\Omega_Z^0 = \left\{ Z | \sum_{i=1}^n z_i^2 \le 2C_0, \sum_{i=1}^n \|\tilde{W}_i\|^2 \le \frac{2C_0}{\lambda_{\min}(\Gamma_i^{-1})}, \bar{x}_{di} \in \Omega_{di}, i = 2, \ldots, n, z_i \notin \Omega_{c_{z_i}}, i = 1, \ldots, n \right\} \quad (43)$$

where $C_0 > 0$ is a constant whose size depends on the initial conditions (as will be defined later in the proof).

ii) The closed-loop signal $z(t) = [z_1, \ldots, z_n]^T \in R^n$ will eventually converge to a compact set defined by

$$\Omega_S := \{ z \mid \|z\|^2 \le \mu \} \quad (44)$$

where $\mu > 0$ is a constant related to the design parameters and will be defined later in the proof, and $\Omega_S$ can be made as small as desired by an appropriate choice of the design parameters.

*Proof:* Consider the following Lyapunov function candidate:

$$V_n(t) = \sum_{i=1}^n \left[ V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}\tilde{W}_i^T \Gamma_i^{-1} \tilde{W}_i \right] \quad (45)$$

where $V_{z_i}(t)$ and $V_{U_i}(t)$ are defined in (8) and (9), respectively, and $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$. The following three cases are considered.

Case 1) $z_i \in \Omega_{c_{z_i}}, i = 1, \ldots, n$. In this case, the controls $\alpha_i = 0$, $i = 1, \ldots, n-1$, $u = 0$ and $\dot{\hat{W}}_i = 0$, $i = 1, \ldots, n$. Since $z_1 = x_1 - y_d$ and $y_d$ is bounded, $x_1$ is bounded. For $i = 2, \ldots, n$, $x_i$ is bounded as $x_i = z_i + \alpha_{i-1}$ and $\alpha_{i-1} = 0$. In addition, $\hat{W}_i$ is kept unchanged in a bounded value, $i = 1, \ldots, n$. Observing the definition for $V_{z_i}(t)$ and $V_{U_i}(t)$, and noting that $g_{i\gamma}(\cdot)$, $\varrho_{ij}(\cdot)$ are smooth functions, we know that for bounded $x_i$, $z_i$ and $\hat{W}_i$, $V_{z_i}(t)$ and $V_{U_i}(t)$ are bounded, i.e., there exists a finite $C_B$ such that

$$V_n(t) \le C_B. \quad (46)$$

Case 2) $z_i \in \Omega_{Z_i}^0$, $i = 1, \ldots, n$. From (42), we have $\dot{V}_n(t) \le -C_1 V_n(t) + C_2$ where $C_1 = \sum_{i=1}^n c_i$ and

$$C_2 = \min\left\{ \frac{g_{10}}{\varepsilon_{10}}, \ldots, \frac{g_{n0}}{\varepsilon_{n0}}, \frac{1}{\varepsilon_{10}}, \ldots, \frac{1}{\varepsilon_{n0}}, \frac{\sigma_1}{\lambda_{\min}\left(\Gamma_1^{-1}\right)}, \ldots, \frac{\sigma_n}{\lambda_{\min}\left(\Gamma_n^{-1}\right)} \right\}.$$

Let $\rho = C_2/C_1$, it follows that

$$0 \le V_n(t) \le [V_n(0) - \rho]e^{-C_1 t} + \rho \le V_n(0) + \rho \quad (47)$$

where constant

$$V_n(0) = \sum_{i=1}^n \left[ \int_0^{z_i(0)} \sigma g_{i\gamma}^{-1}(\bar{x}_{i-1}(0), \sigma + \alpha_{i-1}(0)) d\sigma + \frac{1}{2}\tilde{W}_i^T(0)\Gamma_i^{-1}\tilde{W}_i(0) \right]$$

with $g_{i\gamma}^{-1}(\bar{x}_{i-1}(0), \sigma + \alpha_{i-1}(0)) = g_{1\gamma}^{-1}(\sigma)$ for $i = 1$.

