

# Output Feedback NN Control for Two Classes of Discrete-Time Systems With Unknown Control Directions in a Unified Approach

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**Abstract**—In this paper, output feedback adaptive neural network (NN) controls are investigated for two classes of nonlinear discrete-time systems with unknown control directions: 1) nonlinear pure-feedback systems and 2) nonlinear autoregressive moving average with exogenous inputs (NARMAX) systems. To overcome the noncausal problem, which has been known to be a major obstacle in the discrete-time control design, both systems are transformed to a predictor for output feedback control design. Implicit function theorem is used to overcome the difficulty of the nonaffine appearance of the control input. The problem of lacking *a priori* knowledge on the control directions is solved by using discrete Nussbaum gain. The high-order neural network (HONN) is employed to approximate the unknown control. The closed-loop system achieves semiglobal uniformly-ultimately-bounded (SGUUB) stability and the output tracking error is made within a neighborhood around zero. Simulation results are presented to demonstrate the effectiveness of the proposed control.

**Index Terms**—Discrete Nussbaum gain, discrete-time system, nonlinear autoregressive moving average with exogenous inputs (NARMAX) systems, neural networks (NNs), pure-feedback system, unknown control directions.

## I. INTRODUCTION

IN recent years, control design for complex nonlinear systems has attracted an ever increasing interest, and control design based on neural network (NN) has been drawing much attention owing to NN's excellent function approximation ability [1]. NN control has been extensively studied for both continuous-time and discrete-time systems. For continuous-time systems, much research work has been carried out on affine nonlinear systems through feedback linearization, and recently, some research papers devoted to nonaffine systems have been reported in the literature. There are fewer analysis tools for nonaffine systems compared with affine systems, e.g., the feedback linearization used for many affine systems are

not directly applicable to nonaffine systems. It makes control problem of nonaffine systems more challenging.

In continuous time, to overcome the design difficulty for nonaffine systems, adaptive NN control using implicit function theorem was presented with mathematical rigor in [2], and combined with backstepping design, implicit-function-based adaptive NN control has been proposed for a class of pure-feedback continuous-time system in [3]. The tracking-error-observer-based pseudoinverse control has been proposed in [4] where the pseudoinverse control consists of a linear dynamic compensator and an adaptive NN compensator.

In comparison with continuous-time control design, NN control design of discrete-time systems is more difficult due to lack of mathematical tools in discrete time. For instance, Lyapunov design for nonlinear discrete-time systems becomes much more intractable than for continuous-time systems because the linearity property of the derivative of a Lyapunov function in continuous time is not present in the difference of Lyapunov function in discrete time [5]. Though not as intensive as that in the continuous time, NN control of the discrete-time systems has also drawn much attention [6]. Using passivity property, adaptive NN control has been designed for a class of discrete-time nonlinear system in normal form [7]. This result is further extended in [8] with constrained input.

Recently, the discrete-time systems in lower triangular structure have attracted much research interest. For strict-feedback form nonlinear system, after states prediction and system transformation, backstepping design has been applied using NNs approximating both virtual and real controls [9]. The result has also been extended to multiple-input-multiple-output (MIMO) systems in [10] and [11]. For pure-feedback systems in which the control is of nonaffine appearance, adaptive NN control with a single NN was studied in [12].

The nonlinear autoregressive moving average with exogenous inputs (NARMAX) model proposed in [13] comprises a general nonlinear discrete-time model structure, and it has received much attention in the discrete-time control literature. In [14], NN has been used for identification and control of induction machines represented by a NARMAX model. In [15], based on the NN identified model, a novel linearization method at each step was proposed to deal with the difficulty of nonaffine input and the method was further used in [16] to construct an internal-model-based NN control. It is noted that these results of controlling NARMAX systems are based on offline NN learning and certain approximation accuracy should be guaranteed before control design.

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On the other hand, several adaptive NN control approaches have been proposed for NARMAX systems based on Lyapunov stability theory and they guarantee the closed-loop stability without the requirement of an offline NN training. In [17], for a class of nonlinear system transformable to nonlinear autoregressive moving average (NARMA) model, multimodel adaptive control was proposed with an NN used to compensate the unknown nonlinearity. In [18], NARMA model was represented as a linear part plus a nonlinear part and then a linear generalized minimum variance (GMV) control was designed with an NN to deal with the nonlinearity.

For a class of a single-input–single-output (SISO) affine NARMA system, direct adaptive NN control was proposed in [19]. For NARMAX systems with bounded disturbance, based on the system transformation, adaptive NN control was studied in [20]. For nonaffine NARMA systems, the implicit-function-based NN control was first proposed in [21] and [22] where the implicit function theory was used to assert the existence of a desired control. In [21], the control design was constructed with offline NN training, while in [22], adaptive NN control with online tuning was designed. The implicit-function-based discrete-time adaptive NN control has been further developed in [23] using multilayer neural network (MNN) to approximate the desired control.

In this paper, we will study output-feedback control of the discrete-time systems in both pure-feedback form and NARMAX form in the presence of unknown control directions. The control directions (the signs of “control variable” gains in affine systems, or the signs of partial derivatives over “control variables” in nonaffine systems) are normally required to be known *a priori* in adaptive control. When the control directions are unknown, the control problem becomes much more difficult, because in this case, we cannot decide the direction along which the control operates. It may be noted that for adaptive NN control in [22], the control direction was not assumed to be known, but the stability result was proved using NN weights convergence results, which cannot be guaranteed without the persistent exciting condition. In the adaptive control literature, the unknown control direction problem has received much attention in the past two decades.

In continuous time, the breakthrough solution to counteract the lack of *a priori* knowledge of the control directions was first given for a class of first-order linear systems by introducing the Nussbaum gain [24]. For high-order nonlinear systems in strict-feedback form with unknown functions, adaptive NN control design using Nussbaum gain was developed in [25]. Alternative approaches to deal with the unknown control directions can also be found in the literature. In [26], the projected parameter approach was used for adaptive control of the first-order nonlinear systems with unknown control directions. In [27], on-line identification of the unknown control directions was proposed for a class of the second-order nonlinear systems.

In discrete time, to solve the unknown control direction problem, a two-step adaptation law was proposed for a first-order discrete system without knowledge of control gain [28]. However, this procedure is limited to the first-order linear system. For stable adaptive control of high-order linear systems, the discrete Nussbaum gain was first proposed in [29]. The discrete Nuss-

baum gain is more intractable compared to its continuous-time counterpart, and hence, the control design in continuous time is not directly applicable to discrete time. In this paper, the discrete Nussbaum gain will be employed for adaptive NN control for the first time. In contrast to the control design in continuous time, the Nussbaum gain is not explicitly used in the control law but rather it is embedded in the NN weights adaptation law.

The main contributions of the paper are as follows.

- 1) The control designs for a class of pure-feedback system and a class of NARMAX system are unified.
- 2) Implicit function theory is exploited for the control design to assert the existence of an ideal control. It solves the difficulty of nonaffine appearance of the control input.
- 3) By introducing the discrete Nussbaum gain, the proposed adaptive NN control overcomes the unknown control direction problem. In addition, neither the upper bounds nor the lower bounds of the control gains are required to be known.

Throughout this paper, the following notations are in order.

- $\|\cdot\|$  denotes the Euclidean norm of vectors and induced norm of matrices.
- $A := B$  means that  $B$  is defined as  $A$ .
- $[\cdot]^T$  represents the transpose of a vector or a matrix.
- $\mathbf{0}_{[p]}$  stands for  $p$ -dimension zero vector.
- $W^*$  and  $\hat{W}(k)$  denote the ideal NN weights vector and its estimate at the  $k$ th step, respectively. Let  $\tilde{W}(k) = \hat{W}(k) - W^*$  denote the estimate error.

The remainder of this paper is organized as follows. Section II introduces the systems and the control problem to be studied, as well as preliminaries that are necessary for adaptive NN control design. In Section III, the pure-feedback systems are transformed into a class of NARMAX systems, and in Section IV, the NARMAX systems are further transformed by prediction approach into a suitable form for control design. In Section V, adaptive NN control is first synthesized without considering the external disturbance and the NN approximation error, and then, dead zone technique is used to cope with the disturbance and the approximation error. Simulation studies and some remarks are provided in Section VI. This paper ends with conclusion in Section VII.

## II. PROBLEM FORMULATION AND PRELIMINARIES

### A. Pure-Feedback System

Consider the following SISO discrete-time systems in pure-feedback form:

$$\begin{cases} \xi_i(k+1) = f_i(\bar{\xi}_i(k), \xi_{i+1}(k)), & i = 1, 2, \dots, n-1 \\ \xi_n(k+1) = f_n(\bar{\xi}_n(k), u(k), d(k)) \\ y(k) = \xi_1(k) \end{cases} \quad (1)$$

where  $\bar{\xi}_j(k) = [\xi_1(k), \xi_2(k), \dots, \xi_j(k)]^T$ ,  $j = 1, 2, \dots, n$ ,  $n \geq 1$ , are the system states,  $f_i(\cdot, \cdot)$  and  $f_n(\cdot, \cdot, \cdot)$  are the unknown nonlinear functions,  $u(k) \in R$  and  $y(k) \in R$  are the system input and output, respectively, and  $d(k)$  denotes the external disturbance, which is bounded by an unknown constant  $\bar{d}$  so that  $|d(k)| \leq \bar{d}$ .

**Assumption 1:** The system functions  $f_i(\cdot, \cdot)$  and  $f_n(\cdot, \cdot, 0)$  in (1) are continuous with respect to all the arguments and continuously differentiable with respect to the second argument.

**Assumption 2:** There exist constants  $\bar{g}_i > \underline{g}_i > 0$  such that  $0 < \underline{g}_i \leq |g_{1,i}(\cdot)| \leq \bar{g}_i$ ,  $i = 1, 2, \dots, n$ , where  $g_{1,j}(\cdot) = \partial f_j(\xi_j(k), \xi_{j+1}(k)) / \partial \xi_{j+1}(k)$ ,  $j = 1, 2, \dots, n-1$ , and  $g_{1,n}(\cdot) = \partial f_n(\xi_n(k), u(k), d(k)) / \partial u(k)$ .

