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Robust Adaptive Tracking for Time-Varying Uncertain Nonlinear Systems With Unknown Control Coefficients

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***Abstract*—This note presents a robust adaptive control approach for a class of time-varying uncertain nonlinear systems in the strict feedback form with completely unknown time-varying virtual control coefficients, uncertain time-varying parameters and unknown time-varying bounded disturbances. The proposed design method does not require any *a priori* knowledge of the unknown coefficients except for their bounds. It is proved that the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals and disturbance attenuation.**

***Index Terms*—Robust adaptive control, time-varying nonlinear systems.**

I. INTRODUCTION

Adaptive nonlinear control has seen a significant development in the past decade with the appearance of recursive backstepping design [1]–[3]. A great deal of attention has been paid to tackle the uncertain nonlinear systems with unknown constant parameters [1], [4]–[6]. In this note, we consider a class of single-input–single-output (SISO) uncertain time-varying nonlinear systems with time-varying disturbances in the strict feedback form

$$\begin{aligned} \dot{x}_i &= g_i(t)x_{i+1} + \theta_i^T(t)\psi_i(\bar{x}_i) + d_i^T(t)\phi_i(\bar{x}_i) \\ \dot{x}_n &= g_n(t)u + \theta_n^T(t)\psi_n(x) + d_n^T(t)\phi_n(x) \\ y &= x_1 \end{aligned} \tag{1}$$

where $i = 1, \ldots, n-1$, $x = [x_1, \ldots, x_n]^T \in R^n$ is the state vector, $\bar{x}_i = [x_1, \ldots, x_i]^T$, $i = 1, \ldots, n-1$, $u \in R$ is the control, $\theta_i(t) \in R^{pi}$ are vectors of uncertain and time-varying parameters belonging to known compact sets $\Omega_i \subset R^{pi}$, $d_i(t)$ are vectors of unknown time-varying bounded disturbance evolving in R^{qi}, ψ_i and ϕ_i, $i = 1, \ldots, n$ are known dimensionally compatible smooth nonlinear functions, $g_i(t) \neq 0, i = 1, \ldots, n$ are bounded uncertain time-varying piecewise continuous functions, and they are referred to as virtual control coefficients, in particular, $g_n(t)$ is referred to as the high-frequency gain. For simplicity, let Ω_i be a closed ball of known radius r_{Ω_i} centered in the origin.

Based on the cancellation backstepping design, as termed in [7], many well-known results have been developed for systems with constant virtual control coefficients by seeking for a cancellation of the coupling terms related to $z_i z_{i+1}$ in the next step of the cancellation based backstepping design. When virtual control coefficients $g_i = 1$ and $\theta_i(t)$ are unknown constants, robust adaptive control for a class of systems similar to system (1) have been developed in [8]–[10]. In the presence of time-varying parameters and time-varying disturbance, robust adaptive tracking control was presented in [11] and boundedness of all the signals and arbitrary disturbance attenuation can be achieved. When g_i's are unknown constants, several excellent adaptive control algorithms have also been developed in the literature for nonlinear systems. In [1], under the assumption of unknown constants g_i's but with known signs of g_i's, adaptive control was presented for strict feedback nonlinear systems without disturbances. With the aid of neural networks [12], [13] adaptive control is expanded to much larger class of systems, uncertain strict-feedback and pure-feedback nonlinear systems, where the unknown virtual control coefficients g_i's are functions of states and the signs of g_i as well as the upper bounds of g_i are assumed to be known.

When the signs of g_i are unknown, the adaptive control problem becomes much more difficult. The first solution was given in [14] for a class of first-order linear systems, where the Nussbaum-type gain was originally proposed. Using Nussbaum gains, adaptive control was given for first-order nonlinear systems in [15], for a class of strict feedback nonlinear systems with unknown constant parameters and without disturbances in [16] and [17], and nonlinear systems with completely unknown control coefficients, constant parametric uncertainties and uncertainties in [18] using decoupled backstepping (which, in contrast to cancellation based backstepping, decouples z_i from z_{i+1} using Young's inequality and seeks for the boundedness of z_{i+1} in the next step as said in [7]). Thus far, little attention has been paid to the robust adaptive control problem for systems in (1) in the literature, except for the work in [19]. Reference [19] studied the regulation problem for a class of time-varying uncertain nonlinear systems with time-varying unknown control coefficients under the assumption that uncertain system functions satisfy an additive bound condition. However, when nonlinear systems involve time-varying uncertain parameters and disturbances as well as unknown virtual control coefficients, the solution remains open, and the problem becomes much more difficult due to the presence of the time-varying uncertainties.

In this note, robust adaptive control is presented for system (1) with completely unknown time-varying control coefficients, uncertain time-varying parameters with known bounds, and unknown time-varying disturbances. It is proved that the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals and disturbance attenuation. In addition, for systems with unknown constant parameters and without disturbance, asymptotical tracking of the output can be achieved. The main contributions of the note lie in the following aspects: 1) through introduction of a new technical lemma and the use of Nussbaum gain, stable adaptive control is presented for a class of strict feedback nonlinear system with time-varying uncertain parameters and unknown disturbance; 2) asymptotic output tracking control is achieved when the disturbances

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die out and the uncertain parameters are not time varying; and 3) disturbance attenuation can be achieved without the requirement for the bounds of the unknown bounded disturbances in the controller design.

II. Problem Formulation and Preliminaries

Consider the control problem of a single-input–single-output (SISO) nonlinear uncertain system transformable into (1). The control objective is to construct a robust adaptive nonlinear control law such that the output tracking error, $y - y_d$, of system (1) is driven to be bounded, while keeping all signals in the closed-loop system remain bounded, where $y_d(t)$ is the reference signal.

Assumption 1: The reference signal, $y_d(t)$, is a smooth bounded signal with bounded time derivatives, $y_d^{(i)}(t), 1 \le i \le n$.

Assumption 2: The time-varying control coefficients, $g_i(t)$, take values in the closed intervals $I_i := [l_i^-, l_i^+]$ with $0 \notin I_i, 1 \le i \le n$. Let $\bar{g}_i = \max\{|l_i^-|, |l_i^+|\}$.

To cope with the unknown signs of virtual control coefficients, g_i, the Nussbaum gain technique is employed in this note. A function, $N(\zeta)$, is called a Nussbaum-type function if it has the following properties [14]:

$$\lim_{s\to\infty} \sup \int_{s_0}^{s} N(\zeta) d\zeta = +\infty \tag{2}$$

$$\lim_{s\to\infty} \inf \int_{s_0}^{s} N(\zeta) d\zeta = -\infty. \tag{3}$$

Throughout this note, the even Nussbaum function $N(\zeta) = e^{\zeta^2}\cos((\pi/2)\zeta)$ is considered.