Case 3) Some $z_i \in \Omega_{Z_i}^0$ and some $z_j \in \Omega_{c_{z_j}}$. In this case, the corresponding $\alpha_i$ or $u$ and the adaptation law for $\hat{W}_i$ will be invoked for $z_i \in \Omega_{Z_i}^0$ while $\alpha_j = 0$ or $u = 0$ and $\dot{\hat{W}}_j = 0$ for $z_j \in \Omega_{c_{z_j}}$. Let us define $V_I(t) = \sum_i (V_{z_i} + V_{U_i} + 1/2\tilde{W}_i^T \Gamma_i^{-1} \tilde{W}_i)$ and $V_J(t) = \sum_j (V_{z_j} + V_{U_j} + 1/2\tilde{W}_j^T \Gamma_j^{-1} \tilde{W}_j)$. For $z_j \in \Omega_{c_{z_j}}$, we obtain that $V_J(t)$ is bounded, i.e., $V_J(t) \le C_J$ with $C_J$ being finite, and $z_i \in \Omega_{Z_i}^0$, we have that $\dot{V}_I(t) \le -C_1^I V_I(t) + C_2^I$, i.e.,

$$V_I(t) \le [V_I(0) - \rho_I]e^{-C_1^I t} + \rho_I \le V_I(0) + \rho_I \quad (48)$$

where $\rho_I = C_2^I / C_1^I$ with $C_1^I = \sum_i c_i$ and $C_2^I = \min_i\{g_{i0}/\varepsilon_{i0}, 1/\varepsilon_{i0}, \sigma_i/\lambda_{\min}\left(\Gamma_i^{-1}\right)\}$. Therefore, it can be obtained that

$$V_n(t) = V_I(t) + V_J(t) \le V_I(0) + \rho_I + C_J. \quad (49)$$

Thus, from (46), (47), and (49) for Cases 1)–3), we can conclude that

$$V_n(t) \le C_0 \quad (50)$$

where $C_0 = \max\{C_B, V_n(0) + \rho, V_I(0) + \rho_I + C_J\}$. From (50), we know that $V_n(t)$, $z_i$ and $\hat{W}_i$, $i = 1, \ldots, n$, are bounded. Since $z_1 = x_1 - y_d$ and $y_d$ is bounded, $x_1$ is bounded. For $x_2 = z_2 + \alpha_1$, since $\alpha_1$ is function of bounded signals $z_1, Z_1, \hat{W}_1$, $\alpha_1$ is thus bounded, which in turn leads to the boundedness of $x_2$. Following the same way, we can prove one by one that all $\alpha_{i-1}$ and $x_i$, $i = 3, \ldots, n$ are bounded. Therefore, the systems' states $x_i$, $i = 1, \ldots, n$ are bounded.

Considering (45) and the property for $V_{z_i}(t)$ that

$$\frac{1}{2}z_i^2 \le V_{z_i}(t) \le \frac{z_i^2}{g_{i0}} \int_0^1 \theta \bar{g}_i(\bar{x}_{i-1}, \theta z_i + \alpha_{i-1}) d\theta$$

we know that

$$\sum_{i=1}^n z_i^2 \le 2\sum_{i=1}^n V_{z_i}(t) \le 2V_n(t) \quad \sum_{i=1}^n \|\tilde{W}_i\|^2 \le \frac{2V_n(t)}{\lambda_{\min}\left(\Gamma_i^{-1}\right)}. \quad (51)$$

From (50) and (51), we readily have the compact set $\Omega_Z^0$ defined in (43) over which the NN approximation is carried out with its feasibility being guaranteed.

In addition, in Case 1), as $z_i \in \Omega_{c_{z_i}}$, $i = 1, \ldots n$, we know that $\|z\|^2 = \sum_{i=1}^n z_i^2 \le \sum_{i=1}^n c_{z_i}^2$. In Case 2), from (47) and (51), we have that $\lim_{t\to\infty} \|z\|^2 = 2\rho$. In Case 3), from (48) and (51), we have that $\lim_{t\to\infty} \sum_i z_i^2 = 2\rho_I$ and $\sum_j z_j^2 \le \sum_j c_{z_j}^2$. Therefore, as $t \to \infty$, we can conclude that $\|z\|^2 \le \mu$ where $\mu = \max\left\{2\rho, 2\rho_I, \sum_{i=1}^n c_{z_i}^2\right\}$, i.e., the vector $z$ will eventually converge to the compact set $\Omega_S$ defined in (44). This completes the proof.