The partial derivatives  $g_{1,i}(\cdot)$  are actually the control gains of system (1). Assumption 2 implies that the control gains are strictly either positive or negative, but their signs, the control directions, are unknown. For convenience, let us introduce the notations  $\underline{g}' = \prod_{i=1}^n \underline{g}_i$  and  $\bar{g}' = \prod_{i=1}^n \bar{g}_i$ . It should be noted that the constants  $\bar{g}'$  and  $\underline{g}'$  are only used for analysis and are not required to be known in the control design.

**Assumption 3:** The system functions  $f_i(\cdot, 0)$  and  $f_n(\cdot, 0, \cdot)$  are Lipschitz functions.

### B. NARMAX System

Consider the following SISO discrete-time systems in NARMAX form:

$$y(k + \tau) = f(y(k + \tau - 1), \dots, y(k + \tau - n), u(k), \dots, u(k - m + 1), d(k)) \quad (2)$$

where  $\tau \geq 1$ ,  $m \geq 1$ ,  $f(\cdot) : R^{n+m+1} \rightarrow R$  is an unknown nonlinear function,  $u(k) \in R$  and  $y(k) \in R$  are the system input and output, respectively, and  $d(k)$  denotes the external disturbance, which is bounded by an unknown constant  $\bar{d}$ , i.e.,  $|d(k)| \leq \bar{d}$ .

**Assumption 4:** The system function  $f(\cdot) : R^{m+n+1} \rightarrow R$  in (2) is continuous with respect to all the arguments and continuously differentiable with respect to  $u(k)$ .

**Assumption 5:** There exist constants  $\bar{g} > \underline{g} > 0$  such that  $0 < \underline{g} \leq |g(\cdot)| \leq \bar{g}$ , where  $g(\cdot) = \partial f(\cdot) / \partial u(k)$ .

**Assumption 6:** System (2) is inverse stable, i.e., system (2) is bounded-output-bounded-input (BOBI). In addition, the function  $f(y(k + \tau - 1), y(k + \tau - 2), y(k + \tau - n), \mathbf{0}_{[m]}, d(k))$  is a Lipschitz function.

The control objective is to synthesize an adaptive NN control  $u(k)$  for system (1) and system (2), such that all signals in the closed-loop systems are bounded and the output  $y(k)$  tracks a bounded reference trajectory  $y_d(k)$ .

### C. HONN Approximation

There are many well-developed approaches used to approximate an unknown function. NN is one of the most frequently employed approximation method due to the fact that NN is shown to be capable of universally approximating any unknown continuous function to arbitrary precision [30]–[33]. Similar to biological neural networks, NN consists of massive simple processing units that correspond to biological neurons. With the highly parallel structure, NNs are of powerful computing ability and intelligence to learn and adapt with respect to the fresh and unknown data. Higher order neural network (HONN) is a kind of linearly parametrized neural network (LPNN) [1], and it has been shown to have a strong storage capacity, approximation, and learning capability. HONN satisfies the conditions of the Stone–Weierstrass theorem and can, therefore, approximate any

continuous function over a compact set [34], [35]. It is pointed out in [36] that by utilizing *a priori* information, HONN is very efficient in solving problems because the order or structure of HONN can be tailored to the order or structure of a given problem. The structure of HONN is expressed as follows:

$$\phi(W, z) = W^T S(z) \quad W, S(z) \in R^l$$

$$S(z) = [s_1(z), s_2(z), \dots, s_l(z)]^T \quad (3)$$

$$s_i(z) = \prod_{j \in I_i} [s(z_j)]^{d_j(i)}, \quad i = 1, 2, \dots, l \quad (4)$$

where  $z \in \Omega_z \subset R^m$  is the input to HONN,  $l$  is the NN nodes number,  $\{I_1, I_2, \dots, I_l\}$  is a collection of  $l$  not-ordered subsets of  $\{1, 2, \dots, m\}$ , e.g.,  $I_1 = \{1, 3, m\}$ ,  $I_2 = \{2, 4, m\}$ ,  $d_j(i)$ 's are the nonnegative integers,  $W$  is an adjustable synaptic weight vector, and  $s(z_j)$  is a monotonically increasing and differentiable sigmoidal function. In this paper, it is chosen as a hyperbolic tangent function, i.e.,  $s(z_j) = (e^{z_j} - e^{-z_j}) / (e^{z_j} + e^{-z_j})$ .

For a smooth function  $\varphi(z)$  over a compact set  $\Omega_z \subset R^m$ , given a small constant real number  $\mu^* > 0$ , if  $l$  is sufficiently large, there exists a set of ideal bounded weights  $W^*$  such that

$$\max |\varphi(z) - \phi(W^*, z)| < \mu(z), \quad |\mu(z)| < \mu^*. \quad (5)$$

From the universal approximation results for NNs [37], it is known that the constant  $\mu^*$  can be made arbitrarily small by increasing the NN nodes number  $l$ .

**Lemma 1 [20]:** Consider the basis functions of HONN (3) with  $z$  being the input vector. The following properties of HONN will be used in the proof of the closed-loop system stability:

$$\lambda_{\max} [S(z)S^T(z)] < 1, \quad S^T(z)S(z) < l \quad (6)$$

where  $\lambda_{\max}(M)$  denotes the max eigenvalue of  $M$ .

### D. Preliminaries

The following lemmas and definitions will be used for control design and system stability analysis in the remainder of this paper.

**Definition 1 [20]:** The future output of a discrete-time control system is said to be a semidetermined future output (SDFO) at time instant  $k$ , if it can be predicted based on the available system information up to the time instant  $k$ , and it controls up to the time instant  $k - 1$  without considering the unknown uncertainties.

**Definition 2:** Let  $U$  be an open subset of  $R^{i+1}$ . A mapping  $f(\omega) : U \rightarrow R$  is said to be Lipschitz on  $U$  if there exists a positive constant  $L$  such that

$$|f(\omega_a) - f(\omega_b)| \leq L \|\omega_a - \omega_b\|$$

for all  $(\omega_a, \omega_b) \in U$ .

**Definition 3 [10]:** A trajectory  $x(k)$  of the closed-loop system is said to be semiglobally uniformly-ultimately-bounded (SGUUB), if for any *a priori* given compact set, there exist a feedback control, a bound  $\mu \geq 0$ , and a number  $N(\mu, x(0))$ , such that the trajectory of the closed-loop system starting from the compact set satisfies  $|x(k)| \leq \mu$  for all  $k \geq k_0 + N$ .

**Lemma 2 [38]:** Consider a  $C^r$  function  $f : R^{k+n} \rightarrow R^n$  with  $f(a, b) = \mathbf{0}_{[n]}$  and  $\text{rank}(Df(a, b)) = n$  where  $Df(a, b) = \partial f(x, y) / \partial y|_{(x, y) = (a, b)} \in R^{n \times n}$ . Then, there exist a neighborhood  $A$  of  $a$  in  $R^k$  and a unique  $C^r$  function  $g : A \rightarrow R^n$  such that  $g(a) = b$  and  $f(x, g(x)) = \mathbf{0}_{[n]}$ ,  $\forall x \in A$ .

**Lemma 3 [39]:** For some given real scalar sequences  $s(k)$ ,  $b_1(k)$ , and  $b_2(k)$  and vector sequence  $\sigma(k)$ , if the following conditions hold:

- i)  $\lim_{k \rightarrow \infty} (s^2(k) / (b_1(k) + b_2(k) \sigma^T(k) \sigma(k))) = 0$ ;
- ii)  $0 < b_1(k) < K$  and  $0 \leq b_2(k) < K$  with a finite  $K$ ;
- iii)  $\|\sigma(k)\| \leq C_1 + C_2 \max_{0 \leq k' \leq k} |s(k')|$ , where  $C_1$  and  $C_2$  are some finite constants;

then we have a)  $\lim_{k \rightarrow \infty} s(k) = 0$ , and b)  $\sigma(k)$  is bounded.

### E. The Discrete Nussbaum Gain

**Definition 4:** Consider a discrete nonlinear function  $N(x(k))$  defined on a sequence  $x(k)$  with  $x_s(k) = \sup_{k' \leq k} \{x(k')\}$ .  $N(x(k))$  is a discrete Nussbaum gain if and only if it satisfies the following two properties:

- i) if  $x_s(k)$  increases without bound, then

$$\begin{aligned} \sup_{x_s(k) \geq \delta_0} \frac{1}{x_s(k)} S_N(x(k)) &= +\infty \\ \inf_{x_s(k) \geq \delta_0} \frac{1}{x_s(k)} S_N(x(k)) &= -\infty; \end{aligned} \quad (7)$$

- ii) if  $x_s(k) \leq \delta_1$ , then  $|S_N(x(k))| \leq \delta_2$  with some positive constants  $\delta_1$  and  $\delta_2$ ;

where  $S_N(x(k))$  is defined as

$$S_N(x(k)) = \sum_{k'=0}^k N(x(k')) \Delta x(k') \quad (8)$$

with  $\Delta x(k) = x(k+1) - x(k)$ .

In summary, for a discrete Nussbaum gain, if  $x_s(k)$  is unbounded, then  $S_N(x(k))$  oscillates between positive infinity and negative infinity, but if  $x_s(k)$  is bounded, then  $S_N(x(k))$  is bounded as well.

The first discrete Nussbaum gain was proposed in [29], in which it is pointed out that it is essential for the discrete sequence  $x(k)$  to satisfy

$$x(0) = 0, \quad x(k) > 0, \quad |\Delta x(k)| \leq \delta_0 \quad \forall k \quad (9)$$

where  $\delta_0$  is a positive constant. Then, the discrete Nussbaum gain proposed in [29] is defined on the sequence  $x(k)$  as

$$N(x(k)) = x_s(k) s_N(x(k)) \quad (10)$$

where  $s_N(x(k))$  is the sign function of the discrete Nussbaum gain, i.e.,  $s_N(x(k)) = \pm 1$ . The initial value is set as  $s_N(x(0)) = +1$ . Thereafter, the sign function  $s_N(x(k))$  will be chosen by comparing the summation  $S_N(x(k))$  with a pair of switching curves defined by  $f(x_s(k)) = \pm x_s^{3/2}(k)$ . The details are as follows.