Lemma 1: [17]Let $V(\cdot)$ and $\zeta(\cdot)$ be smooth functions defined on $[0, t_f)$ with $V(t) \ge 0$, $\forall t \in [0, t_f)$, and $N(\cdot)$ be an even smooth Nussbaum-type function. If the following inequality holds:

$$V(t) \le c_0 + \int_0^t (gN(\zeta) + 1)\dot{\zeta} d\tau, \ \forall t \in [0, t_f)$$

where g is a nonzero constant and c_0 represents some suitable constant, then $V(t)$, $\zeta(t)$ and $\int_0^t (gN(\zeta)+1)\dot{\zeta} d\tau$ must be bounded on $[0, t_f)$.

Lemma 2: Let $V(\cdot)$ and $\zeta(\cdot)$ be smooth functions defined on $[0, t_f)$ with $V(t) \ge 0, \forall t \in [0, t_f)$. For $t \in [0, t_f)$, if the following inequality holds:

$$V(t) \le c_0 + e^{-c_1 t} \int_0^t g(\tau) N(\zeta)\dot{\zeta} e^{c_1 \tau} d\tau + e^{-c_1 t} \int_0^t \dot{\zeta} e^{c_1 \tau} d\tau \tag{4}$$

where constant $c_1 > 0$, $g(t)$ is a time-varying parameter which takes values in the unknown closed intervals $I := [l^-, l^+]$ with $0 \notin I$, and c_0 represents some suitable constant, then $V(t)$, $\zeta(t)$ and $\int_0^t g(\tau)N(\zeta)\dot{\zeta} d\tau$ must be bounded on $[0, t_f)$.

Proof: See Appendix A.

According to [21, Prop. 2], if the solution of the resulting closed-loop is bounded, then $t_f = \infty$.

Definition 1: [20] Let $\theta(t) \in \Omega$ be an unknown time-varying parameter vector, $\hat{\theta}$ be the estimate, and $\Omega \subset R^p$ be a closed ball of known radius r_Ω. The projection algorithm $\mathrm{Proj}(y, \hat{\theta})$ is given by

$$\mathrm{Proj}(y, \hat{\theta}) = \begin{cases} y, & \text{if } p(\hat{\theta}) \le 0 \\ y, & \text{if } \left(p(\hat{\theta}) \ge 0 \text{ and } \frac{\partial p}{\partial \hat{\theta}} y \le 0\right) \\ y - \frac{p(\hat{\theta}) \frac{\partial p}{\partial \hat{\theta}} y}{\left\|\frac{\partial p}{\partial \hat{\theta}}\right\|^2} \frac{\partial p}{\partial \hat{\theta}}^T, & \text{otherwise} \end{cases} \tag{5}$$

where $p(\hat{\theta}) = (\hat{\theta}^T\hat{\theta} - r_\Omega^2)/(\epsilon^2 + 2\epsilon r_\Omega)$, ϵ is an arbitrary positive real. From (5), if $\hat{\theta}(0) \in \Omega$, the following nice properties follow immediately: 1) $\|\hat{\theta}(t)\| \le r_\Omega + \epsilon$, $\forall t \ge 0$; 2) $\|\mathrm{Proj}(y, \hat{\theta})\| \le \|y\|$; and 3) $\tilde{\theta}^T \mathrm{Proj}(y, \hat{\theta}) \ge \tilde{\theta}^T y$, $(\tilde{\theta} = \theta - \hat{\theta})$.

Remark 1: Using the projection algorithm, the resulting controller may seem to have a "technical problem"—the intermediate controls are not differentiable on the sphere $S = \{\hat{\theta} : \hat{\theta}^T\hat{\theta} = r_\Omega^2\}$ for $n \ge 3$, because the computation of α_3 (u, if $n = 3$) requires the computation of $\dot{\alpha}_2$, and subsequently that of $\ddot{\alpha}_1$, accordingly, the differentiation of (5), which does not exist on S. However, projection algorithm (5) is practically differentiable because it is differentiable everywhere except on the sphere S which is isolated, and the "energy" is finite. In fact, the sphere is a lower dimensional submanifold of Ω, it is of zero Lebesgue measure, the probability of such an event happening is zero though practically it may happen. In practice, we can simply set the differentiation of the projection algorithm on S to be any finite value, say 0, without any problem, and then every signal in the closed-loop system can be shown to be bounded.

III. Robust Adaptive Control and Main Results

In this section, robust adaptive control design for nonlinear system (1) and the stability of the closed-loop adaptive control system are presented.

Our design consists of n steps. Both the control law and the adaptive laws are based on a change of coordinates $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}(\bar{x}_{i-1}, y_d, \ldots, y_d^{(i-1)}, , \bar{\zeta}_{i-1}\hat{\theta}_{a,1}, \ldots, \hat{\theta}_{a,i-1})$, $i = 2, \ldots, n$, where the functions α_i, $i = 1, \ldots, n-1$, are referred to as intermediate control functions which will be designed iteratively, $\bar{\zeta}_{i-1} = [\zeta_1, \ldots, \zeta_{i-1}]^T$, and $\hat{\theta}_{a,i}$ represent the estimates of unknown parameters $\theta_{a,i}$ with

$$\theta_{a,1} = \theta_1 \quad \theta_{a,i} = \left[g_{i-1}, \theta_i^T\right]^T, \qquad i = 2, \ldots, n$$

At each intermediate step i, we design the intermediate control functions, α_i, using an appropriate Lyapunov functions, V_i, and give the parameters update laws

$$\dot{\hat{\theta}}_{a,i} = \gamma \mathrm{Proj}(z_i \psi_{a,i}, \hat{\theta}_{a,i}) \tag{6}$$

where projection algorithm $\mathrm{Proj}(z_i\psi_{a,i}, \hat{\theta}_{a,i})$ is given by (5) which is practically differentiable as discussed in Remark 1, $\gamma > 0$ is a design parameter, and $\psi_{a,i}$ is a vector of known functions to be given later. At the nth step, the actual control u appears and the design is completed. For clarity, define $\tilde{\theta}_{a,i} = \theta_{a,i} - \hat{\theta}_{a,i}$, $d_{b,i} = \sum_{j=1}^{i} \tilde{\theta}_{a,j}^T \tilde{\theta}_{a,j} + d_{a,i}^T d_{a,i}$, $d_{a,i} = [d_1^T, \ldots, d_i^T]^T$, $i = 1, \ldots, n$, and constants $k > 0$, $k_n > 0$. For $i = 1, \ldots, n-1$, define $k_i > (1/4)$ and $k_{i0} = k_i - (1/4) > 0$ and $2k_{i0} > c_i > 0$.