*Remark 3.1:* Note that the choices of $\gamma_i(\bar{x}_i)$ are not unique. By choosing $\gamma_i(\bar{x}_i) = 1$, we have $g_{i\gamma}^{-1}(\bar{x}_i) = 1/g_i(\bar{x}_i)$ [17] and $V_{z_i} = \int_0^{z_i} \sigma/g_i(\bar{x}_{i-1}, \sigma + \alpha_{i-1}) d\sigma$, $i = 1, \ldots, n$. By the mean value theorem, $V_{z_i}$ can be rewritten as $V_{z_i} = \lambda_s z_i^2 / g_i(\bar{x}_{i-1}, \lambda_s z_i + \alpha_{i-1})$, $\lambda_s \in (0, 1)$. From Assumption 2.2, $0 \le g_{i0} \le g_i(\bar{x}_i)$, we know that $0 < V_{z_i} \le \lambda_s / g_{10} z_i^2$. The adaptive control laws are given by

$$\alpha_i = \frac{p_i(z_i)}{\bar{g}_i(\bar{x}_i)} \left[ -\bar{g}_{i-1}(\bar{x}_{i-1}) z_{i-1} - k_i(t) z_i - \hat{W}_i^T S(Z_i) - \frac{1}{2z_i} \sum_{j=1}^i x_j^2 \varrho_{ij}^2(\bar{x}_i) \right]$$

$$u = \frac{p_n(z_n)}{\bar{g}_n(\bar{x}_n)}\left[-\bar{g}_{n-1}(\bar{x}_{n-1})z_{n-1} - k_n(t)z_n - \hat{W}_n^T S(Z_n) - \frac{1}{2z_n}\sum_{j=1}^{n} x_j^2 \varrho_{nj}^2(\bar{x}_n)\right]$$

$$\dot{\hat{W}}_i = p_i(z_i)\Gamma_i\left[S(Z_i)z_i - \sigma_i\left(\hat{W}_i - W_i^0\right)\right]$$

$$k_i(t) = \frac{1}{\varepsilon_{i0}}\left[1 + \lambda_s + \frac{1}{z_i^2}\int_{t-\tau_{\max}}^{t}\frac{1}{2}\sum_{j=1}^{i} x_j^2(\tau)\varrho_{ij}^2(\bar{x}_i(\tau))d\tau\right]$$

where $0 < \varepsilon_{i0} \le 2$. For bounded initial conditions, all closed-loop signals remain bounded and the tracking error converges to a small neighborhood around zero by appropriately choosing design parameters.

*Remark 3.2:* Note that the size of the compact set $\Omega_Z^0$ is characterized by $C_0$, which depends on system initial conditions $x_i(0)$ and $\hat{W}_i(0)$ as well as the design parameters $\sigma_i$, $\Gamma_i$, $W_i^0$ and $\varepsilon_{i0}$, $i = 1,\ldots,n$. For the compact set $\Omega_S$ to which the closed-loop signals eventually converge, its size only depends on the design parameters. Therefore, it can be seen that large initial errors $z_i(0)$ and $\tilde{W}_i(0)$, $i = 1,\ldots,n$ may lead to a large transient tracking error during the initial period of adaptation, but will not affect the final convergence of the closed-loop signals.

*Remark 3.3:* Since the function approximation property (3) of neural networks is only guaranteed within a compact set, the stability result proposed is semiglobal in the following sense: Given any bounded initial compact set such that $z_i(0), \tilde{W}_i(0) \in \Omega_I$, the proposed NN controller with sufficiently large number of nodes guarantees that all the closed-loop signals will stay within the compact set, i.e., $\Omega_Z^0$ in the note, if the compact set $\Omega_{Zc}^0$, over which the neural network approximation is constructed, satisfies that $\Omega_Z^0 \subseteq \Omega_{Zc}^0$, and eventually all the closed-loop signals will converge to the steady state compact set, i.e., $\Omega_S$ in the note. The relationships among the sets are as: $\Omega_I, \Omega_S \subseteq \Omega_Z^0 \subseteq \Omega_{Zc}^0$. It is apparent that the larger the compact set $\Omega_{Zc}^0$ over which the NN controller is built upon, the more relaxed the initial compact set $\Omega_I$ is.

## IV. Conclusion

An adaptive neural-based control has been addressed for a class of strict-feedback nonlinear systems with unknown time delays. The unknown time delays has been compensated for through the use of appropriate Lyapunov–Krasovskii functionals. As a result, the iterative backstepping design can be carried out. In addition, the controller is free from singularity problem by using the integral Lyapunov function and practical robust neural network control. The proposed systematic backstepping design method has been proven to be able to guarantee SGUUB of closed-loop signals and the output of the system has been proven to converge to an arbitrarily small neighborhood of the origin.

