On Parameter Settings of Hopfield Networks Applied to Traveling Salesman Problems

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***Abstract*—Parameter setting is a critical step in the Hopfield networks solution of the traveling salesman problem (TSP), which is often prone to extraneous solutions. This paper presents some stability criteria that ensure the convergence of valid solutions and suppression of infeasible solutions. Our theory is based on an enhanced parametric formulation that maps TSP onto a continuous-time Hopfield network (CHN), which is more advantageous than the Hopfield–Tank (H-T) formulation. A set of analytical conditions for optimal parameter settings of the CHN is then derived, and the resulting performance is validated by simulations.**

***Index Terms*—Hopfield neural networks, traveling salesman problem (TSP), parameter setting, stability.**

I. Introduction

The traveling salesman problem (TSP) is a classical combinatorial optimization problem widely studied in the field of mathematics and artificial intelligence. The problem can be formulated as follows: given a set of N cities, find the shortest path linking all the cities such that all cities are visited exactly once. Since the seminal work of Hopfield [1], there has been an increased interest in applying Hopfield neural network to classical combinatorial problems [2]–[4]. However, the difficulty of parameter settings in Hopfield network often leads to the convergence of the network to invalid states, let alone the optimal one, which makes it a nontrivial work in order to find the optimal range of the network parameters.

Several methods for convergence of Hopfield network to valid states have been proposed. Aiyer *et al.* [2] analyzed the behavior of the continuous-time Hopfield network (CHN) based on the eigenvalues of connection matrix and derived the parameter settings for TSP. Abe [3] obtained the convergence and suppression conditions based on a piecewise-linear activation function by comparing the values of the energy at vertices of a unit hypercube. Peng *et al.* [4] suggested the local minimum escape (LME) algorithm, which improves the local minima of CHN by combining the network disturbing technique with Hopfield network's local minima searching property. Cooper [5] developed the method of higher order neural networks (HONN) for mapping TSP and studied the stability conditions of valid solutions. A number of approaches based upon chaotic neural networks have also been proposed to solve the TSP, and excellent results with less local minima have been obtained due to the global search capability of the chaotic networks [6]–[10].

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It has been widely recognized that the Hopfield–Tank (H-T) formulation [1] of energy function often causes infeasible solutions in solving the TSP [11]. The inter-relationship among the parameters suggests that the H-T formulation for TSP does not have a good scaling property and only a small range of parameter combinations will result in valid and stable solutions, as indicated by the small percentage of valid tours in approaches based on the H-T formulation [12], [13]. Therefore, a better representation of mapping TSP onto CHN and an effective way of optimally selecting the penalty parameters for the network are essential. Some useful results have been reported in literature (see [14] and the references therein), e.g., Talaván *et al.* [15] presented a parameter setting procedure based on the stability conditions of CHN energy function. The approach gives the analytical conditions that guarantee the equilibria to be valid solutions for the TSP.

This paper proposes an enhanced formulation of mapping the TSP onto CHN. A systematic method is then developed to ensure the convergence of valid solutions and the suppression of spurious steady states of the network by analyzing the dynamical stability of the network on a unit hypercube. Subsequently, a theoretical method for selecting the penalty parameters is obtained. A brief description of TSP and CHN is given in Section II. The enhanced formulation of mapping TSP onto CHN is described in Section III. The dynamical stability analysis of the network to ensure the convergence of valid solutions and the suppression of spurious states is given in Sections IV and V, respectively. A set of parameter setting criterion for various size of TSP is presented in Section VI. The simulations that illustrate the theoretical findings are given in Section VII. Finally, conclusions are drawn in Section VIII.

II. TSP Mapping and CHN Model

The TSP is known as the nondeterministic polynomial (NP) hard problem [16], which is difficult to deal with or even intractable when the number of cities becomes large. For N cities, the TSP can be mapped into a fully connected neural network consisting of $N \times N$ neurons, which are grouped into N groups of N neurons, and each group of the N neurons represents the position in the tour of a particular city. The network's output is a vector of the form, for example, $\mathbf{v} = (0\,1\,0\,1\,0\,0\,0\,0\,1)^t$ or in a two-dimensional (2-D) form of

$$\mathbf{v} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which represents a tour of three cities. Let $u_{x,i}$ and $v_{x,i}$ denotes the current state and the activity (output) of the neuron (x, i)

respectively, where $x, i \in 1, \ldots, N$ is the city index and the visit order, respectively. Each neuron is also subject to a bias of $i^b_{x,i}$. The strength of connection between neuron (x, i) and neuron (y, j) is denoted by $T_{xi,yj}$. Let $\mathbf{u}, \mathbf{v}, \mathbf{i^b}$ be the vector of neuron states, outputs and biases, respectively. The dynamics of CHN can be described by a system of differential equations

$$\frac{d\mathbf{u}}{dt} = -\frac{\mathbf{u}}{\tau} + Tv + \mathbf{i^b} \tag{1}$$

where $T = T^t \in \Re^{N^2 \times N^2}$ is a constant matrix and τ is a time constant. The activation function is a hyperbolic tangent

$$v_{x,i} = \phi(u_{x,i}) = \frac{1}{2}\left(1 + \tanh\left(\frac{u_{x,i}}{u_0}\right)\right), \qquad u_0 > 0. \tag{2}$$

An energy function exists such that

$$\mathcal{E} = -\frac{1}{2}\mathbf{v}^t T v - (\mathbf{i}^b)^t \mathbf{v} \tag{3}$$

and the existence of equilibrium states for the CHN can be ensured.