Step 1: To start, let us study the first equation of (1)

$$\dot{x}_1 = g_1(t)x_2 + \theta_1^T(t)\psi_1(x_1) + d_1^T(t)\phi_1(x_1) \tag{7}$$

where x_2 is taken as a virtual control input. To design a stabilizing adaptive control law for system (7), consider a Lyapunov function candidate $V_1(x_1) = (1/2)z_1^2$. The time derivative of V_1 along the solution of (7) satisfies

$$\dot{V}_1 = z_1 \left[g_1(t)x_2 + \theta_1^T(t)\psi_1(x_1) + d_1^T(t)\phi_1(x_1) - \dot{y}_d\right]. \tag{8}$$

For notational consistence, let $\psi_{a,1} = \psi_1$, $d_{a,1} = d_1$, $\phi_{a,1} = \phi_1$. Choose the intermediate control function α_1 as

$$\alpha_1(x_1, \dot{y}_d, \zeta_1, \hat{\theta}_{a,1}) = N(\zeta_1)\eta_1 \tag{9}$$

where

$$\eta_1 = k_1 z_1 + \hat{\theta}_{a,1}^T \psi_{a,1}(x_1) - \dot{y}_d + v_1(x_1) \tag{10}$$

$$\dot{\zeta}_1 = z_1 \eta_1 \tag{11}$$

$$v_1 = \frac{1}{4} k z_1 \left(\psi_{a,1}^T \psi_{a,1} + \phi_{a,1}^T \phi_{a,1}\right). \tag{12}$$

Using (9), a direct substitution of $x_2 = z_2 + \alpha_1$ and α_1 in (9) into (8) gives

$$\dot{V}_1 = g_1 z_1 z_2 + g_1 N(\zeta_1) z_1 \eta_1 + z_1 \theta_{a,1}^T \psi_{a,1} + z_1 d_{a,1}^T \phi_{a,1} - z_1 \dot{y}_d. \tag{13}$$

Adding and subtracting $\dot{\zeta}_1$ on the right-hand side of (13), and noting (10) and (11), we have

$$\dot{V}_1 = -k_1 z_1^2 + g_1 z_1 z_2 + g_1 N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + z_1 \tilde{\theta}_{a,1}^T \psi_{a,1} + z_1 d_{a,1}^T \phi_{a,1} - \frac{1}{4} k z_1^2 \left(\psi_{a,1}^T \psi_{a,1} + \phi_{a,1}^T \phi_{a,1}\right). \quad (14)$$

By completing the squares, we have

$$\begin{aligned} -\frac{1}{4} k z_1^2 \psi_{a,1}^T \psi_{a,1} + z_1 \tilde{\theta}_{a,1}^T \psi_{a,1} &\leq \frac{1}{k} \tilde{\theta}_{a,1}^T \tilde{\theta}_{a,1} \\ -\frac{1}{4} k z_1^2 \phi_{a,1}^T \phi_{a,1} + z_1 d_{a,1}^T \phi_{a,1} &\leq \frac{1}{k} d_{a,1}^T d_{a,1} \end{aligned}.$$

Further noting Young's inequality $g_1 z_1 z_2 \leq (1/4) z_1^2 + g_1^2 z_2^2$, (14) can be rewritten as

$$\dot{V}_1 \leq -k_{10} z_1^2 + g_1 N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + \frac{d_{b,1}}{k} + g_1^2 z_2^2 \quad (15)$$

$$\leq -c_1 V_1 + g_1 N(\zeta_1)\dot{\zeta}_1 + \frac{d_{b,1}}{k} + \dot{\zeta}_1 + g_1^2 z_2^2. \quad (16)$$

Remark 2: In the cancellation based backstepping design, the coupling term $g_1 z_1 z_2$ is left as it is as it will be cancelled in the next step by augmenting the Lyapunov candidate then. In decoupled backstepping design, we will not seek for the cancellation of the coupling term $g_1 z_1 z_2$, but for the boundedness of z_2 in the next step. According to Lemma 2, if we could prove that z_2 is bounded, then the stability of z_1 is apparent and easy. It is this fundamental change that makes control system design for this problem solvable.

Upon multiplication of (16) by $e^{c_1 t}$, it becomes

$$\frac{d}{dt}\left(V_1(t) e^{c_1 t}\right) \leq g_1 N(\zeta_1)\dot{\zeta}_1 e^{c_1 t} + \dot{\zeta}_1 e^{c_1 t} + \frac{d_{b,1}}{k} e^{c_1 t} + g_1^2 z_2^2 e^{c_1 t}.$$

Integrating it over $[0, t]$, we have

$$0 \leq V_1(t) \leq V_1(0) + e^{-c_1 t} \int_0^t g_1 N(\zeta_1)\dot{\zeta}_1 e^{c_1 \tau} d\tau + e^{-c_1 t} \int_0^t \dot{\zeta}_1 e^{c_1 \tau} d\tau + \int_0^t \frac{d_{b,1}}{k} e^{-c_1 (t-\tau)} d\tau + \int_0^t g_1^2 z_2^2 e^{-c_1 (t-\tau)} d\tau. \quad (17)$$

Remark 3: In (17), if there are no extra terms $\int_0^t (d_{b,1}/k) e^{-c(t-\tau)} d\tau$ and $\int_0^t g_1^2 z_2^2 e^{-c_1(t-\tau)} d\tau$ within the inequality, we can conclude that $V_1(t)$, ζ_1 and z_1 are all bounded on $[0, t_f)$ according to Lemma 2. Thus, no finite-time escape phenomenon may happen and $t_f = \infty$, and we claim that z_1 is uniformly ultimately bounded. Due to the presence of terms $\int_0^t (d_{b,1}/k) e^{-c(t-\tau)} d\tau$ and $\int_0^t g_1^2 z_2^2 e^{-c_1(t-\tau)} d\tau$ in (17), Lemma 2 cannot be applied directly yet. From the adaptation law (6) for $\hat{\theta}_{a,1}$, it is obvious that $\|\hat{\theta}_{a,1}(t)\| \leq r_{\Omega_1} + \epsilon$ by property 1) for operator $\mathrm{Proj}(\cdot,\cdot)$. Because $\|\theta_{a,1}\| \leq r_{\Omega_1}$, we obtain $\|\tilde{\theta}_{a,1}(t)\| \leq 2r_{\Omega_1} + \epsilon$. Together with the boundedness of $\|d_{a,1}(t)\|$, we have

$$\begin{aligned} 0 &\leq \int_0^t \frac{1}{k} d_{b,1} e^{-c_1(t-\tau)} d\tau \\ &= \frac{1 - e^{-c_1 t}}{k c_1} \left[(2r_{\Omega_1} + \epsilon)^2 + \left(\sup_{\tau \in [0,t]} \|d_{a,1}(\tau)\|^2\right)\right] \\ &\leq \frac{1}{k c_1} \left[(2r_{\Omega_1} + \epsilon)^2 + \left(\sup_{\tau \in [0,t]} \|d_{a,1}(\tau)\|^2\right)\right] \end{aligned} \quad (18)$$

which leads to the boundedness of $\int_0^t (d_{b,1}/k) e^{-c_1(t-\tau)} d\tau$. Thus, if z_2 can be regulated as bounded such that $\int_0^t g_1^2 z_2^2 e^{-c_{11}(t-\tau)} d\tau$ is bounded, then, according to Lemma 2, the uniformly ultimate boundedness of $z_1(t)$ can be guaranteed. The boundedness of $\int_0^t g_1^2 z_2^2 e^{-c_{11}(t-\tau)} d\tau$ will be dealt with in the following steps.