- Step 1) At  $k = k_1$ , measure the output  $y(k_1)$  and compute  $\Delta x(k_1)$  and  $x(k_1 + 1) = x(k_1) + \Delta x(k_1)$  and  $S_N(x(k_1)) = S_N(x(k_1 - 1)) + N(x(k_1)) \Delta x(k_1)$ . In this paper,  $\Delta x(k_1)$  is calculated from (33) for NN

control in the absence of disturbance and approximation error, and from (40) for NN robust control.

Case ( $s_N(x(k_1)) = +1$ ):

- ⎧ If  $S_N(x(k_1)) \leq x_s^{3/2}(k_1)$ , then go to Step 2)
- ⎧ If  $S_N(x(k_1)) > x_s^{3/2}(k_1)$ , then go to Step 3)

Case ( $s_N(x(k_1)) = -1$ ):

- ⎧ If  $S_N(x(k_1)) < -x_s^{3/2}(k_1)$ , then go to Step 2)
- ⎧ If  $S_N(x(k_1)) \geq -x_s^{3/2}(k_1)$ , then go to Step 3)

Step 2) Set  $s_N(x(k_1 + 1)) = 1$ , go to Step 4).

Step 3) Set  $s_N(x(k_1 + 1)) = -1$ , go to Step 4).

Step 4) Return to Step 1) and wait for the measurement of output.

In the following lemma, a family of discrete Nussbaum gain is proposed based on the one defined above. The continuous-time counterpart lemma is reported in [40].

**Lemma 4:** Consider the discrete Nussbaum gain defined in (10).

- i) Given an arbitrary bounded function  $g(k) : R \rightarrow R$ , and  $g_1 \leq |g(k)| \leq g_2$ , where  $g_1$  and  $g_2$  are unknown positive constants, then  $N'(x(k)) = g(k)N(x(k))$  is also a discrete Nussbaum gain if  $\Delta x(k) \geq 0$ .
- ii) Given an arbitrary function  $-\epsilon_0 \leq C(k) \leq \epsilon_0$ , then  $N'(x(k)) = N(x(k)) + C(k)$  is still a discrete Nussbaum gain if  $\Delta x(k) \geq 0$ .

*Proof:* See the Appendix. ■

**Remark 1:** It should be emphasized here that in contrast to continuous-time Nussbaum gain, there is a strong restriction on the argument of the discrete Nussbaum gain,  $x(k)$ , i.e., i) it is a nonnegative sequence, ii) the magnitude of the increment  $|\Delta x(k)|$  is bounded by some constant, and iii) for the discrete Nussbaum gain obtained from Lemma 4, there is one more restriction  $\Delta x(k) \geq 0$ . These constraints make the design based on the discrete Nussbaum gain more challenging than the continuous-time case.

The following lemma will be used for stability analysis in this paper.

**Lemma 5 [41]:** Let  $V(k)$  be a positive-definite function defined  $\forall k$ ,  $N(\cdot)$  be discrete Nussbaum gain, and  $\theta$  be a nonzero constant. If the following inequality holds,  $\forall k$ :

$$V(k) \leq \sum_{k'=k_1}^k (c_1 + \theta N(x(k'))) \Delta x(k') + c_2 x(k) + c_3 \quad (11)$$

where  $c_1$ ,  $c_2$ , and  $c_3$  are some constants and  $k_1$  is a positive integer, then  $V(k)$ ,  $x(k)$ , and  $\sum_{k'=k_1}^k (c_1 + \theta N(x(k'))) \Delta x(k') + c_2 x(k) + c_3$  must be bounded  $\forall k$ .

### III. TRANSFORMATION OF PURE-FEEDBACK SYSTEM

In this section, using the same transformation procedure as in [12], it is shown that system (1) under Assumptions 1–3 is transformable to system (2) under Assumptions 4–6. For convenience, we define

$$\underline{y}(k) = [y(k), y(k-1), \dots, y(k-n+1)]^T. \quad (12)$$

The first equation of (1) can be rewritten as

$$\xi_1(k+1) - f_1(\xi_1(k), \xi_2(k)) = 0.$$

According to Assumption 2, the derivative of the left-hand side of the above equation over  $\xi_2(k)$  is not zero, thus, according to Lemma 2, there exists an implicit function  $p'_2(\cdot)$  asserted by Lemma 2 such that  $\xi_2(k)$  can be seen as a function of  $\xi_1(k+1)$  and  $\xi_1(k)$  as follows:

$$\begin{aligned}\xi_2(k) &= p'_2(\xi_1(k+1), \xi_1(k)) \\ &:= p_2(y(k+1), y(k)).\end{aligned}\quad (13)$$

Similarly, from the second equation of (1), there exists an implicit function  $p'_3(\cdot)$  asserted by Lemma 2 such that  $\xi_3(k)$  can be expressed as a function of  $\xi_2(k+1)$ ,  $\xi_2(k)$ , and  $\xi_1(k)$  as

$$\begin{aligned}\xi_3(k) &= p'_3(\xi_2(k+1), \xi_2(k), \xi_1(k)) \\ &= p'_3(p_2(y(k+2), y(k+1)), p_2(y(k+1), y(k)), y(k)) \\ &:= p_3(y(k+2), y(k+1), y(k)).\end{aligned}\quad (14)$$

Continuing the procedure consistently, we see that for  $\xi_i(k)$ ,  $i = 1, 2, \dots, n$ , there exists a function  $p_i(\cdot)$  such that  $\xi_i(k) = p_i(y(k+i-1), y(k+i-2), \dots, y(k))$ . Then, let us introduce a vector function  $P_i(\cdot)$  as follows:

$$P_i(y(k+i-1), y(k+i-2), \dots, y(k)) = \begin{bmatrix} y(k) \\ p_2(y(k+1), y(k)) \\ \vdots \\ p_i(y(k+i-1), y(k+i-2), \dots, y(k)) \end{bmatrix}$$

which leads to

$$\bar{\xi}_i(k) = P_i(y(k+i-1), y(k+i-2), \dots, y(k)), \quad i = 1, 2, \dots, n. \quad (15)$$

To transform system (1) to the NARMAX form, we combine the  $n$  system equations in (1) together. Let us consider rewriting the  $i$ th equation in system (1) as follows:

$$\begin{aligned}\xi_i(k+n+1-i) &= f_i(\bar{\xi}_i(k+n-i), \xi_{i+1}(k+n-i)), \\ &\quad i = 1, 2, \dots, n-1 \\ \xi_n(k+1) &= f_n(\bar{\xi}_n(k), u(k), d(k)).\end{aligned}$$

Combining with  $\bar{\xi}_i(k+n-i) = P_i(y(k+n-1))$  derived from (15), we obtain

$$\begin{aligned}\xi_i(k+n+1-i) &= f_i(P_i(y(k+n-1), \xi_{i+1}(k+n-i)), \\ &\quad i = 1, 2, \dots, n-1 \\ \xi_n(k+1) &= f_n(P_n(y(k+n-1), u(k), d(k)))\end{aligned}$$

Then, let us substitute the  $(i+1)$ th equation into the  $i$ th equation,  $i = 1, 2, \dots, n-1$ , so that we obtain (16), shown at the bottom of the page. It is easy to check that

$$\frac{\partial f(\cdot)}{\partial u(k)} = \Pi_{i=1}^n g_{1,i}(\cdot) := g(\cdot), \quad \underline{g}' \leq |g(\cdot)| \leq \bar{g}'. \quad (17)$$

According to Assumption 1, it is easy to show that the system function  $f(\cdot)$  in (16) is continuous with respect to all the arguments and continuously differentiable with respect to  $u(k)$ .

*Remark 2:* Assume that the output  $y(k)$  is bounded, then according to (16),  $u(k)$  must also be bounded because  $\underline{g} \leq |g(\cdot)| \leq \bar{g}$ . According to Lemma 3 in [12], the output boundedness guarantees the states boundedness for system (1). Then, it is easy to check that after transformation from the original system (1) under Assumptions 1–3, the transformed system (16) satisfies Assumptions 4–6.

At this stage, the pure-feedback system (1) is transformed to the NARMAX system (2) with  $\tau = n$  and  $m = 1$ , and the control objective for both systems (1) and (2) becomes unified.

#### IV. FUTURE OUTPUTS PREDICTION OF NARMAX SYSTEM

The difficulty in controlling system (2) lies in the existence of future outputs  $y(k+1), \dots, y(k+\tau-1)$ , which are not available at the current step. However, by carefully examining (2), it can be seen that control input  $u(k)$  only affects future output  $y(k+\tau)$  and those beyond, which means that the future outputs  $y(k+1), \dots, y(k+\tau-1)$  are independent of  $u(k)$ . When the external disturbance  $d(k)$  is ignored, the future outputs on the right-hand side of (2) can be predicted at the current step. Let us consider the output prediction approach in [20].

Moving back  $(\tau-1)$  steps in (2), we obtain

$$\begin{aligned}y(k+1) &= f(y(k), \dots, y(k-n+1), u(k-\tau+1), \dots, \\ &\quad u(k-m-\tau+2), d(k-\tau+1)) \\ &:= F_1(y(k), u(k-\tau+1), \dots, \\ &\quad u(k-m-\tau+2), d(k-\tau+1)).\end{aligned}\quad (18)$$

$$\begin{aligned}y(k+n) &= f_1(y(k+n-1), f_2(P_2(y(k+n-1)), \xi_3(k+n-2))) \\ &= f_1(y(k+n-1), f_2(P_2(y(k+n-1)), f_3(P_3(y(k+n-1)), \xi_4(k+n-3)))) \\ &= f_1(y(k+n-1), f_2(P_2(y(k+n-1)), \\ &\quad f_3(P_3(y(k+n-1)), \dots, f_{n-1}(P_{n-1}(y(k+n-1)), f_n(P_n(y(k+n-1)), u(k), d(k)))))) \\ &:= f(y(k+n-1), u(k), d(k)).\end{aligned}\quad (16)$$

It implies that the output  $y(k+1)$  is an SDFO according to Definition 1. Assuming that  $\tau \geq 2$ , by moving a step forward, we obtain the following equation from (18):

$$y(k+2) = F_1(\underline{y}(k+1), u(k-\tau+2), \dots, u(k-m-\tau+3), d(k-\tau+2)). \quad (19)$$