The H-T formulation of the energy function for mapping the TSP is described by

$$\begin{aligned} E = {} & \frac{A_0}{2}\sum_x \sum_i \sum_{j \neq i} v_{x,i} v_{x,j} + \frac{B_0}{2} \sum_i \sum_x \sum_{y \neq x} v_{x,i} v_{y,i} \\ & + \frac{C_0}{2}\left(\sum_x \sum_i v_{x,i} - N\right)^2 \\ & + \frac{D_0}{2} \sum_x \sum_{y \neq x} \sum_i d_{xy} v_{x,i}(v_{y,i+1} + v_{y,i-1}) \end{aligned} \tag{4}$$

where d_{xy} is the distance from City x to City y, and the scaling parameters A_0, B_0, C_0, D_0 are positive constants. The first and second term represents the constraint that at most one neuron of the array $\mathbf{v}$ is on fire at each row and column, respectively. The third term represents the constraint that the total number of neurons on fire is exactly N. The fourth term measures the tour length corresponding to a given tour $\mathbf{v}$, where the two terms inside the parenthesis stand for two neighboring visiting cities of $v_{x,i}$, implying the tour length is calculated twice. The energy function reaches a local minimum when the network is at a valid tour state.

With this formulation, the Hopfield network has the connection strengths and external input given as

$$\begin{aligned} T_{xi,yj} = {} & -\{A_0 \delta_{x,y}(1 - \delta_{i,j}) + B_0(1 - \delta_{x,y})\delta_{i,j} \\ & + C_0 + D_0(\delta_{i,j-1} + \delta_{i,j+1})d_{x,y}\} \end{aligned} \tag{5}$$

$$i^b_{x,i} = C_0 N \tag{6}$$

where $\delta_{i,j}$ is equal to 1 $(i = j)$ or 0 (otherwise).

It is known that the H-T formulation does not work well for the TSP since the network often converges to infeasible solutions. Therefore, the formulation (4) was modified slightly in [1] and the works thereafter by replacing the bias $i^b = C_0 N$ with an effective bias $\widetilde{i^b} = C_0 \tilde{N}$, where $\tilde{N}$ is larger enough than N. However, the replacing of N with $\tilde{N}$ also forces the network to violate the constraints, e.g., only a few percentage of trials yield valid tours, and the percentage of valid solutions reaches a maximum for some value of $\tilde{N}$ and drops to zero when $\tilde{N}$ is further increased [12].

III. Enhanced Lyapunov Function for Mapping TSP

From the view of mathematical programming, the TSP can be described as a quadratic 0-1 programming problem with linear constraints

$$\text{minimize} \quad \mathcal{E}^{\text{obj}}(\mathbf{v}) \tag{7}$$

$$\text{subject to} \quad S_i = \sum_{x=1}^{N} v_{xi} = 1 \qquad \forall i \in \{1, \ldots, N\} \tag{8}$$

$$S_x = \sum_{i=1}^{N} v_{xi} = 1 \qquad \forall x \in \{1, \ldots, N\} \tag{9}$$

and a redundant constraint $S = \sum_x \sum_i v_{x,i} = N; v_{xi} \in \{0, 1\}$ and $\mathcal{E}^{\text{obj}}$ is the tour length described by a valid 0-1 solution $\mathbf{v}$.

Bearing this in mind for mapping the TSP onto the Hopfield network, the Lyapunov function is set such that

$$\mathcal{E}^{\text{lyap}}(\mathbf{v}) = \mathcal{E}^{\text{obj}}(\mathbf{v}) + \mathcal{E}^{\text{cns}}(\mathbf{v}). \tag{10}$$

The modified penalty function is given as

$$\mathcal{E}^{\text{cns}}(\mathbf{v}) = \frac{A}{2}\sum_x \left(\sum_i v_{x,i} - 1\right)^2 + \frac{B}{2}\sum_i \left(\sum_x v_{x,i} - 1\right)^2 \tag{11}$$

where the first term represents the constraint that each row has exactly one neuron on fire, and the second term represents the constraint that each column has exactly one neuron on fire. The scaling parameters A and B play the role of balancing the constraints.

In order to force trajectories toward the vertices of the hypercube, it is necessary to add to $\mathcal{E}^{\text{lyap}}$ with an additional penalty function of the form

$$\begin{aligned} \mathcal{E}^{\text{drv}}(\mathbf{v}) & = \frac{C}{2}\sum_{x=1}^{N}\sum_{i=1}^{N}(1 - v_{x,i})v_{x,i} \\ & = \frac{C}{2}\left(\frac{N^2}{4} - \sum_{x=1}^{N}\sum_{i=1}^{N}\left(v_{x,i} - \frac{1}{2}\right)^2\right) \end{aligned} \tag{12}$$

which induces a gradient component to drive the output states of the network outward from $(1/2)$, and thus the trajectories will tend toward the vertices of the hypercube. Finally, the enhanced Lyapunov function for TSP is formulated as

$$\begin{aligned} \mathcal{E}^{\text{lyap}}(\mathbf{v}) = {} & \frac{A}{2}\sum_x \left(\sum_i v_{x,i} - 1\right)^2 + \frac{B}{2}\sum_i \left(\sum_x v_{x,i} - 1\right)^2 \\ & + \frac{C}{2}\sum_x \sum_i (1 - v_{x,i})v_{x,i} \\ & + \frac{D}{2}\sum_x \sum_{y \neq x} \sum_i d_{xy} v_{x,i}(v_{y,i+1} + v_{y,i-1}). \end{aligned} \tag{13}$$

With this new formulation, the connection matrix and external input of the Hopfield network are computed as follows:

$$\begin{aligned} T_{xi,yj} = {} & -\{A\delta_{x,y} + B\delta_{i,j} - C\delta_{x,y}\delta_{i,j} \\ & + D(\delta_{i,j-1} + \delta_{i,j+1})d_{x,y}\} \end{aligned} \tag{14}$$

$$i^b_{x,i} = A + B - \frac{C}{2}. \tag{15}$$

Although a similar energy formulation was proposed in [13] and [17], where A and B are simply set to 1 and only two parameters (C and D) are allowed to change, there is a lack of a systematic way for parameter settings. Some advantages of the modified Lyapunov function have been confirmed by Brandt *et al.* [13]. As stated in their work, the new formulation has relatively sparse connections of $(4N-3)N^2$ as contrast to the H-T formulation having N^4 connections. It is clear that the reduction of the connection becomes significant as the size of the TSP is increased. Using a simple method of dynamical stability analysis, Kamgar-Parsi *et al.* [12] showed that the H-T formulation is not stable, which explains the reason that Hopfield and Tank had to choose a larger value $\tilde{N}$ in their formulation instead of the city number N. Following the same analysis approach, it is easy to show that the modified formulation is stable.