Remark 4: To show that the existing Lemma, Lemma 1, cannot be used to solve the problem in the note, let us integrate (16) over $[0, t]$ under the assumption that g_1 is a constant

$$\begin{aligned} V_1(t) &\leq V_1(0) + \int_0^t [g_1 N(\zeta_1) + 1] \dot{\zeta}_1 d\tau + \int_0^t \left[\frac{d_{b,1}}{k} + g_1^2 z_2^2\right] d\tau \\ &\leq V_1(0) + \int_0^t [g_1 N(\zeta_1) + 1] \dot{\zeta}_1 d\tau + c_d t + \int_0^t g_1^2 z_2^2 d\tau \end{aligned}$$

where $c_d = \sup_{0 \leq \tau \leq t} (d_{b,1}(t)/k)$. Due to the presence of $c_d t$, no conclusion about the boundedness of V_1 or ζ can be drawn by applying Lemma 1. However, by using the new technical Lemma 2 of the note, boundedness of the signals can be established with ease.

Step i $(2 \leq i \leq n-1)$: A similar procedure is employed recursively for each step, $i = 2, \ldots, n-1$. The derivative of $(1/2) z_i^2$ is

$$\begin{aligned} z_i \dot{z}_i &= z_i \Bigg[g_i x_{i+1} + \theta_i^T \psi_i + d_i^T \phi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \times \left(g_j x_{j+1} + \theta_j^T \psi_j + d_j^T \phi_j\right) - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}_{a,j}} \dot{\hat{\theta}}_{a,j} \\ &\quad - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_d^{(j)}} y_d^{(j+1)} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \zeta_{i-1}} \dot{\zeta}_{i-1}\Bigg] \\ &= z_i \Bigg[g_i x_{i+1} - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} g_j x_{j+1} + \theta_i^T \psi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \theta_j^T \psi_j + \beta_{i-1}\Bigg] + z_i d_{a,i}^T \phi_{a,i} \end{aligned} \quad (19)$$

where

$$\begin{aligned} d_{a,i} &= [d_1^T, \ldots, d_{i-1}^T, d_i^T]^T \\ \beta_{i-1} &= -\sum_{j=1}^{i-1} (\partial \alpha_{i-1}/\partial \hat{\theta}_{a,j}) \dot{\hat{\theta}}_{a,j} - \sum_{j=1}^{i-1} (\partial \alpha_{i-1}/\partial y_d^{(j)}) y_d^{(j+1)} \sum_{j=1}^{i-1} - (\partial \alpha_{i-1}/\partial \zeta_{i-1}) \dot{\zeta}_{i-1} \end{aligned}$$

and

$$\phi_{a,i} = [-(\partial \alpha_{i-1}/\partial x_1) \phi_1^T, \ldots, -(\partial \alpha_{i-1}/\partial x_{i-1}) \phi_{i-1}^T, \phi_i^T]^T.$$

Equation (19) can be further written as

$$z_i \dot{z}_i = z_i \left[g_i x_{i+1} + \theta_{a,i}^T \psi_{a,i} + \sum_{j=1}^{i-1} \theta_{a,j}^T \psi_{j,i}\right] + z_i d_{a,i}^T \phi_{a,i} + z_i \beta_{i-1} \quad (20)$$

where

$$\begin{aligned} \psi_{a,i} &= [-(\partial \alpha_{i-1}/\partial x_{i-1}) x_i, \psi_i^T]^T \\ \psi_{1,i} &- (\partial \alpha_{i-1}/\partial x_1) \psi_1 \\ \psi_{1,2} &= -(\partial \alpha_1/\partial x_1) \psi_1 \end{aligned}$$

and $\psi_{j,i} = [-(\partial \alpha_{i-1}/\partial x_{j-1}) x_j, -(\partial \alpha_{i-1}/\partial x_j) \psi_j^T]^T$ for $2 \leq j \leq i-1$, $3 \leq i \leq n-1$.

Consider the Lyapunov function candidate $V_i = (1/2) z_i^2$, let the intermediate control function α_i be

$$\alpha_i = N(\zeta_i) \eta_i \quad (21)$$

where

$$\eta_i = k_i z_i + \hat{\theta}_{a,i}^T \psi_{a,i} + \beta_{i-1} + \sum_{j=1}^{i-1} \hat{\theta}_{a,j}^T \psi_{j,i} + v_i \quad (22)$$

$$\dot{\zeta}_i = z_i \eta_i \quad (23)$$

$$v_i = \frac{1}{4}kz_i\left(\psi_{a,i}^T\psi_{a,i} + \phi_{a,i}^T\phi_{a,i}\right) + \frac{1}{4}kz_i\sum_{j=1}^{i-1}\psi_{j,i}^T\psi_{j,i}. \quad (24)$$

Similarly, substituting $x_{i+1} = z_{i+1} + \alpha_i$, noting α_i in (21), adding and subtracting $\dot{\zeta}_i$ on the right-hand side of (20), and noting (22) and (23), we obtain

$$\begin{aligned}\dot{V}_i \leq & -k_i z_i^2 + g_i z_i z_{i+1} + g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i \\ & - \frac{1}{4}kz_i^2\left(\psi_{a,i}^T\psi_{a,i} + \phi_{a,i}^T\phi_{a,i}\right) - \frac{1}{4}kz_i^2\sum_{j=1}^{i-1}\psi_{j,i}^T\psi_{j,i} \\ & + z_i\tilde{\theta}_{a,i}^T\psi_{a,i} + z_i d_{a,i}^T\phi_{a,i} + z_i\sum_{j=1}^{i-1}\tilde{\theta}_{a,j}^T\psi_{j,i}. \quad (25)\end{aligned}$$

By completing the squares, we have

$$\begin{aligned}-\frac{1}{4}kz_i^2\psi_{a,i}^T\psi_{a,i} + z_i\tilde{\theta}_{a,i}^T\psi_{a,i} &\leq \frac{1}{k}\tilde{\theta}_{a,i}^T\tilde{\theta}_{a,i} \\ -\frac{1}{4}kz_i^2\phi_{a,i}^T\phi_{a,i} + z_i d_{a,i}^T\phi_{a,i} &\leq \frac{1}{k}d_{a,i}^T d_{a,i}\end{aligned}.$$

Further noting Young's inequality $g_i z_i z_{i+1} \leq (1/4)z_i^2 + g_i^2 z^2 z_{i+1}$, we obtain

$$\begin{aligned}\dot{V}_i &\leq -k_i z_i^2 + g_i z_i z_{i+1} + g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + \frac{d_{b,i}}{k} \\ &\leq -k_{i0}z_i^2 + g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + \frac{d_{b,i}}{k} + g_i^2 z_{i+1}^2 \\ &\leq -c_i V_i + g_i N(\zeta_i)\dot{\zeta}_i + \frac{d_{b,i}}{k} + \dot{\zeta}_i + g_i^2 z_{i+1}^2. \quad (26)\end{aligned}$$

Remark 5: Though the decoupled backstepping is simple conceptually (at **Step** *i*, it is only required to guarantee the stability of z_i rather than $z_1, \ldots, z_i$), it does not mean that the actual derivation and resulting controller is simple. In fact, it may be more involved because the virtual control needs to carry the term that could have been left untouched for cancellation in the next step in the cancellation based backstepping.