Substituting (18) into (19), we see that there exists a function  $F_2(\cdot)$  such that

$$y(k+2) = F_2(\underline{y}(k), u(k-\tau+2), \dots, u(k-m-\tau+2), d(k-\tau+2), d(k-\tau+1)) \quad (20)$$

which implies that  $y(k+2)$  is also an SDFO. Continuing the substituting recursively, it is easy to show  $y(k+j)$ ,  $j = 1, 2, \dots, \tau-1$ , are all SDFOs. Finally, we see that there must exist a function  $F_\tau(\cdot)$  such that

$$y(k+\tau) = F_\tau(\underline{z}(k), u(k), \underline{d}(k)) \quad (21)$$

where

$$\begin{aligned} \underline{z}(k) &= [\underline{y}^T(k), \underline{u}^T(k-1)]^T \\ \underline{u}(k-1) &= [u(k-1), \dots, u(k-\tau-m+2)]^T \\ \underline{d}(k) &= [d(k), d(k-1), \dots, d(k-\tau+1)]^T \end{aligned} \quad (22)$$

if  $\tau+m > 2$  and if  $\tau+m = 2$ ,  $\underline{z}(k) = \underline{y}(k)$ . It can be easily shown that function  $F_\tau(\cdot)$  is continuous and continuously differentiable with respect to  $u(k)$  according to Assumption 4. Rewrite system (21) as

$$y(k+\tau) = \phi_o(\underline{z}(k), u(k)) + d_o(k) \quad (23)$$

where

$$\begin{aligned} \phi_o(\underline{z}(k), u(k)) &= F_\tau(\underline{z}(k), u(k), \mathbf{0}_{[\tau]}) \\ d_o(k) &= F_\tau(\underline{z}(k), u(k), \underline{d}(k)) - F_\tau(\underline{z}(k), u(k), \mathbf{0}_{[\tau]}). \end{aligned} \quad (24)$$

Note that  $F_\tau(\cdot)$  is obtained by iterative substitution of the system function  $f(\cdot)$ , which satisfies Lipschitz condition in Assumption 6. Then, there exists a finite constant  $\bar{d}_o$  such that  $|d_o(k)| \leq \bar{d}_o$ .

## V. ADAPTIVE NN CONTROL DESIGN

Without loss of generality, we will assume that  $\bar{g}' = \bar{g}$  and  $\underline{g}' = \underline{g}$  in the rest of this paper. From the derivation of  $F_\tau(\cdot)$ , we see that

$$\frac{\partial \phi_o(\underline{z}(k), u(k))}{\partial u(k)} = \frac{\partial F_\tau(\cdot)}{\partial u(k)} = \frac{\partial f(\cdot)}{\partial u(k)} = g(\cdot) \neq 0.$$

The dynamics of the tracking error  $e(k) = y(k) - y_d(k)$  is given by

$$e(k+\tau) = \phi_o(\underline{z}(k), u(k)) - y_d(k+n) + d_o(k). \quad (25)$$

It is easy to show that

$$\frac{\partial (\phi_o(\underline{z}(k), u(k)) - y_d(k+n))}{\partial u(k)} \neq 0.$$

Therefore, according to Lemma 2, there exists an ideal control input  $u^*(\bar{z}(k))$  such that

$$\begin{aligned} \phi_o(\underline{z}(k), u^*(\bar{z}(k))) - y_d(k+n) &= 0 \\ \bar{z}(k) &= [\underline{z}^T(k), y_d(k+n)]^T. \end{aligned} \quad (26)$$

Using the ideal control  $u^*(\bar{z}(k))$ , we have  $e(k) = 0$  after  $\tau$  steps if  $d_o(k) = 0$ . It implies that the ideal control  $u^*(\bar{z}(k))$  is a  $\tau$ -step deadbeat control.

As mentioned in Section II-C, there exists an ideal constant weights vector  $W^* \in R^l$ , such that

$$\begin{aligned} u_{nn}^*(\bar{z}(k)) &= W^{*T} S(\bar{z}(k)), \quad S(\bar{z}(k)) \in R^l \\ u^*(\bar{z}(k)) &= u_{nn}^*(\bar{z}(k)) + \mu(\bar{z}(k)) \quad \forall \bar{z} \in \Omega_{\bar{z}} \end{aligned} \quad (27)$$

where  $\mu(\bar{z}(k))$  is the NN approximation error and  $\Omega_{\bar{z}}$  is a sufficient large compact set.

*Remark 3:* Because system (23) is transformed from the original system (2), Assumption 6 still holds for (23). Considering that we input  $u^*(\bar{z}(k))$  to system (23), then the output  $y(k)$  catches up  $y_d(k)$  in  $\tau$  steps. This implies the boundedness of output  $y(k)$  because the reference signal  $y_d(k)$  is bounded. Then, from the BOBI property in Assumption 6, the boundedness of  $u^*(\bar{z}(k))$  is guaranteed.

Using HONN as an approximator of  $u^*(\bar{z}(k))$ , then the output feedback adaptive NN control is given as

$$u(k) = \hat{W}^T(k) S(\bar{z}(k)). \quad (28)$$

Adding and subtracting  $\phi_o(\bar{z}(k), u^*(\bar{z}(k)))$  on the right-hand side of (25) leads to

$$\begin{aligned} e(k+\tau) &= \phi_o(\underline{z}(k), u(k)) - \phi_o(\underline{z}(k), u^*(\bar{z}(k))) + d_o(k) \\ &= g(\underline{z}(k), u^c(k)) (u(k) - u^*(\bar{z}(k))) + d_o(k) \end{aligned} \quad (29)$$

where

$$g(\underline{z}(k), u^c(k)) = \frac{\partial \phi_o(\underline{z}(k), u^c(k))}{\partial u^c(k)}$$

with  $u^c(k) \in [\min\{u^*(\bar{z}(k)), u(k)\}, \max\{u^*(\bar{z}(k)), u(k)\}]$ , according to the mean value theorem. For convenience, let us introduce the following notations:

$$g(k) = g(\underline{z}(k), u^c(k)), \quad S(k) = S(\bar{z}(k)), \quad \mu(k) = \mu(\bar{z}(k)) \quad (30)$$

and it is obvious that  $\underline{g} \leq g(k) \leq \bar{g}$ . Substituting (27) into (29) and noting that  $\tilde{W}(k) = \hat{W}(k) - W^*$ , we obtain

$$e(k+\tau) = g(k) \tilde{W}^T(k) S(k) + d^*(k) \quad (31)$$

where

$$d^*(k) = -g(k) \mu(k) + d_o(k)$$

and it is trivial to show that  $|d^*(k)| \leq \bar{g} \mu^* + \bar{d}_0 := d_0^*$ .

### A. NN Control Without Disturbance and Approximation Error

In this section, to clearly demonstrate the control design, let us first study the NN control in the ideal case, where neither external disturbance nor NN approximation error exists, i.e.,  $d^*(k) = 0$ . It will be shown that in the ideal case, the output

tracking error will converge to zero ultimately. Then, in the next section, we will take into consideration the disturbance and approximation error to design a robust control law.

In the ideal case, (31) becomes as follows:

$$e(k + \tau) = g(k)\tilde{W}^T(k)S(k). \quad (32)$$

Consider the following adaptation law for the NN weights:

$$\begin{aligned} \epsilon(k) &= \frac{\gamma e(k)}{G(k)} \\ \hat{W}(k) &= \hat{W}(k - \tau) - \gamma N(x(k))S(k - \tau) \frac{\epsilon(k)}{D(k)} \\ \Delta x(k) &= x(k + 1) - x(k) = \frac{G(k)\epsilon^2(k)}{D(k)}, \quad x(0) = 0 \\ G(k) &= 1 + |N(x(k))| \\ D(k) &= 1 + \|S(k - \tau)\|^2 + |N(x(k))| + \epsilon^2(k) \\ \hat{W}(j) &= \mathbf{0}_l, \quad j = -\tau + 1, \dots, 0 \end{aligned} \quad (33)$$

where  $N(x(k))$  is the discrete Nussbaum gain defined in (10), and  $\epsilon(k)$  is introduced as an augmented error, then the tuning rate  $\gamma > 0$  can be an arbitrary positive constant to be specified by the designer. It should be mentioned that the requirement on the sequence  $x(k)$  in (9) is satisfied, and furthermore,  $\Delta x(k) \geq 0$ .

**Theorem 1:** Consider the adaptive closed-loop system consisting of system (1) under Assumptions 1–3 or system (2) under Assumptions 4–6, control (28) with NN weights adaptation law (33). Assuming there is no external disturbance and approximation error, i.e.,  $d^*(k) = 0$ , all the signals in the closed-loop system are SGUUB and the tracking error  $e(k)$  will converge to zero ultimately.

*Proof:* First, let us assume that the NN is constructed to cover a large enough compact set  $\Omega$  such that the inputs  $u(k)$  and the outputs  $y(k)$  are within the NN approximation range  $\Omega$ , while we will show that it is indeed the case, if we initially construct the NN with approximation range covering a prescribed compact set, and the so-called circular argument does not apply here in this very proof.