Given the above preliminary investigation, it is preferable to choose the modified Lyapunov function for solving the TSP. In the following sections, we present the theoretical results that give an analytical method to optimally set the scaling parameters A, B, C, and D based on the stability analysis of the network. As a consequence, the convergence to valid solutions is ensured at a higher percentage. Additionally, the new parameter setting scheme also avoids the drawback such as the connection weights are dependent on the distances between the inter-cities [12], [13], and thus the obtained results could be easily applied to various size of TSP's.

IV. Stability Analysis of CHN

In this section, we derive the conditions under which all neurons' activities are either firing at state 1 or off at state 0, i.e., the network output vector $\mathbf{v} = (v_{x,i})$ converges to the vertices satisfying the constraints. There are three main approaches to determine the convergence states of a Hopfield network: eigenvalue analysis of the connection matrix [2], comparison of the energy values at vertices of the unit hypercube [3], and Lyapunov stability analysis [15], [18], [5].

The Lyapunov function for CHN mapping of TSP is a quadratic function of the neuron's activity $v_{x,i}$ defined on the vertices of a unit hypercube in $\Re^{N^2}$. Therefore the stability of the equilibria of the network is determined by the derivatives. The vector of activities $\mathbf{v} = (v_{x,i})$ corresponds to an equilibrium of the motion equation (1) of the network if and only if $(\partial \mathcal{E}^{\text{lyap}})/(\partial v_{x,i}) \geq 0 (\forall v_{x,i} = 0)$, $(\partial \mathcal{E}^{\text{lyap}})/(\partial v_{x,i}) = 0 (\forall v_{x,i} \in (0,1))$ and $(\partial \mathcal{E}^{\text{lyap}})/(\partial v_{x,i}) \leq 0 (\forall v_{x,i} = 1)$.

Lemma 1: [18] The activities vector that converges to a vertex of the hypercube is asymptotically stable if it is verified

$$\frac{\partial \mathcal{E}^{\text{lyap}}}{\partial v_{x,i}} = \begin{cases} > 0 (v_{x,i} = 0) \\ < 0 (v_{x,i} = 1) \end{cases} \tag{16}$$

for all the components (x,i). Conversely, it is unstable if

$$\frac{\partial \mathcal{E}^{\text{lyap}}}{\partial v_{x,i}} = \begin{cases} < 0 (v_{x,i} = 0) \\ > 0 (v_{x,i} = 1) \end{cases} \tag{17}$$

holds for all the components (x,i).

Hence, the properties of the equilibria of the network can be investigated by the first partial derivative of the Lyapunov function with respect to $v_{x,i}$. Let $\mathcal{E}_{x,i}(\mathbf{v}) = (\partial \mathcal{E}^{\text{lyap}})/(\partial v_{x,i})$ and from (13), it is obtained

$$\mathcal{E}_{x,i}(\mathbf{v}) = A(S_x - 1) + B(S_i - 1) + \frac{C}{2}(1 - 2v_{x,i}) + D\sum_{y \neq x} d_{x,y}(v_{y,i-1} + v_{y,i+1}). \tag{18}$$

Define the lowerbound and the upperbound of the derivative components of the neurons at fire state and off state as $\underline{\mathcal{E}}^0(\mathbf{v}) = \min_{v_{x,i}=0} \mathcal{E}_{x,i}(\mathbf{v})$ and $\bar{\mathcal{E}}^1(\mathbf{v}) = \max_{v_{x,i}=1} \mathcal{E}_{x,i}(\mathbf{v})$, respectively. Consequently, the inequalities of (16) is reformulated by a more compact form which gives stronger conditions for the stability

$$\begin{cases} \underline{\mathcal{E}}^0(\mathbf{v}) > 0 \\ \bar{\mathcal{E}}^1(\mathbf{v}) < 0. \end{cases} \tag{19}$$

Let $d_L = \min_{x,y} d_{x,y}$ and $d_U = \max_{x,y} d_{x,y}$ be the lowerbound and the upperbound of the Euclidean distance between any two cities, respectively.

V. Suppression of Spurious States

To represent a valid tour, there must be one and only one neuron firing in each row x and each column i of $\mathbf{v}$. In other words, the network activities representing a valid tour exist on the vertices of a unit hypercube. Define the hypercube by $H = \{v_{x,i} \in [0,1]^{N\times N}\}$, its vertex set $H_C = \{v_{x,i} \in \{0,1\}^{N\times N}\}$, then the valid tour set is $H_T = \{v_{x,i} \in H_C \,|\, S_x = 1, S_i = 1\}$. Obviously, $H_T \subset H_C \subset H$.

To ensure that the network does not converge to invalid solutions (i.e., *spurious steady states*), all the spurious states need to be suppressed for both cases of $\mathbf{v} \in H_C - H_T$ and $\mathbf{v} \in H - H_C$.

Definition 1: Given a vertex point $\mathbf{v} \in H_C$, its *adjacent vertex point* $\mathcal{A}_{x,i}$ is defined by changing only one element $v_{x,i}$ with its complementary $1 - v_{x,i}$.