Similarly, we have

$$\begin{aligned}0 \leq V_i(t) \leq V_i(0) &+ e^{-c_i t}\int_0^t g_i N(\zeta_i)\dot{\zeta}_i e^{c_i\tau}d\tau \\ &+ e^{-c_i t}\int_0^t \dot{\zeta}_i e^{c_i\tau}d\tau + \int_0^t \frac{d_{b,i}}{k}e^{-c_i(t-\tau)}d\tau \\ &+ \int_0^t g_i^2 z_{i+1}^2 e^{-c_i(t-\tau)}d\tau. \quad (27)\end{aligned}$$

Remark 6: Similarly, the adaptation law (6) for $\hat{\theta}_{a,i}$ can guarantee the boundedness of $\hat{\theta}_{a,i}$, therefore the boundedness of $\tilde{\theta}_{a,i}$. Together with the boundedness of $\|d_{a,i}\|$ and $\hat{\theta}_{a,j}$, $j = 1, \ldots, i-1$, the boundedness of $\int_0^t (d_{b,i}/k)e^{-c_i(t-\tau)}d\tau$ can be guaranteed. Thus, if z_{i+1} can be regulated as bounded such that $\int_0^t g_i^2 z_{i+1}^2 e^{-c_i(t-\tau)}d\tau$ is bounded at the following steps, then, according to Lemma 2, the boundedness of $z_i(t)$ can be guaranteed.

Step *n*: In this final step, the actual control u appears. Similarly, we have

$$\begin{aligned}z_n\dot{z}_n &= z_n\left[g_n u + \theta_n^T\psi_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\left(g_j x_{j+1} + \theta_j^T\psi_j\right)\right] \\ &\quad + z_n d_{a,n}^T\phi_{a,n} + z_n\beta_{n-1} \\ &= z_n\left[g_n u + \theta_{a,n}^T\psi_{a,n} + \sum_{j=1}^{n-1}\theta_{a,j}^T\psi_{j,n}\right] \\ &\quad + z_n d_{a,n}^T\phi_{a,n} + z_n\beta_n \quad (28)\end{aligned}$$

where

$$\begin{aligned}\beta_{n-1} &= -\sum_{j=1}^{n-1}(\partial\alpha_{n-1}/\partial\hat{\theta}_{a,j})\dot{\hat{\theta}}_{a,j} \\ &\quad - \sum_{j=1}^{n-1}(\partial\alpha_{i-1}/\partial y_d^{(j)})y_d^{(j+1)} - \sum_{j=1}^{i-1}(\partial\alpha_{i-1}/\partial\zeta_{n-1})\dot{\zeta}_{n-1} \\ d_{a,n} &= [d_1^T, \ldots, d_{n-1}^T, d_n^T]^T \\ \phi_{a,n} &= [-(\partial\alpha_{n-1}/\partial x_1)\phi_1^T, \ldots, \\ &\quad -(\partial\alpha_{n-1}/\partial x_{n-1})\phi_{n-1}^T, \phi_n^T]^T \\ \psi_{a,n} &= [-(\partial\alpha_{n-1}/\partial x_{n-1})x_n, \psi_n^T]^T\end{aligned}$$

and

$$\begin{aligned}\psi_{1,n} &= -(\partial\alpha_{n-1}/\partial x_1)\psi_1 \\ \psi_{j,n} &= [-(\partial\alpha_{n-1}/\partial x_{j-1})x_j, -(\partial\alpha_{n-1}/\partial x_j)\psi_j^T]^T \\ 2 &\leq j \leq n-1.\end{aligned}$$

Let the control input be designed as

$$u = N(\zeta_n)\eta_n \quad (29)$$

where

$$\eta_n = k_n z_n + \hat{\theta}_{a,n}^T\psi_{a,n} + \sum_{j=1}^{n-1}\hat{\theta}_{a,j}^T\psi_{j,n} + \beta_{n-1} + v_n \quad (30)$$

$$\dot{\zeta}_n = z_n\eta_n \quad (31)$$

$$v_n = \frac{1}{4}kz_n\left(\psi_{a,n}^T\psi_{a,n} + \phi_{a,n}^T\phi_{a,n}\right) + \frac{1}{4}kz_n\sum_{j=1}^{n-1}\psi_{j,n}^T\psi_{j,n}. \quad (32)$$

Consider the Lyapunov function candidate $V_n = (1/2)z_n^2$. Similarly, the time derivative of V_n satisfies

$$\dot{V}_n \leq -k_n z_n^2 + g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n + \frac{d_{b,n}}{k}. \quad (33)$$

This yields

$$\begin{aligned}0 \leq V_n(t) \leq V_n(0) &+ e^{-c_n t}\int_0^t g_n N(\zeta_n)\dot{\zeta}_n e^{c_n\tau}d\tau \\ &+ e^{-c_n t}\int_0^t \dot{\zeta}_n e^{c_n\tau}d\tau + \int_0^t \frac{d_{b,n}}{k}e^{-c_n(t-\tau)}d\tau \quad (34)\end{aligned}$$

where $0 < c_n \leq 2k_n$. Due to the utilization of adaptation law (6) for $\hat{\theta}_{a,n}$, the boundedness of $\hat{\theta}_n$, therefore, the boundedness of $\tilde{\theta}_n$ can be guaranteed. Together with the boundedness of $\|d_{a,n}\|$ and $\hat{\theta}_j$, $j = 1, \ldots, n-1$, the boundedness of $\int_0^t (d_{b,n}/k)e^{-c_n(t-\tau)}d\tau$ can be guaranteed. Thus, noting (34), and using Lemma 2, we can conclude that $\zeta_n(t)$ and $V_n(t)$, hence $z_n(t)$ are globally uniformly ultimately bounded. From the boundedness of $z_n(t)$, the boundedness of the extra term $\int_0^t g_{n-1}^2 z_n^2 e^{-c_{n-1}(t-\tau)}d\tau$ at Step $(n-1)$ is readily obtained. Applying Lemma 2 $(n-1)$ times backward, it can be seen from the aformentioned design procedures that $V_i(t)$, $z_i(t)$, and hence $x_i(t)$ are globally uniformlyv ultimately bounded. According to [21, Prop. 2], if the solution of the closed-loop is bounded, then $t_f = \infty$.

Theorem 1 (Stability): For the time-varying uncertain strict feedback nonlinear system (1) with completely unknown virtual control coefficients $g_i(t)$, time-varying uncertain parameters $\theta_i(t) \in \Omega_i$ and unknown disturbances $d_i(t)$, given any smooth bounded reference trajectory $y_d(t)$ with bounded time derivative $y_d^{(1)}, \ldots, y_d^{(n)}$ and for any initial condition $x(0) \in R^n$, $\hat{\theta}_i(0) \in \Omega_i$, if we apply the controller (29) with the parameter adaptation law (6), then the solution of the resulting closed-loop adaptive system is globally uniformly ultimately bounded.