Choose a positive-definite function  $V(k)$  as

$$V(k) = \sum_{j=1}^{\tau} \tilde{W}^T(k - \tau + j)\tilde{W}(k - \tau + j). \quad (34)$$

The first difference equation of  $V(k)$  is given as

$$\begin{aligned} \Delta V(k) &= V(k) - V(k - 1) \\ &= \tilde{W}^T(k)\tilde{W}(k) - \tilde{W}^T(k - \tau)\tilde{W}(k - \tau) \\ &= \left(\tilde{W}(k) - \tilde{W}(k - \tau)\right)^T \left(\tilde{W}(k) - \tilde{W}(k - \tau)\right) \\ &\quad + 2\tilde{W}^T(k - \tau) \left(\tilde{W}(k) - \tilde{W}(k - \tau)\right) \\ &= \gamma^2 \frac{N^2(x(k))S^T(k - \tau)S(k - \tau)}{D^2(k)} \epsilon^2(k) \\ &\quad - 2N(x(k)) \frac{\gamma \tilde{W}^T(k - \tau)S(k - \tau)}{D(k)} \epsilon(k) \\ &\leq \gamma^2 \frac{G(k)\epsilon^2(k)}{D(k)} - \frac{2}{g(k - \tau)} N(x(k)) \frac{G(k)\epsilon^2(k)}{D(k)} \\ &= \gamma^2 \Delta x(k) - \frac{2}{g(k - \tau)} N(x(k)) \Delta x(k). \end{aligned} \quad (35)$$

Denote  $N'(x(k)) = (1/g(k - \tau))N(x(k))$  and then, noting  $1/\bar{g} \leq 1/g(k - \tau) \leq 1/\underline{g}$  and according to Lemma 4, we can see that  $N'(x(k))$  is still a discrete Nussbaum gain. Taking summation on both sides of (35) and noting  $0 \leq \Delta x(k) \leq 1$ , we have

$$V(k) \leq -2 \sum_{k'=0}^k N'(x(k')) \Delta x(k') + \gamma^2 x(k) + \gamma^2. \quad (36)$$

Applying Lemma 5 to (36) results in the boundedness of  $V(k)$  and  $x(k)$ . Noting the definition of  $V(k)$ , we obtain the boundedness of  $\hat{W}(k)$  immediately. From the definition of  $N(x(k))$ , it is seen that  $|N(x(k))| = |x_s(k)|$ . Thus, the boundedness of  $x(k)$  implies the boundedness of  $N(x(k))$  and  $G(k) = 1 + |N(x(k))|$ . In (33), we see that  $\Delta x(k) = G(k)\epsilon^2(k)/D(k) \geq 0$ , and therefore,  $x(k)$  is a nondecreasing sequence. Thus, the boundedness of  $x(k)$  results in

$$\lim_{k \rightarrow \infty} \Delta x(k) = 0. \quad (37)$$

According to the definition of  $\Delta x(k)$  in (33), we have

$$\lim_{k \rightarrow \infty} \frac{G(k)\epsilon^2(k)}{1 + \|S(k - \tau)\|^2 + |N(x(k))| + \epsilon^2(k)} = 0. \quad (38)$$

From Lemma 1, we see  $\|S(k - \tau)\|^2$  is bounded. Then, applying Lemma 3 to (38), we obtain

$$\lim_{k \rightarrow \infty} \epsilon(k) = \lim_{k \rightarrow \infty} \gamma \frac{e(k)}{G(k)} = 0. \quad (39)$$

Noting the boundedness of  $G(k)$ , we immediately have  $e(k) \rightarrow 0$ . Next, because  $e(k) = y(k) - y_d(k)$ , where the reference signal  $y_d(k)$  is bounded, the boundedness of output  $y(k)$  is obvious. According to Remark 2, the boundedness of control  $u(k)$  and states of system (1) is guaranteed.

So far, we have proved that given any initial condition  $\bar{z}(0) \in \Omega_0$ , there is a corresponding bounding compact set  $\Omega_{\bar{z}}$  so that  $\bar{z}(k) \in \Omega_{\bar{z}}, \forall k$ , if the NN approximation range is initialized to cover  $\Omega_{\bar{z}}$ .

Next, let us consider that the initial condition  $\Omega_0$  and control parameters to be chosen are known at the beginning. It implies the bounding set  $\Omega_{\bar{z}}$  is determined. Then, if initially the NN approximation range  $\Omega$  is constructed to cover the bounding set  $\Omega_{\bar{z}}$ , the boundedness of all the closed-loop signals is guaranteed. According to Definition 3 (given any initial condition, there is a corresponding control such that the all the closed-loop signals are bounded), the proposed adaptive NN control achieves SGUUB stability. This completes the proof. ■

## B. Robust NN Control

In this section, the proposed control in Section V-A will be improved to deal with the external disturbance and NN approximation error. The dead zone method will be studied in the NN weights adaptation law. The disturbance is assumed to be bounded but we do not know the upper bound of the disturbance. The basic idea in this section is that the NN weights adaptation process is only proceeded when the augmented error is larger than a threshold, i.e.,  $|\epsilon(k)| > \lambda$ , where  $\lambda > 0$  can be arbitrary positive constant to be set by the designer.

The control law still assumes the form in (28) but the adaptation law of NN weights has been modified including dead zone as follows:

$$\begin{aligned}\epsilon(k) &= \gamma \frac{e(k)}{G(k)} \\ \hat{W}(k) &= \hat{W}(k-\tau) - \gamma N(x(k)) S(k-\tau) \frac{a(k)\epsilon(k)}{D(k)} \\ \Delta x(k) &= x(k+1) - x(k) = \frac{a(k)G(k)\epsilon^2(k)}{D(k)}, \quad x(0) = 0 \\ G(k) &= 1 + |N(x(k))| \\ D(k) &= 1 + \|S(k-\tau)\|^2 + |N(x(k))| + \epsilon^2(k) \\ a(k) &= \begin{cases} 1, & \text{if } |\epsilon(k)| > \lambda \\ 0, & \text{others} \end{cases} \\ \hat{W}(j) &= \mathbf{0}_{[l]}, \quad j = -\tau + 1, \dots, 0 \end{aligned} \quad (40)$$

where  $N(x(k))$  is the discrete Nussbaum gain defined in (10).

**Theorem 2:** Consider the adaptive closed-loop system consisting of system (1) under Assumptions 1–3 or system (2) under Assumptions 4–6, NN control (28) with NN weights adaptation law (40). All the signals in the closed-loop system are SGUUB and the discrete Nussbaum gain  $N(x(k))$  will converge to a constant ultimately. Denote  $C = \lim_{k \rightarrow \infty} G(k)$ , then the tracking error satisfies  $\lim_{k \rightarrow \infty} \sup\{|e(k)|\} < C\lambda/\gamma$ , where the tuning rate  $\gamma > 0$  and the threshold value  $\lambda > 0$  can be arbitrary constants to be specified by the designer.

*Proof:* The proof is also shown in two parts as the proof of Theorem 1. First, we assume the NN is constructed to cover a large enough compact set  $\Omega$  such that the inputs and the outputs are within the NN approximation range  $\Omega$ . Substituting the error dynamics (31) into the augmented error  $\epsilon(k)$  and taking disturbance into account, we have

$$\begin{aligned}\gamma \tilde{W}^T(k-\tau) S(k-\tau) &= \frac{1}{g(k-\tau)} G(k) \epsilon(k) \\ &\quad - \frac{1}{g(k-\tau)} \gamma d^*(k-\tau). \end{aligned} \quad (41)$$

Choose the same positive-definite function  $V(k)$  as

$$V(k) = \sum_{j=1}^{\tau} \tilde{W}^T(k-\tau+j) \tilde{W}(k-\tau+j) \quad (42)$$

and note that

$$\frac{2a(k)}{g(k-\tau)} N(x(k)) d^*(k-\tau) \epsilon(k) \leq a(k) \left| \frac{2d_0^*}{g\lambda} \right| G(k) \epsilon^2(k) \quad (43)$$

and  $a^2(k) = a(k)$ . Then, we see that the difference equation of  $V(k)$  can be written as follows by using the same technique as in Section V-A:

$$\begin{aligned}\Delta V(k) &= V(k) - V(k-1) \\ &= \frac{\gamma^2 a^2(k) N^2(x(k)) S^T(k-\tau) S(k-\tau)}{D^2(k)} \epsilon^2(k) \\ &\quad - 2N(x(k)) \frac{a(k) \gamma \tilde{W}^T(k-\tau) S(k-\tau)}{D(k)} \epsilon(k) \\ &\leq \gamma^2 \frac{a(k) G(k) \epsilon^2(k)}{D(k)} + \left| \frac{2d_0^*}{g\lambda} \right| \frac{a(k) G(k) \epsilon^2(k)}{D(k)} \\ &\quad - \frac{2}{g(k-\tau)} N(x(k)) \frac{a(k) G(k) \epsilon^2(k)}{D(k)}. \end{aligned} \quad (44)$$

Note that

$$\frac{a(k) G(k) \epsilon^2(k)}{D(k)} = \Delta x(k).$$

Then, by denoting  $N'(x(k)) = (1/g(k-\tau)) N(x(k))$ , we have

$$\Delta V(k) \leq c_1 \Delta x(k) - 2N'(x(k)) \Delta x(k) \quad (45)$$

where  $c_1 = \gamma^2 + |2d_0^*/g\lambda|$ . The following inequality follows immediately:

$$V(k) \leq -2 \sum_{k'=0}^k N'(x(k')) \Delta x(k') + c_1 x(k) + c_1. \quad (46)$$

Following the same procedure as in Section V-A, we conclude the boundedness of  $\hat{W}(k)$ ,  $G(k)$ ,  $x(k)$ , and  $N(x(k))$ , and in addition, from the boundedness of  $x(k)$ , we have

$$\lim_{k \rightarrow \infty} \Delta x(k) = \lim_{k \rightarrow \infty} \frac{a(k) G(k) \epsilon^2(k)}{D(k)} = 0. \quad (47)$$

Let us define a time interval as  $Z_1 = \{k | a(k) = 1\}$  and suppose that  $Z_1$  is an infinite set. Then, we have

$$\lim_{k \rightarrow \infty, k \in Z_1} \epsilon(k) = \lim_{k \rightarrow \infty, k \in Z_1} a(k) \epsilon(k) = 0$$

which conflicts with  $a(k) = 1, k \in Z_1$ , because  $|\epsilon(k)| \geq \lambda$ , when  $a(k) = 1$ . Therefore,  $Z_1$  must be a finite set and then, we have

$$\lim_{k \rightarrow \infty} a(k) = 0, \quad \lim_{k \rightarrow \infty} \sup |\epsilon(k)| \leq \lambda$$

which implies that  $N(x(k))$  will converge to a constant ultimately. By denoting the limit of  $G(k)$  as  $C$ , it can be derived from the definition of  $\epsilon(k)$  in (40) that

$$\lim_{k \rightarrow \infty} \sup \{|e(k)|\} \leq \frac{C\lambda}{\gamma}.$$

Then, following the same procedure as in Section V-A, the SGUUB of other closed-loop signals can be concluded. This completes the proof. ■