Hence, the adjacent vertices set of $\mathbf{v}$ can be defined by $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$, where

$$\mathcal{A}^+ = \{\mathcal{A}_{x,i} \in H_C \,|\, v_{x,i} = 0 \forall x, i \in \{1,\ldots,N\}\},$$
$$\mathcal{A}^- = \{\mathcal{A}_{x,i} \in H_C \,|\, v_{x,i} = 1 \forall x, i \in \{1,\ldots,N\}\}.$$

Definition 2: Any point $\mathbf{v} \in H$ of the hypercube $\Re^N$ is an *interior point* $\mathbf{v} \in H - H_C$ if and only if a component $v_{x,i} \in (0,1)$ exists. An interior point $\mathbf{v} \in H - H_C$ becomes an *edge point* if

$$|\{(x,i) \in \{1,2,\ldots,N\}^2 \,|\, v_{x,i} \in (0,1)\}| = 1$$

where $|\cdot$ denotes the cardinal operation.

Definition 3: Let $\mathbf{v} \in H - H_C$ be an edge point with $v_{x^0,i^0} \in (0,1)$, its *adjacent vertex point* $\mathbf{v}' = (v'_{x,i}) \in H_C$ is defined by increasing or decreasing the nonintegral component v_{x^0,i^0}, i.e.,

$$v'_{x,i} = \begin{cases} 1 \text{ or } 0, & \text{if } (x,i) = (x^0, i^0) \\ v_{x,i}, & \text{otherwise.} \end{cases}$$

Fig. 1 illustrates the vertex point, edge point and interior point.

As pointed out by Park [19], an interior point that is not an edge point has a null probability of being the convergence point

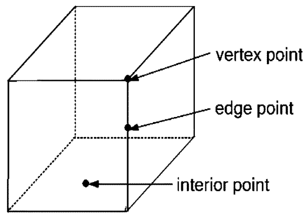

Fig. 1. Vertex point, edge point, and interior point.

of the continuous Hopfield neural networks. Therefore, we need to suppress the states of the network output being an edge point.

Lemma 2: The vertex $\mathbf{v} \in H_C - H_T$ is unstable if the derivatives of the Lyapunov function satisfy

$$\max_{S=N-1} \underline{\mathcal{E}}^0(\mathbf{v}) < 0 < \min_{S=N} \bar{\mathcal{E}}^1(\mathbf{v}) \tag{20}$$

or

$$\max_{S=N} \underline{\mathcal{E}}^0(\mathbf{v}) < 0 < \min_{S=N+1} \bar{\mathcal{E}}^1(\mathbf{v}). \tag{21}$$

Proof: Since the energy always increases if more than N neurons are fired, the spurious states (invalid tours) only occur when N neurons or less are fired. Let $k \in \{0, 1, 2, \ldots, N\}$, given an invalid tour $\mathbf{v}^k \in H_C - H_T$ such that $S = \sum_x \sum_i v_{x,i}^k = k$ and suppose

$$\underline{\mathcal{E}}^0(\mathbf{v}^k) = \max_{S=k} \underline{\mathcal{E}}^0(\mathbf{v}) \tag{22}$$

there exists a row x' and a column i' such that $v_{x',i'}^k = 0$, and thus another invalid tour can be chosen from its neighborhood set $\mathcal{A}^+$, i.e., $\mathbf{v}^+ = \mathcal{A}_{x',i'}(\mathbf{v}^k)$. Obviously, $S = \sum_x \sum_i v_{x,i}^+ = k + 1$ and

$$\max_{S=k+1} \underline{\mathcal{E}}^0(\mathbf{v}) \geq \underline{\mathcal{E}}^0(\mathbf{v}^+). \tag{23}$$

To show the derivation, three exhaustive cases are considered in the following.

Case 1) $v_{x,i} = 1$, and $v_{x',i'}$ changes from 0 to 1, for $x' \neq x, i' \neq i$. For example

$$\begin{pmatrix} v_{x,i} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} v_{x,i} & * & a & * \\ \star & \circ & \diamond & \circ \\ b & d & c & d \\ \star & \circ & \diamond & \circ \end{pmatrix}.$$

The above matrix can be divided into four nonoverlapped regions a, b, c, and d according to their relationships with $v_{x,i}$. For example, region a represents the row $(*, a, *)$. Region d indicates either one of its two consecutive columns of $v_{x,i}$, while region c denotes a column that is not next to $v_{x,i}$. As observed from (18), the value of $\mathcal{E}_{x,i}$ can only be changed by the elements in regions a, b, and d. Thus, we can determine how the energy derivative $\mathcal{E}_{x,i}$ changes by applying (18). If $v_{x',i'}$ is in region a, then the first term in (18) will be increased, due to the increment of S_x. If $v_{x',i'}$ is in region b, then the second term in (18) will be increased, due to the increment of S_i. If $v_{x',i'}$ is in region c, then $\mathcal{E}_{x,i}$ remains the same. If $v_{x',i'}$ is in region d, then the fourth term in (18) will be increased, due to the increment of either $v_{y,i-1}$ or $v_{y,i+1}$. Thus

$$\mathcal{E}_{x,i}(\mathbf{v}^+) \geq \mathcal{E}_{x,i}(\mathbf{v}^k) \qquad \forall (x,i) \in \{1, 2, \cdots, N\}^2.$$

Case 2) $v_{x,i} = 0$, and $v_{x',i'}$ changes from 0 to 1, for $x' \neq x, i' \neq i$. Similarly, the energy derivative will be increased due to the parameters A, B, D, if $v_{x',i'}$ is in region a, b, and d respectively. It remains the same if $v_{x',i'}$ is in region c. Thus,

$$\mathcal{E}_{x,i}(\mathbf{v}^+) \geq \mathcal{E}_{x,i}(\mathbf{v}^k) \qquad \forall (x,i) \in \{1, 2, \cdots, N\}^2.$$

Case 3) $v_{x,i}$ changes from 0 to 1.