Proof: The proof can be easily completed by following the previous design procedures from **Step 1** to **Step** *n*. ◇

When $\theta_i(t)$ and $g_i(t)$ are constants and there is no disturbance, i.e., $d_i(t) = 0$, a much stronger conclusion can be drawn as stated in Corollary 1.

Corollary 1: For the time-varying uncertain strict feedback nonlinear system (1) with completely unknown constant control coefficients g_i, unknown constant parameters θ_i and $d_i(t) = 0$, given any smooth bounded reference trajectory $y_d(t)$ with bounded time derivative $y_d^{(1)}, \ldots, y_d^{(n)}$ and for any initial condition $x(0) \in R^n$, $\hat{\theta}_i(0) \in \Omega_i$, if we apply the controller (29) with the parameters updating law (6), then we have $\lim_{t\to\infty} |y(t) - y_d(t)| = 0$.

Proof: First, let us consider Lyapunov function candidate $V_1 = (1/2)(z_1^2 + (1/\gamma)\tilde{\theta}_{a,1}^T\tilde{\theta}_{a,1})$. Because $\theta_{a,1}$ is constant, we have $\dot{\theta}_{a,1} = 0$. Using (9) and (6), we obtain

$$\begin{aligned}\dot{V}_1 \le &- k_1 z_1^2 + g_1 z_1 z_2 + g_1 N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 \\ &- \frac{1}{4} k z_1^2 \left(\psi_{a,1}^T\psi_{a,1} + \phi_{a,1}^T\phi_{a,1}\right) + z_1\tilde{\theta}_{a,1}^T\psi_{a,1} \\ &- \tilde{\theta}_{a,1}^T \mathrm{Proj}(z_1\psi_{a,1}, \hat{\theta}_{a,1}). \end{aligned} \tag{35}$$

From property 4) of the operator $\mathrm{Proj}(\cdot,\cdot)$, it follows that

$$\dot{V}_1 \le -k_{10} z_1^2 + g_1 N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + g_1^2 z_2^2 \tag{36}$$

Similarly, by considering Lyapunov function candidate $V_i = (1/2)(z_i^2 + (1/\gamma)\tilde{\theta}_{a,i}^T\tilde{\theta}_{a,i})$, we can obtain

$$\begin{aligned}\dot{V}_i \le &- k_{i0} z_i^2 + g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + g_i^2 z_{i+1}^2 \\ & i = 2, \ldots, n-1 \end{aligned} \tag{37}$$

$$\dot{V}_n \le - k_n z_n^2 + g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n. \tag{38}$$

Integrating (38) over $[0,t]$, $\forall t \in [0, t_f]$,we have the following inequality:

$$V_n(t) + \int_0^t k_n z_n^2(\tau) d\tau \le V_n(0) + \int_0^t \left[g_n N\left(\zeta_n(\tau)\right) + 1\right] \dot{\zeta}_n(\tau) d\tau. \tag{39}$$

Since $\int_0^t k_n z_n^2(\tau) d\tau \ge 0$, we further have

$$V_n(t) \le V_n(0) + \int_0^t \left[g_n N\left(\zeta_n(\tau)\right) + 1\right] \dot{\zeta}_n(\tau) d\tau. \tag{40}$$

Using Lemma 1, it follows that $\zeta_n(t)$, $V_n(t)$ and $\int_0^t [g_n N(\zeta_n) + 1]\dot{\zeta}_n d\tau$ are GUUB, hence, $z_n(t)$ and $\hat{\theta}_{a,n}(t)$ are GUUB. The boundedness of $\hat{\theta}_{a,n}(t)$ can also be guaranteed by the project algorithm. According to [21, Prop. 2], if the solution of the closed-loop is bounded, then $t_f = \infty$. From (39), $z_n(t)$ is square integrable. Noting (37), and applying Lemma 1 $(n-1)$ times backward, it can be obtained that $V_i(t)$, $z_i(t)$, $\hat{\theta}_{a,i}(t)$ and, hence, $x_i(t)$ are GUUB. Since $z_{i+1}(t)$ (e.g., $i = n-1$) has been regulated such that it is square integrable and we have the following inequality by integrating (37) over $[0,t]$, $\forall t \in [0, t_f]$

$$\begin{aligned} V_i(t) + \int_0^t k_{i0} z_i^2(\tau) d\tau \le V_i(0) &+ \int_0^t \left[g_i N(\zeta_i) + 1\right] \dot{\zeta}_i(\tau) d\tau \\ &+ \int_0^t g_i^2 z_{i+1}^2(\tau) d\tau \end{aligned} \tag{41}$$

we can directly conclude that $z_i(t)$ is square integrable. In addition, as an immediate result, $\dot{z}_i(t)$, $1 \le i \le n$ are bounded, and $z_i(t)$ are square integrable. Hence, a direct application of Barbalat's lemma [22], [23] gives that $\lim_{t\to\infty} \|z_i(t)\| = 0$, which implies, in particular, that $\lim_{t\to\infty} |y(t) - y_d(t)| = 0$. $\diamond$

Remark 7: The proposed method is essentially a robust adaptive one. In fact, we can fix $\hat{\theta}_{a,i}$ to be arbitrarily any bounded values without adaptation by setting adaptation gain $\gamma = 0$ in (6), the global uniform ultimate boundedness of the closed-loop system signals and disturbance attenuation can still be ensured, but at the expense of not achieving asymptotic tracking even when disturbances die out and unknown parameters $\theta_{a,i}$ are not time-varying any more. The proposed robust adaptive control provides a unified approach to handle nonlinear system (1) with time varying or constant $\theta_i(t) \in R^{p_i}$. The reason for the use of adaptation law (6) for the unknown time-varying parameters $\theta_{a,i}$ based on the project algorithm is that the proposed adaptation laws are robust with respect to the external disturbances, and at the same time, the design guarantees the boundedness of all the signals in the closed-loop system. In fact, asymptotic output tracking performance is achieved when the unknown parameters become constant and the disturbances die out as stated in Corollary 1.

Remark 8: It is of interest to note that for the strict-feedback nonlinear systems with more general uncertainties

$$\begin{aligned} \dot{x}_i &= g_i(t)x_{i+1} + \theta_i^T(t)\psi_i(\bar{x}_i) + \Delta_i(x,t),\ i = 1, \ldots, n-1 \\ \dot{x}_n &= g_n(t)u + \theta_n^T(t)\psi_n(x) + \Delta_n(x,t) \\ y &= x_1 \end{aligned}$$

where uncertainties $\Delta_i(x,t)$ satisfy a triangularity condition $|\Delta_i(x,t)| \le d_i^T(t)\phi_i(\bar{x}_i)$, with $\phi_i(\bar{x}_i)$ being a known smooth vector function, and unknown vector $d_i(t) \in R^{q_i}$ is the bounded output of a nonlinear exosystem [24], the robust adaptive control in this note is still applicable while with a modification for the construction of a smooth $\phi_{a,i}$ in order to handle the nonsmooth term $|(\partial\alpha_{i-1}/\partial x_j)|$ at each design step due to $|\Delta_i(x,t)|$.