## References


[1] D. G. Taylor, P. V. Kokotović, R. Marino, and I. Kanellakopoulos, "Adaptive regulation of nonlinear systems with unmodeled dynamics," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 405–412, Apr. 1989.

[2] I. Kanellakopoulos, P. V. Kokotović, and R. Marino, "An extended direct scheme for robust adaptive nonlinear control," *Automatica*, vol. 27, no. 2, pp. 247–255, 1991.

[3] S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 1123–1131, Nov. 1989.

[4] I. Kanellakipoulos, P. V. Kokotović, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 1241–1253, Nov. 1991.

[5] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, "Adaptive nonlinear control without overparsmetrization," *Syst. Control Lett.*, vol. 19, no. 3, pp. 177–185, 1992.

[6] D. Seto, A. M. Annaswamy, and J. Baillieul, "Adaptive control of nonlinear systems with a triangular structure," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1411–1428, July 1994.

[7] S. S. Ge, C. C. Hang, and T. Zhang, "Stable adaptive control for nonlinear multivariable systems with a triangular control structure," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 1221–1225, June 2000.

[8] M. M. Polycarpou and P. A. Ioannou, "A robust adaptive nonlinear control design," *Automatica*, vol. 32, no. 3, pp. 423–427, 1996.

[9] Z. Pan and T. Basar, "Adaptive controller design for tracking and disturbance attenuation in parametric strict-feedback nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 1066–1083, Aug. 1998.

[10] A. M. Annaswamy, S. Evesque, S. Niculescu, and A. P. Dowling, "Adaptive control of a class of tinse-delay systems in the presence of saturation," in *Adaptive Control of Nonsmooth Dynamic Systems*, G. Tao and F. Lewis, Eds. New York: Springer-Verlag, 2001, pp. 289–310.

[11] F.-H. Hsiao and J.-D. Hwang, "Stabilization of nonlinear singularly perturbed multiple time-delay systems by dither," *J. Dyna. Syst., Measure., Control*, vol. 118, no. 1, pp. 176–181, 1996.

[12] J. Hale, *Theory of Functional Differential Equations*, 2nd ed. New York: Springer-Verlag, 1977.

[13] S. K. Nguang, "Robust stabilization of a class of time-delay nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 756–762, Apr. 2000.

[14] S. Zhou, G. Feng, and S. K. Nguang, "Comments on robust stabilization of a class of time-delay nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 47, pp. 1586–1586, Sept. 2002.

[15] F. L. Lewis, S. Jagannathan, and A. Yesilidrek, *Nerual Network Control of Robot Manipulators and Nonlinear Systems*. Philadelphia, PA: Taylor & Francis, 1999.

[16] K. S. Narendra and K. Parthasanithy, "Identification and control of dynamic systems using neural networks," *IEEE Trans. Neural Networks*, vol. 1, pp. 1–23, Feb. 1990.

[17] S. S. Ge, C. C. Hang, and T. Zhang, "A direct adaptive controller for dynamic systems with a class of nonlinear parameterizations," *Automatica*, vol. 35, no. 4, pp. 74l–747, 1999.

[18] S. S. Ge, C. C. Hang, T. H. Lee, and I. Zhang, *Stable Adaptive Neural Network Control*. Norwell, MA: Kluwer, 2002.

[19] S.-L Niculescu, *Delay Effects on Stability: A Robust Control Approach*. London, U.K.: Springer-Verlag, 2001.

[20] R. M. Sanner and J. E. Slotine, "Gaussian networks for direct adaptive control," *IEEE Trans. Neural Networks*, vol. 3, pp. 837–863, Dec. 1992.

[21] E. B. Kosmatopoulos, M. M. Polycarpou, M. A. Christodoulou, and P. A. Ioannou, "High-order neural network structures for identification of dynamical systems," *IEEE Trans. Neural Networks*, vol. 6, pp. 422–431, Apr. 1995.

[22] K. Hornik, M. Stinchcombe, and H. White, "Multilayer feedforward networks are universal approximators," *Neural Networks*, vol. 2, no. 5, pp. 359–366, 1989.