In this case, by applying the energy derivative formulation (18), we obtain

$$\mathcal{E}_{x,i}(\mathbf{v}^k) = -A - B + \frac{C}{2} + D(d_{xy} + d_{yz})$$
$$\mathcal{E}_{x,i}(\mathbf{v}^+) = -\frac{C}{2} + D(d_{xy} + d_{yz}).$$

To satisfy $\mathcal{E}_{x,i}(\mathbf{v}^+) \geq \mathcal{E}_{x,i}(\mathbf{v}^k)$, we obtain $A + B \geq C$. This criterion is necessary for the nonconvergence of valid solutions and thus must be considered in the parameter settings of A, B, C.

From the above three cases, we obtain

$$\underline{\mathcal{E}}^0(\mathbf{v}^+) \geq \underline{\mathcal{E}}^0(\mathbf{v}^k). \tag{24}$$

By applying (22) and (23), it is obtained for large enough A, B that

$$\max_{S=k+1} \underline{\mathcal{E}}^0(\mathbf{v}) \geq \underline{\mathcal{E}}^0(v^k) = \max_{S=k} \underline{\mathcal{E}}^0(\mathbf{v}) \qquad \forall \mathbf{v} \in H_C - H_T. \tag{25}$$

It can be proven analogously that

$$\min_{S=k+1} \bar{\mathcal{E}}^1(\mathbf{v}) \leq \min_{S=k} \bar{\mathcal{E}}^1(\mathbf{v}) \qquad \forall \mathbf{v} \in H_C - H_T. \tag{26}$$

From the conditions (17) for instability, any spurious steady state $\mathbf{v} \in H_C - H_T$ is unstable if the conditions (20) or (21) are satisfied. It completes the proof. ∎

Theorem 1: Any invalid tour $\mathbf{v} \in H_C - H_T$ is not a stable equilibrium point for CHN if the following criteria are satisfied:

$$\min\{B, A + Dd_L, (N-1)A\} - \frac{C}{2} > 0 \tag{27}$$
$$-A - B + \frac{C}{2} + 3Dd_U < 0 \tag{28}$$
$$A + B \geq C. \tag{29}$$

Proof: It is already obtained that $A + B \geq C$ from the Proof of Lemma 2. Then by applying Lemma 2 and Propositions 1 and 2 in the Appendix, the remaining conditions can be obtained. ∎

Lemma 3: Suppose the edge point $\mathbf{v} \in H - H_C$ and $v_{x^0,i^0} \in (0, 1)$ be an equilibrium of the CHN with $T_{xi,xi} < 0$ $\forall (x,i) \in$

$\{1, \ldots, N\}^2$, then it is unstable if the following condition is satisfied:

$$\mathcal{E}_{x,i} \notin (0, A+B-C) \qquad \forall (x,i) \,|\, v_{x,i} = 1 \tag{30}$$

or

$$\mathcal{E}_{x,i} \notin (-A-B+C, 0) \qquad \forall (x,i) \,|\, v_{x,i} = 0. \tag{31}$$

Proof: According to the result in [15], the stability of any edge point is avoided if

$$\mathcal{E}_{x,i}(\mathbf{v}') \notin (0, -T_{xi,xi}) \qquad \forall (x,i) \,|\, v'_{x,i} = 1 \qquad \forall \mathbf{v}' \in H_C \tag{32}$$

or

$$\mathcal{E}_{x,i}(\mathbf{v}') \notin (T_{xi,xi}, 0) \qquad \forall (x,i) \,|\, v'_{x,i} = 0 \qquad \forall \mathbf{v}' \in H_C. \tag{33}$$

Derived from (14), we obtain

$$T_{xi,xi} = -A - B + C. \tag{34}$$

Hence, for $T_{xi,xi} < 0$, the following condition should be met:

$$A + B > C. \tag{35}$$

The proof is straightforward. ∎

Therefore, the conditions for nonconvergence of edge point $\mathbf{v} \in H - H_C$ can be obtained easily.

Theorem 2: Any invalid solution $\mathbf{v} \in H - H_C$ is not a stable equilibrium point if

$$\min\{B, A + Dd_L, (N-1)A\} - \frac{C}{2} > A + B - C \tag{36}$$

$$3Dd_U - \frac{C}{2} < 0 \tag{37}$$

$$A + B > C. \tag{38}$$

Proof: According to Lemma 3 and Propositions 1 and 2 in the Appendix, it can derive the conditions easily. ∎

Remark 1: By making all spurious states unstable, the valid states will necessarily be stable since the Hopfield network is convergent by design, i.e., the convergence condition of valid solutions can be omitted. Consider the valid state, we obtain

$$\mathcal{E}_{x,i}(\mathbf{v}) = \frac{C}{2} + Dd_{x,y} \geq \frac{C}{2} + Dd_L$$

for $v_{x,i} = 0$ and

$$\mathcal{E}_{x,i}(\mathbf{v}) = -\frac{C}{2} + D(d_{x,y} + d_{yz}) \leq -\frac{C}{2} + 2Dd_U$$

for $v_{x,i} = 1$. Recall the condition (19), any valid state $\mathbf{v} \in H_T$ is stable if $-(C/2) + 2Dd_U < 0$. Obviously, it can be omitted as implied by the condition (37) in the above theorem.