## VI. SIMULATION RESULTS

### A. Pure-Feedback System

In this section, the following second-order nonlinear pure-feedback plant is used for simulation studies:

$$\begin{aligned}\xi_1(k+1) &= f_1(\xi_1(k), \xi_2(k)) \\ \xi_2(k+1) &= f_2(\xi_1(k), \xi_2(k), u(k)) + d(k) \end{aligned} \quad (49)$$

where system functions are

$$\begin{aligned}f_1(\xi_1(k), \xi_2(k)) &= 1.4 \frac{\xi_1^2(k)}{1 + \xi_1^2(k)} + 0.1 \xi_2^3(k) + 0.5 \xi_2(k) \\ f_2(\xi_2(k), u(k)) &= \frac{\xi_1(k)}{1 + \xi_1^2(k) + \xi_2^2(k)} + gu(k) \end{aligned}$$

where  $g = \pm 1$  and the disturbance is  $d(k) = 0.1 \cos(0.05k) \cos(\xi_1(k))$ . The control objective is to make the output  $y(k)$  track the desired reference trajectory  $y_d(k) = (1/2) \sin((\pi/5)kT) + (1/2) \cos((\pi/10)kT)$ , where



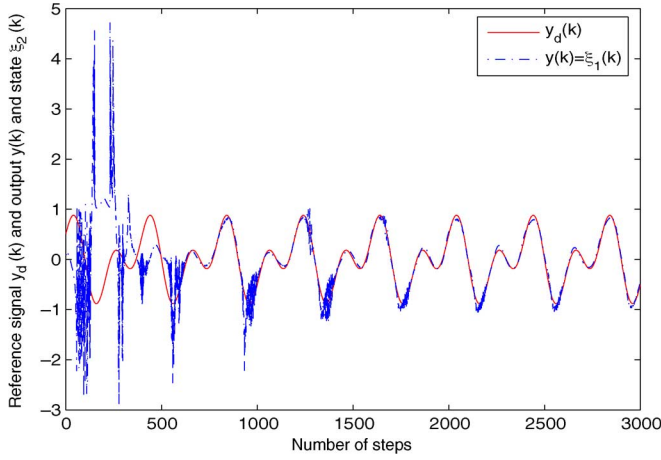


Fig. 1. Reference signal and system states.

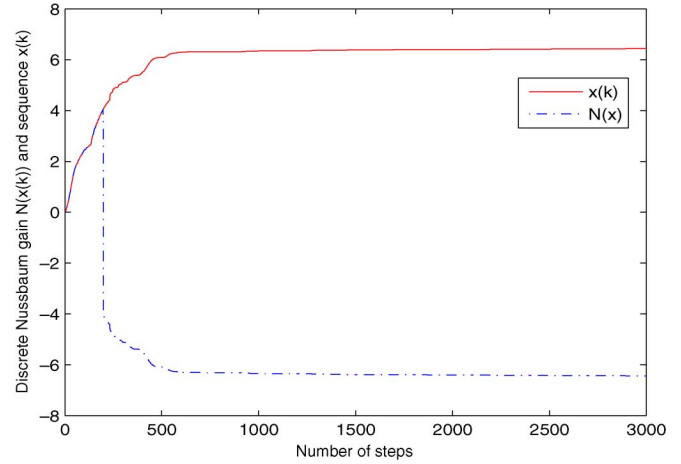
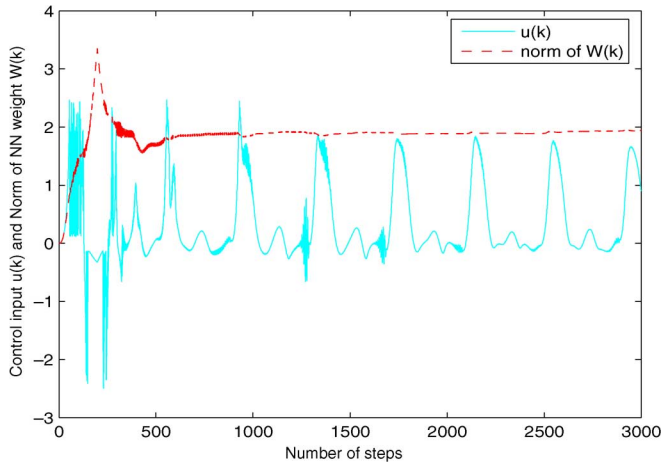

 Fig. 3. Discrete Nussbaum gain  $N(x(k))$  and its argument  $x(k)$ .


Fig. 2. Control signal and NN weights norm.

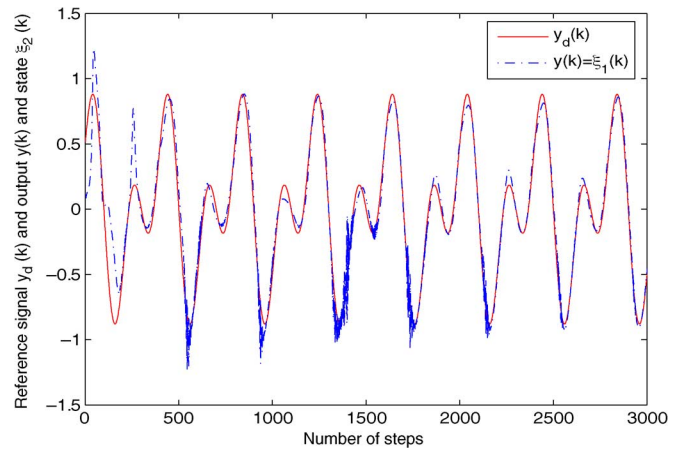


Fig. 4. Reference signal and system states.

$T = 0.05$ , and guarantee the boundedness of all the closed-loop signals. The system initial states are  $\bar{\xi}_2(0) = [0.1, 0.1]^T$ . The HONN is constructed with  $l = 27$  neurons. The tuning rate and the threshold value are chosen as  $\gamma = 0.42$  and  $\lambda = 0.001$ .

First, we choose  $g = -1$  and the simulation results are presented in Figs. 1–3. Fig. 1 shows the reference signal  $y_d(k)$  and system output  $y(k)$ . Fig. 2 illustrates the boundedness of the control input  $u(k)$  and the NN weights vector estimate  $\hat{W}(k)$ . Fig. 3 shows the discrete sequence  $x(k)$  and discrete Nussbaum gain  $N(x(k))$ .

It can be checked that  $g_{1,1}(\cdot) = \partial f_1(\cdot)/\partial \xi_2(k) = 0.5 + 0.3\xi_2^2(k) > 0$  and  $g_{1,2}(\cdot) = \partial f_2(\cdot)/\partial u(k) = -1 < 0$  such that Assumption 2 is satisfied. Following the transformation procedure, we define

$$\begin{aligned} y(k+2) &= f_1(\xi_1(k+1), \xi_2(k+1)) \\ &= f_1(f_1(\xi_1(k), \xi_2(k)), f_2(\xi_1(k), \xi_2(k), u(k)) + d(k)) \\ &:= f(\xi_1(k), \xi_2(k), u(k), d(k)) \end{aligned} \quad (50)$$

and it is easy to check that  $g(\cdot) = \partial f(\cdot)/\partial u(k) = g_{1,1}(\cdot)g_{1,2}(\cdot) < 0$ . According to (46), the discrete Nussbaum gain  $N(x(k))$  should turn to be ultimately negative

because only when it is negative,  $N'(x(k))$  is positive and  $V(k)$  decreases. In Fig. 3, it can be seen that the discrete Nussbaum gain turns to be negative from positive at about step 200, and thereafter, it remains negative. This explains how discrete Nussbaum gain works. It will change its sign accordingly such that we do not need to know the sign of the control gain  $g(\cdot)$  of the system.

In Fig. 1, it can be seen that the initial tracking performance is not good. The output goes to an opposite direction compared with the reference signal. It is the same with the control signal in Fig. 2. This is due to the discrete Nussbaum gain still tuning, and only when sufficient tracking error accumulation is achieved, it will change to be in the correct sign. After the discrete Nussbaum gain turns to be negative at about step 200, it can be seen that the tracking performance improves to be much better.

To further demonstrate that the proposed NN control is insensitive to control direction, we choose  $g = 1$  for simulation. With employment of the same control law and NN weights adaptation law, the simulation results are shown in Figs. 4–6. It can be seen from Fig. 5 that the discrete Nussbaum gain is always positive and from Fig. 4 that the initial tracking errors are much smaller than those in Fig. 1. It is also noted by comparison between Figs. 2 and 5 that the control signals are reversed to each other.

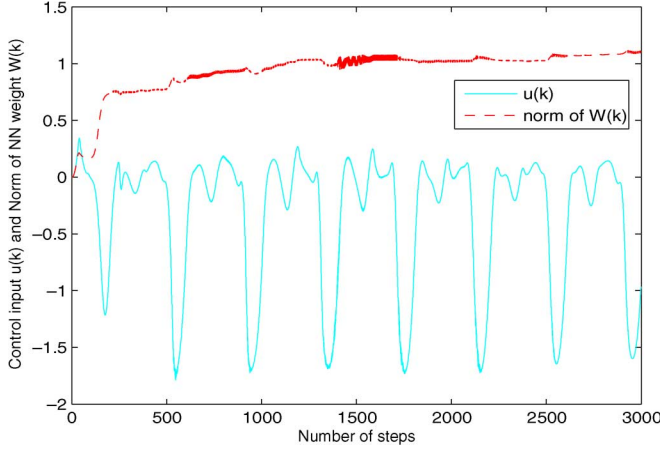


Fig. 5. Control signal and NN weights norm.