IV. Conclusion

In this note, robust adaptive control has been presented for a class of time-varying uncertain strict feedback nonlinear systems with unknown virtual control coefficients and time-varying parametric uncertainties as well as unknown bounded disturbances. It has been proved that the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals and disturbance attenuation. Asymptotical output tracking control is achieved when the system parameters become constant and the disturbances die out.

Appendix

A. Proof of Lemma 2

Proof: For convenience, define $g_{\max} = \max\{|l^-|, |l^+|\}$, $g_{\min} = \min\{|l^-|, |l^+|\}$, and

$$V_g(y_i, y_j) = \int_{y_i}^{y_j} \left[g(\tau) N\left(\zeta(\tau)\right) + 1\right] e^{-c_1(t_j - \tau)} d\zeta(\tau) \tag{42}$$

with an understanding that $V_g(y_i, y_j) = V_g(y(t_i), y(t_j)) = V_g(t_i, t_j)$ for notation convenience, $y_i \le y_j$ and $\tau \in [t_i, t_j]$.

Using integral inequality $(b-a)m_{f1} \le \int_a^b f(x)dx \le (b-a)m_{f2}$ with $m_{f1} = \inf_{a \le x \le b} f(x)$ and $m_{f2} = \sup_{a \le x \le b} f(x)$, and noting the facts that $|g(\tau)| \le g_{\max}$, $0 < e^{-c_1(t-\tau)} \le 1$ for $\tau \in [0,t]$, we have

$$\begin{aligned} |V_g(\zeta_0, \zeta)| &\le (\zeta - \zeta_0) \sup_{\tau \in [t_0, t], \zeta \in [\zeta_0, \zeta]} |g(\tau) N(\zeta) + 1| \\ &\le (\zeta - \zeta_0) \left[g_{\max} \sup_{\zeta \in [\zeta_0, \zeta]} |N(\zeta)| + 1\right]. \end{aligned} \tag{43}$$

For the Nussbaum function $N(\zeta) = e^{\zeta^2}\cos((\pi/2)\zeta)$, we know that it is positive for $\zeta \in (4m-1, 4m+1)$ and negative for $\zeta \in (4m+1, 4m+3)$ with m an integer.

Rewrite (4) as

$$0 \le V(t) \le c_0 + \int_0^t [g(\tau)N(\zeta)+1]\,\dot{\zeta}e^{-c_1(t-\tau)}d\tau = c_0 + V_g(\zeta(0), \zeta(t)) \qquad \forall t \in [0, t_f). \tag{44}$$

We first show that $\zeta(t)$ is bounded on $[0, t_f)$ by seeking a contradiction. Suppose that $\zeta(t)$ is unbounded and two cases should be considered: 1) $\zeta(t)$ has no upper bound and 2) $\zeta(t)$ has no lower bound.

Case 1): $\zeta(t)$ *has no upper bound on* $[0, t_f)$. In this case, there must exist a monotone increasing variable $\{\zeta_i = \zeta(t_i)\}$ with $\zeta_0 = |\zeta(t_0)| > 0$, $\lim_{i\to\infty} t_i = t_f$, and $\lim_{i\to\infty}\zeta_i = \infty$.

First, let us consider the case $g(t) < 0$. From (43), we know, for $[\zeta_0, \zeta_{m_1}] = [\zeta_0, 4m-1]$, that

$$|V_g(\zeta_0, \zeta_{m_1})| \le (\zeta_{m_1} - \zeta_0)\left[g_{\max}e^{(4m-1)^2} + 1\right] = l_{m_1}g_{\max}e^{(4m-1)^2} + l_{m_1} \tag{45}$$

where $l_{m_1} = (4m-1-\zeta_0)$. Further noting that $N(\zeta) > 0$, $\forall \zeta \in [\zeta_{m_1}, \zeta_{m_2}] = [4m-1, 4m+1]$, we have

$$V_g(\zeta_{m_1}, \zeta_{m_2}) \le \int_{4m-c_{m_1}}^{4m+c_{m_1}} [g(\tau)N(\zeta(\tau))+1]\,e^{-c_1(t_{m_2}-\tau)}d\zeta(\tau)$$

where $c_{m_1} \in (0,1)$ strictly, say, $1/2$, for example. Using the integral inequality and noting that $g(t) \le -g_{\min}$, $e^{-c_1(t_{m_2}-\tau)} \ge e^{-c_1(t_{m_2}-t_{m_1})} > 0$ for $\tau \in [t_{m_1}, t_{m_2}]$, we have

$$V_g(\zeta_{m_1}, \zeta_{m_2}) \le 2c_{m_1}\left[-g_{\min}\inf_{\zeta\in[\zeta_{m_1},\zeta_{m_2}]} N(\zeta) + 1\right] \times e^{-c_1(t_{m_2}-t_{m_1})} = -b_{m_1}e^{(4m-c_{m_1})^2} + b_{e_1} \tag{46}$$

where $b_{m_1} = 2c_{m_1}g_{\min}e^{-c_1(t_{m_2}-t_{m_1})}\cos(\pi c_{m_1}/2) > 0$, and $b_{e_1} = 2c_{m_1}e^{-c_1(t_{m_2}-t_{m_1})}$. It is known that if $|f_1(x)| \le a_1$ and $f_2(x) \le a_2$, then $f_1(x) + f_2(x) \le a_1 + a_2$. Thus, we know from (45) and (46), that

$$V_g(\zeta_0, \zeta_{m_2}) = V_g(\zeta_0, \zeta_{m_1}) + V_g(\zeta_{m_1}, \zeta_{m_2}) \le e^{(4m-1)^2}\left[-b_{m_1}e^{\left[2(4m-1)(1-c_{m1})+(1-c_{m1})^2\right]} + l_{m_1}g_{\max} + \frac{l_{m_1}+b_{e_1}}{e^{(4m-1)^2}}\right] \tag{47}$$

Because $1 - c_{m_1} > 0$ strictly, e^m grows much faster than m, and l_{m_1} is linear in m, we know, from (47), that $V_g(\zeta_0, \zeta_{m_2}) = V_g(\zeta_0, 4m+1) \to -\infty$ as $m \to \infty$.