VI. Setting of Parameters

It is noted that the conditions in Theorem 1 can be omitted since they are implied by the conditions in Theorem 2. Thus, the following conditions ensure nonconvergence of all spurious states $v \in H - H_T$

$$3Dd_U - \frac{C}{2} < 0 \tag{39}$$

$$A + B > C \tag{40}$$

$$\min\{B, A + Dd_L, (N-1)A\} - \frac{C}{2} > A + B - C. \tag{41}$$

TABLE I
Performance of Parameter Settings Obtained From Talaván

C	Good	Invalid	Min length	Ave length
$100K$	16	411	18.1421	25.2547
$10K$	44	405	19.0618	25.6066
$1K$	13	394	19.0618	25.5345
100	16	351	17.3137	25.3033
10	52	55	18.8089	24.1888
1	135	14	17.3137	22.8208
0.1	195	0	22.0837	23.4533
0.01	192	14	16.4853	22.1017
0.001	267	29	16.4853	21.6479

TABLE II
Performance of New Parameter Settings

C	Good	Invalid	Min length	Ave length
$100K$	220	22	16.4853	22.1737
$10K$	226	27	16.4853	22.1040
$1K$	227	3	15.3137	21.9492
100	233	2	15.3137	21.7767
10	232	2	15.3137	21.93333
1	215	1	15.3137	22.0452
0.1	223	0	16.4853	21.7484
0.01	208	5	16.4853	22.0157
0.001	204	11	16.4853	22.1432

Based upon the above three criteria, the following settings are proposed for the parameters A, B, C, D (note that any other parameter values satisfying the above criteria are also equally valid):

$$\begin{aligned} D &= C/(10d_U), \\ A &= C/2 - Dd_L/10 \\ B &= A + Dd_L \\ C &= \text{any desired value.} \end{aligned}$$

This allows the parameter C to be set at any arbitrary value while ensuring the criteria (39)–(41) are satisfied.

VII. Simulation Results and Discussions

In this section, we validate the criteria (39)–(41) by simulations and compare the performance with that obtained by the parameter settings of Talaván *et al.* [15]. Here, we adopted the algorithm introduced in [15]. The continuous Hopfield network may encounter stability problems when it is discretized in implementation (readers may refer to the work of Wang [20], [21] for more details).

Ten-City TSP: A 10-city example is designed to illustrate the performance of both the energy functions. In each case, parameter C was varied and 1000 trials were carried out. The initial states were randomly generated by $v_{x,i} = 0.5 + \alpha u, \forall (x,i) \in \{1, 2, \ldots, N\}^2$ where α is a small value and u is a uniform random value in $[-0.5, 0.5]$.

The simulation results are shown in Tables I and II, which are listed in terms of the number of good and invalid solutions as well as the minimum and average tour length. The 10-city configuration of optimum state and near-optimum state is shown in Figs. 2 and 3, respectively. It is noted that the number of good solutions includes the optimum and near-optimum states (within 25% more than the optimum length).

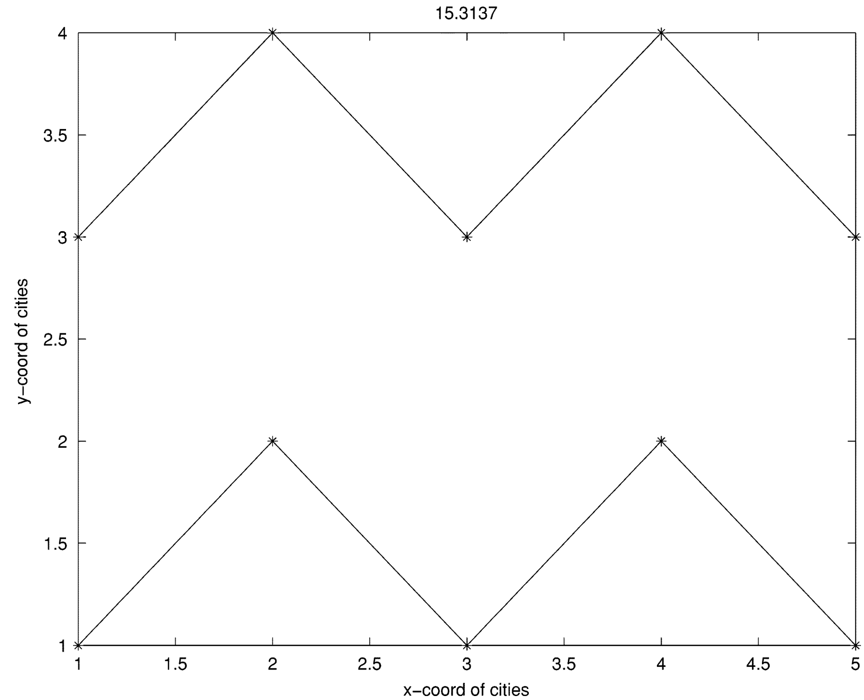

Fig. 2. Optimum tour state.