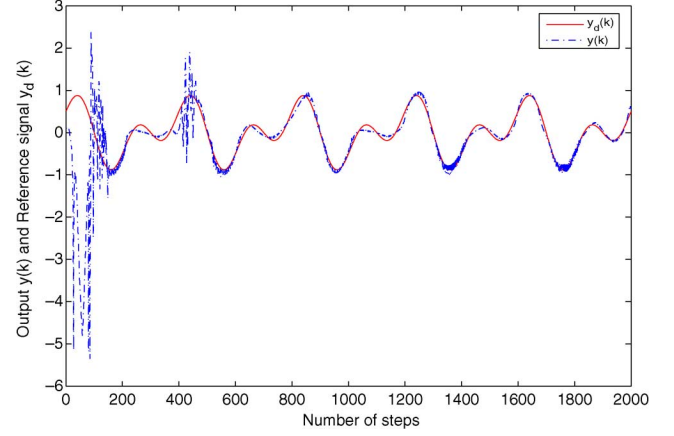
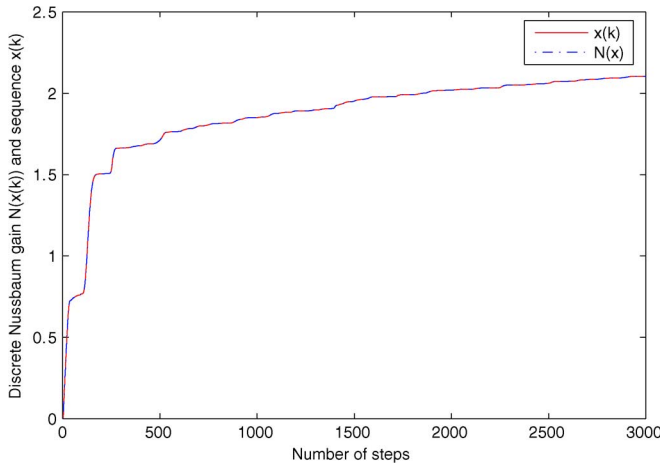


Fig. 7. Reference signal and system output.

Fig. 6. Discrete Nussbaum gain  $N(x(k))$  and its argument  $x(k)$ .

### B. NARMAX System

In this section, the following NARMAX system studied in [42] is used for simulation:

$$y(k+1) = \frac{y(k)y(k-1)[y(k)+2.5]}{1+y^2(k)+y^2(k-1)} + gu(k-2) + d(k) \quad (51)$$

where the control gain is chosen to be  $g = \pm 1.25$  and the disturbance is  $d(k) = 0.1 \cos(0.05k) \cos(\xi_1(k))$ . The control objective is to make the output  $y(k)$  track the desired reference trajectory  $y_d(k) = (1/2) \sin((\pi/5)kT) + (1/2) \cos((\pi/10)kT)$ , with  $T = 0.05$ , and guarantee the boundedness of all the closed-loop signals. The initial condition is  $y(-1) = y(-2) = y(0) = 0.1$ . The NN control is constructed in the same manner as in Section VI-A. The tuning rate and the threshold value are chosen as  $\gamma = 0.9$  and  $\lambda = 0.02$ . The simulation results for  $g = -1.25$  are presented in Figs. 7–9. Fig. 7 shows the reference signal  $y_d(k)$  and system output  $y(k)$ . Fig. 8 illustrates the boundedness of the control input  $u(k)$  and the NN weights vector estimate  $\hat{W}(k)$ . Fig. 9 shows the discrete sequence  $x(k)$  and discrete Nussbaum gain  $N(x(k))$ .

It can be checked that  $g(\cdot) = \partial f(\cdot)/\partial u(k) = g < 0$ . Therefore, it is seen in Fig. 9 that the Nussbaum gain changes to be negative after step 150 and remains to be so. Accordingly, the

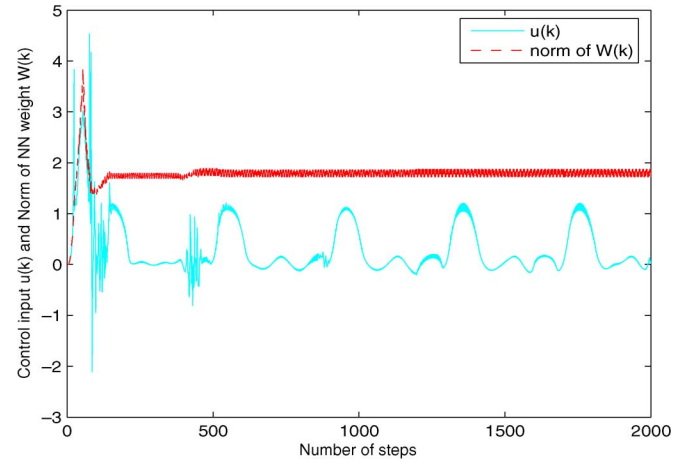
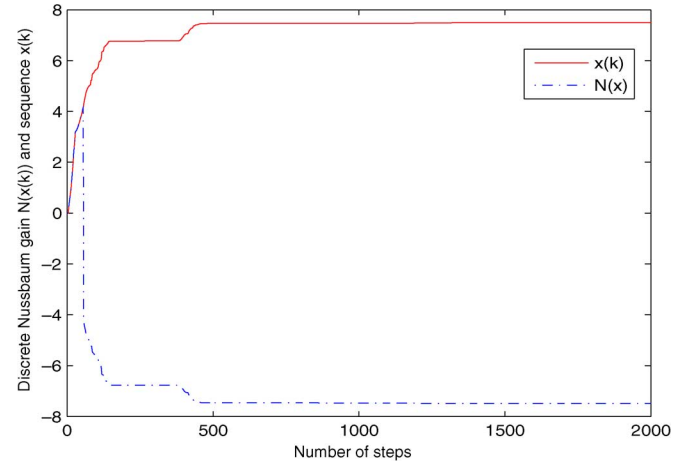


Fig. 8. Control signal and NN weights norm.

Fig. 9. Discrete Nussbaum gain  $N(x(k))$  and its argument  $x(k)$ .

output and the control signal go to a wrong direction at initial stage as shown in Figs. 7 and 8. After the discrete Nussbaum gain turns to be negative, the output tracking performance improves to be much better.

Next, let us change  $g = -1.25$  to  $g = 1.25$ . The simulation results by the same control law and NN weights adaptation law are shown in Figs. 10–12.

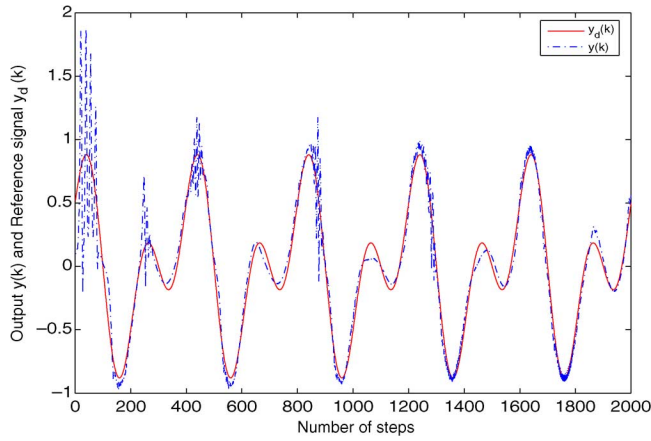


Fig. 10. Reference signal and system output.

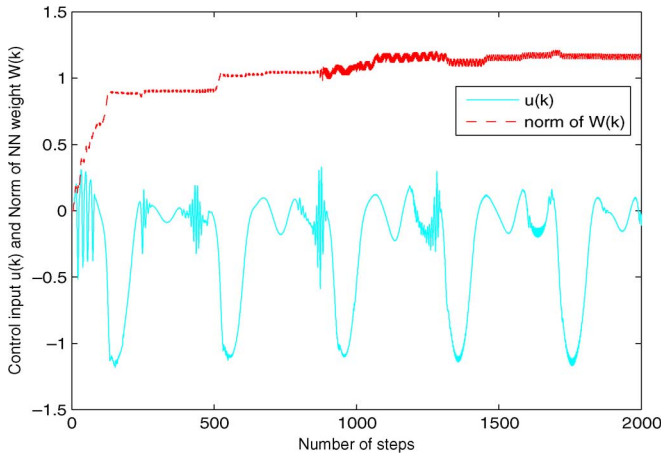
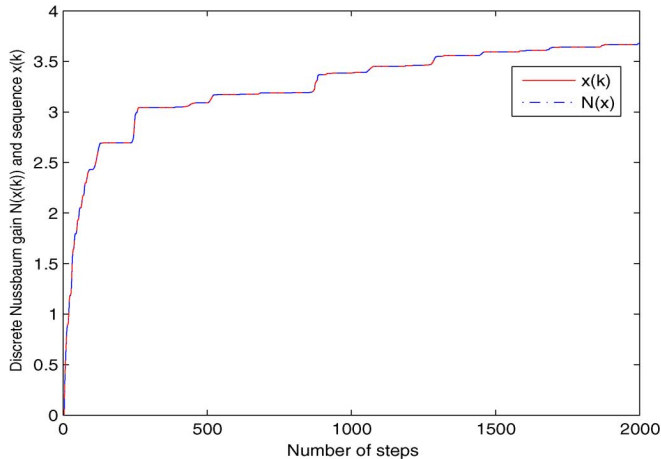


Fig. 11. Control signal and NN weights norm.


 Fig. 12. Discrete Nussbaum gain  $N(x(k))$  and its argument  $x(k)$ .

In summary, the adaptive NN control with discrete Nussbaum gain adapts by searching alternately in the two directions. The adaptive NN control will be able to reverse its direction of adaptation if initially the adaptation is in the wrong direction. However, we also noted that while the boundedness of all the signals in the adaptive system was maintained, during those intervals when the adaptation is in the wrong direction, the bounds may be

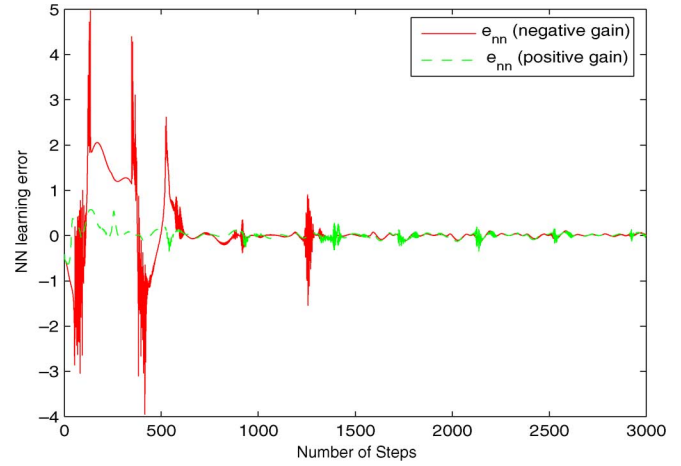


Fig. 13. NN learning errors of pure-feedback system (49).