Then, let us consider the case $g(t) > 0$. Similar to the derivation of (45), in the interval $[\zeta_0, \zeta_{m_2}] = [\zeta_0, 4m+1]$, we have

$$|V_g(\zeta_0, \zeta_{m_2})| = \left|\int_{\zeta_0}^{\zeta_{m_2}} [g(\tau)N(\zeta)+1]\,e^{-c_1(t_{m_2}-\tau)}d\zeta(\tau)\right| \le (\zeta_{m_2} - \zeta_0)\left[g_{\max}\sup_{\zeta\in[\zeta_0,\zeta_{m_2}]}|N(\zeta)| + 1\right] = l_{m_2}g_{\max}e^{(4m+1)^2} + l_{m_2} \tag{48}$$

where $l_{m_2} = 4m + 1 - \zeta_0$. By noting that $N(\zeta) < 0$, $\forall \zeta \in [\zeta_{m_2}, \zeta_{m_3}] = [4m+1, 4m+3]$, we have

$$V_g(\zeta_{m_2}, \zeta_{m_3}) \le \int_{4m+2-c_{m_1}}^{4m+2+c_{m_1}} \times [g(\tau)N(\zeta(\tau))+1]\,e^{-c_1(t_{m_3}-\tau)}d\zeta(\tau).$$

Similarly, using the integral inequality and noting that $g(t) \ge g_{\min}$, $e^{-c_1(t_{m_3}-\tau)} \ge e^{-c_1(t_{m_3}-t_{m_2})} > 0$ for $\tau \in [t_{m_2}, t_{m_1}]$, we have

$$V_g(\zeta_{m_3}, \zeta_{m_3}) \le 2c_{m_1}\left[-g_{\min}\inf_{\zeta\in[\zeta_{m_2},\zeta_{m_3}]} N(\zeta) + 1\right] \times e^{-c_1(t_{m_3}-t_{m_2})} = -b_{m_2}e^{(4m+2-c_{m_1})^2} + b_{e_2} \tag{49}$$

where $b_{m_2 2} = 2c_{m_1}g_{\min}e^{-c_1(t_{m_3}-t_{m_2})}\cos(\pi c_{m_1}/2) > 0$, and $b_{e_2} = 2c_{m_1}e^{-c_1(t_{m_3}-t_{m_2})} > 0$. Thus, we know from (48) and (49), that

$$V_g(\zeta_0, \zeta_{m_3}) = V_g(\zeta_0, \zeta_{m_2}) + V_g(\zeta_{m_2}, \zeta_{m_3}) \le e^{(4m+1)^2}\left[-b_{m_2}e^{\left[2(4m+1)(1-c_{m_1})+(1-c_{m_1})^2\right]} + g_{\max}(4m+1-\zeta_0) + \frac{l_{m_2}+b_{e_2}}{e^{(4m+1)^2}}\right]. \tag{50}$$

From (50), we know that that $V_g(\zeta_0, \zeta_{m_3}) = V_g(\zeta_0, 4m+3) \to -\infty$ as $m \to \infty$. In summary, we can always find a sequence that leads to a contradiction in (44) whether $g(t)$ is positive or negative. Thus, we know that ζ is upper bounded.

Case 2): $\zeta(t)$ *has no lower bound on* $[0, t_f)$. Define $\zeta = -\omega$. Accordingly, $\omega(t)$ has no upper bound. Further noting that $N(\cdot)$ is an even function, (44) becomes

$$V(t) \le c_0 - \int_0^t [g(\tau)N(\omega)\dot{\omega}+1]\,e^{-c_1(t-\tau)}d\omega = c_0 - V_g(\omega(0), \omega(t)) \qquad \forall t \in [0, t_f). \tag{51}$$

Thus, there must exist a monotone increasing variable $\{\omega_i = \omega(t_i)\}$ with $\omega_0 = |\omega(t_0)| > 0$, $\lim_{i\to\infty} t_i = t_f$, and $\lim_{i\to\infty}\omega_i = \infty$. Following the procedure as in Case 1), we can always construct a sequence that leads to a contradiction, accordingly, we can claim that $\omega(t)$ is upper bounded on $[0, t_f)$. Since $\zeta(t) = -\omega(t)$, we know that $\omega(t)$ is lower bounded on $[0, t_f)$.

Therefore, $\zeta(t)$ must be bounded on $[0, t_f)$. In addition, $V(t)$ and $\int_0^t g(\tau)N(\zeta)\dot{\zeta}d\tau$ are bounded on $[0, t_f)$. ■

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Corrections to “Robust H_∞ Control for Linear Discrete-Time Systems With Norm-Bounded Nonlinear Uncertainties”

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***Abstract*—This note provides some corrections to the aforementioned paper.**

In [1, eq. (7)], C and aM_kE_1 cannot be added because the dimensions of the matrices are different. Therefore, the consequent derivations have to be adjusted. We propose two methods that can be used to correct this problem without harming the main idea of [1].

The first method is zero padding to $z_k \in \Re^r$ to make $\overline{z}_k \in \Re^n$. This can be written as follows:

$$\overline{z}_k = \begin{bmatrix} z_k \\ 0 \end{bmatrix}, \qquad 0 \in \Re^{n-r}. \tag{1}$$

The over bar represents a modification and the modified vector or matrix replace the original vector or matrix.

Consequently, the matrices C, D_1, D_2 are also changed as follows:

$$\overline{C} = \begin{bmatrix} C \\ 0 \end{bmatrix} \quad \overline{D}_1 = \begin{bmatrix} D_1 \\ 0 \end{bmatrix} \quad \overline{D}_2 = \begin{bmatrix} D_2 \\ 0 \end{bmatrix}, \qquad 0 \in \Re^{r\times(n-r)}. \tag{2}$$

However, this method can introduce unnecessary conservatism.

The second method is as follows. [1, eq. (7)] can be modified as

$$z_k = (C + a\overline{M}_kE_1)x_k + (D_1 + b\overline{M}_kE_2)w_k + (D_2 + c\overline{M}_kE_3)u_k \tag{3}$$

$$\overline{M}_k = HM_k, \qquad H \in \Re^{r\times n} \quad \rho(H) \le 1. \tag{4}$$

Consequently, [1, eq. (9)] is modified as

$$\tilde{z}_k = \begin{bmatrix} C \\ \varepsilon^{-1}aE_1 \end{bmatrix} x_k + \begin{bmatrix} D_1 & \varepsilon H \\ \varepsilon^{-1}bE_2 & 0 \end{bmatrix} \tilde{w}_k. \tag{5}$$

With (5), [1, eqs. (10), (11), and (13)] have to be written as (6), (7), and (8), respectively

$$Y = \begin{bmatrix} -P^T & 0 & 0 & A & B_\gamma & \varepsilon I \\ 0 & -I & 0 & C & D_1 & \varepsilon H \\ 0 & 0 & -I & \varepsilon^{-1}aE_1 & \varepsilon^{-1}bE_2 & 0 \\ A^T & C^T & \varepsilon^{-1}aE_1^T & -P & 0 & 0 \\ B_\gamma^T & D_1^T & \varepsilon^{-1}bE_2^T & 0 & -I & 0 \\ \varepsilon I & \varepsilon H^T & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{6}$$

$$S^TYS = \begin{bmatrix} -P^T & 0 & A & B_\gamma & 0 & \varepsilon I \\ 0 & -I & C & D_1 & 0 & \varepsilon H \\ A^T & C^T & -P & 0 & \varepsilon^{-1}aE_1 & 0 \\ B_\gamma^T & D_1^T & 0 & -I & \varepsilon^{-1}bE_2^T & 0 \\ 0 & 0 & \varepsilon^{-1}aE_1^T & \varepsilon^{-1}bE_2^T & -I & 0 \\ \varepsilon I & \varepsilon H^T & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{7}$$

$$\tilde{I} = \begin{bmatrix} I \\ H \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

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