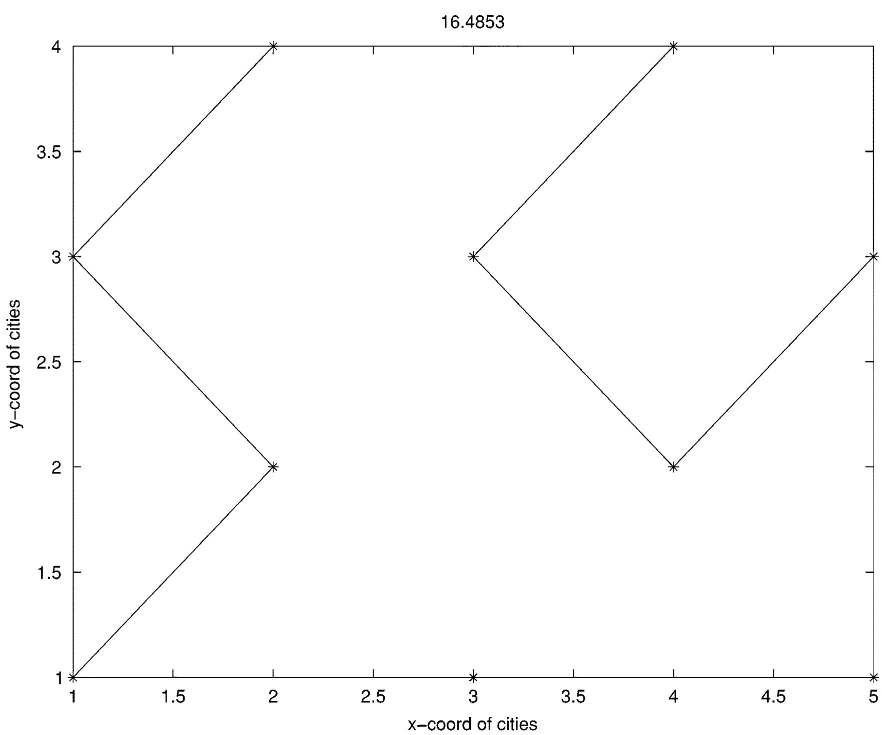

Fig. 3. Near-optimum tour state.

Remark 2: Ideally, it is fair to have all neurons starting from the same initial state. However, this cannot be implemented in practice. All neurons with the same initial state will lead to nonconvergence of the network, as all neurons on each row will have the same potential. Therefore, a small random value $\alpha \in [-10^{-3}, 10^{-3}]$ is added here to break the symmetry. The value of α is chosen to be small so that the randomness introduced does not play a significant role in determining the final state.

Remark 3: The new parameter settings of the network suppress the number of invalid states generated by the parameter settings of [15] significantly. As shown in Table I, the results of the parameter settings with H-T formulation of [15] depend heavily on the value of C. It shows that the performance of the network associated with H-T formulation is considered good only when C ranges from 0.1 to 1. On the contrary, the new parameter settings give consistently good results for a wide range of values of C, ranging from 0.1 to 10^5. It can be observed from

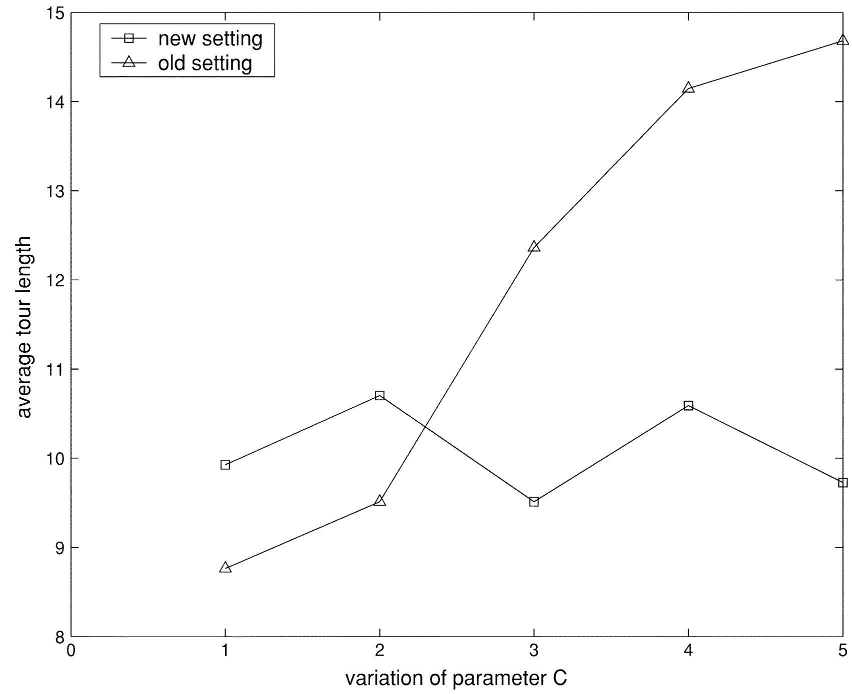

Fig. 4. Comparison of average tour length between the modified formulation (new setting) and H-T formulation (old setting).

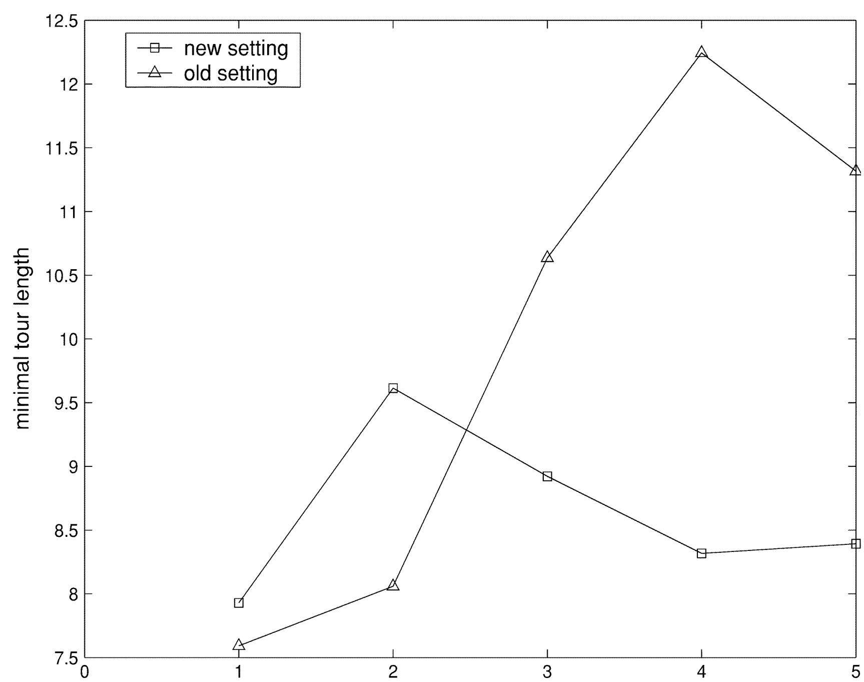

Fig. 5. Comparison of minimal tour length between the modified formulation (new setting) and H-T formulation (old setting).