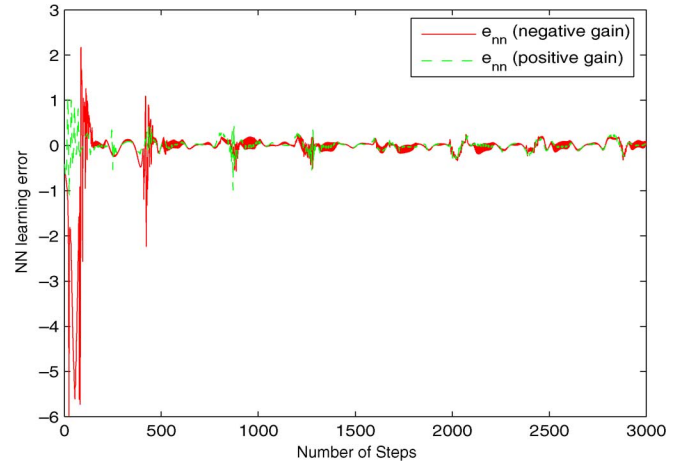


Fig. 14. NN learning errors of NARMAX systems (51).

very large. This appears to be a limitation of the proposed control. Actually, when the control direction is unknown, no matter what approach is used, if the adaptive NN control is initialized to start in the bad regime where it adapts in the wrong direction, it must at least remain in that regime until the errors become correspondingly large. Only then can the adaptive NN control determine that the direction of the adaptation is wrong so that it can reverse its direction of the adaptation.

### C. NN Learning Performance

To demonstrate the NN learning performance, we define the following NN learning error:

$$e_{nn}(k) = \phi_o(\bar{z}(k), u(\bar{z}(k))) - y_d(k+n) \quad (52)$$

as the measurement of NN learning performance. According to (26) and (27), the better the NN approximation is (the smaller the NN approximation error  $u(k) - u^*(k)$  is), the smaller  $e_{nn}(k)$  is. If  $u(k) - u^*(k) = 0$ , we have  $e_{nn}(k) = 0$ .

The NN learning errors are demonstrated in Figs. 13 and 14. It is noted that the NN learning performance is satisfactory, i.e., the defined NN learning error  $e_{nn}(k)$  is ultimately bounded in a neighborhood of zero.

## VII. CONCLUSION

In this paper, it has been shown that a class of nonlinear discrete-time systems in pure-feedback form is transformable to a class of inverse stable NARMAX system, and the control design for both systems can be synthesized in a unified frame. By prediction approach, the original NARMAX system is transformed to a suitable form for control design without noncausal problem. Implicit function theorem has been exploited to identify the existence of an ideal deadbeat control, while HONN has been used to approximate the ideal control, and discrete Nussbaum gain has been employed to counter the lack of knowledge on control gain. The resulted adaptive NN control guarantees the SGUUB of all the closed-loop signals. The performance of the adaptive NN control has been investigated by simulation, which shows that the adaptive NN control using a discrete Nussbaum gain works as predicted in the analysis. The results in this paper can be further extended to another linearly parametrized approximator that also has the property as in (6), such as radius basis function (RBF) NN, fuzzy systems, polynomial, splines, and wavelet networks.

### APPENDIX PROOF OF LEMMA 4

*Proof:* Case i) According to the prerequisite that  $g_1 \leq |g(k)| \leq g_2$ ,  $g(k)$  is either strict positive or negative. Only proof with positive  $g(k)$  is given here and the proof with negative  $g(k)$  is omitted because they are quite similar. It should be noted that because  $\Delta x(k)$  is nonnegative, we have  $x(k) = x_s(k)$  and  $f(x_s(k)) = \pm x^{3/2}(k)$ .

First, let us consider that  $x(k)$  grows without bound. If the sign of  $s_N(x(k))$  changes infinite times, then the switching curve  $f(x_s(k)) = \pm x^{3/2}(k)$  will be crossed infinite number of times. Then, the first property in Definition 4 is satisfied. In the following, we prove that  $s_N(x(k))$  definitely changes its sign for infinite number of times if  $x(k)$  grows without bound. Suppose that  $s_N(x(k)) = 1$  remains positive in an interval  $\{l_1 \leq k \leq l_2\}$ , where  $x(l_1) > \delta_0$ , and noting that  $x(k) \geq 0$ , we have

$$\begin{aligned} S'_N(x(l_2)) &= \sum_{k=0}^{l_2} N'(x(k)) \Delta x(k) \\ &= c_1 + \sum_{k=l_1}^{l_2} x(k)g(k)\Delta x(k) \\ &\geq c_1 + g_1 \sum_{k=l_1}^{l_2} x(k)\Delta x(k) \end{aligned} \quad (53)$$

where  $c_1 = \sum_{k=0}^{l_1-1} N'(x(k))\Delta x(k)$ . It is noted that in (53), the inequality cannot be obtained without  $\Delta x(k) \geq 0$ . This is why the restriction  $\Delta x(k) \geq 0$  is indispensable.

Because  $x(k) > \delta_0 \geq \Delta x(k)$ ,  $\forall k \in \{k|l_1 \leq k \leq l_2\}$ , we have

$$\begin{aligned} \Delta \{x(k)\}^2 &= x^2(k+1) - x^2(k) \\ &= 2x(k)\Delta x(k) + \{\Delta x(k)\}^2 \\ &\leq 2x(k)\Delta x(k) + x(k)\Delta x(k) \\ &= 3x(k)\Delta x(k). \end{aligned} \quad (54)$$

Substituting (54) into (53), we have

$$S'_N(x(l_2)) \geq \frac{g_1}{3}x^2(l_2+1) - \frac{g_1}{3}x^2(l_1) + c_1 \quad (55)$$

which implies that when  $s_N(x(k)) = 1$ ,  $S'_N(x(l_2))$  increases at least as fast as  $(g_1/3)x^2(l_2+1)$  as  $l_2$  increases. Therefore, it is obvious that the switching curve  $f(x_s(k)) = x^{3/2}(k)$  will be crossed as  $l_2$  increases if  $x(k)$  is unbounded.

On the other hand, suppose that  $s_N(x(k)) = -1$  remains on the interval  $\{k|l_1 \leq k \leq l_2\}$ , then, by the similar approach, we have

$$\begin{aligned} S'_N(x(l_2)) &= \sum_{k=0}^{l_2} N'(x(k)) \Delta x(k) \\ &= c_1 - \sum_{k=l_1}^{l_2} x(k)g(k)\Delta x(k) \\ &\leq c_1 - g_1 \sum_{k=l_1}^{l_2} x(k)\Delta x(k) \\ &= -\frac{g_1}{3}x^2(l_2+1) + \frac{g_1}{3}x^2(l_1) + c_1. \end{aligned} \quad (56)$$

It implies  $S'_N(x(k))$  decreases at least as fast as  $-(g_1/3)x^2(l_2+1)$  when  $l_2$  increases so that the switching curve of  $f(x_s(k)) = -x^{3/2}(k)$  will always be crossed as  $l_2$  increases if  $x(k)$  is unbounded.

According to the above analysis, it is impossible for  $s_N(x(k))$  to keep its sign unchanged as  $x(k)$  grows unbounded. Therefore,  $s_N(x(k))$  will change infinite times as  $k \rightarrow \infty$ . It is equivalent to the fact that  $S_N(x(k))$  grows unbounded in both positive direction and negative direction as  $x(k)$  grows unbounded. By now, it is proved that the first property in Definition 4 is satisfied.

Second, let us consider that  $x(k)$  is bounded, i.e.,  $x(k) \leq \delta_1$ . Let us denote  $\lim_{k \rightarrow \infty} \sup\{x(k)\} = \bar{x}$ . Note that  $x(k)$  is a monotonic nondecreasing sequence, and we have  $x(k) \leq \bar{x}$ . According to the definition of  $N(x(k))$ , we have  $\lim_{k \rightarrow \infty} |N(x(k))| = \bar{x}$  and  $|N(x(k))| \leq \bar{x}, \forall k$ .

Then, it is easy to derive

$$\begin{aligned} |S'(x(k))| &= \left| \sum_{k'=0}^k g(k')N(x(k')) \Delta x(k') \right| \\ &\leq \sum_{k'=0}^k |g(k')| |N(x(k'))| \Delta x(k') \\ &\leq g_2 \bar{x} \sum_{k'=0}^k \Delta x \leq g_2 \bar{x}^2. \end{aligned} \quad (57)$$

Because the two properties in the definition of discrete Nussbaum gain are satisfied, it is concluded that  $g(k)N(x(k))$  is also a discrete Nussbaum gain.

Case ii) Noting that  $-\epsilon_0 \leq C(k) \leq \epsilon_0$  and  $\Delta x(k) \geq 0$ , then we have

$$\begin{aligned} S_N(x(k)) - \epsilon_0 x(k) - \epsilon_0 &\leq \sum_{k'=0}^k N'(x(k')) \Delta x(k') \\ &\leq S_N(x(k)) + \epsilon_0 x(k) + \epsilon_0 \end{aligned} \quad (58)$$

where  $S_N(x(k))$  is defined in (8). It is noted in (58) that the inequality will not hold without  $\Delta x(k) \geq 0$ . This is the reason why the restriction  $\Delta x(k) \geq 0$  is indispensable. According to the properties of discrete Nussbaum gain  $N(x(k))$ , when  $x(k)$  increases without bound, it is easy to obtain the following:

$$\begin{aligned} & \lim_{k \rightarrow \infty} \sup_{x(k) \geq \delta_0} \{S_N(x(k)) \pm \epsilon_0 x(k) \pm \epsilon_0\} \\ &= \lim_{k \rightarrow \infty} \sup_{x(k) \geq \delta_0} \left[ x(k) \left\{ \pm \epsilon_0 \pm \frac{\epsilon_0}{x(k)} + \frac{1}{x(k)} S_N(x(k)) \right\} \right] \\ &= +\infty \end{aligned} \quad (59)$$

and, similarly

$$\lim_{k \rightarrow \infty} \inf_{x(k) \geq \delta_0} \{S_N(x(k)) \pm \epsilon_0 x(k) \pm \epsilon_0\} = -\infty. \quad (60)$$

Then, from (58), we conclude that  $N'(x(k))$  satisfies the first property in Definition 4. When  $x(k)$  is bounded, from the property of  $N(x(k))$ , it is obvious that  $S_N(x(k))$  is bounded. Therefore, it is easy to see from (58) that  $N'(x(k))$  also satisfies the second property in Definition 4. This completes the proof. ■

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