Table II that the average tour length is about 22, the minimum tour length is about 16, and the percentage of good solutions is about 22%, regardless of the values of C.

Thirty-City TSP: The 30-city TSP was used in the work [1], [3], where the cities were generated randomly from $[0, 1]$. The new formulation and H-T formulation were validated on different values of parameter C (e.g., 0.1, 1, 10, 100, 1000). For each value of C, 50 valid tours were found and results for the 30-city TSP were generated. Figs. 4 and 5 show the comparison of average tour length and minimal tour length, respectively. The results again confirm the advantages of the modified energy function proposed in this paper. It is shown that good results can only be obtained for the H-T formulation using parameters limited to a small range. The modified energy function with the new parameter settings, however, is able to produce good results for a wide range of parameter values.

VIII. Conclusion

In order to guarantee the feasibility of CHN in solving the TSP, the dynamical stability conditions of CHN defined on a unit hypercube have been proved in this paper. The features of an enhanced Lyapunov function mapping TSP onto CHN have been studied, which confirm its advantages over the H-T formulation. The results reveal the relationship between the network parameters and solution feasibility, under which a systematic approach for parameter settings in solving the TSP has been obtained.

With the proposed parameter settings, the quality of the solutions has been improved in the sense that the network gives consistently good performance for a wide range of parameter values as compared to the recent work of [15]. The proposed criteria also ensure the convergence of valid solutions and the suppression of spurious states, which facilitate the procedure of parameter settings for various size of TSP's.

Appendix

Proposition 1: Any invalid tour $\mathbf{v} \in H_C - H_T, S = \sum_x \sum_i v_{x,i} = N - 1$, the following bound is obtained

$$\max_{S=N-1} \mathcal{E}^0(\mathbf{v}) \leq -A - B + \frac{C}{2} + 3Dd_U.$$

Proof: For any point $\mathbf{v} \in H_C$, let I_0 denote the index set of its columns whose elements are equal to 0, i.e., $I_0 = \{i \in \{1, 2, \ldots, N\} : S_i = 0\}$ and $N_0 = |I_0|$ denotes the cardinal. Consider $S = N - 1, I_0 \neq \emptyset$, and $N_0 \geq 1$, then x' and i' exist such that $v_{x',i'} = 0$ and $S_{x'} = 0$, e.g.,

$$\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & v_{x',i'} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, from (18)

$$\begin{aligned} \mathcal{E}_{x',i'}(\mathbf{v}) &= -A - B + \frac{C}{2} + D \sum_{y \neq x} d_{x,y}(v_{y,i'-1} + v_{y,i'+1}) \\ &\leq -A - B + \frac{C}{2} + Dd_U(S_{i'-1} + S_{i'+1}). \end{aligned}$$

Taking minimum over all $i \in I_0$, it is obtained that

$$\mathcal{E}^0(\mathbf{v}) \leq -A - B + \frac{C}{2} + Dd_U \min_{i \in I_0}(S_{i-1} + S_{i+1}).$$

Applying the technical result [15, Lemma A. 2]

$$\min_{i \in I_0}(S_{i-1} + S_{i+1}) \leq 4 - \frac{2(N - S)}{N_0} \tag{42}$$

and taking into account that $S_{i-1} + S_{i+1} \in \mathcal{N}$, the following is obtained, $\min_{i \in I_0}(S_{i-1} + S_{i+1}) \leq [4 - (2(N - S))/(N_0)] \leq 3$. Hence, it proves that $\mathcal{E}^0(\mathbf{v}) \leq -A - B + (C/2) + 3Dd_U$. ■

Proposition 2: Any invalid tour $\mathbf{v} \in H_C - H_T, S = \sum_x \sum_i v_{x,i} = N$, the following bound is obtained:

$$\min_{S=N} \bar{\mathcal{E}}^1(\mathbf{v}) \geq \min\{B, A + Dd_L, (N - 1)A\} - \frac{C}{2}.$$

Proof: To prove $\min_{S=N} \mathcal{E}^1(\mathbf{v}) \geq \min\{B, A + Dd_L, (N - 1)A\} - (C/2)$, we consider the following 3 exhaustive cases for $v_{x,i} = 1$.

Case 1) $\exists v_{x',i'} = 1 : S_{x'} \geq 1, S_{i'} \geq 2$, e.g.,

$$\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

From (18), we have

$$\begin{aligned} \mathcal{E}_{x',i'} &= A(S_{x'} - 1) + B(S_{i'} - 1) - \frac{C}{2} \\ &\quad + D \sum_{y \neq x} d_{x,y}(v_{y,i'-1} + v_{y,i'+1}) \\ &\geq B - \frac{C}{2}. \end{aligned} \tag{43}$$

Case 2) $\exists v_{x',i'} = 1 : S_{x'} \geq 2, S_{i'} = 1$, e.g.,

$$\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} \mathcal{E}_{x',i'} &= A(S_{x'} - 1) - \frac{C}{2} + Dd_{x,y} \\ &\geq A - \frac{C}{2} + Dd_L. \end{aligned} \tag{44}$$

Case 3) $\exists v_{x',i'} = 1 : S_{x'} = N, S_{i'} = 1$, e.g.,

$$\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that

$$\mathcal{E}_{x',i'}(\mathbf{v}) \geq A(N - 1) - \frac{C}{2}. \tag{45}$$

Considering (43)–(45), we obtain

$$\min_{S=N} \mathcal{E}^1(\mathbf{v}) \geq \min\{B, A + Dd_L, A(N - 1)\} - \frac{C}{2}.$$

This completes the proof. ■

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